Interpolation as Cut-Introduction

On the Computational Content of Craig-Lyndon Interpolation

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Abstract

Analyzing Maehara's method for proving Craig's interpolation theorem, we extract a "proof relevant" interpolation theorem for first-order LL in the sense that if π is a cut-free sequent proof of $A \vdash B$, we can find a formula C in the common vocabulary of A and B and proofs π_1, π_2 of $A \vdash C$ and $C \vdash B$ respectively such that π_1 composed with π_2 cut-reduces to π. As a direct corollary, we get similar proof relevant interpolation results for LJ and LK using linear translations. This refined interpolation is then rephrased in terms of *a cut-introduction process synthetizing the interpolant*.

Finally, we analyze the computational content of interpolation by proving an interpolation result for Curien and Herbelin's *Duality of Computation*.

2012 ACM Subject Classification Theory of computation → Proof theory; Theory of computation → Linear logic; Theory of computation → Lambda calculus

Keywords and phrases Classical Logic, Interpolation, Cut Elimination, Linear Logic, Sequent calculus, System L

Digital Object Identifier 10.4230/LIPIcs.FSCD.2025.32

Related Version *Color-blind-friendly and printable version*: https://www.irif.fr/_media/users/saurin/pub/interpolation_as_cut_introduction-cb.pdf

Funding Partially funded by the ANR project RECIPROG ANR-21-CE48-019.

Acknowledgements This study was motivated by a question raised by Hugo Férée and Sam van Gool during Scalp working group annual meeting at CIRM, France, in February 2023. I would like to thank them as well as Esaïe Bauer, Anupam Das, Abhishek De, Farzad Jafarrahmani, Hugo Herbelin, Roman Kuznets, Paul-André Melliès, Daniel Osorio and Elaine Pimentel for valuable discussions and suggestions. Many thanks to Adrien Deloro for uncountable discussions and references.

1 "– Why Not a Proof-Relevant Interpolation Theorem? – Introduce Cuts, Of Course!"

In the words of Solomon Feferman, *"though deceptively simple and plausible on the face of it, Craig's interpolation theorem (...) has proved to be a central logical property that has been used to reveal the deep harmony between the syntax and semantics of first order logic"* [17]. Indeed, Craig's interpolation (which states that in the predicate calculus, if $A \vdash B$, there exists a formula C built from the relation symbols occurring both in A and B such that $A \vdash C$ and $C \vdash B$) and its developments suggest far deeper connections between models and proofs than the simple correspondence between provability and validity given by Gödel completeness theorem. This could be argued to be in line with and pursue structural proof-theoretic proofs of Gödel completeness theorem such as Schütte proof [35] or the more recent analysis by Basaldella and Terui of completeness in Ludics [3, 2].

First of all, one should recall that while the original proof of interpolation by Craig [7, 8] was proof-theoretic as well as Maehara's method [25] its most striking applications were model-theoretic results whether we consider standard model-theoretic results that are consequences of interpolation, such as Beth definability theorem [4] or Robinson's consistency theorem [33] or if we consider modern uses of interpolation in model-checking [21, 27]. Among the proof-theoretical methods for proving interpolation, the success of Maehara's method is probably due to its wide applicability to a range of logics and proof-systems, from intuitionistic logic [28, 35] to modal logics [18, 24, 1, 36] or in infinitary logics and abstract model theory [17, 15].

While in most proof theory textbooks [20, 35, 39, 40] Craig's interpolation theorem is presented as an application of cut-elimination, one shall see here that it also has in fact much to do with cut-introduction, giving a proof-relevant and computational content to Interpolation theorem. This opens the way to an analysis of interpolation in terms of the denotational semantics of proofs and programs.

Contributions and organization of the paper. More precisely, after providing background on linear logic sequent calculus in Section 2, we state in Section 3 the following result for first-order linear logic (LL in the following): For any first-order LL formulas A, B, if π proves $A \vdash B$, there exists a formula C in the common vocabulary (that refers to the subset of the first-order language occurring in a formula, in terms of relation symbols, possibly taking into account polarity constraints as in Lyndon's interpolation) of A and B and proofs π_1, π_2 of $A \vdash C$ and $C \vdash B$ respectively such that $(\mathsf{Cut})(\pi_1, \pi_2)$, that is π_1 composed with π_2, is equivalent to π (by $=_{\mathsf{cut}}$ we refer to the congruence generated by the cut-reduction rules):

$$\dfrac{A \overset{\pi_1}{\vdash} C \qquad C \overset{\pi_2}{\vdash} B}{A \vdash B}\ (\mathsf{Cut}) \quad =_{\mathsf{cut}} \quad A \overset{\pi}{\vdash} B.$$

Interpolation can therefore be achieved while preserving the computational / denotational content of proofs, while factoring the computation through an interfacing, interpolant type made only of the base types used in both the input and output types.

This result is then extended to classical and intuitionistic logics simply by means of linear translations, which sheds an interesting light on the relationship between Lyndon and Craig's interpolation.

Then, by a further analysis of Maehara's method, we show in Section 4 that the interpolation process for a cut-free proof deriving $A \vdash B$ can be in fact decomposed in two phases: (i) an ascending phase which equips each sequent of π with a splitting is followed by (ii) a descending phase which solves the interpolation problem. This latter descending phase happens to be a cut-introduction phase which provides a proof of the proof-relevant interpolation result of the previous section. In particular, the resulting proof is, *by construction*, denotationally equal to π.

Finally, in Section 5, we consider the computational content of interpolation and address this question using the framework of the Duality of Computation introduced by Curien and Herbelin: we consider the linear system L, a term calculus with typing rules very close to that of the sequent calculus[1], and prove a computational interpolation result.

Related works. Surprisingly we could not find any occurrence in the literature analyzing Maehara's method in terms of cut-elimination (or rather, cut-introduction) even though *all ingredients* were there since Maehara's seminal work.

[1] System L can be viewed as achieving a Curry-Howard correspondence with sequent calculus proofs – while the λ-calculus achieves a correspondence with natural deduction proofs

On the other hand, another early proof-theoretic proof of interpolation theorem was proposed by Prawitz for natural deduction [31]. Just like for Maehara's method the strengthened, proof-relevant interpolation result was at hand in this work as well and Čubrić actually showed this in the setting of the simply typed λ-calculus as well as a corresponding factorization result for bicartesian closed categories in the early 90s [9, 10]. Sadly, Čubrić's paper as well as his PhD thesis supervised by Makkai, received too little attention and very few following works refer to his results: we could only find less than 10 references to these works among them only three truly consider the interpolation aspect [16, 26, 23][2]. We hope that the present work can contribute to foster interest in Čubrić results.

Another related work is that of Carbone [6] where she establishes a strengthened form of Maehara's interpolation paying great attention to the ancestor relation (formulated in terms of flow graphs in that work) which allows her to get bounds on the complexity of the interpolant but did not lead her to a study of proof-relevant interpolation, invariance by cut-elimination nor interpolation as cut-introduction. Only few works consider interpolation in (fragments of) linear logic, starting with Roorda [34]. In the framework of the calculus of structure, Straßburger proves a decomposition theorem for MELL [37] that is advocated to correspond to an interpolation theorem and may have a more fine-grained proof-theoretical content. More recently, several papers investigated and formalized interpolation theorems in substructural logics, including exponential-free linear logic [5, 14, 32].

2 Background on LL proof theory

In the following, we provide the necessary background on first-order LL.

As usual, we assume a first-order language $\mathcal{L}$ (without equality nor function symbols). We assume the set of atomic formulas to be equipped with an involution, inducing a partitioning between ***positive atomic formulas***, written a, and ***negative atomic formulas***, written $a^\perp$. We introduce a language of first-order LL formulas:

▶ **Definition 1.** *The grammar of first-order* LL *formulas is defined inductively as:*

$$\begin{array}{rllllllll} F & ::= & a^\perp & \mid \bot & \mid \top & \mid F \parr F & \mid F \& F & \mid \forall x.F & \textit{\textbf{negative}}\ \textbf{MALL}\ \textit{\textbf{formulas}} \\ & \mid & a & \mid 1 & \mid 0 & \mid F \otimes F & \mid F \oplus F & \mid \exists x.F & \textit{\textbf{positive}}\ \textbf{MALL}\ \textit{\textbf{formulas}} \\ & \mid & ?F & \mid !F & & & & & \textit{\textbf{exponential formulas}} \end{array}$$

Linear negation *is defined as usual as an involution on* LL *formulas (changing $a^\perp$, $\bot$, $\top$, $\parr$, $\&$, $\forall$, $?$ into a, 1, 0, $\otimes$, $\oplus$, $\exists$, $!$ respectively). A one-sided* LL *sequent is an ordered list of* LL *formulas usually written $\vdash \Gamma$ to emphasize the sequent* **judgments***.*

▶ **Definition 2** (LL sequent calculus)**.** *The inference rules of* LL *sequent calculus are given in Figure 1. The inferences of Figure 1 show a relation between conclusion and premises formulas, the* ancestor relation *(or sub-occurrence relation) that is extended by transitivity to formulas of non consecutive sequents of a derivation. An* LL **proof** *of a sequent $\vdash \Gamma$ is a finite tree rooted in $\vdash \Gamma$ generated by the rules of Figure 1.*

[2] Matthes [26] extends Čubrić' results to a natural deduction with general elimination rules and a corresponding term calculus while Kanazawa [23] considers interpolation in purely implicational fragments of intuitionistic logic and finds workarounds for the lack of interpolation in this setting. On the way, he considers various sequent calculi and proves a result which has some similarities with our result but is weaker both in terms of the logical language – which is restricted to implicative LJ– and of the characterization of the equivalence between the interpolating proofs and the original proof.

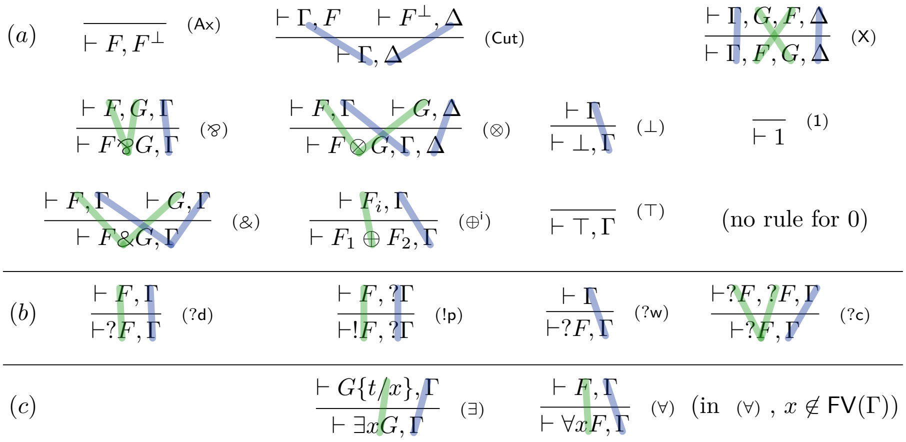

(a)
⊢ F, F^⊥ (Ax)
⊢ Γ, F ⊢ F^⊥, Δ / ⊢ Γ, Δ (Cut)
⊢ Γ, G, F, Δ / ⊢ Γ, F, G, Δ (X)
⊢ F, G, Γ / ⊢ F⅋G, Γ (⅋)
⊢ F, Γ ⊢ G, Δ / ⊢ F⊗G, Γ, Δ (⊗)
⊢ Γ / ⊢ ⊥, Γ (⊥)
⊢ 1 (1)
⊢ F, Γ ⊢ G, Γ / ⊢ F&G, Γ (&)
⊢ F_i, Γ / ⊢ F_1 ⊕ F_2, Γ (⊕^i)
⊢ ⊤, Γ (⊤)
(no rule for 0)
(b)
⊢ F, Γ / ⊢?F, Γ (?d)
⊢ F, ?Γ / ⊢!F, ?Γ (!p)
⊢ Γ / ⊢?F, Γ (?w)
⊢?F, ?F, Γ / ⊢?F, Γ (?c)
(c)
⊢ G{t/x}, Γ / ⊢ ∃xG, Γ (∃)
⊢ F, Γ / ⊢ ∀xF, Γ (∀) (in (∀), x ∉ FV(Γ))

Figure 1 (a) Propositional MALL Inferences; (b) LL Exponential Inferences; (c) First-order Inferences.

▶ Remark 3. The ancestor relation defined above is (implicitly) used in designing a cut-reduction system and plays a crucial role in expressing the validity condition for non-wellfounded and circular proofs as well as in the extension of Maehara's method we will show next, in order to propagate sequent splittings from conclusions to premises.

To state Craig-Lyndon interpolation, we need to make clear the required notion to express the interpolation condition on the vocabulary:

▶ **Definition 4** ((polarized) vocabulary of a formula). *Given an* LL *formula* F*,* $\mathbb{V}\mathsf{oc}(F)$ *is the set of all predicate symbols occurring in* F **together with their linear negations***, while* $\mathbb{V}\mathsf{oc}^+(F)$ *(resp.* $\mathbb{V}\mathsf{oc}^-(F)$*) is the set of positive (resp. negative) predicate symbols occurring in* F*. This is naturally lifted to lists of formulas.*

In the present paper, we shall heavily rely on LL cut-reduction rules (and their symmetric, cut-introduction rules). The cut-reduction relation (resp. its reflexive and transitive and its reflexive, transitive and symmetric closures) is generically written $\longrightarrow_{\mathsf{cut}}$ (resp. $\longrightarrow^{\star}_{\mathsf{cut}}$ and $=_{\mathsf{cut}}$).

3 Proof-relevant interpolation theorem

The following proof-relevant interpolation theorem can be established by refining Maehara's proof in order to take seriously into account the relations, not only between the interpolation sequents, but also between the interpolating proofs.

▶ **Theorem 5.** *Let* Γ, Δ *be lists of* LL *formulas and* $\pi \vdash \Gamma, \Delta$ *cut-free. There exists a* LL *formula* C *such that* $\mathbb{V}\mathsf{oc}^+(C) \subseteq \mathbb{V}\mathsf{oc}^-(\Gamma) \cap \mathbb{V}\mathsf{oc}^+(\Delta)$ *and* $\mathbb{V}\mathsf{oc}^-(C) \subseteq \mathbb{V}\mathsf{oc}^+(\Gamma) \cap \mathbb{V}\mathsf{oc}^-(\Delta)$ *and two cut-free proofs* π_1, π_2 *of* $\vdash \Gamma, C$ *and* $\vdash C^\perp, \Delta$ *respectively such that*

$$\frac{\overset{\pi_1}{\vdash \Gamma, C} \qquad \overset{\pi_2}{\vdash C^\perp, \Delta}}{\vdash \Gamma, \Delta}\ (\mathsf{Cut}) \quad \longrightarrow^{\star}_{\mathsf{cut}} \quad \pi.$$

In the following sections, we will follow two approaches which we consider to be more instructive from the computer science logic viewpoint, as they reveal the computational content of proof-relevant interpolation. We will proceed by "introducing cuts" so that we preserve the denotational equivalence of the *interpolated proof* with its interpolation (proofs).

But first, let us extend the proof-relevant interpolation result to LK and LJ. These results could be obtained directly, by refining Maehara's method or by following the cut-introduction or computational approach of the following section. On the other hand, one can directly deduce this result by using the linear embeddings of LK and LJ [19, 13] as a corollary of the above theorem.

▶ **Corollary 6.** *Let $\Gamma, \Gamma', \Delta, \Delta'$ be lists of* LK *formulas (resp. of* LJ *formulas, with Δ being empty and Δ' of length at most 1) and π be a cut-free* LK *(resp.* LJ*) proof of $\Gamma, \Gamma' \vdash \Delta, \Delta'$.*

There exists a LK *(resp.* LJ*) formula C such that $\mathbb{V}\mathsf{oc}^+(C) \subseteq \mathbb{V}\mathsf{oc}^+(\Gamma, \Delta) \cap \mathbb{V}\mathsf{oc}^+(\Gamma', \Delta')$ and $\mathbb{V}\mathsf{oc}^-(C) \subseteq \mathbb{V}\mathsf{oc}^-(\Gamma, \Delta) \cap \mathbb{V}\mathsf{oc}^-(\Gamma', \Delta')$ and two cut-free* LK *(resp.* LJ*) proofs π_1, π_2 of $\Gamma \vdash C, \Delta$ and $\Gamma', C \vdash \Delta'$ respectively such that* $\dfrac{\overset{\pi_1}{\Gamma \vdash C, \Delta} \qquad \overset{\pi_2}{\Gamma', C \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ {\scriptstyle (\mathsf{Cut})} \quad \longrightarrow^{\star}_{\mathsf{cut}} \ \pi.$

Proof. We sketch this here in the case of LK.

First, it is a matter of a simple check that proof-relevant interpolation presented in the above theorem also holds for the two-sided LL sequent calculus.

Second, a cut-free LK proof π (it would be similar for LJ) can be decorated with exponential modalities and inferences in order to turn it into a cut-free LL proof π'.

After interpolating this proof (obtaining I', π'_1 and π'_2), one can erase the linear information of the interpolants and the two interpolating proofs (that is, taking the classical *skeleton* of the proofs) and get back a pair of LK (resp. LJ) proofs π_1, π_2 together with a formula I in LK (resp. LJ) that lives in the appropriate vocabulary.

The properties of the linear embeddings ensure that the skeleton of a cut-free proof obtained from $\mathsf{Cut}(\pi'_1, \pi'_2)$ can be obtained by eliminating cuts from $\mathsf{Cut}(\pi_1, \pi_2)$. ◀

4 Interpolation as cut-introduction

We will now show how the proof-relevant interpolation theorem stated in the previous section can be established by exploiting the dynamics of cut-introduction that is, more precisely, *how the synthesis of the interpolant is in fact a cut-introduction process.*

For this, we need to analyze and refine Maehara's proof method.

4.1 Refining Maehara's method

The usual statement and proof method for interpolation, made more informative in the previous section, actually obfuscate the fact that the interpolating formula and proofs are synthetized using a cut-introduction mechanism: this is due to the structure of the inductive reasoning used to establish the interpolation theorem under Maehara's method. Indeed, the inductive structure of the reasoning amounts to inspecting the structure of the sequent proofs until one reaches the base cases (the axiom and unit cases), after which the call to the induction hypotheses synthetize the interpolant.

Analyzing what happens in constructing the interpolating formula and proofs, for a cut-free proof π, is made clearer by structuring the process in two phases, a bottom-up phase and a top-down phase:

Ascending phase. This first phase consists in traversing the initial proof π bottom-up, from root (conclusion) to leaves (axioms), and building, for each visited sequent Γ, a splitting (Γ', Γ'') inherited from the splitting of the conclusion of the proof by the ancestor relation.

In this way, each node of the proof is ultimately decorated with some additional information on how to split the sequent labelling the node. We will use the red-blue colors throughout the paper to represent such splittings, and from now on, enrich sequents with such splitting information.

Ultimately, for each logical axiom rule $\vdash A^\perp, A$, we are in one of the following situations: (i) $(\{A^\perp, A\}, \emptyset)$; (ii) $(\{A^\perp\}, \{A\})$; (iii) $(\{A\}, \{A^\perp\})$; (iv) $(\emptyset, \{A^\perp, A\})$ which are summed up with the color code as: $\vdash A^\perp, A$; $\vdash A^\perp, A$; $\vdash A^\perp, A$; $\vdash A^\perp, A$. (and similarly for each axiom corresponding to some unit, $\top$ or 1.) This corresponds to the various base cases of the inductive proof of the previous section.

Once every axiom has been reached, we switch to the descending phase, traversing again the proof, top to bottom, in an asynchronous manner.

Descending phase. Equipped with the sequents splitting information one shall now apply cut-introduction rules to axioms, progressively moving the cuts down and merging them in such a way, ultimately, to reach the root sequent of the original proof. We call *active* a sequent such that all its premises are concluded with cut inferences. (Initially, since π is cut-free, only the conclusions of logical axioms or 0-ary unit rules are *active sequents*.)

We apply cut-introductions to active sequents, maintaining the following two invariants:

- when a sequent is active with splitting (Γ', Γ''), the cut formulas of its premises are interpolants for the premise sequents wrt. their splitting (Note that this condition is trivially satisfied initially since the active axioms have no premise).
- when an inference (r) has conclusion $\mathcal{S}$ which is active, we apply a (sequence of) cut-introduction step(s) on this inference, in such a way that (i) $\mathcal{S}$ becomes the conclusion of the introduced cut and (ii) the premises of this cut correspond to the splitting associated with sequent $\mathcal{S}$.

The descending phase therefore terminates when the cut reaches the root.

Sequencing these phases, $\vdash \overset{\pi}{\Gamma}, \Delta$ is interpolated as some $\dfrac{\vdash \overset{\pi^L}{\Gamma}, I \qquad \vdash \overset{\pi^R}{I^\perp}, \Delta}{\vdash \Gamma, \Delta}\,(\mathsf{Cut})$.

Ultimately, one therefore builds a cut formula I and two cut-free proofs π_1 and π_2 such that I is an interpolant of the conclusion sequent with respect to the original splitting and such that $(\mathsf{Cut})(\pi_1, \pi_2) \longrightarrow^\star_{\mathsf{cut}} \pi$, a condition which is satisfied *by construction.*

To state precisely the properties of the two phases, we introduce the following notions:

- A ***decorated proof*** is an LL proof such that each sequent is equipped with a splitting.
- A ***coherent*** decorated proof is such that for each node, the splitting of the conclusion and of its premises is coherent with respect to the ancestor relation: a formula belonging to the left (resp. right) component of the splitting has all its ancestors belonging to the left (resp. right) component of the splitting. We shall now consider only such coherent proofs.

With the above notions, the following lemma is clear by induction on the proof:

▶ **Lemma 7.** *For any* LL *proof and any splitting of its conclusion sequent, the ascending phase terminates with a coherent decorated proof.*

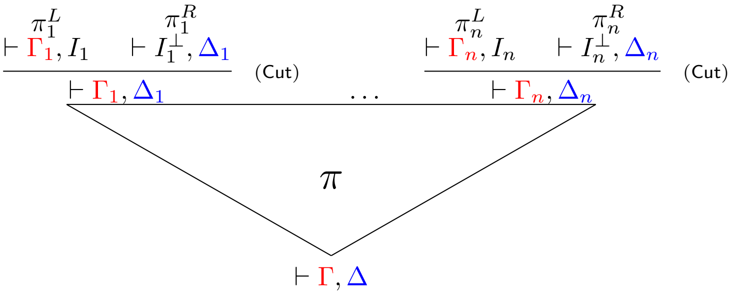


Figure 2 A PRIS– Proof-relevant Interpolation Situation.

4.2 PRIS: Proof-Relevant Interpolation Situation

We shall now focus on the top-down phase and, in order to make formal the discussion above, we shall introduce a useful class of coherent decorated proofs, called PRIS for proof-relevant interpolation situation; they are essentially partially solved interpolation problems:

▶ **Definition 8** (Proof-relevant Interpolation Situation). *A* PRIS *for* (Γ, Δ) *is the data of:*

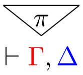
- **the goal**, *that is a cut-free* LL *proof* π *of conclusion* $\vdash \Gamma, \Delta$ *and with* $n \geq 0$ *open premises* $(\vdash \Gamma_i, \Delta_i)_{1 \leq i \leq n}$ *such that for each* $1 \leq i \leq n$ *the formulas in* Γ_i *(resp.* Δ_i*) are ancestors of formulas in* Γ *(resp. of* Δ*);*
- **the partial interpolants,** *that is for each* $1 \leq i \leq n$*, a formula* I_i *st.* $\mathbb{V}\mathsf{oc}^+(I_i) \subseteq \mathbb{V}\mathsf{oc}^-(\Gamma_i) \cap \mathbb{V}\mathsf{oc}^+(\Delta_i)$ *and* $\mathbb{V}\mathsf{oc}^-(I_i) \subseteq \mathbb{V}\mathsf{oc}^+(\Gamma_i) \cap \mathbb{V}\mathsf{oc}^-(\Delta_i)$ *and;*
- **the partial solutions**, *that is, for each* $1 \leq i \leq n$*, derivations* π_i^L *(resp.* π_i^R*) of conclusion* $\vdash \Gamma_i, I_i$ *(resp.* $\vdash I_i^\perp, \Delta_i$*).*

A PRIS as given by the above definition will be graphically represented as in Figure 2.

In order to make clear our methodology of exploiting PRIS, we define some special cases of PRIS which have a specific role in the forthcoming development:

▶ **Definition 9** (Initial, solved, and elementary PRIS). *We define the following cases of* PRIS*:*

- *An initial* PRIS *is a* PRIS *with* $n = 0$*. It thus has the following form:*



with π *being a cut-free coherent decorated proof. This is the initial situation of an interpolation problem.*

- *A solved* PRIS *is a* PRIS *with* $n = 1$ *and* π *being reduced to the trivial derivation constituted only of an open premise node* $\vdash \Gamma, \Delta$*, of the form:*

$$\pi = \dfrac{\overset{\pi_1^L}{\vdash \Gamma, I_1} \qquad \overset{\pi_1^R}{\vdash I_1^\perp, \Delta}}{\vdash \Gamma, \Delta}\ (\mathsf{Cut}) \ .$$

A solved PRIS *therefore corresponds to the solution of an interpolation problem, with* I_1 *being the interpolant and* π_1^L, π_1^R *being the interpolating proofs.*

- *An elementary* PRIS *is a* PRIS *such that its goal is reduced to an instance of a single n-ary inference rule* (r)*, together with n open premises: it is a* PRIS *where there remains a single inference rule to solve in order to obtain a solution to an interpolation problem. It thus has the form:*

$$\pi = \dfrac{\dfrac{\overset{\pi_1^L}{\vdash \Gamma_1, I_1} \qquad \overset{\pi_1^R}{\vdash I_1^\perp, \Delta_1}}{\vdash \Gamma_1, \Delta_1}\,(\mathsf{Cut}) \qquad \dots \qquad \dfrac{\overset{\pi_n^L}{\vdash \Gamma_n, I_n} \qquad \overset{\pi_n^R}{\vdash I_n^\perp, \Delta_n}}{\vdash \Gamma_n, \Delta_n}\,(\mathsf{Cut})}{\vdash \Gamma, \Delta}\,(\mathsf{r})$$

4.3 Solving PRIS

Our proof-relevant interpolation problem can therefore be rephrased as: *How to relate initial and solved* PRIS *via cut-introduction?* The crucial step lies in the following lemma:

▶ **Lemma 10.** *For any n-ary inference rule* (r) *and any elementary* PRIS π *of the form:*

$$\dfrac{\dfrac{\dfrac{\pi_1^L}{\vdash \Gamma_1, I_1} \qquad \dfrac{\pi_1^R}{\vdash I_1^\perp, \Delta_1}}{\vdash \Gamma_1, \Delta_1}\,(\mathsf{Cut}) \qquad \dots \qquad \dfrac{\dfrac{\pi_n^L}{\vdash \Gamma_n, I_n} \qquad \dfrac{\pi_n^R}{\vdash I_n^\perp, \Delta_n}}{\vdash \Gamma_n, \Delta_n}\,(\mathsf{Cut})}{\vdash \Gamma, \Delta}\,(\mathsf{r})$$

there exist I, π^L, π^R such that $\pi' = \dfrac{\overset{\pi^L}{\vdash \Gamma, I} \qquad \overset{\pi^R}{\vdash I^\perp, \Delta}}{\vdash \Gamma, \Delta}\,(\mathsf{Cut})$ *is a solved* PRIS *and* $\pi \longleftarrow^{\star}_{\mathsf{cut}} \pi'$*.*

Since, each application of the above lemma reduces by one inference the size of the *goal* part of an interpolation situation, each sequence of cut-introduction obtained by application of Lemma 10 terminates in a solved PRIS, that is an interpolated proof. As a conclusion, we obtain the following corollary:

▶ **Corollary 11.** *Any initial* PRIS *can be reduced, by cut-expansions, to a solved* PRIS*.*

As a conclusion, we obtain the expected main theorem (Theorem 5) by working in a *reversed way*, using cut-introduction:

▶ **Theorem 12.** *Let A, B be* LL *formulas and π be a cut-free* LL *proof of $A \vdash B$.*

There exists a LL *formula C such that $\mathbb{V}\mathsf{oc}^+(C) \subseteq \mathbb{V}\mathsf{oc}^+(A) \cap \mathbb{V}\mathsf{oc}^+(B)$ and $\mathbb{V}\mathsf{oc}^-(C) \subseteq \mathbb{V}\mathsf{oc}^-(A) \cap \mathbb{V}\mathsf{oc}^-(B)$ and two cut-free* LL *proofs π_1, π_2 of $A \vdash C$ and $C \vdash B$ respectively such that* $\dfrac{\overset{\pi_1}{A \vdash C} \qquad \overset{\pi_2}{C \vdash B}}{A \vdash B}\,(\mathsf{Cut}) \quad \longrightarrow^{\star}_{\mathsf{cut}} \pi.$

4.4 Proof of Lemma 10

We now prove the main lemma (missing cases are treated in Section A) which proceed by a case analysis on the elementary PRIS under consideration.

4.4.1 Axiom case

- If $\pi = \dfrac{}{\vdash F, F^\perp}\,(\mathsf{Ax})$, take $I = F^\perp$, $\pi_1^l = \dfrac{}{\vdash F, F^\perp}\,(\mathsf{Ax})$ and $\pi_1^r = \dfrac{}{\vdash F, F^\perp}\,(\mathsf{Ax})$.
  The cut between π_1^l and π_1^r reduces to π by one cut-axiom reduction step.
- If $\pi = \dfrac{}{\vdash F, F^\perp}\,(\mathsf{Ax})$, one takes $I = \bot$, $\pi_1^l = \dfrac{\pi}{\vdash F, F^\perp, \bot}\,(\bot)$ and $\pi_1^r = \dfrac{}{\vdash 1}\,(1)$.
  The cut of π_1^l and π_1^r reduces to π by a key $1/\bot$ case.
- The two other cases are treated similarly.

4.4.2 Logical rules

We analyze the possible cases for a logical rule involved in an elementary PRIS. Note that in each case, the principal formula may be part of the left or right part of the splitting; we treat only one case each time since the other is symmetrical be taking the dual interpolant and exchanging π^L and π^R.

If the last rule is $(\parr)$, ie. if $\pi = \dfrac{\dfrac{\overset{\pi_1^L}{\vdash \Gamma, I} \qquad \overset{\pi_1^R}{\vdash I^\perp, \Delta, A, B}}{\vdash \Gamma, \Delta, A, B}\,(\mathsf{Cut})}{\vdash \Gamma, \Delta, A \parr B}\,(\parr)$ then taking $I' = I$, $\pi^L = \pi_1^L$ and $\pi^R = \dfrac{\overset{\pi_1^R}{\vdash I^\perp, \Delta, A, B}}{\vdash I^\perp, \Delta, A \parr B}\,(\parr)$ we obtain a solved PRIS π' such that $\pi \longleftarrow_{\mathsf{cut}} \pi'$ by a commutative reduction of (Cut).

If the last rule is $(\otimes)$, ie. if $\pi = \dfrac{\dfrac{\overset{\pi_1^L}{\vdash \Gamma_1, I_1} \qquad \overset{\pi_1^R}{\vdash I_1^\perp, \Delta_1, A}}{\vdash \Gamma_1, \Delta_1, A}\,(\mathsf{Cut}) \qquad \dfrac{\overset{\pi_2^L}{\vdash \Gamma_2, I_2} \qquad \overset{\pi_2^R}{\vdash I_2^\perp, \Delta_2, B}}{\vdash \Gamma_2, \Delta_2, B}\,(\mathsf{Cut})}{\vdash \Gamma_1, \Gamma_2, \Delta_1, \Delta_2, A \otimes B}\,(\otimes)$,

then setting $I = I_1 \otimes I_2$, $\pi^L = \dfrac{\overset{\pi_1^L}{\vdash \Gamma_1, I_1} \qquad \overset{\pi_2^L}{\vdash \Gamma_2, I_2}}{\vdash \Gamma_1, \Gamma_2, I_1 \otimes I_2}\,(\otimes)$, $\pi^R = \dfrac{\dfrac{\overset{\pi_1^R}{\vdash I_1^\perp, \Delta_1, A} \qquad \overset{\pi_2^R}{\vdash I_2^\perp, \Delta_2, B}}{\vdash I_1^\perp, I_2^\perp, \Delta_1, \Delta_2, A \otimes B}\,(\otimes)}{\vdash (I_1 \otimes I_2)^\perp, \Delta_1, \Delta_2, A \otimes B}\,(\parr)$ one gets a solved PRIS π' such that $\pi \longleftarrow_{\mathsf{cut}}^{\star} \pi'$ by a commutative reduction of (Cut) and a key $(\otimes)/(\parr)$ case.

If the last rule is $(!\mathsf{p})$, that is $\pi = \dfrac{\dfrac{\overset{\pi_1^L}{\vdash F, ?\Gamma, I_1} \qquad \overset{\pi_1^R}{\vdash I_1^\perp, ?\Delta}}{\vdash F, ?\Gamma, ?\Delta}\,(\mathsf{Cut})}{\vdash !F, ?\Gamma, ?\Delta}\,(!\mathsf{p})$, then by setting $I = ?I_1$,

$\pi^L = \dfrac{\dfrac{\overset{\pi_1^L}{\vdash F, ?\Gamma, I_1}}{\vdash F, ?\Gamma, ?I_1}\,(?\mathsf{d})}{\vdash !F, ?\Gamma, ?I_1}\,(!\mathsf{p})$ and $\pi^R = \dfrac{\overset{\pi_1^R}{\vdash I_1^\perp, ?\Delta}}{\vdash !I_1^\perp, ?\Delta}\,(!\mathsf{p})$, one gets that π cut-expands to π':

$$\dfrac{\overset{\pi_1^L}{\vdash !F, ?\Gamma, ?I_1} \qquad \overset{\pi_1^R}{\vdash !I_1^\perp, ?\Delta}}{\vdash !F, ?\Gamma, ?\Delta}\,(\mathsf{Cut}) \quad \longrightarrow_{\mathsf{cut}} \quad \dfrac{\dfrac{\dfrac{\overset{\pi_1^L}{\vdash F, ?\Gamma, I_1}}{\vdash F, ?\Gamma, I}\,(?\mathsf{d}) \qquad \dfrac{\overset{\pi_1^R}{\vdash I_1^\perp, ?\Delta}}{\vdash I^\perp, ?\Delta}\,(!\mathsf{p})}{\vdash F, ?\Gamma, ?\Delta}\,(\mathsf{Cut})}{\vdash !F, ?\Gamma, ?\Delta}\,(!\mathsf{p}) \quad \longrightarrow_{\mathsf{cut}} \pi.$$

If the last rule is $(?\mathsf{c})$, ie. $\pi = \dfrac{\dfrac{\overset{\pi_1^L}{\vdash ?F, ?F, \Gamma, I_1} \qquad \overset{\pi_1^R}{\vdash I_1^\perp, \Delta}}{\vdash ?F, ?F, \Gamma, \Delta}\,(\mathsf{Cut})}{\vdash ?F, \Gamma, \Delta}\,(?\mathsf{c})$, by setting $I = I_1$, $\pi^R = \pi_1^R$ and $\pi^L = \dfrac{\overset{\pi_1^L}{\vdash ?F, ?F, \Gamma, I_1}}{\vdash ?F, \Gamma, I}\,(?\mathsf{c})$ one gets that π cut-expands to the solved PRIS π':

$$\pi' = \dfrac{\overset{\pi^L}{\vdash ?F, \Gamma, I} \qquad \overset{\pi^R}{\vdash I^\perp, \Delta}}{\vdash ?F, \Gamma, \Delta}\,(\mathsf{Cut}) \quad \longrightarrow_{\mathsf{cut}} \quad \dfrac{\dfrac{\overset{\pi_1^L}{\vdash ?F, ?F, \Gamma, I_1} \qquad \overset{\pi_1^L}{\vdash I_1^\perp, \Delta}}{\vdash ?F, ?F, \Gamma, \Delta}\,(\mathsf{Cut})}{\vdash ?F, \Gamma, \Delta}\,(?\mathsf{c}) \quad = \pi.$$

If the last rule is $(\exists)$, ie. $\pi = \cfrac{\cfrac{\overset{\pi_1^L}{\vdash F\{y/x\},\Gamma,I_1} \qquad \overset{\pi_1^R}{\vdash {I_1}^{\perp},\Delta}}{\vdash F\{y/x\},\Gamma,\Delta}\,(\mathsf{Cut})}{\vdash \exists xF,\Gamma,\Delta}\,(\exists)$ with the vocabulary condition $\mathbb{V}\mathsf{oc}(I_1) \subseteq \mathbb{V}\mathsf{oc}(F\{y/x\},\Gamma) \cap \mathbb{V}\mathsf{oc}(\Delta)$. In this case, note that we treat only the case of a FO language containing no function symbols. We reason by case on whether y occurs in Γ,Δ:

- If y occurs in both, then we simply take $I = I_1$ as interpolant, $\pi^L = \cfrac{\pi_1^L}{\vdash \exists xF,\Gamma}\,(\exists)$ and $\pi^R = \pi_1^R$. Since $\mathbb{V}\mathsf{oc}(I) = \mathbb{V}\mathsf{oc}(I_1) \subseteq \mathbb{V}\mathsf{oc}(F,\Gamma) \cap \mathbb{V}\mathsf{oc}(\Delta) = \mathbb{V}\mathsf{oc}(\exists xF,\Gamma) \cap \mathbb{V}\mathsf{oc}(\Delta)$, we have that $\pi' = \cfrac{\overset{\pi^L}{\vdash \exists xF,\Gamma,I} \qquad \overset{\pi^R}{\vdash I^{\perp},\Delta}}{\vdash \exists xF,\Gamma,\Delta}\,(\mathsf{Cut})$ is a solved PRIS to which π cut-expands via a cut-commutation rule.
- If y occurs in Γ but not in Δ, then we set $I = \exists yI_1$, $\pi^L = \cfrac{\cfrac{\pi_1^L}{\vdash \exists xF,\Gamma,I_1}\,(\exists)}{\vdash \exists xF,\Gamma,\exists yI_1}\,(\exists)$ and $\pi^R = \cfrac{\overset{\pi_1^R}{\vdash {I_1}^{\perp},\Delta}}{\vdash \forall y{I_1}^{\perp},\Delta}\,(\forall)$ we have that $\pi' = \cfrac{\overset{\pi^L}{\vdash \exists xF,\Gamma,I} \qquad \overset{\pi^R}{\vdash I^{\perp},\Delta}}{\vdash \exists xF,\Gamma,\Delta}\,(\mathsf{Cut})$ is a solved PRIS to which π cut-expands via a cut-commutation rule and a key $(\exists)/(\forall)$ rule:

$$\pi' \longrightarrow_{\mathsf{cut}} \cfrac{\cfrac{\overset{\pi_1^L}{\vdash F\{y/x\},\Gamma,I_1}}{\vdash \exists xF,\Gamma,I_1}\,(\exists) \qquad \pi_1^R}{\vdash \exists xF,\Gamma,\Delta}\,(\mathsf{Cut}) \longrightarrow_{\mathsf{cut}} \cfrac{\cfrac{\overset{\pi_1^L}{\vdash F\{y/x\},\Gamma,I_1} \qquad \overset{\pi_1^R}{\vdash {I_1}^{\perp},\Delta}}{\vdash F\{y/x\},\Gamma,\Delta}\,(\mathsf{Cut})}{\vdash \exists xF,\Gamma,\Delta}\,(\exists)\ .$$

- If y occurs in Δ but not in Γ, then we set $I = \forall yI_1$, $\pi^L = \cfrac{\cfrac{\overset{\pi_1^L}{\vdash F\{y/x\},\Gamma,I_1}}{\vdash \exists xF,\Gamma,I_1}\,(\exists)}{\vdash \exists xF,\Gamma,\forall yI_1}\,(\forall)$ and $\pi^R = \cfrac{\overset{\pi_1^R}{\vdash {I_1}^{\perp},\Delta}}{\vdash \exists y{I_1}^{\perp},\Delta}\,(\exists)$. One gets that $\pi' = \cfrac{\overset{\pi^L}{\vdash \exists xF,\Gamma,\forall yI_1} \qquad \overset{\pi^R}{\vdash \exists y{I_1}^{\perp},\Delta}}{\vdash \exists xF,\Gamma,\Delta}\,(\mathsf{Cut})$ is a solved PRIS to which π cut-expands:

$$\pi' \longrightarrow_{\mathsf{cut}} \cfrac{\cfrac{\overset{\pi_1^L}{\vdash F\{y/x\},\Gamma,I_1}}{\vdash \exists xF,\Gamma,I_1}\,(\exists) \qquad \overset{\pi_1^R}{\vdash {I_1}^{\perp},\Delta}}{\vdash \exists xF,\Gamma,\Delta}\,(\mathsf{Cut})$$

$$\longrightarrow_{\mathsf{cut}} \cfrac{\cfrac{\overset{\pi_1^L}{\vdash F\{y/x\},\Gamma,I_1} \qquad \overset{\pi_1^R}{\vdash {I_1}^{\perp},\Delta}}{\vdash F\{y/x\},\Gamma,\Delta}\,(\mathsf{Cut})}{\vdash \exists xF,\Gamma,\Delta}\,(\exists) = \pi.$$

Other cases are treated similarly.

4.5 Some consequences and extensions

In the same way as in the previous section, a similar development of LK-PRIS and LJ-PRIS can be done, either from scratch or by means of linear embeddings which relates linear PRIS to classicial or intuitionistic ones, this results in the following proposition:

▶ **Proposition 13.** *Any initial* LK*-PRIS (resp.* LJ*-PRIS) can be reduced, by cut-expansions, to a solved* LK*-PRIS (resp. solved* LJ*-PRIS).*

▶ Remark 14. In fact, the above theorem does not rely on the cut-freeness of π, only the notion of interpolant does: the cut-introduction performed in solving PRIS can indeed be extended to handle the cut-case, as long as one relaxes the conditions on the language for the interpolant. This suggests an immediate generalization of proof-relevant Craig and Lyndon's interpolation for proofs with cuts in which each (pair of) cut-formula is assigned to the left/right component of a splitting (in a consistent way for a given cut inference): the language of the interpolant is then constructed by taking into account the language of the cut-formulas depending on the choice of splitting. In that case, an additional degree of freedom appears in the choice of the interpolant when assigning each cut to a component of the splitting: strategies for optimizing the language of the interpolant could therefore be investigated.

▶ Remark 15. Complementing on the previous remark, analytic cuts (that is cut formulas that are subformula of some formula of the conclusion sequent) are of particular interest here. Indeed, the above described procedure to work with PRIS containing cuts can be readily applied to proofs with analytic cuts.

In a similar way, it would be interesting to consider extending our procedure to proofs with cuts which are globally analytic (ie. simply requiring that every cut-formula is a subformula of some formula in the conclusion sequent). While the vocabulary constraint on interpolant *is locally violated* when encountering a globally-analytic cut, it is globally satisfied when reaching a solved PRIS. Other weakenings on the conditions on cut-formulas car be considered, such as the semi-analyticity conditions by Jalali and Tabatakai [38].

5 On the computational significance of the result: Interpolating System L

Stated purely in terms of interpolating validity judgements, interpolation is not very meaningful computationally of course. On the other hand, the refined statement investigated in the present paper opens new perspectives: the interpolant formula can be viewed as a an interface type I through which a computation can from data type A to data type B be factored via type I. Let us focus on intuitionistic statement (say, in ILL or LJ):

If π proves $\Gamma, \Delta \vdash C$, there exists some formula I such that $\mathbb{V}\mathsf{oc}(I) \subseteq \mathbb{V}\mathsf{oc}(\Gamma) \cap \mathbb{V}\mathsf{oc}(\Delta, C)$ and proofs π_1, π_2 of $\Gamma \vdash I$ and $I, \Delta \vdash C$ respectively, such that $(\mathsf{Cut})(\pi_1, \pi_2) \longrightarrow^{\star}_{\mathsf{cut}} \pi$.

Rephrased in terms of the λ-calculus, that would mean in particular that: for any closed term $\lambda x.t : A \to B$ there is a type C such that $\mathbb{V}\mathsf{oc}(C) \subseteq \mathbb{V}\mathsf{oc}(A) \cap \mathbb{V}\mathsf{oc}(B)$ and terms $u : A \to C$ and $v : C \to B$ such that $\lambda x.t =_\beta \lambda x.(v(ux))$. This is the content of the analysis by Čubrić. Data type C can therefore be viewed as an interface between types A and B while processing computation t, involving only pieces of data involved in both A and B.

Of course, this interface depends in t itself in the above proof, which would be different if uniform interpolation is lifted to this proof-relevant framework. Indeed, in that case, the data type for the interpolant only depend on A and the common vocabulary with the output type. In that way, interpolation would express some generic preprocessing from A to the interpolant type $UI(A, \mathcal{V})$ depending only on formula A and type variables $\mathcal{V}$, from which any computation to a type B (sharing vocabulary $\mathcal{V}$ with A) can be performed. In the same spirit, more precise results on interpolation (especially on polarity of occurrences of relational symbols, etc.) would provide more information on data usage.

We shall now outline a rephrasing of our previous results in a term calculus which corresponds to sequent calculus as the λ-calculus does to natural deduction. For uniformity, this version is essentially inspired from the calculus designed by Munch-Maccagnoni [29, 30], presented in a two-sided manner rather than one-sided to emphasize the input/output behaviour and term/context duality, that we restrict to the purely linear (ie multiplicative and additive setting).

▶ **Definition 16** (Linear System L). *As usual, the grammar and type system of linear system L consists of three syntactic categories: commands, terms and contexts, defined by mutual recursion:*

- $c ::= \langle t \mid e \rangle$
- $t, u ::= x \mid \mu\alpha.c \mid (s, t) \mid \lambda x.t \mid \mu[\alpha, \beta].c \mid () \mid \mu[].c \mid \iota_1(t) \mid \iota_2(t) \mid \mu(\pi_1(\alpha) \mapsto c \mid \pi_2(\beta) \mapsto d) \mid \texttt{tp}$
- $e, f ::= \alpha \mid \tilde{\mu}x.c \mid \tilde{\mu}(x, y).c \mid t \cdot e \mid [e, f] \mid \tilde{\mu}().c \mid [] \mid \pi_1(e) \mid \pi_2(e) \mid \tilde{\mu}(\iota_1(x) \mapsto c \mid \iota_2(x) \mapsto d) \mid \texttt{stop}$

This gives rise to three kinds of typing judgments: $c : (\Gamma \vdash \Delta)$; $\quad \Gamma \vdash t : A \mid \Delta$; $\quad \Gamma \mid e : A \vdash \Delta$ *and the typing rules given in Figure 3.*

▶ Remark 17. Notice that we use very similar notation for term and context constructs (ι/π, $(u, v)/[e, f]$, $()/[]$, $\texttt{tp}/\texttt{stop}$), in order to emphasize the deep symmetry of the System L framework we work with. This symmetry is exploited in the proof of our main theorem.

The calculus could actually be presented with one-sided typing judgments, see [29].

We provide some of the main reduction rules of this calculus, keeping in mind the invariant that one reduces ***commands***:

- $\langle \mu\alpha.c \mid e \rangle \longrightarrow_{\mu} c\{e/\alpha\}$
- $\langle t \mid \tilde{\mu}x.c \rangle \longrightarrow_{\tilde{\mu}} c\{t/x\}$
- $\langle \lambda x.t \mid u \cdot e \rangle \longrightarrow_{\lambda} \langle \tilde{\mu}x.\langle t \mid e \rangle \mid u \rangle$
- $\langle (t, u) \mid \tilde{\mu}(x, y).c \rangle \longrightarrow_{\otimes} c\{t/x, u/y\}$
- $\langle \mu[\alpha, \beta].c \mid [e, f] \rangle \longrightarrow_{⅋} c\{e/\alpha, f/\beta\}$
- $\langle \iota_j(t) \mid \tilde{\mu}(\iota_1(x_1) \mapsto c_1 \mid \iota_2(x_2) \mapsto c_2) \rangle \longrightarrow_{\oplus} c_j\{t/x_j\}$
- $\langle \mu(\pi_1(\alpha_1) \mapsto c_1 \mid \pi_2(\alpha_2) \mapsto c_2) \mid \pi_j(e) \rangle \longrightarrow_{\&} c_j\{e/\alpha_j\}$
- $\langle () \mid \tilde{\mu}().c \rangle \longrightarrow_{1} c$
- $\langle \mu[].c \mid [] \rangle \longrightarrow_{\perp} c$

Moreover, one shall consider in the development below the η rules for μ and $\tilde{\mu}$ (which respectively reduce terms and contexts):

- $\mu\alpha.\langle t \mid \alpha \rangle \longrightarrow_{\eta_\mu} t \qquad$ (if α is not free in t)
- $\tilde{\mu}x.\langle x \mid e \rangle \longrightarrow_{\eta_{\tilde{\mu}}} e \qquad$ (if x is not free in e)

The above rules satisfy the expected subject reduction property. Moreover, we need to reflect the splitting of sequents which will lead us to define a system L with a split typing system to be detailed now.

▶ **Definition 18** (System L with split typing judgements). *We introduce split typing judgements where types are enriched with a splitting information in the form of a label $l \in \{L, R\}$ carried by each type occurring in the judgment, which are then of one of the following shape:*

- $c : (x_1 : A_1^{k_1}, \ldots, x_m : A_m^{k_m} \vdash \alpha_1 : B_1^{l_1}, \ldots, \alpha_n : B_n^{l_n})$
- $x_1 : A_1^{k_1}, \ldots, x_m : A_m^{k_m} \vdash t : B^l \mid \alpha_1 : B_1^{l_1}, \ldots, \alpha_n : B_n^{l_n}$
- $x_1 : A_1^{k_1}, \ldots, x_m : A_m^{k_m} \mid e : A^k \vdash \alpha_1 : B_1^{l_1}, \ldots, \alpha_n : B_n^{l_n}$

with $k, l, k_1, \ldots, k_m, l_1, \ldots, l_n \in \{L, R\}$.

Identity rules

$$\frac{}{x : A \vdash x : A \mid \emptyset}\ \mathrm{Axt} \qquad \frac{c : (\Gamma \vdash \alpha : A, \Delta)}{\Gamma \vdash \mu\alpha.c : A \mid \Delta}\ \mu \qquad \textit{Term formation}$$

$$\frac{}{\emptyset \mid \alpha : A \vdash \alpha : A}\ \mathrm{Axc} \qquad \frac{c : (\Gamma, x : A \vdash \Delta)}{\Gamma \mid \tilde{\mu}x.c : A \vdash \Delta}\ \tilde{\mu} \qquad \textit{Context formation}$$

$$\frac{\Gamma \vdash t : A \mid \Delta \qquad \Gamma' \mid e : A \vdash \Delta'}{\langle t \mid e \rangle : (\Gamma, \Gamma' \vdash \Delta, \Delta')}\ \mathsf{cut} \qquad \textit{Command formation}$$

Multiplicative rules

$$\frac{\Gamma \vdash s : A \mid \Delta \qquad \Gamma' \vdash t : B \mid \Delta'}{\Gamma, \Gamma' \vdash (s, t) : A \otimes B \mid \Delta, \Delta'}\ \otimes^r \qquad \frac{\Gamma, x : A \vdash t : B \mid \Delta}{\Gamma \vdash \lambda x.t : A \multimap B \mid \Delta}\ \multimap^r \qquad \textit{Term formation}$$

$$\frac{c : (\Gamma \vdash \alpha : A, \beta : B, \Delta)}{\Gamma \vdash \mu[\alpha, \beta].c : A \mathbin{⅋} B \mid \Delta}\ ⅋^r \qquad \frac{}{\emptyset \vdash () : 1 \mid \emptyset}\ 1^r \qquad \frac{c : (\Gamma \vdash \Delta)}{\Gamma \vdash \mu[].c : \bot \mid \Delta}\ \bot^r$$

$$\frac{c : (\Gamma, x : A, y : B \vdash \Delta)}{\Gamma \mid \tilde{\mu}(x, y).c : A \otimes B \vdash \Delta}\ \otimes^l \qquad \frac{\Gamma \vdash t : A \mid \Delta \qquad \Gamma' \mid e : B \vdash \Delta'}{\Gamma, \Gamma' \mid t \cdot e : A \multimap B \vdash \Delta, \Delta'}\ \multimap^l \qquad \textit{Context formation}$$

$$\frac{\Gamma \mid e : A \vdash \Delta \qquad \Gamma' \mid f : B \vdash \Delta'}{\Gamma, \Gamma' \mid [e, f] : A \mathbin{⅋} B \vdash \Delta, \Delta'}\ ⅋^l \qquad \frac{c : (\Gamma \vdash \Delta)}{\Gamma \mid \tilde{\mu}().c : 1 \vdash \Delta}\ 1^l \qquad \frac{}{\emptyset \mid [] : \bot \vdash \emptyset}\ \bot^l$$

Additive rules

$$\frac{\Gamma \vdash t : A \mid \Delta}{\Gamma \vdash \iota_1(t) : A \oplus B \mid \Delta}\ \oplus_1^r \qquad \frac{\Gamma \vdash t : B \mid \Delta}{\Gamma \vdash \iota_2(t) : A \oplus B \mid \Delta}\ \oplus_2^r \qquad \textit{Term formation}$$

$$\frac{c : (\Gamma \vdash \alpha : A, \Delta) \qquad d : (\Gamma \vdash \beta : B, \Delta)}{\Gamma \vdash \mu(\pi_1(\alpha) \mapsto c \mid \pi_2(\beta) \mapsto d) : A \& B \mid \Delta}\ \&^r \qquad \frac{}{\Gamma \vdash \mathtt{tp} : \top \mid \Delta}\ \top^r$$

$$\frac{\Gamma \mid e : A \vdash \Delta}{\Gamma \mid \pi_1(e) : A \& B \vdash \Delta}\ \&_1^l \qquad \frac{\Gamma \mid e : B \vdash \Delta}{\Gamma \mid \pi_2(e) : A \& B \vdash \Delta}\ \&_2^l \qquad \textit{Context formation}$$

$$\frac{c : (\Gamma, x : A \vdash \Delta) \qquad d : (\Gamma', x : B \vdash \Delta')}{\Gamma \mid \tilde{\mu}(\iota_1(x) \mapsto c \mid \iota_2(y) \mapsto d) : A \oplus B \vdash \Delta}\ \oplus^l \qquad \frac{}{\Gamma \mid \mathtt{stop} : 0 \vdash \Delta}\ 0^l$$

Figure 3 Type derivation rules for System L.

By convention, to save space and ease the reading of judgements, we may use colors to represent the labels: a labelled type T^L (resp. U^R) may be written ${\color{red}T}$ (resp. ${\color{blue}U}$), shorthand which will be extended to whole typing contexts sharing the same label: ${\color{red}\Gamma}$ or ${\color{blue}\Gamma}$.

▶ **Theorem 19.** *In what follows, t (resp. e, resp. c) is a normal L-term (resp. normal L-context, resp. normal L-command). The following interpolating results hold:*

1. *If $c : ({\color{red}\Gamma_1}, {\color{blue}\Gamma_2} \vdash {\color{red}\Delta_1}, {\color{blue}\Delta_2})$, there exist a type $I \in \mathbb{V}\mathtt{oc}(\Gamma_1, \Delta_1) \cap \mathbb{V}\mathtt{oc}(\Gamma_2, \Delta_2)$ and t, e such that ${\color{red}\Gamma_1} \vdash t : {\color{red}I} \mid {\color{red}\Delta_1}$ and ${\color{blue}\Gamma_2} \mid e : {\color{blue}I} \vdash {\color{blue}\Delta_2}$, and $\langle t \mid e \rangle \longrightarrow^\star c$.*
2. *If ${\color{red}\Gamma_1}, {\color{blue}\Gamma_2} \vdash t : {\color{red}A} \mid {\color{red}\Delta_1}, {\color{blue}\Delta_2}$, there exist a type $I \in \mathbb{V}\mathtt{oc}(\Gamma_1, \Delta_1, A) \cap \mathbb{V}\mathtt{oc}(\Gamma_2, \Delta_2)$ and α, t', e' such that ${\color{red}\Gamma_1} \vdash t' : {\color{red}A} \mid \alpha : {\color{red}I}, {\color{red}\Delta_1}$ and ${\color{blue}\Gamma_2} \mid e' : {\color{blue}I} \vdash {\color{blue}\Delta_2}$, and $t'\{e'/\alpha\} \longrightarrow^\star t$.*
3. *If ${\color{red}\Gamma_1}, {\color{blue}\Gamma_2} \mid e : {\color{red}A} \vdash {\color{red}\Delta_1}, {\color{blue}\Delta_2}$, there exist a type $I \in \mathbb{V}\mathtt{oc}(\Gamma_1, \Delta_1, A) \cap \mathbb{V}\mathtt{oc}(\Gamma_2, \Delta_2)$ and α, e', e'' such that ${\color{red}\Gamma_1} \mid e' : {\color{red}A} \mid \alpha : {\color{red}I}, {\color{red}\Delta_1}$ and ${\color{blue}\Gamma_2} \mid e'' : {\color{blue}I} \vdash {\color{blue}\Delta_2}$, and $e'\{e''/\alpha\} \longrightarrow^\star e$.*
4. *If ${\color{red}\Gamma_1}, {\color{blue}\Gamma_2} \vdash t : {\color{blue}A} \mid {\color{red}\Delta_1}, {\color{blue}\Delta_2}$, there exist a type $I \in \mathbb{V}\mathtt{oc}(\Gamma_1, \Delta_1) \cap \mathbb{V}\mathtt{oc}(\Gamma_2, \Delta_2, A)$ and α, t', t'' such that ${\color{red}\Gamma_1} \vdash t'' : {\color{red}I} \mid {\color{red}\Delta_1}$ and ${\color{blue}\Gamma_2}, x : {\color{blue}I} \vdash t' : {\color{blue}A} \mid {\color{blue}\Delta_2}$, and $t'\{t''/x\} \longrightarrow^\star t$.*
5. *If ${\color{red}\Gamma_1}, {\color{blue}\Gamma_2} \mid e : {\color{blue}A} \vdash {\color{red}\Delta_1}, {\color{blue}\Delta_2}$, there exist a type $I \in \mathbb{V}\mathtt{oc}(\Gamma_1, \Delta_1) \cap \mathbb{V}\mathtt{oc}(\Gamma_2, \Delta_2, A)$ and x, t', e' such that ${\color{red}\Gamma_1} \vdash t' : {\color{red}I} \mid {\color{red}\Delta_1}$ and ${\color{blue}\Gamma_2}, x : {\color{blue}I} \mid e' : {\color{blue}A} \vdash {\color{blue}\Delta_2}$, and $e'\{t'/x\} \longrightarrow^\star e$.*

Proof. The result is proved by mutual induction on the structure of terms, contexts and commands. We detail cases 1–3, case 4 (resp. 5) being essentially similar to cases 3 (resp. 2).

Case 1. If c is a command in normal form, it is either of the form $\langle z \mid e\rangle$ (with z being declared in Γ_1 or Γ_2) or $\langle t \mid \beta\rangle$ (with α being declared in Δ_1 or Δ_2). Depending on the case and whether the variable is declared in the left or right component of the typing contexts, we apply induction hypotheses for one of cases 2–4.

Let us assume for instance that $c = \langle z \mid e\rangle$ and $\langle z \mid e\rangle : (\Gamma_1, \Gamma_2, z : A \vdash \Delta_1, \Delta_2)$. Then, e being structurally smaller than c, the induction hypothesis applied on $\Gamma_1, \Gamma_2 \mid e : A \vdash \Delta_1, \Delta_2$, ensures the existence of a type $I \in \mathbb{V}\mathsf{oc}(\Gamma_1, \Delta_1) \cap \mathbb{V}\mathsf{oc}(\Gamma_2, \Delta_2, A)$ as well as x, t', e' such that $\Gamma_1 \vdash t' : I \mid \Delta_1$ and $\Gamma_2, x : I \mid e' : A \vdash \Delta_2$, and $e'\{t'/x\} \longrightarrow^\star e$.

Therefore $\langle t' \mid \tilde{\mu}x.\langle z \mid e'\rangle\rangle \longrightarrow^\star c$ and we indeed have $\Gamma_1 \vdash t' : I \mid \Delta_1$ and $\Gamma_2, z : A \mid \tilde{\mu}x.\langle z \mid e'\rangle : I \vdash \Delta_2$. The other cases are similar.

Case 2. In that case, we make a case distinction based on the structure of t.

If $t = \mu\alpha.c$ and its typing judgment has shape $\Gamma_1, \Gamma_2 \vdash \mu\alpha.c : A \mid \Delta_1, \Delta_2$; It follows that c has typing judgment $\Gamma_1, \Gamma_2 \vdash \alpha : A, \Delta_1, \Delta_2$ and therefore by induction hypothesis, there exist an interpolant type I, a term t and a context e such that $\Gamma_1 \vdash t : I \mid \alpha : A, \Delta_1$ and $\Gamma_2 \mid e : I \vdash \Delta_2$ such that $\langle t \mid e\rangle \longrightarrow^\star c$. Let us then set $I' = I$, $t' = \mu\alpha.\langle t \mid \beta\rangle$ and $e' = e$ and one straightforwardly gets that $t'\{e'/\beta\} \longrightarrow^\star \mu\alpha.c$.

If $t = \lambda x.u$ and its typing judgment has shape $\Gamma_1, \Gamma_2 \vdash \lambda x.u : A \multimap B \mid \Delta_1, \Delta_2$.
By induction hypothesis we find an interpolant type I for u which can be used to interpolate t as well: we have α, t', e' such that $\Gamma_1, x : A \vdash t' : B \mid \alpha : I, \Delta_1$ and $\Gamma_2 \mid e' : I \vdash \Delta_2$, and $t'\{e'/\alpha\} \longrightarrow^\star u$. Therefore $\lambda x.t'\{e'/\alpha\} \longrightarrow^\star \lambda x.u = t$.

If $t = (u, v)$ with typing judgment of shape $\Gamma_1, \Gamma'_1, \Gamma_2, \Gamma'_2 \vdash (u, v) : A \otimes B \mid \Delta_1, \Delta'_1, \Delta_2, \Delta'_2$ with $\Gamma_1, \Gamma_2 \vdash u : A \mid \Delta_1, \Delta_2$ and $\Gamma'_1, \Gamma'_2 \vdash v : B \mid \Delta'_1, \Delta'_2$
By induction hypothesis we find an interpolant type J for u and K for v that we can combine to interpolate t as well: we have α, u', e' such that $\Gamma_1 \vdash u' : A \mid \alpha : J, \Delta_1$ and $\Gamma_2 \mid e' : J \vdash \Delta_2$, and $u'\{e'/\alpha\} \longrightarrow^\star u$ and we have β, v', f' such that $\Gamma'_1 \vdash v' : B \mid \beta : K, \Delta'_1$ and $\Gamma'_2 \mid f' : K \vdash \Delta'_2$, and $v'\{f'/\alpha\} \longrightarrow^\star v$. Therefore we set $t' = \mu\gamma.\langle \mu[\alpha, \beta].\langle (u', v') \mid \gamma\rangle \mid \delta\rangle$ and we have $t'\{[e', f'])/\delta\} = \mu\gamma.\langle \mu[\alpha, \beta].\langle (u', v') \mid \gamma\rangle \mid \delta\rangle\{[e', f'])/\delta\} \longrightarrow^\star \mu\gamma.\langle (u'\{e'/\alpha\}, v'\{f'/\beta\}) \mid \gamma\rangle \longrightarrow^\star \mu\gamma.\langle (u, v) \mid \gamma\rangle \longrightarrow_{\eta_\mu} (u, v)$.

Other cases are treated similarly.

Case 3. The only case to consider is that of the applicative context, all the other case can be directly retrieve from the previous item, dualizing terms and contexts.

If $e = u \cdot f$ with typing judgment of shape $\Gamma_1, \Gamma'_1, \Gamma_2, \Gamma'_2 \mid u \cdot f : A \multimap B \vdash \Delta_1, \Delta'_1, \Delta_2, \Delta'_2$.
Then we have $\Gamma_1, \Gamma_2 \vdash u : A \mid \Delta_1, \Delta_2$ and $\Gamma'_1, \Gamma'_2 \mid f : B \vdash \Delta'_1, \Delta'_2$.
By induction hypothesis we find an interpolant type J, together with u', g, α for u and an interpolant type K, together with f', f'', β for f which satisfy:

- $\Gamma_1 \vdash u' : A \mid \Delta_1, \alpha : J$;
- $\Gamma_2 \mid g : J \vdash \Delta_2$;
- $\Gamma'_1 \mid f' : B \vdash \Delta'_1, \beta : K$;
- $\Gamma'_2 \mid f'' : K \vdash \Delta'_2$;
- $u'\{g/\alpha\} \longrightarrow^\star u$;
- $f'\{f''/\beta\} \longrightarrow^\star f$.

Therefore, we have $\Gamma_1, \Gamma_1', \mid u' \cdot f' : A \multimap B \vdash \Delta_1, \Delta_1', \alpha : J, \beta : K$ and $\Gamma_2, \Gamma_2' \mid [g, f''] : J \parr K \vdash \Delta_2, \Delta_2'$. As a consequence, we have $\Gamma_1, \Gamma_1' \mid \tilde{\mu}x.\langle \mu[\alpha, \beta].\langle x \mid u' \cdot f' \rangle \mid \gamma \rangle : A \multimap B \vdash \Delta_1, \Delta_1', \gamma : J \parr K$. We thus set $I = J \parr K$, $e' = \tilde{\mu}x.\langle \mu[\alpha, \beta].\langle x \mid u' \cdot f' \rangle \mid \gamma \rangle$ and $e'' = [g, f'']$ and we observe that
$e'\{e''/\gamma\} \longrightarrow^\star \tilde{\mu}x.\langle x \mid u'\{g/\alpha\} \cdot f'\{f''/\beta\} \rangle \longrightarrow^\star \tilde{\mu}x.\langle x \mid u \cdot f \rangle \longrightarrow_{\eta_{\tilde{\mu}}} u \cdot f = e.$

Case 4. These cases are treated similarly to 2, but for the fact that the interpolant lives in the world of terms rather than of contexts but since we did not use connective $\multimap$ to build an interpolant in 2, the symmetry of system L does the job.

Case 5. These cases are treated similarly to 3, but for the fact that the interpolant lives in the world of terms rather than of contexts and the same justification as for the previous case applies. ◀

▶ Remark 20. The above results could be extended to full LL, even though it would be more adequate formally to move to the original calculus by Curien and Herbelin and prove a corresponding result to proof-relevant interpolation for LK to avoid complexities in dealing with structural rules.

In Curien-Herbelin's setting, a particular attention should be paid to the reduction rules that are needed in order to investigate to what account one can restrict to specific evaluation strategies, such as call-by-name and call-by-value, in interpolating, in order to avoid the computational critical-pair. Note that in the above development, this critical pair is not problematic due to the linearity discipline that is enforced: the sequent calculus non confluence does not induce a trivial equational theory (as is known from proof nets for instance which precisely perform this quotient).

6 Conclusion

In this paper, we established a refined, proof-relevant, version of Craig-Lyndon interpolation theorems for first-order linear logic and then deduced it, using completely standard tools of LL proof theory, to LK and LJ. A most striking fact, in our opinion, is that the result was *almost there* for decades, since the early proofs by Maehara (and its broad dissemination in proof theory textbooks, not to speak of applications to broader logical frameworks) and Prawitz. Borrowing Feferman's words, *"though deceptively simple and plausible on the face of it"*, we think that this approach to proof-relevant interpolation in sequent calculus emphasizes a deep duality between interpolation and cut elimination : more specifically, the process of synthesis of the interpolant and the two interpolating proofs is reformulated as *a cut-introduction process*. Finally, we considered the computational content of the results by an analysis of the results in system L.

While we think that interpolation as cut-introduction is both a new conceptual and technical contribution of this work, a proof-relevant interpolation theorem has already been established by Čubrić [9, 10] in the early 90's for propositional intuitionistic natural deduction in the form of an interpolation for the typed λ-calculus and for bicartesian closed categories. Similarly, our approach is subject to a computational interpretation in system L [11, 12] that we plan to develop further in a future work. In fact we also hope that our interpolation-as-cut-introduction can pave the way for a broader analysis of the computational content of interpolation as a manner to factor computation through interfacing (that is, interpolating) types. Indeed, while the computational interpretation of Čubrić's result, stated in the

λ-calculus, is certainly more transparent than the sequent calculus that we presented here, it has not been extended in more than 30 years, except once by Matthes [26]. A reason for this might be that while both allows for a proof-relevant phrasing, Maehara's method is more modular and easily extensible than Prawitz as it rests on a logical framework, the sequent calculus, that is inherently more modular that natural deduction. For instance, the results of Section 5 can be easily extended to a computational version of interpolation holding for classical System L (that is in a classical framework featuring continuations) and not only it linear, multiplicative and additive restriction as discussed here, while it is not clear how Čubrić' results can be extended beyond simply typed λ-calculus, eg. to Parigot's $\lambda\mu$-calculus or λ-calculi with dependent types.

Among ongoing and future works that we plan to tackle, one can list the following directions.

- Extend our treatment of interpolation in System L from the multiplicative-additive fragment to full LL as well as LK (with a focus on evaluation strategies in this latter case).
- While Maehara's method strongly relies on the inductive and tree structures of sequent proofs as well as on the cut-freeness assumption, we believe that those assumptions can be weakened to some extent, still allowing proof-relevant interpolation to go through. We have several extensions in mind that will witness this fact:
  - **Circular proofs** First we are working at extending proof-relevant interpolation to provide a treatment to circular proofs for LL with least and greatest fixed-points.
  - **Proof-nets** Second, we are investigating proof-relevant interpolation for LL proof nets, which are a proof system satisfying strong canonicity properties akin to those of intuitionistic natural deduction. In particular, it seems that interpolation as cut-introduction in (multiplicative) proof-nets can be reformulated in terms of correctness criteria.
  - **Proof with cuts** As discussed in Section 4.5, the assumption that is truly at work in Maehara's method is not really cut-freeness, nor the subformula property, but simply a consequence of those properties, amounting to a sort of "sublanguage property"; it is indeed possible to perform Maehara's interpolation for proofs having cuts satisfying some structural properties. In a similar way, our proof-relevant interpolation should be extendable for PRIS with cuts of certain forms. In particular, it was shown recently by Hetzl and Jalali [22] that interpolating proofs with cuts has useful consequences. We therefore plan to follow the directions outlined in Remarks 14 and 15.
- There is of course a denotational counter-part to proof-relevant interpolation, that was already investigated by Čubrić for models of the simply typed lambda-calculus with pairs. We plan to extend this to models of linear logic, establishing factorization properties for models of LL.
- An intrinsic advantage of sequent calculus over natural deduction, and therefore of our approach over Čubrić's, is that many more logics can be formulated as sequent calculi than in natural deduction. Can we extend our results to other logics having cut-elimination?
- We plan to reconsider Čubrić's results from the cut-introduction perspective and try and extend them to various extensions of the simply typed λ-calculus.
- Finally, an important and difficult question that we would like to explore is whether such a proof-relevant approach to interpolation can be extended to uniform interpolation. That would mean that all computations that can be performed from a piece of data u in a type A to data sharing with A only a fixed set $\mathcal{L}$ of primitive datatypes can be factored through a program that computes a value v in the uniform interpolant datatype build from $\mathcal{L}$ such that everything that can be computed from u can be computed from v as well.

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A Details on the Proof of the Main Lemma

We now provide the missing cases for the proof of Lemma 10 given in Section 4.4.

A.1 Axiom case

- If $\pi = \dfrac{}{\vdash F, F^\perp}\,(\mathsf{Ax})$, one simply takes $I = F^\perp$, $\pi_1^l = \dfrac{}{\vdash F, F^\perp}\,(\mathsf{Ax})$ and $\pi_1^r = \dfrac{}{\vdash F, F^\perp}\,(\mathsf{Ax})$.
  The cut between π_1^l and π_1^r reduces to π by one cut-axiom reduction step.
- If $\pi = \dfrac{}{\vdash F, F^\perp}\,(\mathsf{Ax})$, the case is symmetrical taking $I = F$.
- If $\pi = \dfrac{}{\vdash F, F^\perp}\,(\mathsf{Ax})$, one takes $I = \bot$, $\pi_1^l = \dfrac{\pi}{\vdash F, F^\perp, \bot}\,(\bot)$ and $\pi_1^r = \dfrac{}{\vdash 1}\,(1)$.
  The cut of π_1^l and π_1^r reduces to π by a key $1/\bot$ case.
- If $\pi = \dfrac{}{\vdash F, F^\perp}\,(\mathsf{Ax})$, the case is symmetrical to the previous one, taking $I = 1$: $\pi_1^l = \dfrac{}{\vdash 1}\,(1)$ and $\pi_1^r = \dfrac{\pi}{\vdash F, F^\perp, \bot}\,(\bot)$.

A.2 Logical rules

We analyze the possible cases for a logical rule involved in an elementary PRIS. Note that in each case, the principal formula may be part of the left or right part of the splitting; we treat only one case each time since the other is symmetrical be taking the dual interpolant and exchanging π^L and π^R.

If the last rule is $(\bot)$, that is if $\pi = \dfrac{\dfrac{\overset{\pi_1^L}{\vdash \Gamma, I_1} \quad \overset{\pi_1^R}{\vdash I_1{}^\perp, \Delta}}{\vdash \Gamma, \Delta}\,(\mathsf{Cut})}{\vdash \bot, \Gamma, \Delta}\,(\bot)$. By setting $I = I_1$, $\pi^L = \dfrac{\overset{\pi_1^L}{\vdash \Gamma, I_1}}{\vdash \bot, \Gamma, I_1}\,(\bot)$ and $\pi^R = \pi_1^R$, one gets a solved PRIS $\pi' = \dfrac{\overset{\pi^L}{\vdash \bot, \Gamma, I} \quad \overset{\pi^R}{\vdash I^\perp, \Delta}}{\vdash \bot, \Gamma, \Delta}\,(\mathsf{Cut})$ such that $\pi' \longrightarrow_{\mathsf{cut}} \pi$ by a cut-commutation case.

If the last rule is (1), that is if $\pi = \dfrac{}{\vdash 1}\,(1)$ (which is indeed both an initial PRIS and an elementary PRIS), let $I = 1$ and $\pi_1^L = \dfrac{}{\vdash 1}\,(1)$ and $\pi_1^R = \dfrac{}{\vdash \bot, 1}\,(\mathsf{Ax})$. One gets a solved PRIS π' such that $\pi \longleftarrow^{\star}_{\mathsf{cut}} \pi'$ by a key $1/\bot$ case.

If the last rule is $(\oplus^{\mathsf{i}}), i \in \{1, 2\}$, that is if $\pi = \dfrac{\dfrac{\overset{\pi_1^L}{\vdash F_i, \Gamma, I_1} \quad \overset{\pi_1^R}{\vdash I_1{}^\perp, \Delta}}{\vdash F_i, \Gamma, \Delta}\,(\mathsf{Cut})}{\vdash F_1 \oplus F_2, \Gamma, \Delta}\,(\oplus^{\mathsf{i}})$, then setting $I = I_1$, $\pi^L = \dfrac{\overset{\pi_1^L}{\vdash F_i, \Gamma, I_1}}{\vdash F_1 \oplus F_2, \Gamma, I_1}\,(\oplus^{\mathsf{i}})$ and $\pi^R = \pi_1^R$, one gets a solved PRIS π' such that $\pi \longleftarrow^{\star}_{\mathsf{cut}} \pi'$ by a commutative reduction of (Cut).

If the last rule is $(\&)$, that is if π is has the following form

$$\dfrac{\dfrac{\overset{\pi_1^L}{\vdash F,\Gamma,I_1}\qquad\overset{\pi_1^R}{\vdash {I_1}^{\perp},\Delta}}{\vdash F,\Gamma,\Delta}\,(\mathsf{Cut})\qquad\dfrac{\overset{\pi_2^L}{\vdash G,\Gamma,I_2}\qquad\overset{\pi_2^R}{\vdash {I_2}^{\perp},\Delta}}{\vdash G,\Gamma,\Delta}\,(\mathsf{Cut})}{\vdash F\&G,\Gamma,\Delta}\,(\&)$$

then setting $I = I_1 \oplus I_2$, $\pi^L = \dfrac{\dfrac{\overset{\pi_1^L}{\vdash F,\Gamma,I_1}}{\vdash F,\Gamma,I_1\oplus I_2}\,(\oplus^1)\qquad\dfrac{\overset{\pi_2^L}{\vdash G,\Gamma,I_2}}{\vdash G,\Gamma,I_1\oplus I_2}\,(\oplus^2)}{\vdash F\&G,\Gamma,I}\,(\&)$ and $\pi^R = \dfrac{\overset{\pi_1^R}{\vdash {I_1}^{\perp},\Delta}\qquad\overset{\pi_2^R}{\vdash {I_2}^{\perp},\Delta}}{\vdash I_1^{\perp}\&{I_2}^{\perp},\Delta}\,(\&)$, one observes that

$$\dfrac{\overset{\pi^L}{\vdash\Gamma,I}\qquad\overset{\pi^R}{\vdash I^{\perp},\Delta}}{\vdash\Gamma,\Delta}\,(\mathsf{Cut})$$

$$\longrightarrow_{\mathsf{cut}}\ \dfrac{\dfrac{\dfrac{\overset{\pi_1^L}{\vdash F,\Gamma,I_1}}{\vdash F,\Gamma,I}\,(\oplus^1)\qquad\overset{\pi^R}{\vdash I^{\perp},\Delta}}{\vdash F,\Gamma,\Delta}\,(\mathsf{Cut})\qquad\dfrac{\dfrac{\overset{\pi_2^L}{\vdash G,\Gamma,I_2}}{\vdash G,\Gamma,I}\,(\oplus^2)\qquad\overset{\pi^R}{\vdash I^{\perp},\Delta}}{\vdash G,\Gamma,\Delta}\,(\mathsf{Cut})}{\vdash F\&G,\Gamma,\Delta}\,(\&)$$

$$\longrightarrow_{\mathsf{cut}}\ \dfrac{\dfrac{\overset{\pi_1^L}{\vdash F,\Gamma,I_1}\qquad\overset{\pi_1^R}{\vdash {I_1}^{\perp},\Delta}}{\vdash F,\Gamma,\Delta}\,(\mathsf{Cut})\qquad\dfrac{\dfrac{\overset{\pi_2^L}{\vdash G,\Gamma,I_2}}{\vdash G,\Gamma,I}\,(\oplus^2)\qquad\overset{\pi^R}{\vdash I^{\perp},\Delta}}{\vdash G,\Gamma,\Delta}\,(\mathsf{Cut})}{\vdash F\&G,\Gamma,\Delta}\,(\&)$$

$$\longrightarrow_{\mathsf{cut}}\ \dfrac{\dfrac{\overset{\pi_1^L}{\vdash F,\Gamma,I_1}\qquad\overset{\pi_1^R}{\vdash {I_1}^{\perp},\Delta}}{\vdash F,\Gamma,\Delta}\,(\mathsf{Cut})\qquad\dfrac{\overset{\pi_2^L}{\vdash G,\Gamma,I_2}\qquad\overset{\pi_2^R}{\vdash {I_2}^{\perp},\Delta}}{\vdash G,\Gamma,\Delta}\,(\mathsf{Cut})}{\vdash F\&G,\Gamma,\Delta}\,(\&)\ = \pi$$

so that π indeed cut-expands to π'.

If the last rule is $(\top)$, that is if $\pi = \dfrac{}{\vdash \top,\Gamma,\Delta}\,(\top)$, by setting $I = 0$, $\pi^L = \dfrac{}{\vdash \top,\Gamma,0}\,(\top)$ and $\pi^R = \dfrac{}{\vdash \top,\Delta}\,(\top)$. Then $\pi' = \dfrac{\pi^L\qquad\pi^R}{\vdash \top,\Gamma,\Delta}\,(\mathsf{Cut})$ is a solved PRIS to which π cut-expands by a $(\top)$ commutation case.

If the last rule is $(?\mathsf{d})$, that is $\pi = \dfrac{\dfrac{\overset{\pi_1^L}{\vdash F,\Gamma,I_1}\qquad\overset{\pi_1^R}{\vdash {I_1}^{\perp},\Delta}}{\vdash F,\Gamma,\Delta}\,(\mathsf{Cut})}{\vdash ?\,F,\Gamma,\Delta}\,(?\mathsf{d})$, then setting $I = I_1$, $\pi^L = \dfrac{\overset{\pi_1^L}{\vdash F,\Gamma,I_1}}{\vdash ?\,F,\Gamma,I_1}\,(?\mathsf{d})$ and $\pi^R = \pi_1^R$ one gets:

$$\dfrac{\overset{\pi^L}{\vdash ?\,F,\Gamma,I}\qquad\overset{\pi^R}{\vdash I^{\perp},\Delta}}{\vdash ?\,F,\Gamma,\Delta}\,(\mathsf{Cut})\ \longrightarrow_{\mathsf{cut}}\ \dfrac{\dfrac{\dfrac{\pi_1^L}{\vdash F,\Gamma,I_1}\qquad\dfrac{\pi_1^R}{\vdash {I_1}^{\perp},\Delta}}{\vdash F,\Gamma,\Delta}\,(\mathsf{Cut})}{\vdash ?\,F,\Gamma,\Delta}\,(?\mathsf{d})\ = \pi$$

so that π indeed cut-expands to π'.

If the last rule is $(?\mathsf{w})$, that is $\pi = \dfrac{\dfrac{\overset{\pi_1^L}{\vdash \Gamma, I_1} \qquad \overset{\pi_1^R}{\vdash {I_1}^{\perp}, \Delta}}{\vdash \Gamma, \Delta}\,(\mathsf{Cut})}{\vdash ?\,F, \Gamma, \Delta}\,(?\mathsf{w})$, then by setting $I = I_1$, $\pi^L = \dfrac{\overset{\pi_1^L}{\vdash \Gamma, I_1}}{\vdash ?\,F, \Gamma, I_1}\,(?\mathsf{w})$ and $\pi^R = \pi_1^R$, we ensure that, since $\mathbb{V}\mathsf{oc}(I) \subseteq \mathbb{V}\mathsf{oc}(\Gamma) \cap \mathbb{V}\mathsf{oc}(\Delta)$, $\pi' = \dfrac{\overset{\pi^L}{\vdash ?\,F, \Gamma, I} \qquad \overset{\pi^R}{\vdash I^{\perp}, \Delta}}{\vdash ?\,F, \Gamma, \Delta}\,(\mathsf{Cut})$ is indeed a solved PRIS and that

$$\pi' \quad \longrightarrow_{\mathsf{cut}} \quad \dfrac{\dfrac{\overset{\pi_1^L}{\vdash \Gamma, I_1} \qquad \overset{\pi_1^R}{\vdash {I_1}^{\perp}, \Delta}}{\vdash \Gamma, \Delta}\,(\mathsf{Cut})}{\vdash ?\,F, \Gamma, \Delta}\,(?\mathsf{w}) \quad = \quad \pi.$$

by a cut commutation rule so that π indeed cut-expands to π' as expected.

If the last rule is $(?\mathsf{c})$, that is $\pi = \dfrac{\dfrac{\overset{\pi_1^L}{\vdash ?\,F, ?\,F, \Gamma, I_1} \qquad \overset{\pi_1^R}{\vdash {I_1}^{\perp}, \Delta}}{\vdash ?\,F, ?\,F, \Gamma, \Delta}\,(\mathsf{Cut})}{\vdash ?\,F, \Gamma, \Delta}\,(?\mathsf{c})$ by setting $I = I_1$, $\pi^L = \dfrac{\overset{\pi_1^L}{\vdash ?\,F, ?\,F, \Gamma, I_1}}{\vdash ?\,F, \Gamma, I}\,(?\mathsf{c})$ and $\pi^R = \pi_1^R$ one gets:

$$\pi' = \dfrac{\overset{\pi^L}{\vdash ?\,F, \Gamma, I} \qquad \overset{\pi^R}{\vdash I^{\perp}, \Delta}}{\vdash ?\,F, \Gamma, \Delta}\,(\mathsf{Cut}) \qquad \longrightarrow_{\mathsf{cut}} \qquad \dfrac{\dfrac{\overset{\pi_1^L}{\vdash ?\,F, ?\,F, \Gamma, I_1} \qquad \overset{\pi_1^L}{\vdash {I_1}^{\perp}, \Delta}}{\vdash ?\,F, ?\,F, \Gamma, \Delta}\,(\mathsf{Cut})}{\vdash ?\,F, \Gamma, \Delta}\,(?\mathsf{c}) \;= \pi$$

so that π' is indeed a solved PRIS to which π cut-expands.

If the last rule is $(\forall)$, that is $\pi = \dfrac{\dfrac{\overset{\pi_1^L}{\vdash F, \Gamma, I_1} \qquad \overset{\pi_1^R}{\vdash {I_1}^{\perp}, \Delta}}{\vdash F, \Gamma, \Delta}\,(\mathsf{Cut})}{\vdash \forall x F, \Gamma, \Delta}\,(\forall)$ with $x \notin \mathsf{FV}(\Gamma, \Delta)$ and $\mathbb{V}\mathsf{oc}(I_1) \subseteq \mathbb{V}\mathsf{oc}(F, \Gamma) \cap \mathbb{V}\mathsf{oc}(\Delta)$. By setting $I = \exists x I_1$, $\pi^L = \dfrac{\dfrac{\overset{\pi_1^L}{\vdash F, \Gamma, I_1}}{\vdash F, \Gamma, \exists x I_1}\,(\exists)}{\vdash \forall x F, \Gamma, \exists x.I_1}\,(\forall)$ and $\pi^R = \dfrac{\overset{\pi_1^R}{\vdash {I_1}^{\perp}, \Delta}}{\vdash \forall x {I_1}^{\perp}, \Delta}\,(\forall)$, one gets that: $\pi' = \dfrac{\overset{\pi^L}{\vdash \forall x F, \Gamma, \exists x.I_1} \qquad \overset{\pi^R}{\vdash \forall x {I_1}^{\perp}, \Delta}}{\vdash \forall x F, \Gamma, \Delta}\,(\mathsf{Cut})$ is a solved PRIS to which π cut-expands: $\pi' \longrightarrow_{\mathsf{cut}} \dfrac{\dfrac{\dfrac{\overset{\pi_1^L}{\vdash F, \Gamma, I_1}}{\vdash F, \Gamma, \exists x.I_1}\,(\exists) \qquad \dfrac{\overset{\pi_1^R}{\vdash {I_1}^{\perp}, \Delta}}{\vdash \forall x {I_1}^{\perp}, \Delta}\,(\forall)}{\vdash F, \Gamma, \Delta}\,(\mathsf{Cut})}{\vdash \forall x F, \Gamma, \Delta}\,(\forall) \longrightarrow_{\mathsf{cut}} \pi.$