Symmetry Classes of Hamiltonian Cycles

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─── **Abstract** ───

We initiate the study of Hamiltonian cycles up to symmetries of the underlying graph. Our focus lies on the extremal case of *Hamiltonian-transitive graphs*, i.e., Hamiltonian graphs where, for every pair of Hamiltonian cycles, there is a graph automorphism mapping one cycle to the other. This generalizes the extensively studied uniquely Hamiltonian graphs. In this paper, we show that Cayley graphs of abelian groups are not Hamiltonian-transitive (under some mild conditions and some non-surprising exceptions), i.e., they contain at least two structurally different Hamiltonian cycles. To show this, we reduce Hamiltonian-transitivity to properties of the prime factors of a Cartesian product decomposition, which we believe is interesting in its own right. We complement our results by constructing infinite families of regular Hamiltonian-transitive graphs and take a look at the opposite extremal case by constructing a family with many different Hamiltonian cycles up to symmetry.

1 Introduction

Our work is motivated by a line of research studying the existence of Hamiltonian cycles in graphs that fulfill some symmetry condition. One of the most intriguing open problems in this area is the *Lovász Conjecture* [25]: It asserts that every vertex-transitive graph, apart from five small counterexamples, contains a Hamiltonian cycle. Despite considerable efforts over the last decades [5, 7, 23, 24, 26, 29, 28], this conjecture remains wide open. Another prominent open problem in this area is *Sheehan's Conjecture* [34]. Together with results in [35, 36], its assertion is equivalent to the following: Cycles are the only regular graphs that are uniquely Hamiltonian, i.e., contain precisely one Hamiltonian cycle. It was proven in special cases [9, 13, 32, 33], but remains open in general.

One of the most important special cases for these conjectures is Cayley graphs [1, 7, 11, 30, 37]. While the Lovász conjecture for general Cayley graphs remains open, it is well known to hold for abelian groups (e.g., follows from [4]). In fact, Cayley graphs of abelian groups even "exceed it" in the sense that they are not uniquely Hamiltonian (except for

cycles) [32], i.e., they satisfy Sheehan's conjecture. This raises the question of whether the multiple Hamiltonian cycles only exist due to many symmetries of Cayley graphs. This is the central question of this paper and we solve it under some mild conditions.

We study Hamiltonian cycles in finite simple graphs up to symmetry. More precisely, we consider two Hamiltonian cycles of a graph G to be equivalent, if there is an automorphism of G that maps one cycle to the other. Here we identify a cycle with its set of edges. The focus of this paper lies on the extremal case of graphs that only contain a unique Hamiltonian cycle up to symmetry. We introduce the following notion.

▶ **Definition 1.** *A graph G is* Hamiltonian-transitive *if it is Hamiltonian and, for every pair of Hamiltonian cycles, there is an automorphism of G that maps one cycle to the other. We denote the class of all Hamiltonian-transitive graphs by $\mathcal{H}$.*

Note that every uniquely Hamiltonian graph is also Hamiltonian-transitive. The intuitive difference is that we only distinguish Hamiltonian cycles that are *structurally* different, which fits into a broader framework of studying combinatorial objects up to isomorphism/symmetry [21, 27]. For example, the complete graph is Hamiltonian-transitive, but not uniquely Hamiltonian. This raises the question of to what extent Sheehan's conjecture generalizes to Hamiltonian transitivity.

We prove that a large family of Cayley graphs of abelian groups are not Hamiltonian-transitive, i.e., that Sheehan's conjecture for this family generalizes. The starting point for this is studying product decompositions of graphs. More precisely, we study how Hamiltonian transitivity behaves with respect to Cartesian products, which we believe is interesting in its own right because it reduces Hamiltonian transitivity to properties of the prime factors of a graph. Opposed to this, we give a simple construction for an infinite family of d-regular Hamiltonian-transitive graphs for any $d \geq 2$. Through the lens of Sheehan's conjecture, this highlights one of the key differences between uniquely Hamiltonian and Hamiltonian-transitive graphs.

Our results

We characterize the Hamiltonian-transitive graphs within a large family of Cayley graphs of abelian groups and Cartesian products.

Our main result is a full characterization of Hamiltonian-transitive Cayley graphs of abelian groups of odd order. In this paper, all graphs are finite and simple, which means in the context of Cayley graphs that we only consider finite groups with a generating set that is closed under taking inverses and does not contain the identity element. In the following, we denote by K_n and C_n the complete graph and the cycle on n vertices. For a graph $G = (V, E)$, we write $|G| := |V|$ for its order.

▶ **Theorem 2.** *Let G be the Cayley graph of an abelian group w.r.t. some inverse-closed generating set and assume that $n := |G| \geq 3$ is odd. Then $G \in \mathcal{H}$ if and only if $G \in \{K_n, C_n\}$.*

For Cayley graphs of abelian groups of even order, we require somewhat different techniques and rely on a mild non-redundancy assumption on the generating set. We obtain the following characterization, where the notation $G \square H$ denotes the *Cartesian product* and $K_{n,n}$ denotes the complete bipartite graph in which both parts of the bipartition have size n.

▶ **Theorem 3.** *Let G be the Cayley graph of an abelian group w.r.t. some inverse-closed generating set S and assume that $n := |G| \geq 4$ is even. Moreover, assume that there is some $s \in S$ such that $S \setminus \{s, -s\}$ is non-generating. Then $G \in \mathcal{H}$ if and only if $G \in \{C_n, C_4 \square K_2, K_{4,4}\}$ or $G = C_k \square K_2$ where $k = n/2$ is odd.*

A crucial ingredient for our proofs of Theorems 2 and 3 is the study of Cartesian products, which are interesting in their own right. First, observe that the Cartesian product of two Hamiltonian graphs is Hamiltonian again. We show that most non-prime graphs (graphs decomposable into Cartesian products, formally defined in Section 2.1) have, due to their symmetric structure, at least two structurally different Hamiltonian cycles.

▶ **Theorem 4.** *Let G and H be Hamiltonian graphs.*
- *If G and H are relatively prime, then $G \,\square\, H \notin \mathcal{H}$.*
- *If G is prime, then $G^{\square\, n} \in \mathcal{H}$ if and only if $n = 1$ and $G \in \mathcal{H}$.*

To extend our study of Cartesian products, we also investigate the special case of products with K_2 and give a full characterization of the Hamiltonian-transitive ones (Theorem 8).

While these results give the impression that there are not many highly symmetric Hamiltonian-transitive graphs, we complement our findings by giving constructions of regular Hamiltonian-transitive graphs in Section 4. More precisely, we give a simple construction of an infinite class of Hamiltonian-transitive d-regular graphs for every $d \geq 2$ (Remark 30). Further, we show that truncations of cubic graphs preserve Hamiltonian transitivity, providing an interesting class of cubic 3-connected Hamiltonian-transitive graphs (Proposition 31).

These results focus on graphs with only one Hamiltonian cycle up to symmetry, and in the final part of this paper, we touch on the opposite extremal case by studying graphs with many Hamiltonian cycles up to symmetry. We provide a construction of graphs on n vertices with $2^{\Theta(n \log(n))}$ symmetry classes of Hamiltonian cycles (Proposition 34), which matches a trivial upper bound on the number of structurally different Hamiltonian cycles.

Related work

There is a vast amount of work studying uniquely Hamiltonian graphs and Sheehan's Conjecture. The arguably most important result for our work is that the conjecture was proven for graphs with a large automorphism group in [32], including vertex-transitive graphs, and in particular, Cayley graphs. Further research includes the degree sequences that uniquely Hamiltonian graphs can have (see e.g., [2, 3, 10]) or verification of the conjecture for special graph classes, such as claw-free graphs of order not divisible by 6 [9], claw-free graphs of minimum degree at least 3 [33], and graphs of order up to 21 [13]. Bounds on the number of distinct Hamiltonian cycles in regular Hamiltonian graphs were given in [20].

Uniquely Hamiltonian graphs as well as graphs with few Hamiltonian cycles have also been investigated from a computational point of view. Algorithms to compute a second Hamiltonian cycle in a graph were developed in [6]. An algorithm for the exhaustive generation of graphs with precisely k Hamiltonian cycles is given in [13]. In [19], the authors consider symmetry breaking for uniquely Hamiltonian graphs. In [12], the authors study the number of Hamiltonian cycles in k-regular and (k, l)-regular graphs and disprove a lower bound on the number of Hamiltonian cycles in k-regular graphs conjectured by Haythorpe [18].

Motivated by the application for Gray codes, there has recently been interest in finding Hamiltonian cycles that are particularly symmetric in the sense that they can be rotated by a graph automorphism. To this end, the *Hamilton compression*, a parameter measuring the rotation symmetry of a Hamiltonian cycle, was introduced [14] and investigated across various families of vertex-transitive graphs [22, 23]. While the Hamilton compression measures the symmetries within a Hamiltonian cycle, we focus on the symmetries between different Hamiltonian cycles. However, the Hamilton compression is the same for all Hamiltonian cycles if the graph is Hamiltonian-transitive.
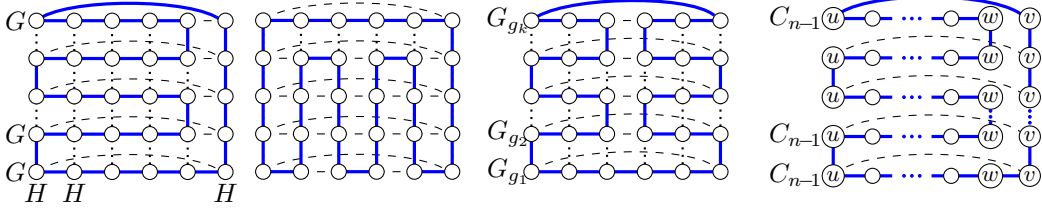
Figure 1 Three Hamiltonian cycles of $G \square H$. The first has $2(|H|-1)$ vertical edges, while the second has $2(|H|-1) + (|G|-2)(|H|-2)$ vertical edges. In case $G = H$, the cycle on the left is C_2 from the construction of Lemma 6 and the one on the right is $\hat{C}_2$.

Figure 2 The Hamiltonian cycle C_n of $G^{\square n}$ constructed from the Hamiltonian cycle C_{n-1} of $G^{\square n-1}$.

2   Cartesian products

Cartesian products provide a natural product decomposition of graphs, so that we begin with studying how Hamiltonian transitivity behaves with respect to such a decomposition.

More formally, the *Cartesian product* of two graphs G and H, denoted by $G \square H$, is defined as the graph with vertex set $V(G) \times V(H)$ and two vertices (g, h) and (g', h') being adjacent if and only if $g = g'$ and $\{h, h'\} \in E(H)$, or $h = h'$ and $\{g, g'\} \in E(G)$.

2.1   Cartesian products of Hamiltonian graphs

In this subsection, we focus on Hamiltonian-transitive Cartesian products of Hamiltonian graphs. Here we rely on the assumption that the factors are Hamiltonian because this ensures that their product is Hamiltonian. In particular, we assume that each factor has order at least three. Finding conditions under which $G \square H$ is Hamiltonian for general graphs G and H is an independent research direction and remains wide open.

Next, let us introduce some of the notation in the context of Cartesian products. A non-trivial graph is called *prime* if it cannot be decomposed into a Cartesian product of two non-trivial graphs. Every connected graph G has a unique decomposition

$$G = H_1^{\square\, r_1} \square \ldots \square H_k^{\square\, r_k} \tag{1}$$

for pairwise distinct prime graphs $H_1, \ldots, H_k$ and $r_1, \ldots, r_k \in \mathbb{N}$ (see [16, Chapter 6]), where $H_i^{\square\, r_i}$ denotes the Cartesian product of r_i copies of H_i. In this case, we call the graphs $H_1, \ldots, H_k$ *prime factors* of G. Two graphs G and H are *relatively prime*, if they do not share a prime factor.

With this, we can formulate the main result of this subsection.

▶ **Theorem 4.** *Let G and H be Hamiltonian graphs.*
- *If G and H are relatively prime, then $G \square H \notin \mathcal{H}$.*
- *If G is prime, then $G^{\square n} \in \mathcal{H}$ if and only if $n = 1$ and $G \in \mathcal{H}$.*

We split the proof of Theorem 4 into two lemmata, beginning with the case that G and H are relatively prime.[1]

▶ **Lemma 5.** *Let G and H be Hamiltonian graphs. If G and H are relatively prime, then $G \square H \notin \mathcal{H}$.*

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[1] All missing proofs (and full proofs for sketches) are deferred to the full version.

Proof sketch. We consider the Hamiltonian cycle on the left of Figure 1 and show that it cannot be mapped to the one in the middle, which is obtained by exchanging the roles of G and H. For this, we show that every automorphism of $G \,\square\, H$ preserves the set of vertical edges and the set of horizontal edges, using that the graphs are relatively prime. The assertion then follows because the two cycles have a different number of vertical edges. ◀

After dealing with relatively prime graphs, the next step towards a classification is to consider the Cartesian product of a prime graph with itself.

▶ **Lemma 6.** *Let G be a Hamiltonian and prime graph. Then $G^{\square\, n} \notin \mathcal{H}$ for all $n \geq 2$.*

Proof. Note that vertices of $G^{\square\, n}$ are n-tuples of vertices of G and two vertices u and v are adjacent if and only if they only differ in one component i and $\{u_i, v_i\}$ is an edge of G. Let E_i be the set of edges $\{u, v\}$ of $G^{\square\, n}$ where u and v differ in their i-th component. Then $E_1, \ldots, E_n$ is a partition of the edges of $G^{\square\, n}$. Since G is prime, every automorphism of $G^{\square\, n}$ permutes the sets $E_1, \ldots, E_n$ ([16, Theorem 6.10]), i.e., if $e, e' \in E_i$ and the automorphism maps e to E_j, then e' is also mapped to E_j.

Let $k := |G|$. First, we consider the case that $k \geq 4$. Our strategy is as follows: We inductively construct cycles C_n and $\hat{C}_n$ (for $n \geq 2$) that cannot be mapped to each other. For this, we also make use of the auxiliary statement that the constructed cycles fulfill $|E(C_n) \cap E_i| \geq 4$ and $|E(\hat{C}_n) \cap E_i| \geq 4$ for all $i \in \{1, \ldots, n\}$, where by slight abuse of notation, E_i is also w.r.t. n.

We start with the base case, i.e., $n = 2$. For simplicity, we call the edges in E_1 and E_2 *vertical* and *horizontal* edges, respectively. Let $(g_1, \ldots, g_k)$ be a Hamiltonian cycle of G. From this cycle, we can construct the two Hamiltonian cycles C_2 and $\hat{C}_2$ of $G^{\square\, 2}$ depicted on the left and right of Figure 1. Note that the constructions slightly change depending on the parity of $|G|$, but the following arguments hold for both. For the sake of contradiction, assume there is an automorphism $\varphi \in \mathrm{Aut}(G \,\square\, G)$ mapping C_2 to $\hat{C}_2$. Since every vertical edge in $\hat{C}_2$ is preceded and followed by horizontal edges (here we use $k \geq 4$), but all horizontal edges in C_2 are either preceded or followed by another horizontal edge, φ cannot swap the sets E_1 and E_2, so it preserves each. But this leads to a contradiction because there are $k - 1$ consecutive vertical edges in C_2, but not in $\hat{C}_2$. Also, note that both cycles contain at least $2(k-1) \geq 4$ edges from E_1 and from E_2 so that this completes the base case.

For the induction step, let $n \geq 3$. Let u, v and w be three consecutive vertices of C_{n-1}. Let C_n be the cycle obtained from C_{n-1} using u, v, and w as depicted in Figure 2. Note here that the vertical edges in Figure 2 are precisely the edges in E_n. Analogously, construct $\hat{C}_n$ from $\hat{C}_{n-1}$ and three consecutive vertices $\hat{u}, \hat{v}$ and $\hat{w}$ in $\hat{C}_{n-1}$. Note that

$$|E(C_n) \cap E_n| = 2(k-1) \geq 4. \tag{2}$$

Further, since C_n contains k copies of the cycle C_{n-1} with two edges removed and, by the induction hypothesis, C_{n-1} uses at least 4 edges from E_i, we have

$$|E(C_n) \cap E_i| \geq 2k \geq 4. \tag{3}$$

for all $i \in \{1, \ldots, n-1\}$. We obtain the equations (2) and (3) similarly for the cycle $\hat{C}_n$. It remains to prove that the cycles cannot be mapped to each other. Assume there is an automorphism $\varphi_n \in \mathrm{Aut}(G^{\square\, n})$ mapping C_n to $\hat{C}_n$. Using (2) and (3), we obtain that $|E(C_n) \cap E_n| = 2(k-1) < 2k \leq |E(\hat{C}_n) \cap E_i|$ for all $i \neq n$, so that φ has to preserve the set E_n. Thus, φ_n is of the form (φ_{n-1}, α) for some $\varphi_{n-1} \in \mathrm{Aut}(G^{\square\, n-1})$ and $\alpha \in \mathrm{Aut}(G)$. Note that both cycles C_n and $\hat{C}_n$ contain precisely one path of edges in E_n of length $k - 1$,

so that φ maps one to the other, i.e., the path $((v, g_1), \ldots, (v, g_k))$ in C_n is mapped to the path $((\hat{v}, g_1), \ldots, (\hat{v}, g_k))$ in $\hat{C}_n$. Hence, α either maps g_1 to g_1 or, in case α reverses the order of the path, to g_k. In both cases, φ_{n-1} maps C_{n-1} to $\hat{C}_{n-1}$, which contradicts the induction hypothesis. Therefore, C_n cannot be mapped to $\hat{C}_n$ by an automorphism. This completes the proof of the Lemma in case $k \geq 4$.

The case $|G| = 3$ can be proven by a similar analysis and is deferred to the full version. ◄

Now Theorem 4 follows immediately by combining Lemma 5 and Lemma 6. In case a graph decomposes into a Cartesian product of Hamiltonian graphs, we can deduce the following characterization of Hamiltonian-transitive graphs.

▶ **Corollary 7.** *Let G be a graph with decomposition $G = H_1^{\Box\, r_1} \Box \ldots \Box H_k^{\Box\, r_k}$ for pairwise distinct prime graphs $H_1, \ldots, H_k$. Assume that $H_1, \ldots, H_k$ are Hamiltonian. Then $G \in \mathcal{H}$ if and only if $G = H_1 \in \mathcal{H}$.*

2.2   Cartesian products with K_2

In the previous subsection, we assumed that all factors of the Cartesian product are Hamiltonian to ensure that the obtained graph is Hamiltonian as well. In this subsection, we extend our study to Cartesian products with K_2, which later turns out to be a crucial case for our work on Cayley graphs. More precisely, we prove the following result.

▶ **Theorem 8.** *Let G be a Hamiltonian graph. Then $G \Box K_2 \in \mathcal{H}$ if and only if $G = C_k$ for $k \geq 3$ odd or $G = C_4$.*

Throughout this section, we let $V(G) := \{v_0, \ldots, v_{k-1}\}$ such that $(v_0, \ldots, v_{k-1})$ is a Hamiltonian cycle of G. We denote the Cartesian product by taking an isomorphic copy of G on vertices $\{u_0, \ldots, u_{k-1}\}$ and letting v_i neighboring u_i for all $i \in \{0, \ldots, k-1\}$ (see Figure 3). We call the subgraph induced by $\{v_0, \ldots, v_{k-1}\}$ the *outer layer* and the subgraph induced by $\{u_0, \ldots, u_{k-1}\}$ the *inner layer*.

We split the proof of Theorem 8 into several lemmata. First, we show that, for $G = C_k$ with k odd, the Cartesian product $G \Box K_2$ is indeed Hamiltonian-transitive.

▶ **Lemma 9.** *Let $k \geq 3$ be odd. Then $C_k \Box K_2 \in \mathcal{H}$.*

Proof sketch. We argue that every Hamiltonian cycle contains precisely two edges leading from one layer to the other and that these have to lie next to each other, i.e., that the cycle depicted in the middle of Figure 3 is the only Hamiltonian cycle up to symmetry. To prove this, note first that the number of edges leading from one layer to the other used by a Hamiltonian cycle is even and at least two. We then prove that every Hamiltonian cycle uses either two consecutive such edges or all of them. Since k is odd, the latter is not possible. ◄

In the next two lemmata, we argue that in all other cases (where $|G| \geq 5$), the Cartesian product $G \Box K_2$ is not in $\mathcal{H}$. The difficulty is that we cannot assume any non-adjacency between vertices of G. More precisely, we will only use that $C_k \Box K_2$ is a subgraph of $G \Box K_2$ and that u_i is non-adjacent to v_j if $i \neq j$. First, we settle the case when $|G|$ is even.

▶ **Lemma 10.** *Let G be Hamiltonian with $k := |G| \geq 6$ even. Then $G \Box K_2 \notin \mathcal{H}$.*

Proof sketch. The key idea is to show that the Hamiltonian cycle on the left side in Figure 3 cannot be mapped to both, the cycle in the middle and the cycle on the right via a graph automorphism. For this, we use that the Hamiltonian cycle on the left has the property that every vertex is adjacent to precisely one of the two vertices at distance three in the cycle. ◄
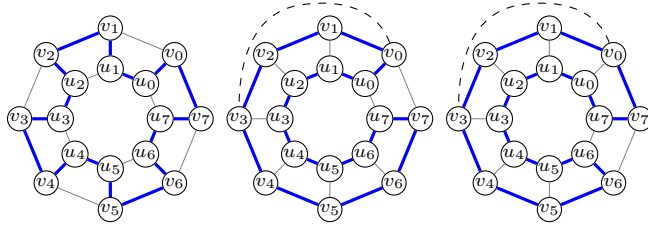
Figure 3 Illustration of the construction in Lemma 10. The possible edges inside a layer are omitted. Every vertex in the left Hamiltonian cycle is adjacent to precisely one of the vertices at distance three in the cycle. It is impossible that v_0 has this property in both, the cycle in the middle and on the right. The cycle on the left only exists when $|G|$ is even.
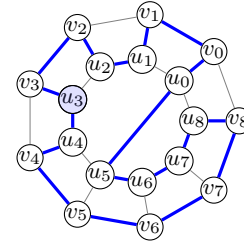
Figure 4 Illustration of the construction in Lemma 11. The cycle uses the existence of one additional edge inside a layer $\{u_0, u_5\}$. The marked vertex u_3 is not adjacent to the vertices at distance 5 along the cycle (v_1 and v_7).

Next, we conisder the case where $|G|$ is odd but not a cycle. Note that we need a different strategy here because the cycle we used in Lemma 10 (left side of Figure 3) only exists when $|G|$ is even.

▶ **Lemma 11.** *Let G be Hamiltonian with $|G| =: k$ odd and $G \neq C_k$. Then $G \square K_2 \notin \mathcal{H}$.*

Proof sketch. First, we show that, if $G \square K_2 \in \mathcal{H}$, then G fulfills some symmetry conditions. Then, we consider the cycle depicted in Figure 4 and use the symmetry conditions to show that it cannot be mapped to the cycle in the middle of Figure 3. ◀

The only case that is left to settle for Theorem 8 is when $|G| = 4$. Note that there are only 3 Hamiltonian graphs on 4 vertices and checking these (either by hand or computation), we obtain that $G = C_4$ is the only case where $G \square K_2 \in \mathcal{H}$. Together with Lemmas 9, 10, and 11, this completes the proof of Theorem 8.

3 Cayley graphs of abelian groups

In this section, we study Cayley graphs of abelian groups with the goal to find a characterization of the Hamiltonian-transitive ones. Recall that it is in general open whether all Cayley graphs of order ≥ 3 are Hamiltonian, but it is well known that this holds for abelian groups. In this section, we extend this result and show that graphs in a large family of Cayley graphs of abelian groups even contain multiple Hamiltonian cycles up to symmetry (Theorem 2 and Theorem 3).

3.1 Preliminaries

In this subsection, we collect some preliminary notation and results. In the following, we write all groups additively. Given a generating set S of an abelian group Γ, we write $\mathrm{Cay}(\Gamma, S)$ for the Cayley graph of Γ with respect to S. As we restrict ourselves to undirected graphs without loops, we always assume S to be closed under taking inverses and $0 \notin S$.

The following lemma follows immediately from the definition of Cayley graphs and Cartesian products.

▶ **Lemma 12.** *Let Γ_1 and Γ_2 be finite groups with generating sets S_1 and S_2, respectively. For $\Gamma = \Gamma_1 \times \Gamma_2$, we have $\mathrm{Cay}(\Gamma, S_1 \cup S_2) \cong \mathrm{Cay}(\Gamma_1, S_1) \square \mathrm{Cay}(\Gamma_2, S_2)$.*

In the result above, $\Gamma_1 \times \Gamma_2$ denotes the direct product of the two groups and, by slight abuse of notation, when S_1 is considered a subset of Γ, this actually refers to the set $\{(s, 0) : s \in S_1\}$, and analogously for S_2.

A useful parameter in the context of Cayley graphs is the *Hamilton compression*, introduced by Gregor et al. in [14]. A Hamiltonian cycle $C = (v_0, \ldots, v_{n-1})$ of a graph G on n vertices is called *k-symmetric* if k divides n and $v_i \mapsto v_{i+n/k \mod n}$ is an automorphism of G. We call this automorphism a *rotation of C by n/k*. Note that, if a cycle is k-symmetric, i.e., can be rotated by n/k, it can also be rotated by multiples of n/k as this corresponds to repeatedly applying the induced automorphism. In particular, a k-symmetric cycle can be rotated in both directions by n/k because a rotation by n/k in one direction corresponds to a rotation by $(k-1)n/k$ in the other direction. The *Hamilton compression of C* is then defined by $\kappa(C) := \max\{k : C \text{ is } k\text{-symmetric}\}$. For a Hamiltonian graph G, the *Hamilton compression of G* is defined by $\kappa(G) := \max\{k : \text{there is a } k\text{-symmetric Hamiltonian cycle of } G\}$. Note that, by definition, $\kappa(G)$ divides n.

Since the Hamilton compression of a Hamiltonian cycle is preserved under applying automorphisms, we immediately obtain the following result.

▶ **Lemma 13.** *Let G be a graph. If $G \in \mathcal{H}$, then all Hamiltonian cycles of G have the same Hamilton compression.*

The next result will be used to split Cayley graphs into Cartesian products.

▶ **Lemma 14.** *Let Γ be an abelian group and p be a prime divisor of $|\Gamma|$. Let S be a generating set of Γ such that the order of all elements in S is either a power of p or coprime to p. Setting $S_p := \{s \in S : p \mid \text{ord}(s)\}$ and $S'_p := S \setminus S_p$, we have $\Gamma = \langle S_p \rangle \times \langle S'_p \rangle$.*

The following statement is a direct consequence of the proof of [14, Theorems 6.4 and 6.6].

▶ **Lemma 15.** *Let $G \in \mathcal{H}$ and assume that $G = \text{Cay}(\Gamma, S)$ for an abelian group Γ and a generating set S.*
1. *Suppose that $|\Gamma|$ is odd. If S contains an element of order pm for a prime p and $m > 1$, then p divides $\kappa(G)$.*
2. *If $|\Gamma|$ is even, then $\kappa(G)$ is even.*

Last, many arguments that we use in this section only work for graphs that are not too small. Therefore, the following result ensures that we do not have to take care of these cases.

▶ **Lemma 16.** *Let G be the Cayley graph of an abelian group Γ with respect to an inverse-closed generating set.*
- *Assume that $n := |\Gamma| \leq 16$. Then $G \in \mathcal{H}$ if and only if one of the following holds*
  - *$G \in \{K_n, C_n, C_4 \square K_2\}$,*
  - *$G = K_{k,k}$ where $k = n/2$,*
  - *$G = C_k \square K_2$ where $k = n/2$ is odd.*
- *Assume that $\Gamma \in \{\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_9 \times \mathbb{Z}_3\}$. Then $G \in \mathcal{H}$ if and only if $G = K_{27}$.*

We confirmed this lemma via exhaustive generation of all Cayley graphs using Sage. For most of the graphs, we could easily find two Hamiltonian cycles which could not be mapped onto one another. Further, we excluded some more graphs by counting all their Hamiltonian cycles and comparing this number to the size of the automorphism group. For the remaining graphs we check by brute-force if they are in $\mathcal{H}$.

3.2 Cyclic groups

In this section, we characterize Hamiltonian-transitive Cayley graphs of cyclic groups whose generating set contains an element whose order equals the order of the group. This later serves as a base for Cayley graphs of abelian groups.

▶ **Theorem 17.** *Let $\Gamma \cong \mathbb{Z}_n$ and S be an inverse-closed generating set of Γ containing an element of order n. Let $G = \mathrm{Cay}(\Gamma, S)$. Then, the following are equivalent:*
1. $G \in \mathcal{H}$,
2. *every Hamiltonian cycle C in G has Hamilton compression $\kappa(C) = n$,*
3. *we have $G \in \{K_n, C_n\}$ or $n = 2m$ is even and $G = K_{m,m}$.*

Proof. We first show that (3) implies (1). Clearly, we have $K_n, C_n \in \mathcal{H}$. To see that also $K_{m,m} \in \mathcal{H}$, let $A \sqcup B$ be the bipartition of $K_{m,m}$ and observe that every Hamiltonian cycle of $K_{m,m}$ alternately uses a vertex of A and B. Since every combination of permutations of A and B is an automorphism of $K_{m,m}$, every Hamiltonian cycle can be mapped to every other Hamiltonian cycle by an automorphism of $K_{m,m}$.

For the remaining implications, let us first argue that we can assume $1 \in S$: by assumption, there exists an element $x \in S$ of order n. Note that the group automorphism $\varphi(kx) := k$ induces an isomorphism between $\mathrm{Cay}(\Gamma, S)$ and $\mathrm{Cay}(\Gamma, \varphi(S))$, so we can assume w.l.o.g. that $x = 1$.

Next, we show that (1) implies (2). Consider the Hamiltonian cycle $C = (0, 1, \ldots, n-1)$ of G. Since, $y \mapsto y + 1$ defines an automorphism of G, the cycle C can be rotated by 1 so that the Hamilton compression of C is n. Since $G \in \mathcal{H}$, by Lemma 13 every cycle of G has Hamilton compression n.

It remains to show that (2) implies (3). Thus from now on, we assume that every Hamiltonian cycle in G has Hamilton compression n and we aim to show that G is isomorphic to K_n, C_n, or $K_{m,m}$. If $G \not\cong C_n$, we have $k \in S$ for some $2 \le k < n - 1$. Then we consider the Hamiltonian cycle $C' = (0, k, k+1, k+2, \ldots, n-1, k-1, k-2, \ldots, 1)$ in G. Since C' has Hamilton compression n, rotating C' by any $y \in \mathbb{N}$ yields an automorphism of G. For every $y \in \{1, \ldots, \min\{k-1, n-k-1\}\}$, this automorphism maps the edge $\{k-1, k\}$ to $\{k-1-y, k+y\}$. The existence of this edge implies $(k+y) - (k-1-y) = 2y+1 \in S$, which shows that $\{1, 3, 5, \ldots, \min\{2k-1, 2(n-k)-1\}\} \subseteq S$.

We have shown that, for every $k \in S$, we have $\{1, 3, 5, \ldots, \min\{2k-1, 2(n-k)-1\}\} \subseteq S$. Applying this repeatedly, we obtain that S contains all odd numbers up to $n - 1$. In particular, if n is odd, we obtain that $S = \{1, \ldots, n-1\}$ because S is inverse-closed. This means that $G \cong K_n$.

Now assume that n is even. In this case, we obtain that $A := \{1, 3, 5, \ldots, n-1\} \subseteq S$. If equality holds, we have $G \cong K_{m,m}$. Otherwise, G is a supergraph of $K_{m,m}$ and S contains an element $2l$ for some $l < \frac{n}{2}$. We show that, for every $l' \le \frac{n}{2}$, we also have $2l' \in S$: Since S contains all elements in A, note that every permutation of $(0, 1, 2, \ldots, n-1)$, where every second element is in A, is a Hamiltonian cycle of G. Let C' be the Hamiltonian cycle obtained from $(0, \ldots, n-1)$ by replacing $2l$ and $2l'$. Rotating C' by $n - 1$ (i.e., by 1 in the other direction) maps the edge $\{1, 2l+1\}$ to $\{0, 2l'\}$, and hence, $2l' \in S$. This shows that $S = \Gamma \setminus \{0\}$, i.e., $G \cong K_n$.   ◀

3.3 Layer structure theorem

In this subsection, we investigate a general framework for graphs containing a grid-like structure and having non-trivial Hamilton compression. As we explain later, Cayley graphs of abelian groups fulfill these conditions and thus fall within the scope of this subsection.
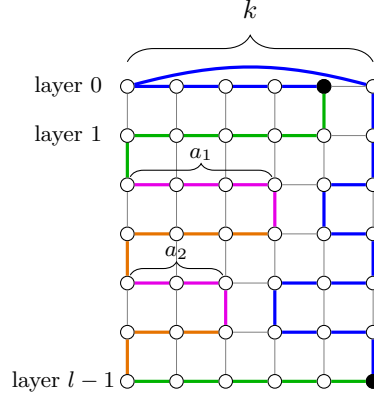
Figure 5 Zigzag cycle for $l = 7$ odd and $\boldsymbol{a} = (3, 2)$. The filled vertices are the endpoints of the described segment of length $\mu(C)$.

Note that despite the fact that Cartesian products do have a layer structure, they do not necessarily have non-trivial Hamilton compression, so the results from this section cannot be applied in that case.

We begin by formalizing the grid-like structure with which we work.

▶ **Definition 18** (Layer structure). *Let G be a graph and $l \in \mathbb{N}_{\geq 1}$. If there is a decomposition $V(G) = V_1 \,\dot\cup\, \ldots \,\dot\cup\, V_l$ such that*

1. *for $i = 1, \ldots, l - 1$, G contains a perfect matching between V_i and V_{i+1} that defines an isomorphism between $G[V_i]$ and $G[V_{i+1}]$, and*

2. *the graph $K := G[V_1] \cong \ldots \cong G[V_l]$ is Hamiltonian,*

we say that G has K-l-layer structure. In this case, we call $G[V_1], \ldots, G[V_l]$ the layers *of G.*

Note that, if G has K-l-layer structure, then G has a spanning subgraph isomorphic to $K \,\square\, P_l$. Given some ordering of the vertices of K (in the following usually given by a Hamiltonian cycle of K), by $v_{j,i}$ we denote the i-th vertex in the j-th layer of G. Next, we construct a class of Hamiltonian cycles that every graph with layer structure contains. Since we will only use this construction for graphs of odd order (in particular, l will always be odd), we restrict ourselves to this case. However, note that this construction can be easily generalized to graphs of even order (but some of the steps later throughout the proofs cannot be generalized for even order).

▶ **Definition 19** (Zigzag cycle). *Assume that G has K-l-layer structure with $l \geq 5$ odd and set $k := |K|$. Let $C_K = (v_0, \ldots, v_{k-1})$ be a Hamiltonian cycle of K. Given a tuple $\boldsymbol{a} = (a_1, a_2, \ldots, a_{(l-3)/2}) \in \{0, \ldots, k-2\}^{(l-3)/2}$, we define the C_K-$\boldsymbol{a}$-zigzag cycle as the Hamiltonian cycle illustrated in Figure 5. For a zigzag cycle C, we define $\mu(C)$ to be the number of edges in the segment between $v_{0,k-1}$ and $v_{l-1,k-1}$ that contains vertex $v_{1,0}$, i.e., the length of the segment consisting of the green, pink, and orange line segments in Figure 5.*

The following lemma is a direct consequence of the definition and shows that, by suitably choosing $\boldsymbol{a}$, many desirable values for $\mu(C)$ for a zigzag cycle C can be achieved.

▶ **Lemma 20.** *Let G be a graph with K-l-layer structure, where $l \geq 5$ is odd, let $k := |K|$, and let C be the C_K-$\boldsymbol{a}$-zigzag cycle for some choice of C_K and $\boldsymbol{a}$. Then we have*

$$\mu(C) = 2k - 1 + 2 \sum_{i=1}^{(l-3)/2} (a_i + 1).$$

In particular, we obtain that $\mu(C)$ is odd. Even further, for any odd number $\mu^* \in \mathbb{N}$ with $2k - 1 + (l - 3) \leq \mu^* \leq 2k - 1 + (l - 3)(k - 1)$, there is a choice for $\boldsymbol{a}$ such that the C_K-$\boldsymbol{a}$-zigzag cycle C fulfills $\mu(C) = \mu^*$.

Now, we have all the prerequisites in place to prove the main result of this subsection.

▶ **Theorem 21.** *Let $G \in \mathcal{H}$ be a graph of odd order that has K-l-layer structure and let $k := |K|$. Assume that one of the following conditions holds*
1. *$k, l \geq 7$ and $\kappa(G) \geq 5$*
2. *$k, l \geq 5$ and, for every prime p with $p|n := |G|$, we have $p|\kappa(G)$.*
Then K is isomorphic to C_k or K_k.

Proof. The strategy is to show that, for every Hamiltonian cycle C_K of K, we have $\kappa(C_K) = k$. By [14, Section 1.4], this implies that K is the Cayley graph of a cyclic group and the generating set contains an element of order k. The assertion then follows from Theorem 17.

Fix a Hamiltonian cycle C_K of K. To prove that $\kappa(C_K) = k$, we will make use of the following claim.

▷ **Claim 22.** There is a choice for $\boldsymbol{a}$ such that the C_K-$\boldsymbol{a}$-zigzgag cycle C fulfills that $\mu(C)$ is a multiple of $\alpha := n/\kappa(G)$.

The proof of this claim is deferred to the full version and we argue here why it implies the Theorem. Since $G \in \mathcal{H}$, every Hamiltonian cycle of G, in particular C, is $\kappa(G)$-symmetric. Therefore, rotating C by a multiple of α defines an automorphism of G. Together with Claim 22, this means that C can be rotated by $\mu(C)$ (in counter-clockwise direction in Figure 5). This maps the first layer of G bijectively to the last, inducing an automorphism of K. More precisely, $v_{0,j}$ is mapped to $v_{l-1,j+1 \mod k}$. The induced automorphism of K then maps v_j to $v_{j+1 \mod k}$, i.e., it rotates C_K by 1. Therefore, $\kappa(C_K) = k$. ◀

3.4 Cayley graphs of odd order

In this subsection, we prove our main result.

▶ **Theorem 2.** *Let G be the Cayley graph of an abelian group w.r.t. some inverse-closed generating set and assume that $n := |G| \geq 3$ is odd. Then $G \in \mathcal{H}$ if and only if $G \in \{K_n, C_n\}$.*

As a first step, let us argue that Cayley graphs of abelian groups have a natural layer structure, which will allow us to use results from the previous subsection. For this, let Γ be an abelian group with generating set S and $G = \mathrm{Cay}(\Gamma, S)$. Assume that $|G| \geq 3$. Let $S' \subseteq S$ such that $\langle S' \rangle \cap S = S'$ and $|\langle S' \rangle| \geq 3$. Then G has K-l-layer structure (Definition 18) with $K := \mathrm{Cay}(\Delta, S')$ and l is the index of $\Delta := \langle S' \rangle$, i.e., $l := |\Gamma : \Delta|$:

To see this, note that the cosets of Δ, given by $\{\gamma + \Delta : \gamma \in \Gamma\}$, provide a partition of the vertices of G into l sets. Since $\langle S' \rangle \cap S = S'$, for each of these sets, the induced subgraph is $G[\gamma + \Delta] \cong \mathrm{Cay}(\Delta, S') = K$. Moreover, since K is the Cayley graph of an abelian group on at least 3 vertices, it is Hamiltonian. Similarly, $G' := \mathrm{Cay}(\Gamma/\Delta, S \setminus S')$ is the Cayley graph of an abelian group and therefore traceable, i.e., it contains a Hamiltonian

path $(\gamma_1 + \Delta, \ldots, \gamma_l + \Delta)$. This means that, for $i \in \{1, \ldots, l-1\}$, there exists some $s_i \in S \setminus S'$ such that $s_i + \gamma_i + \Delta = \gamma_{i+1} + \Delta$. Then the edges of G corresponding to s_i form a perfect matching between $\gamma_i + \Delta$ and $\gamma_{i+1} + \Delta$. Moreover, since $x \mapsto x + s_i$ is an automorphism of G, the matching defines an isomorphism between $\gamma_i + \Delta$ and $\gamma_{i+1} + \Delta$. Therefore, G has K-l-layer structure as stated. We call such a layer structure *group-induced*.

In the next lemma, we make use of this structure and our results from the previous section.

▶ **Lemma 23.** *Let Γ be an abelian group of odd order with generating set S. Suppose that the Cayley graph $G = \mathrm{Cay}(\Gamma, S)$ is Hamiltonian-transitive and $\kappa(G) > 1$. Assume that G has a group-induced K-l-layer structure with $|K|, l \geq 5$. Then K is complete or a cycle.*

Proof sketch. The key idea is to show that condition (2) in Theorem 21 is fulfilled, which then immediately implies the assertion, i.e., it is only left to prove that, for every prime number p that divides $|G|$, p also divides $\kappa(G)$. For this, we distinguish two cases on the prime factors of the order of elements in S. In one case, the assertion follows by Lemma 15. In the other case, we use Lemma 14 to decompose Γ into a direct product, Lemma 12 to decompose G into a Cartesian product and Lemma 5 to deduce that one factor is trivial. ◀

In the following two results, we build on Lemma 23 and further resolve the cases where K is complete or a cycle.

▶ **Lemma 24.** *Let Γ be an abelian group with generating set S. Suppose that the Cayley graph $G = \mathrm{Cay}(\Gamma, S)$ is Hamiltonian-transitive and $\kappa(G) > 1$. Assume that G has a group-induced K-l-layer structure where K is a complete graph with $|K| \geq 5$. Then G is complete.*

Proof sketch. First, we argue that, if two vertices of different layers are adjacent, then all vertices of the layers are adjacent. For this, we construct a suitable Hamiltonian cycle and rotate it using $\kappa(G) > 1$. Second, we argue that every pair of layers has adjacent vertices by constructing structurally different Hamiltonian cycles if this is not the case. ◀

▶ **Lemma 25.** *Let Γ be an abelian group of odd order with generating set S. Suppose that the Cayley graph $G = \mathrm{Cay}(\Gamma, S)$ is Hamiltonian-transitive and $\kappa(G) > 1$. Assume that G has a group-induced K-l-layer structure with $k := |K| \geq 5$ and $l \geq 5$. Then we have $K \not\cong C_k$.*

Proof sketch. For the sake of contradiction, assume that K is a cycle. Since $\kappa(G) > 1$, every Hamiltonian cycle has a non-trivial rotation. By suitably choosing the parameter $\boldsymbol{a}$ of the zigzag cycle (Definition 19), we obtain that this rotation maps two non-adjacent vertices in the first layer to two adjacent vertices in different layers, which is a contradiction. ◀

The last step before concluding the main theorem is to settle the cases where $|K|$ or l is small.

▶ **Lemma 26.** *Let Γ be an abelian group of odd order with generating set S. Suppose that the Cayley graph $G = \mathrm{Cay}(\Gamma, S)$ is Hamiltonian-transitive and $\kappa(G) > 1$. Assume that, for all $s \in S$, $\mathrm{ord}(s) < 5$ or $|\Gamma : \langle s \rangle| < 5$. Then $\Gamma = \mathbb{Z}_3^t$ for $t \geq 4$ or G is complete or a cycle.*

Proof sketch. We distinguish three cases: either S contains a generator; or all elements in S have order 3; or there exists $s \in S$ with $|\Gamma : \langle s \rangle| = 3$. In the first case, the assertion follows from Theorem 17. In the second case, Γ is a power of $\mathbb{Z}_3$ and we conclude with Lemma 16. In the last case, we show that $\kappa(G)$ is a multiple of three and rotating a suitable Hamiltonian cycle by $|G|/3$ yields a contradiction. ◀

Now, we have all the tools at hand to prove Theorem 2.

Proof of Theorem 2. First, assume that $\kappa(G) = 1$. By [14, Theorem 6.4], this implies $S = \{s_{p_1}, \ldots, s_{p_r}\}$ for pairwise distinct prime numbers $p_1, \ldots, p_r$ such that the element s_{p_i} has order p_i for $i = 1, \ldots, r$. By Lemmas 12 and 14, we obtain the Cartesian product decomposition $G = \mathrm{Cay}(\langle s_{p_1}\rangle, \{s_{p_1}\}) \square \ldots \square \mathrm{Cay}(\langle s_{p_r}\rangle, \{s_{p_r}\})$. Theorem 4 then yields $r = 1$, and thus $G = C_{p_1}$ is a cycle.

From now on, assume $\kappa(G) > 1$. By Lemma 26, we obtain that $\Gamma = \mathbb{Z}_3^t$ for $t \geq 4$; or G is complete; or a cycle; or there exists $s \in S$ of order at least 5 and $|\Gamma : \langle s\rangle| \geq 5$. In the second and third cases, there is nothing to do. In the first case, we can choose $s_1, s_2 \in S$ such that $\Delta := \langle S'\rangle := \langle s_1, s_2\rangle$ has order 9 and $|\Gamma : \Delta| \geq 9$. Similarly, in the last case, we can set $S' := \{s\}$, so that $\Delta = \langle S'\rangle$ has order at least 5 and index at least 5. Consider the K-l-layer structure given by S' as described at the beginning of this section. By Lemma 23, $K = \mathrm{Cay}(\Delta, S')$ is a cycle or a complete graph. The first case is excluded by Lemma 25. In the second, G is a complete graph by Lemma 24.   ◀

3.5   Cayley graphs of even order

In the previous subsection, we studied Cayley graphs of odd order. For the even case, we require somewhat different techniques and present them in this subsection. Our main result for Cayley graphs of even order is the following, where we rely on a mild non-redundancy condition on the generating set. However, we conjecture that the statement can be generalized to all Cayley graphs of abelian groups (with the additional exceptions of K_n and $K_{n/2,n/2}$).

▶ **Theorem 3.** *Let G be the Cayley graph of an abelian group w.r.t. some inverse-closed generating set S and assume that $n := |G| \geq 4$ is even. Moreover, assume that there is some $s \in S$ such that $S \setminus \{s, -s\}$ is non-generating. Then $G \in \mathcal{H}$ if and only if $G \in \{C_n, C_4 \square K_2, K_{4,4}\}$ or $G = C_k \square K_2$ where $k = n/2$ is odd.*

We first prove Theorem 3 for the case $l := |\langle S\rangle/\langle S \setminus \{\pm s\}\rangle| \geq 4$. Subsequently, the cases $l \in \{2, 3\}$ will be studied separately.

▶ **Lemma 27.** *Let Γ be an abelian group of even order with generating set S and $G = \mathrm{Cay}(\Gamma, S)$. Suppose that there exists an $s \in S$ such that $l = |\Gamma : \langle S \setminus \{\pm s\}\rangle| \geq 4$. Then $G \in \mathcal{H}$ if and only if $G = C_n$ or $G = C_l \square K_2$, where l is odd or $l = 4$.*

Proof sketch. From Theorems 8 and 17, we already know that $C_n, C_4 \square K_2$, and $C_l \square K_2$ for l odd are in $\mathcal{H}$. For the other direction, assume $G \in \mathcal{H}$. Let $S' := S \setminus \{\pm s\}$ and $K := \mathrm{Cay}(\langle S'\rangle, S')$. The cases of $|K| = 2$ and $|K| = 3$ are dealt with separately. In case $|K| \geq 4$, the key idea is to use the group-induced K-l-layer structure and exploit that only edges corresponding to s and $-s$ connect different layers. Then we consider the Hamiltonian cycles C and $\hat{C}$ of G as depicted on the left and in the middle of Figure 6. Next, we show that every automorphism that maps C to $\hat{C}$ maps the green vertices a, b, c and d in C to $V_2 \cup V_{l-1}$ in $\hat{C}$. Then we exploit the distance and order of the vertices a, b, c and d on the cycle C, as well as the fact that a and d are adjacent, to obtain a contradiction by doing several case distinctions.   ◀

In the next two lemmas, we settle the cases where $l \in \{2, 3\}$.

▶ **Lemma 28.** *Let Γ be an abelian group with generating set S and let $G = \mathrm{Cay}(\Gamma, S)$. Suppose that there exists an $s \in S$ such that $l = |\Gamma : \langle S \setminus \{\pm s\}\rangle| = 2$. Then $G \in \mathcal{H}$ if and only if $G = K_{4,4}$ or $G = C_k \square K_2$, where k is odd or $k = 4$.*
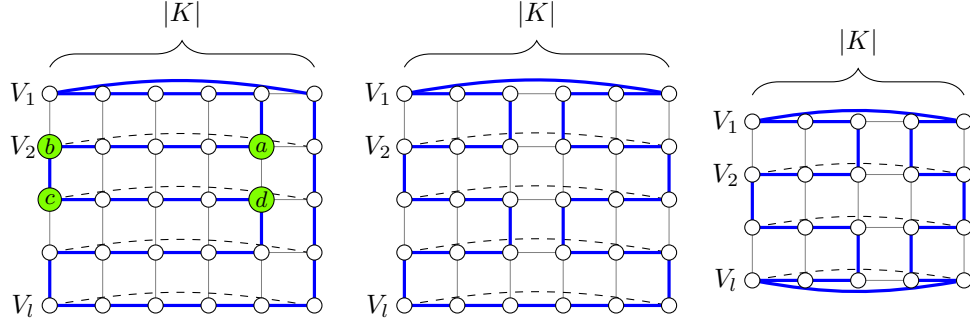
Figure 6 Illustrations for the Hamiltonian cycles in the proof of Lemma 27. The left subfigure depicts the cycle C, the middle one $\hat{C}$ in case $|K|$ even, and the right one $\hat{C}$ for $|K|$ odd. The vertical edges in the middle of $\hat{C}$ are at columns $\lceil |K|/2 \rceil$ and $\lceil |K|/2 \rceil + 1$.

▶ **Lemma 29.** *Let Γ be an abelian group of even order with generating set S and $G = \mathrm{Cay}(\Gamma, S)$. Suppose that there exists an $s \in S$ such that $l = |\Gamma : \langle S \setminus \{\pm s\} \rangle| = 3$. Then $G \notin \mathcal{H}$.*

Proof sketch for Lemmas 28 and 29. We use that G has a group-induced K-l-layer structure (for $l = 2$, respectively $l = 3$) to construct Hamiltonian cycles of G from a Hamiltonian cycle of K. The key idea in both cases is to use that $\kappa(G)$ is even by Lemma 15. This means that every Hamiltonian cycle of G can be rotated by $k := |K|$ in the case $l = 2$, and by $3k/2$ in the case $l = 3$.

In the case $l = 3$, we construct a Hamiltonian cycle of G where this rotation leads to a contradiction. In the case $l = 2$, by suitably choosing Hamiltonian cycles of G, the rotation by k induces symmetries of K. In addition, we use that the case of Cartesian products is already settled in Theorem 8, so we can assume that G consists of two copies of K connected by two disjoint matchings. Using the second matching and the symmetries of K, we then construct Hamiltonian cycles of G that cannot be mapped to each other by an automorphism.   ◀

Theorem 3 now follows immediately by combining Lemmas 27, 28, and 29.

4 Constructions of regular Hamiltonian-transitive graphs

In this section, we give two constructions for families of regular Hamiltonian-transitive graphs. Somewhat in contrast to Section 2 and Section 3, this underlines that there are many regular Hamiltonian-transitive graphs.

▶ Remark 30. For every $d \geq 2$, there are infinitely many d-regular Hamiltonian-transitive graphs in $\mathcal{H}$: We construct such graphs as follows. For an arbitrary $n \geq 3$, take C_n and n disjoint copies of K_{d+1}. Remove one edge from each copy of K_{d+1} and replace each vertex v of C_n by a copy of K_{d+1} such that the two incident edges to v are now incident to the two vertices of degree $d - 1$. See Figure 7 for an illustration. This yields a d-regular Hamiltonian graph. It is Hamiltonian-transitive because every Hamiltonian cycle uses all of the edges of C_n and visits the vertices in each copy of K_{d+1} in some arbitrary order. It is immediate that this order can be permuted by applying an automorphism.

This is in strong contrast to uniquely Hamiltonian graphs, where it is known that no uniquely Hamiltonian d-regular graphs exist for all odd values of d and all $d > 22$ [35, 36, 17], and Sheehan's Conjecture asserts that no such graphs exist for any $d \geq 3$ [34].
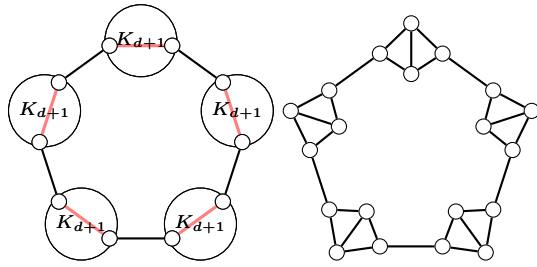
Figure 7 The construction of the graphs in Remark 30 for $n = 5$. In each K_{d+1} the removed edge is indicated in red. On the right you can see the construction for $d = 3$..
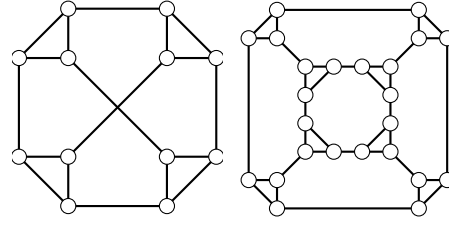
Figure 8 The truncation of K_4 and the cube. These provide further examples of cubic vertex-transitive Hamiltonian-transitive graphs besides the ones from Theorem 3.

Note that the constructed graphs in Remark 30 are not 3-connected and this forces certain edges to be contained in every Hamiltonian cycle. Opposed to this, for cubic graphs, we construct an infinite family of 3-connected Hamiltonian-transitive graphs. This shows that there are many different cubic Hamiltonian-transitive graphs besides the odd prisms from Theorem 3. In particular, this construction yields a family of Hamiltonian-transitive graphs with an unbounded number of edge orbits. An *edge orbit* of a graph is an equivalence class of edges under the action of $\mathrm{Aut}(G)$. More precisely, we show the following.

▶ **Proposition 31.** *There exists a family of cubic 3-connected Hamiltonian-transitive graphs with an unbounded number of edge orbits.*

For this, we use the truncation of a cubic graph, which is a classical concept often used in the literature (e.g., [8, 15, 31]). The *truncation* $\mathrm{T}(G)$ of a cubic graph G is the cubic graph obtained by replacing each vertex v in G with a triangle and adding for each edge $\{u, v\}$ of G an edge between two vertices of the triangles corresponding to u and v. This is illustrated in Figure 8. For a vertex v in G with neighbors $\{x, y, z\}$, we denote the corresponding vertices in $\mathrm{T}(G)$ by $\{v_x, v_y, v_z\}$. Note that the truncation of a 3-connected graph is again 3-connected.

Now we show that the truncation operation preserves Hamiltonian transitivity. Then repeatedly applying it yields a family of Hamiltonian-transitive graphs.

▶ **Lemma 32.** *If G is a cubic graph in $\mathcal{H}$, then $\mathrm{T}(G) \in \mathcal{H}$.*

Proof sketch. We show that every Hamiltonian cycle in $T(G)$ arises from a Hamiltonian cycle in G by mapping consecutive vertices (u, v, w) to $(u_v, v_u, v_x, v_w, w_v)$, where x is the third neighbor of v in G. This construction is compatible with the extension of an automorphism φ of G to an automorphism of $T(G)$ mapping x_y to $\varphi(x)_{\varphi(y)}$. ◀

In the next result, we argue that the truncation operation increases the number of edge orbits. While this is well-known, we include the proof in the full version for self-containment. For an edge e, we denote by $\ell(e)$ the length of the shortest cycle containing e. This clearly is invariant under graph automorphisms.

▶ **Lemma 33.** *Let $k \geq 3$ and G be a cubic graph such that all edges e of G satisfy $\ell(e) = k$. Then the family $\{\mathrm{T}^n(G)\}_{n \in \mathbb{N}}$ has an unbounded number of edge orbits.*

Now Proposition 31 follows from Lemma 32 and Lemma 33 applied to the base graph K_4. Repeatedly applying the truncation operation to other cubic Hamiltonian-transitive base graphs, e.g., to odd prisms, yields even more families of graphs in $\mathcal{H}$. This gives rise to an even broader class of Hamiltonian-transitive cubic graphs.

5   Graphs with many Hamiltonian cycles up to symmetry

In the previous sections, we studied which graphs have a unique Hamiltonian cycle up to symmetry. More generally speaking, we investigated the number of Hamiltonian cycles of a graph up to symmetry, with a focus on the special cases where this number is one. In this section, we take a look at the other end of the spectrum, i.e., we consider graphs that have as many Hamiltonian cycles up to symmetry as possible.

Let G be a graph with n vertices. Then, G has at most $\frac{n!}{2n}$ many Hamiltonian cycles. For complete graphs, this bound is tight. Thus, asymptotically the maximum number of Hamiltonian cycles of a graph on n vertices is in $\Theta((n-1)!) = 2^{\Theta(n \log(n))}$. We show that there are graphs that asymptotically have this many Hamiltonian cycles even up to symmetry.

▶ **Proposition 34.** *For every $n \in \mathbb{N}$, there exists a graph on n vertices with $2^{\Theta(n \log n)}$ many Hamiltonian cycles up to symmetry.*

Proof sketch. We consider the graphs $G_n := K_n \setminus C_n$ and argue that G_n has at least $\Omega((n-4)!) = 2^{\Omega(n \log n)}$ Hamiltonian cycles while the size of the automorphism group is only $|\operatorname{Aut}(K_n \setminus C_n)| = |\operatorname{Aut}(C_n)| = 2n$.   ◀

6   Conclusion and future work

In this paper, we initiated the study of the number of symmetry classes of Hamiltonian cycles in a graph. We mainly focused on graphs with a unique Hamiltonian cycle up to symmetry. We showed that this graph class contains d-regular graphs for every $d \geq 2$ (Remark 30) and broad families of 3-connected cubic graphs (Proposition 31).

Moreover, we studied Cartesian products and provided conditions on the factors for the product to be Hamiltonian-transitive (Theorem 4). If one factor is K_2 and the other Hamiltonian, we gave a full characterization (Theorem 8).

Further, by explicitly constructing Hamiltonian cycles, we proved that most Cayley graphs over abelian groups have more than one symmetry class of Hamiltonian cycles (Theorem 2, Theorem 3). Based on these results, we conjecture the following.

▶ **Conjecture 35.** *Let Γ be a finite abelian group with generating set S and let $G = \operatorname{Cay}(G, S)$. Then $G \in \mathcal{H}$ if and only if $G \in \{C_4 \square K_2, C_n, K_n, K_{n,n}, C_k \square K_2 \colon n, k \in \mathbb{N}, k \text{ odd}\}$.*

Finally, we showed that there are graphs with asymptotically as many symmetry classes of Hamiltonian cycles as possible (Proposition 34).

Overall, we believe that the enumeration and analysis of substructures in graphs up to symmetry is an important topic that should be further pursued. For example, we suggest the following research directions:

- investigate Hamiltonian transitivity for further classes of graphs, for instance, further Cayley graphs that are known to be Hamiltonian,
- study the computational aspects of this problem, for instance, by developing algorithms for enumerating Hamiltonian cycles up to symmetry,
- explore analogous notions of transitivity for other substructures in graphs, such as perfect matchings.

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