

Cops and Robbers for Graphs on Surfaces with Crossings

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Abstract

Cops and Robbers is a game played on a graph where a set of cops attempt to capture a single robber. The game proceeds in rounds, where each round first consists of the cops' turn, followed by the robber's turn. In the first round, the cops place themselves on a subset of vertices, after which the robber chooses a vertex to place himself. From the next round onwards, in the cops' turn, every cop can choose to either stay on the same vertex or move to an adjacent vertex, and likewise the robber in his turn. The robber is considered to be captured if, at any point in time, there is some cop on the same vertex as the robber. The cops win if they can capture the robber within a finite number of rounds; else the robber wins.

A natural question in this game concerns the *cop-number* of a graph – the minimum number of cops needed to capture a robber. It has long been known that graphs embeddable (without crossings) on surfaces of bounded genus have bounded cop-number. In contrast, it was shown recently that the class of 1-planar graphs – graphs that can be drawn on the plane with at most one crossing per edge – does not have bounded cop-number. This paper initiates an investigation into how the distance between crossing pairs of edges influences a graph's cop number. In particular, we look at *Distance d Cops and Robbers*, a variant of the classical game, where the robber is considered to be captured if there is a cop within distance d of the robber.

Let $c_d(G)$ denote the minimum number of cops required in the graph G to capture a robber within distance d . We look at various classes of graphs, such as 1-plane graphs, k -plane graphs (graphs where each edge is crossed at most k times), and even general graph drawings, and show that if every crossing pair of edges can be connected by a path of small length, then $c_d(G)$ is bounded, for small values of d . For example, we show that if a graph G admits a drawing in which every pair of crossing edges is contained in a path of length at most 3, then $c_4(G) \leq 21$. And if the drawing permits a stronger assumption that the endpoints of every crossing induce the complete graph K_4 , then $c_3(G) \leq 9$. The tools and techniques that we develop in this paper are sufficiently general, enabling us to examine graphs drawn not only on the sphere but also on orientable and non-orientable surfaces.

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1 Introduction

Pursuit-evasion is a general mathematical framework for problems that involve a set of *pursuers* attempting to capture a set of *evaders*. Such problems have many applications in robotics [21], network security [15], surveillance [9], etc. The game of *Cops and Robbers*,



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introduced independently by Quilliot [29], and Nowakowski and Winkler [27], belongs to the family of pursuit-evasion problems where a set of cops (the pursuers) attempt to capture a single robber (the evader) on a graph. The game is played in rounds, with each round consisting first of the *cops' turn*, followed by the *robber's turn*. Initially, the cops first position themselves on a set of vertices, after which the robber chooses a vertex to place himself. In subsequent rounds, the cops' turn consists of each cop either staying on the same vertex or moving to a neighbouring vertex, and likewise the robber in his turn. The game ends when the robber is *captured* by a cop; this happens when the robber is on the same vertex as some cop. If the robber has a strategy to permanently evade the cops, then the robber wins; else the cops win. (See the book [14] by Bonato and Nowakowski for an extensive introduction to Cops and Robbers.)

One of the central questions in Cops and Robbers is to identify graph classes that require few cops to capture the robber. Formally, the *cop-number* of a graph G , denoted by $c(G)$, is defined as the minimum number of cops required to capture the robber. A graph class \mathcal{G} is said to be *cop-bounded* if $c(G) \leq p$ for some integer p and all graphs $G \in \mathcal{G}$; else \mathcal{G} is *cop-unbounded*. Many graph classes are known to be cop-bounded; for example, chordal graphs [27, 29], planar graphs [1], graphs of bounded genus [30], H -minor-free graphs [3] and H -(subgraph)-free graphs [26]. On the other hand, examples of graph classes that are cop-unbounded include bipartite graphs [14], d -regular graphs, for all $d \geq 3$ [2], 1-planar graphs [23], and graphs of diameter 2 [12]. One of the deepest open problems on cop-number is Meyniel's conjecture which states that $c(G) \in O(\sqrt{n})$ for all n -vertex graphs G (see [4] for a survey paper on Meyniel's conjecture).

1.1 Survey of Related Results

In this paper, we are interested in the cop-number of graphs drawn on (orientable and non-orientable) surfaces with crossings. Most results in this area are concerned with cop-number of graphs embedded without crossings on surfaces with genus g . For planar graphs, which are graphs embeddable on the sphere ($g = 0$), Aigner and Fromme [1] proved that 3 cops are sufficient, and sometimes necessary. For graphs with (orientable) genus $g > 0$, the best known bound on the cop-number is $\frac{4}{3}g + \frac{10}{3}$ [18]. A long-standing conjecture by Schroeder [31] is that $c(G) \leq g + 3$ for all values of g . For non-orientable surfaces of genus g , Andreae [3] gave the first result that $c(G) \in O(g)$, which was later improved by Nowakowski and Schröder (in an unpublished work [28]) to $c(G) \leq 2g + 1$. The current best result is by Clarke et al. [22] who show that $c(G)$ is at most the maximum of all cop-numbers of graphs embeddable on an orientable surface of genus $g - 1$. (Refer to [13] for a survey of results, conjectures and open problems for cop-numbers of graphs on surfaces.)

Ever since the appearance of the proof that $c(G) \leq 3$ for planar graphs [1], it has been known that the geometrical representation of a graph plays an important role in bounding its cop-number. For instance, the strategy for guarding planar graphs was adapted to show that $c(G) \leq 9$ for unit disk graphs [8], and $c(G) \leq 15$ for string graphs [24]. More recently, the study of cop-numbers for the beyond-planar graph classes was initiated by Durocher et al. [23]. They show that unlike planar graphs, the class of *1-planar graphs* – which allow for one crossing per edge – is cop-unbounded. On the positive side, they show 3 cops are sufficient, and sometimes necessary, for a *maximal 1-planar graph*: a 1-planar graph to which no edge can be added without violating 1-planarity (while staying simple). To prove this, they crucially use the fact that the endpoints of every crossing induce the complete graph K_4 . In a recent paper, Bose et al. [16] showed that even under the relaxed requirement that no crossing of a 1-plane graph is an \times -crossing – a crossing whose endpoints induce a

matching – the cop-number remains bounded, and is at most 21. As a corollary, their result implies that 1-planar graphs with large cop-numbers necessarily embed with a large number of \times -crossings.

■ **Table 1** Summary of results for various graph classes drawn on the sphere with crossings.

Category	Graph Class	Cop Number	Remarks
1-Plane Graphs	Full 1-plane graphs	$c(G) \leq 3$	Corollaries 4.7 and 5.5; generalizes result in [23]
	1-plane graphs without \times -crossings	$c(G) \leq 15$	Corollary 4.7; improves $c(G) \leq 21$ from [16]
	1-plane graphs	$c_{X-1}(G) \leq 6X - 3$ $c_{x-1}(G) \leq 6x + 9$	Corollary 4.7
	Graphs with at most one crossing per face of $sk(G)$	Cop-unbounded	Theorem 3.9; holds even with distant crossings
k -Plane Graphs	Full k -plane graphs	$c_2(G) \leq 12k - 3$	Corollary 4.8
	k -plane graphs without \times -crossings	$c_3(G) \leq 18k - 3$	Corollary 4.8
	k -plane graphs	$c_{X-1}(G) \leq 6k(X+1) - 3$ $c_{x-1}(G) \leq 6k(x+2) - 3$	Corollary 4.8
	k -Framed graphs	$c_{\lceil \frac{k+8}{3} \rceil}(G) \leq 21$	Corollary 4.9
General Graphs	Graphs where all crossings are full	$c_3(G) \leq 9$	Corollary 4.10
	Graphs without \times -crossings	$c_4(G) \leq 21$	Corollary 4.10
	Map graphs	$c(G) \leq 3$	Theorem 5.4; includes full 1-plane and optimal 2-plane graphs
	All graphs	$c_{\lceil \frac{3}{2}(X+1) \rceil}(G) \leq 9$ $c_{\lceil \frac{3}{2}(x+2) \rceil}(G) \leq 15$	Corollary 4.11
	All graphs, $1 < \alpha < 3/2$	$c_{\lceil \alpha(X+1) \rceil}(G) \leq \frac{8}{\alpha-1}$ $c_{\lceil \alpha(x+2) \rceil}(G) \leq \frac{11}{\alpha-1}$	Corollary 4.11
			For all integers $\mu \geq 6$, the set $\{c_{\lfloor X/6 \rfloor - 1}(G) : X(G) \geq \mu\}$ is unbounded (Theorem 4.12)

1.2 Our Contribution

Motivated by [16], we undertake a comprehensive study of how the distance between crossing pairs of edges affects the cop number of graphs. Unlike [16], our aim is to go beyond 1-planar graphs, and look at general graphs drawn on surfaces, under the restriction that every crossing pair of edges can be connected by a path of small length. (A similar approach was undertaken in [10] to generalise results on vertex connectivity from 1-plane graphs without

\times -crossings to arbitrary graphs drawn on the sphere.) We consider two notions of distance between crossing pairs of edges. For any pair of edges $\{e_1, e_2\}$ that cross in G , we say that two endpoints of the crossing are *consecutive* if one endpoint is a vertex incident with e_1 and the other is a vertex incident with e_2 . For any fixed drawing of a graph G on a surface, let $x(G)$ (resp. $X(G)$) be the smallest integer such that for every crossing in the drawing, there is a path of length at most $x(G)$ (resp. $X(G)$) connecting some (resp. every) pair of consecutive endpoints of the crossing. By the definition above, $x(G) \leq X(G)$; furthermore, $X(G) \leq x(G) + 2$ since there exists a path of length at most $x(G) + 2$ at every crossing where the first and last edges of the path are precisely the crossing pair of edges. Notwithstanding the tight relation between $X(G)$ and $x(G)$, we differentiate between the two parameters to capture their varying impact on cop-number. Indeed, one may already notice this for 1-plane graphs. If G is a maximal 1-plane graph, where the endpoints of every crossing induce a K_4 , we have $X(G) = 1$ and $c(G) \leq 3$ [23]. On the other hand, if G is a 1-plane graph without \times -crossings, we have $x(G) = 1$ and $c(G) \leq 21$ [16].

Most results that we present in this paper concern *Distance d Cops and Robbers* – a variant of the classical Cops and Robbers in which the robber is considered to be captured if, at any point in time, there is a cop within distance d of the robber [12]. We use the notation $c_d(G)$ to denote the minimum number of cops required to capture the robber within distance d in the graph G . In this paper, we give several interesting results on distance d cop-numbers of graphs G , where d is parameterised by $X(G)$ and $x(G)$. We restrict our discussion here only to graphs drawn on the sphere, even though they extend to graphs drawn on surfaces with crossings. Our results not only encompass and improve the existing results on 1-plane graphs [23, 16], but also greatly simplify the existing proofs. The tools that we develop in this paper are equipped to handle beyond 1-planar graphs, such as k -plane graphs (where each edge is allowed to cross at most k times), and even arbitrary graphs on surfaces.

1.2.1 Our Results

We show that graphs that embed on the plane with small values of $X(G)$ or $x(G)$ have small cop-numbers. (A summary of our results is given in Table 1.) Let us call a crossing in a graph G a *full crossing* if its endpoints induce the complete graph K_4 , and an \times -*crossing* if its endpoints induce a matching. We show that for any graph where all crossings are full, $c_3(G) \leq 9$, and if no crossing of the graph is an \times -crossing, then $c_4(G) \leq 21$. For larger values of $X(G)$ or $x(G)$, we show that $c_{\lceil \alpha(X+1) \rceil}(G)$ and $c_{\lceil \alpha(x+2) \rceil}(G)$ are in $O(\frac{1}{\alpha-1})$, for any $1 < \alpha < 3/2$. For values of $\alpha \geq 3/2$, the cop-numbers are at most 9 and 15, respectively. In contrast to this, we show that for every integer $\mu \geq 6$, $\max\{c_{\lfloor X/6 \rfloor - 1}(G) : X(G) \geq \mu\}$ is unbounded. We also consider the question of standard cop-numbers $c(G)$ for special classes of graphs. Let $\text{sk}(G)$ denote the subgraph induced by the set of uncrossed edges of G . We show that for the class of *map graphs*, which are precisely graphs that have an embedding on the plane such that all crossed edges are inserted as cliques inside some faces of $\text{sk}(G)$, the cop-number is at most 3. (In fact, this generalises the result $c(G) \leq 3$ for maximal 1-plane graphs [23].) On the other hand, for graphs where there is at most one crossing in each face of $\text{sk}(G)$, we show that $c(G)$ is unbounded, even when no pair of crossings are close to each other. However, if one were to restrict the number of edges that bound each face of $\text{sk}(G)$ by an integer k , then we are in the class of *k -framed graphs*, and we show that $c_{\lceil \frac{k+8}{3} \rceil}(G) \leq 21$.

Organisation. Section 2 provides the necessary preliminaries and formal definitions used throughout the paper. In Section 3, we investigate the distance d variant of Cops and Robbers, and introduce two fundamental operations that simplify studying distance d cop-number

for arbitrary graphs drawn on surfaces to 1-plane drawings of graphs on surfaces. Section 4 focuses on 1-planar graphs drawn on the sphere, and explores the consequences of our result for k -plane graphs and general graphs drawn on the sphere. Section 5 extends our study to graphs drawn on (orientable and non-orientable) surfaces with crossings, and also discussing map graphs on surfaces.

2 Preliminaries

All graphs in this paper are finite and undirected. If H is a subgraph of G , then we write $H \subseteq G$. For any distinct pair of vertices s and t in G , an (s, t) -path is a simple path in G with s and t as its end vertices. If a path P contains vertices u and v , then the subpath of P containing all vertices from u to v will be denoted by $P(u \dots v)$. The length of a path P is the number of edges on P , and is denoted by $|P|$. If G is edge-weighted and P is a path in G , then $|P|$ denotes the sum of weights of all edges in P .

Graph Drawings. A *drawing* of a graph $G = (V, E)$ on a surface is a mapping of its vertex set $V(G)$ to distinct points on the surface and a mapping of each edge $e = (u, v)$ to a non-self-intersecting curve that has u and v as its endpoints and does not pass through any other vertex of G . In all drawings that we consider, any pair of edges can intersect only a finite number of times, no three edges are allowed to intersect at a point that is not a vertex, and no two edges can touch each other tangentially. (This is not a restriction because one can re-draw edges within a local neighbourhood of points on the surface to satisfy the requirements.) A pair of distinct edges are said to *cross* each other if there is a point on the surface interior to both edges. For any graph G drawn on a surface S , the *faces* of G are the connected regions of $S \setminus G$. The *boundary of a face* is a sequence of edges or part-edges (the interior of an edge with at least one end being a crossing point) that bound the face. The *skeleton* of a graph G drawn on some surface is the subgraph $\text{sk}(G)$ induced by all uncrossed edges of G . Hence, any drawing of a graph G can be viewed as the union of $\text{sk}(G)$ together with the set of crossed edges drawn inside the faces of $\text{sk}(G)$.

Crossings and $(\Sigma, 1)$ -Planar Graphs. A graph is *planar* if it has a drawing on the sphere such that no two edges cross each other. If G is a graph drawn on a surface Σ , then the Σ -*planarisation* of G is the graph G^\times obtained by inserting a *dummy vertex* at each crossing point in the drawing. Let G be any graph drawn on a surface. Let e_1 and e_2 be two edges of G that cross each other. The endpoints of e_1 and e_2 are together called *endpoints of the crossing*. Two endpoints (possibly non-distinct) v_1, v_2 of the crossing are said to be *consecutive* if v_1 is incident with e_1 and v_2 with e_2 . Let (u, v) and (w, x) be a pair of edges that cross in a drawing of a graph G . Assume that the crossing has four distinct endpoints, and consider the graph $G[\{u, v, w, x\}]$ induced by these endpoints, up to counting parallel edges as a single edge. If $G[\{u, v, w, x\}]$ is isomorphic to the complete graph K_4 , then we say that the crossing is *full*. If $G[\{u, v, w, x\}]$ consists only of edges (u, v) and (w, x) , then we say that the crossing is an \times -*crossing*. (The nomenclature of crossings based on the subgraphs induced by its endpoints is borrowed from [11].)

A graph G is a (Σ, k) -*planar graph* if it has a drawing on a surface Σ such that each edge is crossed at most k times; a graph G with such a drawing is called a (Σ, k) -*plane graph*. If Σ is a sphere, then we omit mentioning Σ , and say that G is a k -*planar* or a k -*plane graph*. In this paper, we deal extensively with $(\Sigma, 1)$ -plane graphs. For any $(\Sigma, 1)$ -plane graph, we can always assume that no pair of edges incident to the same vertex cross each

other, since the edges can be re-drawn without affecting $(\Sigma, 1)$ -planarity. Hence, every crossing in a $(\Sigma, 1)$ -plane graph has four distinct endpoints. In a $(\Sigma, 1)$ -plane graph, if two edges e_1 and e_2 cross each other, we use the notation $\{e_1, e_2\}$ to denote the crossing. A simple $(\Sigma, 1)$ -planar graph is said to be *maximal* if no edge can be added to the graph while maintaining $(\Sigma, 1)$ -planarity and simplicity. Consider a crossing $\{(u, v), (w, x)\}$ in a $(\Sigma, 1)$ -planar drawing of a graph G , with c as the crossing point. If there is an uncrossed edge connecting two consecutive endpoints of the crossing, say (u, w) , and $\{(u, w), (w, c), (c, u)\}$ bounds a face of the drawing, then we say that (u, w) is a *kite-edge* of the crossing. In a maximal $(\Sigma, 1)$ -plane graph, the endpoints of every crossing induce the complete graph K_4 . More generally, a graph is called *full (Σ, k) -planar* if it has a (Σ, k) -planar drawing such that every crossing is full. If one allows for parallel edges, then these graphs can be drawn such that there is a kite-edge connecting every pair of consecutive endpoints of a crossing.

Cops and Robbers. *Cops and Robbers* is a game played on a graph G where a set of cops try to capture a robber. The game proceeds in *rounds*, where each round consists first of the *cops' turn* and then the *robber's turn*. We assume that the game is played with perfect information – all players know all other players' position at every time instant. In the first round, the cops initially choose to place themselves on a subset of vertices, and thereafter, the robber chooses a vertex for himself. In subsequent rounds, in the cops' turn, every cop can choose to stay on the same vertex or move to an adjacent vertex. Similarly, in the robber's turn, the robber may choose to stay on the same vertex or move to an adjacent vertex. In the classical version of the game, the robber is considered to be *captured* if, at some point, the robber is on the same vertex as a cop. The *cop-number* of a graph G , denoted by $c(G)$, is the minimum number of cops required to capture a robber. A family of graphs \mathcal{G} is said to be *cop-bounded* if there is an integer p such that $c(G) \leq p$ for all graphs $G \in \mathcal{G}$; otherwise, \mathcal{G} is said to be *cop-unbounded*. A variant of this game which we study extensively is *Distance d Cops and Robbers*, where a robber is considered to be *captured* if, at some point, the robber is within distance d of some cop [12]. When $d = 0$, this reduces to the classical Cops and Robbers game. The *distance d cop-number* of G , denoted by $c_d(G)$, is the minimum number of cops required to capture the robber within distance d .

3 Distance d Cops and Robbers

At the outset, we state one of the simplest yet frequently used observation in our study of distance d cop-numbers.

► **Observation 3.1.** *For any graph G and integers $d_1 \geq d_2 \geq 0$, we have $c_{d_1}(G) \leq c_{d_2}(G)$.*

By Observation 3.1, $c_d(G) \leq c(G)$ for all values of $d \geq 0$. However, the two types of cop numbers are connected in a more interesting way. For any integer $k > 0$, let $G^{(k)}$ denote the graph obtained by replacing each edge of G with a path on k edges. Equivalently, the graph $G^{(k)}$ is obtained from G by subdividing each edge $(k - 1)$ times.

► **Theorem 3.2** (Lemma 9 in [12]). *For any graph G and any integer $d \geq 0$, $c(G) \leq c_d(G^{(2d+1)}) \leq c(G) + 1$.*

It is worth mentioning that there is no direct relation between $c(G)$ and $c_d(G)$. In fact, for any pair of integers $d, m \geq 1$, there exists a graph G such that $c_d(G) = 1$ but $c(G) \geq m$ [12]. Despite this, some important results from the classical Cops and Robbers game can be extrapolated to the distance version of the game. For instance, one of the well-known results

concerns the cop-number of a graph G and its retracts. A subgraph $H \subseteq G$ is a *retract* of G if there is a function $f : V(G) \mapsto V(H)$ such that $f(v) = v$ for every $v \in V(H)$ and if $(u, v) \in E(G)$, then $(f(u), f(v)) \in E(H)$. It is well-known that if H is a retract of G , then $c(H) \leq c(G) \leq \max\{c(H), c(G \setminus H) + 1\}$ [7]. One can easily extend this to distance d Cops and Robbers, as shown in Theorem 3.3.

► **Theorem 3.3** (Theorems 3.1 and 3.2 in [7]). *If H is a retract of a graph G , then $c_d(H) \leq c_d(G) \leq \max\{c_d(H), c_d(G \setminus H) + 1\}$.*

The proof for Theorem 3.3 is similar to the proof for the special case $d = 0$. For values of $d > 0$, we make use of the fact that if there is a path P from u to v in G , then H contains a path P' from $f(u)$ to $f(v)$ of length at most $|P|$.

We now state another result that is a distance analogue of the following classical result: $c(G) \leq c(G^{(k)}) \leq c(G) + 1$ for any $k \geq 1$ [26]. Considering that the reader may not find it straight-forward to extend the proof in [26] to Theorem 3.4, and given the importance of this theorem for this paper, we believe that it will be useful to provide a complete proof of Theorem 3.4. (Proof in full-version [17].)

► **Theorem 3.4.** *For any graph G and integers $d \geq 0$, $k \geq 1$, we have $c_{\lceil d/k \rceil}(G) \leq c_d(G^{(k)}) \leq c_{\lfloor d/k \rfloor}(G) + 1$.*

3.1 $(\Sigma, 1)$ -Planarisation and Kite-Augmentation

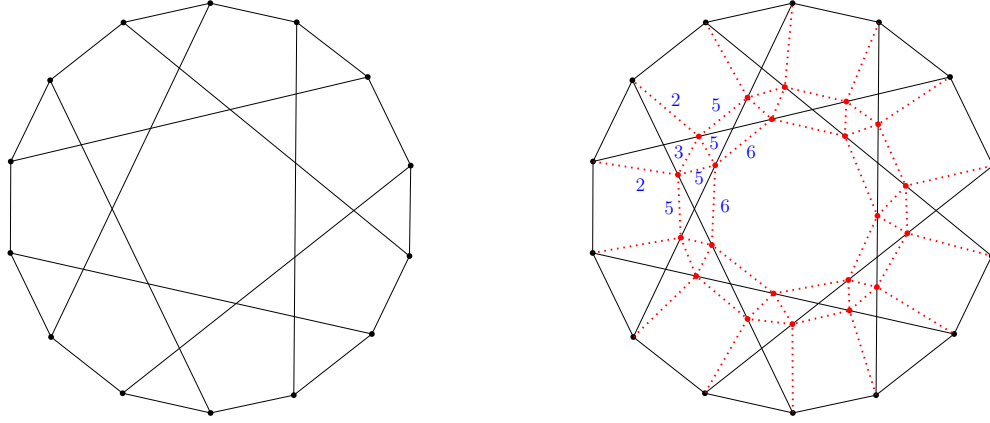
We use Theorems 3.3 and 3.4 to define two simple operations for graphs drawn on surfaces, both of which are foundational to this paper. The two operations, called $(\Sigma, 1)$ -planarisation and kite-augmentation, are described below.

► **Definition 3.5** ($(\Sigma, 1)$ -Planarisation). *For any graph G drawn on a surface Σ , let the $(\Sigma, 1)$ -planarisation of G be a graph $G^{(k)}$ such that $G^{(k)}$ is $(\Sigma, 1)$ -plane.*

Every graph G drawn on a surface Σ has a $(\Sigma, 1)$ -planarisation: set k equal to the minimum integer such that every edge is crossed at most k times, and place $(k-1)$ subdivision vertices on each edge around crossing points so that $G^{(k)}$ is $(\Sigma, 1)$ -planar. Note that $(\Sigma, 1)$ -planarisation of a graph is not unique, since subdivisions of $(\Sigma, 1)$ -plane graphs are also $(\Sigma, 1)$ -plane. When Σ is a sphere, then the resulting graph is 1-plane; in this case we simply write *1-planarisation* instead of $(\Sigma, 1)$ -planarisation. In [23], the process of 1-planarisation was used to prove that 1-planar graphs are cop-unbounded. Briefly, notice that by Theorem 3.4, if $G^{(k)}$ is a 1-planarisation of G , then $c(G) \leq c(G^{(k)}) \leq c(G) + 1$. Since every graph G has a 1-planarisation H , and the universal set of graphs is cop-unbounded, so is the set of all 1-planar graphs.

Recall from Section 2 that a kite edge at a crossing of a $(\Sigma, 1)$ -plane graph is an uncrossed edge connecting two consecutive endpoints of a crossing such that the edge bounds a face incident to the crossing point. In this paper, we shall see that the presence of kite edges, or paths of uncrossed edges connecting consecutive endpoints of crossings, are especially useful to bound cop-numbers of 1-planar graphs. To this end, we introduce a class of $(\Sigma, 1)$ -planar drawings on surfaces called kite-augmented $(\Sigma, 1)$ -planar drawings.

► **Definition 3.6** (Kite-augmented $(\Sigma, 1)$ -plane graph.). *A $(\Sigma, 1)$ -plane graph is said to be kite-augmented if, for every pair of consecutive endpoints u, v of a crossing at a point c , there is a shortest (u, v) -path κ_{uv} such that each edge of κ_{uv} is uncrossed, each internal vertex of κ_{uv} has degree 2, and $\{(c, u)\} \cup \kappa_{uv} \cup \{(v, c)\}$ bounds a face of the drawing.*



■ **Figure 1** On the left is a drawing of a graph G , and on the right is a drawing of G^{\boxtimes} . The numbers alongside the edges indicate the lengths of the paths κ_{uv} . (The lengths of other paths can be inferred by symmetry.)

The above definition lends itself to a procedure called kite-augmentation, defined below.

► **Definition 3.7** (Kite-augmentation). *For any $(\Sigma, 1)$ -plane graph, let kite-augmentation refer to the following process: for every pair of consecutive endpoints u and v of every crossing in the drawing, add a kite edge between u and v (if it does not already exist), and subdivide it to get a path κ_{uv} of length equal to that of a shortest (u, v) -path in the original drawing.*

For any graph G drawn on a surface Σ , we use the notation G^{\boxtimes} to denote a graph obtained by $(\Sigma, 1)$ -planarising G and then kite-augmenting the resulting graph (see Figure 1 for an example). If $G^{(k)}$ is a $(\Sigma, 1)$ -plane graph, and G^{\boxtimes} is obtained by kite-augmenting $G^{(k)}$, then $G^{(k)}$ is a retract of G^{\boxtimes} since there is a homomorphism from each path κ_{uv} to a shortest (u, v) -path in $G^{(k)}$. This observation, together with Theorems 3.3 and 3.4 gives us the following corollary.

► **Corollary 3.8** (Corollary of Theorems 3.3 and 3.4). *Let G be any graph drawn on a surface Σ , $G^{(k)}$ be a $(\Sigma, 1)$ -planarisation of G , and G^{\boxtimes} be the kite-augmentation of $G^{(k)}$. Then $c_{\lceil d/k \rceil}(G) \leq c_d(G^{\boxtimes}) \leq c_{\lfloor d/k \rfloor}(G) + 1$ for any integer $d \geq 0$.*

Proof. From Theorem 3.4, we have $c_{\lceil d/k \rceil}(G) \leq c_d(G^{(k)}) \leq c_{\lfloor d/k \rfloor}(G) + 1$. Since $G^{(k)}$ is a retract of G^{\boxtimes} , we have from Theorem 3.3 that $c_d(G^{(k)}) \leq c_d(G^{\boxtimes}) \leq \max\{c_d(G^{(k)}), c_d(G^{\boxtimes} \setminus G^{(k)}) + 1\}$. As G^{\boxtimes} is obtained by kite-augmenting $G^{(k)}$, the graph $G^{\boxtimes} \setminus G^{(k)}$ is a collection of vertex-disjoint paths; hence $c_d(G^{\boxtimes} \setminus G^{(k)}) = 1$. This implies that $c_d(G^{\boxtimes}) \leq \max\{c_d(G^{(k)}), 2\} \leq \max\{c_{\lfloor d/k \rfloor}(G) + 1, 2\} = c_{\lfloor d/k \rfloor}(G) + 1$. By combining the above inequalities, we get $c_{\lceil d/k \rceil}(G) \leq c_d(G^{\boxtimes}) \leq c_{\lfloor d/k \rfloor}(G) + 1$. ◀

The process of $(\Sigma, 1)$ -planarisation combined with kite-augmentation leads to several interesting consequences. One such consequence is presented in Theorem 3.9. It has long been known that planar graphs have cop-number at most 3 [1]. An interesting question is whether the set of all graphs G that can be drawn with at most one crossing in each face of its skeleton $\text{sk}(G)$ (i.e., the subgraph induced by the set of all uncrossed edges) has a small cop-number. Using the process of 1-planarisation and kite-augmentation, we can show that this is false. Indeed, one can even assume that no two crossings are close to each other in terms of face-distances. For a precise formulation, we need some definitions first. For any graph G drawn on the sphere, consider the graph obtained by first planarising G (i.e.,

adding dummy vertices at crossing points), and then inserting a *face-vertex* inside each face and making it adjacent to all vertices on the face-boundary. Define the *radial graph* of G as the plane bipartite subgraph $R(G)$ induced by all edges incident with face-vertices. Let the *face-distance* between any two crossings be the number of face-vertices on a shortest path in $R(G)$ connecting the two crossing points.

► **Theorem 3.9.** *For any three integers $d, f, m \geq 0$, there exists a 1-plane graph G such that each face of $\text{sk}(G)$ contains at most one crossing, the face-distance between any two crossing points is at least f , and $c_d(G) \geq m$.*

Proof. Consider any graph H with $c(H) \geq m$. From Theorem 3.2, we know that $c_d(H^{(2d+1)}) \geq m$. Let G' be a graph obtained by 1-planarising $H^{(2d+1)}$ and then kite-augmenting it. Therefore, G' is a 1-planar graph such that $\text{sk}(G')$ has at most one crossing per face. Using Corollary 3.8 and Observation 3.1 ($d_1 \leq d_2$ implies $c_{d_1}(G) \geq c_{d_2}(G)$), we can see that $m \leq c_d(H^{(2d+1)}) \leq c_d(G')$. To make the pairwise face-distance between the crossing points at least f , we repeat the following process $\lceil f/2 \rceil$ times: Subdivide each edge twice to get a 1-planar drawing such that the endpoints of every crossing consist only of the newly added subdivision vertices, and then kite-augment the graph. This increases the face-distance between any two crossing points by 2, and after repeating $\lceil f/2 \rceil$ times, the face-distance becomes at least f . Let G be the resulting graph. By repeatedly applying Corollary 3.8 (and Observation 3.1), we can verify that $c_d(G) \geq m$. ◀

3.2 Parameterizing d through Distances Between Crossing Edges

Our study of distance d Cops and Robbers for graphs drawn on surfaces focuses primarily on those values of d that in some way capture how close crossing pairs of edges are in the drawing. To this end, we define two parameters $X(G)$ and $x(G)$:

► **Definition 3.10** ($x(G)$ and $X(G)$). *For any fixed drawing of a graph G on a surface, $x(G)$ is the smallest integer such that for every crossing in the drawing, there is a path of length at most $x(G)$ connecting some pair of consecutive endpoints of the crossing. On the other hand, $X(G)$ is the smallest integer such that for every crossing in the drawing, there is a path of length at most $X(G)$ connecting every pair of consecutive endpoints of the crossing.*

These two parameters are closely related to a structure called ribbon associated with crossings (first defined in [10]): For any crossing pair of edges e_1 and e_2 , a *ribbon* at the crossing is a path with e_1 as the first edge and e_2 as the last edge. Every crossing in a graph has a ribbon of length at most $x(G) + 2$. Since any pair of consecutive endpoints of a crossing can be connected by a sub-path of a ribbon at the crossing, $X(G) \leq x(G) + 2$. Therefore, $X(G) \in \{x(G), x(G) + 1, x(G) + 2\}$. Our intention in differentiating between $x(G)$ and $X(G)$ is to capture the differences arising in cop-numbers due to variations in crossing configurations. For instance, in a maximal 1-planar graph G where the endpoints of every crossing induce K_4 , we have $x(G) = X(G) = 1$, and $c(G) = 3$ [23]. However, in a 1-planar graph without \times -crossings where the endpoints induce at least 3 edges, we have $x(G) = 1$ (and $X(G) \leq 3$), and $c(G) \leq 21$ [16].

In Lemma 3.11 (proof in full-version [17]), we give useful bounds on $X(G^{(k)})$ and $x(G^{(k)})$ in terms of the corresponding parameters for G . These will be used in Theorem 3.12, and in later sections of the paper.

► **Lemma 3.11.** *For any graph G and integer $k \geq 1$, we have $k \cdot X(G) \leq X(G^{(k)}) \leq k \cdot X(G) + 2\lfloor k/2 \rfloor$ and $k \cdot x(G) \leq x(G^{(k)}) \leq k \cdot x(G) + 2(k - 1)$.*

In Theorem 3.12 (proof in full-version [17]), we give a relation between distance d cop-numbers of G and G^\boxtimes , where d is measured through the parameters $x(G)$ and $X(G)$.

► **Theorem 3.12.** *For all graphs G drawn on a surface Σ , all graphs G^\boxtimes obtained by $(\Sigma, 1)$ -planarising and kite-augmenting G , and all real numbers $\alpha > 0$:*

$$(a) \quad c_{\lceil \alpha(x(G)+2) \rceil}(G) \leq c_{\lceil \alpha x(G^\boxtimes) \rceil - 1}(G^\boxtimes) \leq c_{\lfloor \alpha x(G) \rfloor - 1}(G) + 1$$

$$(b) \quad c_{\lceil \alpha(X(G)+1) \rceil}(G) \leq c_{\lceil \alpha X(G^\boxtimes) \rceil - 1}(G^\boxtimes) \leq c_{\lfloor \alpha X(G) \rfloor - 1}(G) + 1$$

4 1-Planar Graphs

In view of Theorem 3.12, it is sufficient to restrict attention to kite-augmented $(\Sigma, 1)$ -plane graphs. In this section, we focus on kite-augmented 1-planar graphs (drawn on the sphere), while Section 5 extends the discussion to kite-augmented $(\Sigma, 1)$ -plane graphs on surfaces. The entirety of this section is devoted to proving Theorem 4.1 and understanding its implications.

► **Theorem 4.1.** *Let G be any kite-augmented 1-plane graph. For $d \in \{X(G), x(G)\}$ and $\alpha \geq 1$, we have $c_{\lceil \alpha d \rceil - 1}(G) \leq 3 \cdot (2\beta + 1)$ where*

$$\beta = \begin{cases} d + 1 & \text{if } d = x(G) \text{ and } \alpha = 1 \\ \left\lceil \frac{1}{2(\alpha-1)} \right\rceil + 1 & \text{if } d = x(G) \text{ and } \alpha > 1 \end{cases} \quad \beta = \begin{cases} d - 1 & \text{if } d = X(G) \text{ and } \alpha = 1 \\ \left\lceil \frac{1}{2(\alpha-1)} \right\rceil & \text{if } d = X(G) \text{ and } \alpha > 1 \end{cases}$$

The proof of Theorem 4.1 is structured along the lines of the proof that the cop-number of a planar graph is at most 3 [1, 14]. For planar graphs, the essential idea is that 3 cops progressively guard larger and larger subgraphs, while maintaining as an invariant that the frontier between the guarded and unguarded subgraph is always either a shortest path or a cycle composed of two shortest paths. Unlike planar graphs that require a single cop to guard a shortest path, we require $2\beta + 1$ cops since the cops must also ensure that the robber does not cross any edge of the shortest path; this coarsely explains why the cop-number increases by a factor of $2\beta + 1$ in Theorem 4.1.

4.1 Guarding Shortest Paths in (s, t) -Subgraphs

The choice of which shortest path or cycle to guard is made by selecting two vertices, say s and t , and then finding a shortest (s, t) -path through the unguarded subgraph. Since we require that such a shortest path include at least one vertex from the unguarded subgraph, it may be necessary to exclude the edge (s, t) when both s and t are already guarded and adjacent in G . In light of this, we find it convenient to view an unguarded subgraph as an (s, t) -subgraph, defined as follows. (For Definition 4.2, recall the notation κ_{ab} from Definition 3.7 for the path resulting from repeated subdivision of kite-edges.)

► **Definition 4.2** ((s, t) -subgraphs of G). *Let G be a kite-augmented 1-planar graph. Let s and t be two distinct vertices of G . A connected subgraph H of G , with $s, t \in V(H)$, is an (s, t) -subgraph of G if, for any pair of distinct vertices $a, b \in V(H)$, except possibly for the pair $\{a, b\} = \{s, t\}$, if $\kappa_{ab} \subseteq G$, then $\kappa_{ab} \subseteq H$.*

In Lemma 4.3, we show that there always exists a shortest (s, t) -path in an (s, t) -subgraph such that no two edges of the shortest path cross each other. For brevity, we say that a subgraph H of G is *self-crossing* if there exist a pair of edges of H that cross each other; otherwise, H is *non-self-crossing*.

► **Lemma 4.3.** *In any (s, t) -subgraph H of a kite-augmented 1-planar graph G , there exists an (s, t) -path P that is a shortest path in H and is non-self-crossing.*

Proof. Let P be a shortest (s, t) -path of H with the minimum number of self-crossings. If P has no self-crossings, then we are already done, so assume otherwise, for the sake of contradiction. We will construct another shortest (s, t) -path P' with fewer self-crossings than P . Enumerate the vertices of P based on their distances from s , and let $\{(u, v), (w, x)\}$ be a crossing of G such that $\{(u, v), (w, x)\} \subseteq E(P)$, and up to renaming, let the four endpoints of the crossing appear in P in the order u, v, w, x . As G is kite-augmented, there exists a path κ_{ux} in G . If $\kappa_{ux} \subseteq H$, then one can substitute the subpath $P(u \dots x)$ with κ_{ux} to get another shortest path; i.e., $P' := P \setminus E(P(u \dots x)) \cup E(\kappa_{ux})$ is a shortest (s, t) -path in H with fewer self-crossings. Since this leads to a contradiction, we must assume that κ_{ux} is not a subgraph of H . Since H is an (s, t) -subgraph and κ_{ux} is a path of uncrossed edges, this can happen only when $\{u, x\} = \{s, t\}$. Let P' be the path $\kappa_{uw} \cup \{(w, x)\}$. Since $|\kappa_{uw}| \leq |P(u \dots w)|$, we have $|P'| \leq |P|$. This again leads to a contradiction as P' has fewer self-crossings than P . ◀

As mentioned before, our intention is to guard paths and cycles in (s, t) -subgraphs of G such that the robber can neither land on a vertex nor cross any edge of the path or cycle. This idea is formalised with the following definition.

► **Definition 4.4** (Crossing-guarded subgraph). *For any integer $d \geq 0$, a graph H is crossing-guarded at distance d by a set \mathcal{U} of cops if:*

- (a) *For any vertex $v \in V(H)$, if the robber lands on v , then he is captured by a cop of \mathcal{U} , and*
- (b) *For any crossing $\{(u, v), (w, x)\}$ of G with $(u, v) \in E(H)$, if the robber crosses the edge (u, v) by moving from w to x , then he is captured by a cop of \mathcal{U} within distance d .*

In Lemma 4.5 (proof in full-version [17]), we show that any shortest path in an (s, t) -subgraph of G is crossing guardable at distance $\lceil \alpha d \rceil - 1$ by $2\beta + 1$ cops for the values of α , d and β as stated in Theorem 4.1.

► **Lemma 4.5.** *Let H be an (s, t) -subgraph of G containing the robber, where the robber is restricted to moving only along the edges of H . Let P be a shortest (s, t) -path in H . For $d \in \{x(G), X(G)\}$ and any $\alpha \geq 1$, P is crossing-guardable at distance $\lceil \alpha d \rceil - 1$ by a set \mathcal{U} of cops where $|\mathcal{U}| \leq 2\beta + 1$ and*

$$\beta = \begin{cases} d + 1 & \text{if } d = x(G) \text{ and } \alpha = 1 \\ \left\lceil \frac{1}{2(\alpha-1)} \right\rceil + 1 & \text{if } d = x(G) \text{ and } \alpha > 1 \end{cases} \quad \beta = \begin{cases} d - 1 & \text{if } d = X(G) \text{ and } \alpha = 1 \\ \left\lceil \frac{1}{2(\alpha-1)} \right\rceil & \text{if } d = X(G) \text{ and } \alpha > 1 \end{cases}$$

4.2 Cop-Strategy for Kite-Augmented 1-Planar Graphs

We now use Lemma 4.5 on guarding shortest paths to prove Theorem 4.1. For this, we use ideas from [14] for the proof that planar graphs have cop-number at most 3, and from [16] for the modifications needed to handle crossings in 1-planar graphs. We first give an overview of the proof, and then provide a formal description in detail in Lemma 4.6.

We consider three sets of $2\beta + 1$ cops, and proceed in iterations, where every iteration begins with a path or a cycle being crossing-guarded at distance $\lceil \alpha d \rceil - 1$. (For the rest of this discussion, we use the term “guard” as a shortcut to saying “crossing-guard at distance $\lceil \alpha d \rceil - 1$ ”.) We ensure that the paths and cycles are non-self-crossing, so that they trace simple paths and simple cycles on the sphere. These paths and cycles serve as the frontiers between the guarded subgraph and the unguarded subgraph of G , and these are the only



■ **Figure 2** The robber territory, denoted by $\mathcal{R}(\cdot)$, is progressively cut down across iterations, where each iteration is marked by either a path P or a cycle C being guarded. For each successive iteration, the new path or cycle to be guarded is determined by choosing a shortest (s, t) -path Q to be guarded in the (s, t) -subgraph (the region enclosed within bold edges).

subgraphs that are actively guarded at any point in time. A path requires only one set of cops to guard, and a cycle requires two sets of cops; hence at least one of three sets of cops is free at the beginning of every iteration.

Every iteration begins by choosing a shortest path to be guarded by the free set of cops. For this, two special vertices s and t are identified that have a neighbour in the robber territory – the maximal subgraph of G within which the robber can move without getting captured. Then, we consider the (s, t) -subgraph formed by the union of the robber territory together with the set of all edges connecting $\{s, t\}$ and the robber territory. Now, we guard a shortest non-self-crossing (s, t) -path in the (s, t) -subgraph identified above. This path may be the only path actively guarded, or may be combined with a previously guarded path to form a new cycle to be actively guarded (Figure 2). Once the path has been guarded, the current iteration ends, at least one set of cops is freed, and the next iteration begins. At the end of every iteration, we shall maintain the invariant that the shortest (s, t) -path chosen to be guarded contains at least one vertex of the robber territory. Therefore, every iteration ends with at least one more vertex of the graph being guarded, and after a finite number of iterations, the robber is captured.

We now give a detailed and formal treatment of the above overview in Lemma 4.6 (proof in full-version [17]).

► **Lemma 4.6.** *Let G be a kite-augmented 1-plane graph. For $d \in \{x(G), X(G)\}$, $\alpha \geq 1$, and any (s, t) -subgraph H of G , let $|\mathcal{U}|$ be the smallest integer such that any shortest (s, t) -path $P \subseteq H$ is crossing-guardable at distance $\lceil \alpha d \rceil - 1$ by a set \mathcal{U} of cops. Then $c_{\lceil \alpha d \rceil - 1}(G) \leq 3|\mathcal{U}|$.*

Lemmas 4.5 and 4.6 together establish Theorem 4.1.

4.3 Implications of Theorem 4.1

In this section, we discuss the implications of Theorem 4.1 for 1-plane graphs, k -plane graphs and general graphs drawn on the sphere. A summary of the results in this section appears in Table 1. To keep notation compact, we let $c_{f(X)}(G)$ and $c_{f(x)}(G)$ be shortcuts to $c_{f(X(G))}(G)$ and $c_{f(x(G))}(G)$, respectively.

4.3.1 1-Plane graphs

One of the first implications Theorem 4.1 for 1-plane graphs is that $c_{X-1}(G)$ and $c_{x-1}(G)$ are at most linear in $X(G)$ and $x(G)$.

► **Corollary 4.7.** *If G is a 1-plane graph, then $c_{X-1}(G) \leq 6X - 3$ and $c_{x-1}(G) \leq 6x + 9$.*

Proof. Consider the graph G^\boxtimes obtained by kite-augmenting G . From Theorem 4.1, setting $\alpha = 1$, we have $c_{X-1}(G^\boxtimes) \leq 3 \cdot (2(X(G^\boxtimes) - 1) + 1) = 6X(G^\boxtimes) - 3$ and $c_{x-1}(G^\boxtimes) \leq 3 \cdot (2(x(G^\boxtimes) + 1) + 1) = 6x(G^\boxtimes) + 9$. As $X(G) = X(G^\boxtimes)$ and $x(G) = x(G^\boxtimes)$, we get the stated result. ◀

Corollary 4.7 corroborates, improves and generalises all existing results on cop-numbers of 1-planar graphs. For a maximal 1-planar graph, it was shown in [23] that $c(G) \leq 3$. In a maximal 1-planar graph, the endpoints of every crossing induce a K_4 . Setting $X(G) = 1$ in Corollary 4.7 gives us the same cop-number. In fact, it shows that 3 cops are sufficient for the larger class of full 1-planar graphs: 1-planar graphs where the endpoints of every crossing induce a K_4 . (We shall give a much simpler proof of this result in Section 5.2.) Another class of 1-planar graphs that has been studied is 1-planar graphs embeddable without \times -crossings (crossings whose endpoints induce a matching). In [16], it was shown that $c(G) \leq 21$ for all such 1-planar graphs G . Corollary 4.7 gives us a better bound: as $x(G) = 1$ for 1-planar graphs without \times -crossings, we get $c(G) \leq 15$.

4.3.2 k -Plane graphs

We now look at k -plane graphs. In Corollary 4.8, we look at distance $X + 1$ and $x + 2$ cop-numbers obtained by setting $\alpha = 1$ in Theorem 3.12.

► **Corollary 4.8.** *If G is a k -plane graph, then $c_{X+1}(G) \leq 6k(X + 1) - 3$ and $c_{x+2}(G) \leq 6k(x + 2) - 3$.*

Proof. From Theorem 3.12, setting $\alpha = 1$, we have $c_{X+1}(G) \leq c_{X-1}(G^\boxtimes)$ and $c_{x+2}(G) \leq c_{x-1}(G^\boxtimes)$. From Lemma 3.11, we get $X(G^\boxtimes) \leq k(X(G) + 1)$ and $x(G^\boxtimes) \leq k(x(G) + 2) - 2$. From Corollary 4.7, we have $c_{X-1}(G^\boxtimes) \leq 6X(G^\boxtimes) - 3 \leq 6k(X(G) + 1) - 3$ and $c_{x-1}(G^\boxtimes) \leq 6x(G^\boxtimes) + 9 \leq 6k(x + 2) - 3$. Hence, the result. ◀

From Corollary 4.8, we see that if G is a full k -plane graph ($X(G) = 1$), then $c_2(G) \leq 12k - 3$. On the other hand, if G is a k -plane graph without \times -crossings ($x(G) = 1$), then $c_3(G) \leq 18k - 3$.

A graph G is a *k -framed graph* if it has a drawing on the sphere such that its skeleton $\text{sk}(G)$ (the subgraph induced by the set of all uncrossed edges) is simple, biconnected, spans all vertices and each face boundary has at most k edges [6]. Clearly, a simple k -framed graph can have at most k^2 edges inside each face. One can draw these graphs such that any two edges cross at most once, hence they are k^2 -planar graphs.

► **Corollary 4.9.** *If G is a k -framed graph, then $c_{\lceil \frac{k+8}{3} \rceil}(G) \leq 21$.*

Proof. Since every face of $\text{sk}(G)$ has at most k edges, for every crossing of G , there is a path of length at most $k/4$ that connects some pair of consecutive endpoints; therefore, $x(G) \leq k/4$. By setting $\alpha = 4/3$ for $k = x(G)$ in Theorem 4.1, we get $c_{\lceil \frac{k+8}{3} \rceil}(G) = c_{\lceil \frac{4}{3}(\frac{k}{4}+2) \rceil}(G) \leq c_{\lceil \frac{4}{3}x \rceil-1}(G^\boxtimes) \leq 21$. ◀

4.3.3 General graphs

We now consider general graphs drawn on the sphere. In Corollary 4.10, we look at graphs where all crossings are full and graphs without \times -crossings.

► **Corollary 4.10.** *For any graph G , if $X(G) = 1$, then $c_3(G) \leq 9$, and if $x(G) = 1$, then $c_4(G) \leq 21$.*

Proof. From Theorem 3.12, we have $c_{\lceil \alpha(X+1) \rceil}(G) \leq c_{\lceil \alpha X \rceil - 1}(G^{\boxtimes})$ and $c_{\lceil \alpha(x+2) \rceil}(G) \leq c_{\lceil \alpha x \rceil - 1}(G^{\boxtimes})$. By setting $\alpha = 3/2$ for $d = X(G^{\boxtimes})$ in Theorem 4.1, we get $\beta = \left\lceil \frac{1}{2(\alpha-1)} \right\rceil = 1$, and setting $\alpha = 4/3$ for $d = x(G^{\boxtimes})$, we get $\beta = \left\lceil \frac{1}{2(\alpha-1)} \right\rceil + 1 = 3$. This implies that $c_3(G) = c_{\lceil \alpha(X+1) \rceil}(G) \leq c_{\lceil \alpha X \rceil - 1}(G^{\boxtimes}) \leq 3$ when $X(G) = 1$, and $c_4(G) = c_{\lceil \alpha(x+2) \rceil}(G) \leq c_{\lceil \alpha x \rceil - 1}(G^{\boxtimes}) \leq 21$ when $x(G) = 1$. ◀

From Theorem 4.1, it is easy to see that for $\alpha > 1$, distance $\lceil \alpha x \rceil - 1$ and $\lceil \alpha X \rceil - 1$ cop-numbers of 1-plane graphs are within a constant factor of $1/(\alpha - 1)$. When combined with Theorem 3.12, one can obtain a similar bound for $c_{\lceil \alpha(X+1) \rceil}(G)$ and $c_{\lceil \alpha(x+2) \rceil}(G)$ for arbitrary graphs G .

► **Corollary 4.11.** *For any graph G ,*

$$c_{\lceil \alpha(X+1) \rceil}(G) \leq \begin{cases} \frac{8}{\alpha-1} & \text{if } 1 < \alpha < \frac{3}{2} \\ 9 & \text{if } \alpha \geq \frac{3}{2} \end{cases} \quad c_{\lceil \alpha(x+2) \rceil}(G) \leq \begin{cases} \frac{11}{\alpha-1} & \text{if } 1 < \alpha < \frac{3}{2} \\ 15 & \text{if } \alpha \geq \frac{3}{2} \end{cases}$$

(See full-version [17] for a proof of Corollary 4.11.) Corollary 4.11 shows that $c_{\lceil \alpha(X+1) \rceil}(G)$ is bounded for all constants $\alpha > 1$. In contrast, the class of diameter 2 graphs are such that $X(G) \leq 2$, but this class of graphs is cop-unbounded [12]. In other words, for every integer $m > 0$, there is a graph G such that $X(G) \in \{1, 2\}$, but $c(G) \geq m$. In Theorem 4.12, we show a similar result for arbitrarily large values of $X(G)$.

► **Theorem 4.12.** *For every pair of integers $m \geq 1$ and $\mu \geq 6$, there exists a graph G with $X(G) \geq \mu$ and $c_{\lfloor X/6 \rfloor - 1}(G) \geq m$.*

Proof. Let H be a graph of diameter 2 such that $c(H) \geq m$. To get the distance d cop-number to be at least m , we use Theorem 3.2, which guarantees that $c(H) \leq c_d(H^{(2d+1)})$. By this, $c(H) \leq c_{(d-1)/2}(H^{(d)})$ when d is odd, and $c(H) \leq c_{d/2-1}(H^{(d-1)})$ when d is even. Substituting $d := \mu + 1$, we get:

Case (a): $\mu + 1$ is odd. Set $G := H^{(\mu+1)}$, and we have $c_{\lfloor \mu/2 \rfloor}(G) \geq c(H)$.

Case (b): $\mu + 1$ is even. Set $G := H^{(\mu)}$, and we have $c_{\lfloor (\mu-1)/2 \rfloor}(G) \geq c(H)$.

As H has diameter 2, in any drawing of H on the sphere, $1 \leq X(H) \leq 2$. By Lemma 3.11, we have $\mu \leq \mu X(H) \leq X(H^{(\mu)}) \leq \mu X(H) + \mu \leq 3\mu$. Likewise, we can show $\mu + 1 \leq X(H^{(\mu+1)}) \leq 3(\mu + 1)$. As $G = H^{(\mu)}$ or $G = H^{(\mu+1)}$, we have $\mu \leq X(G) \leq 3(\mu + 1)$. We now are left with showing that $c_{\lfloor X(G)/6 \rfloor - 1}(G) \geq m$. Note that $\lfloor X(G)/6 \rfloor - 1 = \lfloor \frac{X(G)-6}{6} \rfloor \leq \lfloor \frac{3(\mu-1)}{6} \rfloor = \lfloor \frac{\mu-1}{2} \rfloor \leq \lfloor \frac{\mu}{2} \rfloor$. Hence, for all $\mu \geq 6$, we have $X(G) \geq \mu \geq 6$, and $c_{\lfloor X/6 \rfloor - 1}(G) \geq c_{\lfloor (\mu-1)/2 \rfloor}(G) \geq c_{\lfloor \mu/2 \rfloor}(G) \geq c(H) \geq m$. ◀

5 Graphs on Surfaces

In this section, we study cop-numbers for $(\Sigma, 1)$ -planar drawings of graphs on both orientable and non-orientable surfaces. (We assume that the reader is familiar with basic concepts related to graphs on surfaces; see [25] for a reference.) All drawings are assumed to be *cellular*; i.e, if $\gamma : G \mapsto S$ is a function that maps a graph G onto a surface S , then $S \setminus \gamma(G)$ is a disjoint union of disks. Equivalently, G is cellularly drawn if and only if G^\times (the graph obtained by adding dummy vertices at crossing points) is a cellular embedding on S . For orientable surfaces, we use the term *genus* to refer to the number of handles on the surface, whereas for non-orientable surfaces, *genus* refers to the number of crosscaps on the surface.

5.1 $(\Sigma, 1)$ -Planar Drawings on Orientable and Non-orientable Surfaces

In Theorem 5.1, we extend the results of Theorem 4.1 to $(\Sigma, 1)$ -planar drawings on orientable surfaces. The main ideas for the proof of Theorem 5.1 (in the full-version [17]) are inspired from [30], which shows that any graph embeddable (without crossings) on an orientable surface of genus g has cop-number at most $2g + 3$.

► **Theorem 5.1.** *Let G be any graph that has a kite-augmented $(\Sigma, 1)$ -planar graph drawing on an orientable surface Σ of genus g . For any $\alpha \geq 1$, we have $c_{\lceil \alpha x \rceil - 1}(G) \leq (2g + 3) \cdot (2\beta + 1)$ and $c_{\lceil \alpha X \rceil - 1}(G) \leq (2g + 3) \cdot (2\beta + 1) + g$ where*

$$\beta = \begin{cases} d + 1 & \text{if } d = x(G) \text{ and } \alpha = 1 \\ \left\lceil \frac{1}{2(\alpha-1)} \right\rceil + 1 & \text{if } d = x(G) \text{ and } \alpha > 1 \end{cases} \quad \beta = \begin{cases} d - 1 & \text{if } d = X(G) \text{ and } \alpha = 1 \\ \left\lceil \frac{1}{2(\alpha-1)} \right\rceil & \text{if } d = X(G) \text{ and } \alpha > 1 \end{cases}$$

We now shift our focus to non-orientable surfaces. In [22], it was shown that for any graph G embedded (without crossings) on a non-orientable surface of genus g , there is a graph H embedded (without crossings) on an orientable surface of genus $g - 1$ such that $c(G) \leq c(H)$. The main idea for this stems from the well-known fact that every non-orientable surface Σ_g of genus g has an orientable surface Σ_{g-1} of genus $g - 1$ as a *2-sheeted covering space*. In other words, there is a continuous function $\pi : \Sigma_{g-1} \mapsto \Sigma_g$ such that for every point $x \in \Sigma_g$, there exist an open neighbourhood U_x containing x such that $\pi^{-1}(U_x)$ is a disjoint union of two open sets V_x^1 and V_x^2 where $\pi(V_x^i)$ is homeomorphic to U_x for $i \in \{1, 2\}$. So, if a graph G embedded on Σ_g is lifted into a graph H in Σ_{g-1} , then there are two copies for every vertex, edge and face of G in H (a simple application of Euler's formula then explains the genus $g - 1$ for the covering space). Here, we use ideas from [22] to derive a similar result for kite-augmented 1-planar drawings on non-orientable surfaces (proof in full-version [17]).

► **Theorem 5.2.** *Let G be a graph drawn on a non-orientable surface Σ_g of genus g . Then there exist a $(\Sigma_g, 1)$ -plane graph G^\boxtimes and a $(\Sigma_{g-1}, 1)$ -plane graph H , where Σ_{g-1} is an orientable surface of genus $g - 1$, such that $X(G^\boxtimes) = X(H)$, $x(G^\boxtimes) = x(H)$, and $c_d(G^\boxtimes) \leq c_d(H) \leq 2 \cdot c_d(G^\boxtimes)$ for any integer $d \geq 0$.*

Combining Theorems 3.12, 5.1, and 5.2, we get the following corollary (proof in full-version [17]).

► **Corollary 5.3.** *Let G be any graph drawn on a non-orientable surface Σ of genus g . For any $\alpha \geq 1$, we have $c_{\lceil \alpha(x+2) \rceil}(G) \leq (2g+1) \cdot (2\beta+1)$ and $c_{\lceil \alpha(X+1) \rceil}(G) \leq (2g+1) \cdot (2\beta+1) + (g-1)$ where*

$$\beta = \begin{cases} d + 1 & \text{if } d = x(G) \text{ and } \alpha = 1 \\ \left\lceil \frac{1}{2(\alpha-1)} \right\rceil + 1 & \text{if } d = x(G) \text{ and } \alpha > 1 \end{cases} \quad \beta = \begin{cases} d - 1 & \text{if } d = X(G) \text{ and } \alpha = 1 \\ \left\lceil \frac{1}{2(\alpha-1)} \right\rceil & \text{if } d = X(G) \text{ and } \alpha > 1 \end{cases}$$

5.2 Map Graphs

We now consider a special class of graphs on surfaces called map graphs. A graph G is a *map graph on a surface S* if the vertex set of G can be represented by internally-disjoint closed-disc homeomorphs called *nations* on S , and a pair of vertices are adjacent in G if and only if their corresponding nations touch each other. The above definition generalises *map graphs* originally defined for the sphere [20] to arbitrary surfaces. In [20], it is shown that a graph $G = (V, E)$ is a map graph on the sphere if and only if there is a planar bipartite graph H , called the *witness*, with bipartition $V(H) = \{V, F\}$ such that $G = H^2[V]$. That is, two vertices u and v are adjacent in G if and only if there is a path (u, f, v) in H where $u, v \in V$ and $f \in F$. The witness graph H is obtained from the map representation of G as follows: insert a *nation vertex* in the interior of each nation and a *boundary vertex* at points where two or more nations touch each other; then join a nation vertex with a boundary vertex if and only if the boundary point is part of the nation. Conversely, given a plane bipartite witness graph H with bipartition $V(H) = \{V, F\}$, we draw a star-shaped region around each vertex of V such that its boundary contains all vertices of F that the vertex is adjacent to. This gives us a map representation of $G = H^2[V]$. Clearly, this combinatorial characterisation of map graphs in terms of witness graphs extends to map graphs on surfaces. That is, a graph $G = (V, E)$ is a map graph on a surface S if and only if there is a bipartite *witness* graph H embeddable on S (without crossings), with bipartition $V(H) = \{V, F\}$, such that $G = H^2[V]$.

► **Theorem 5.4.** *If G has a map representation on a surface with a graph H as a witness, then $c(G) \leq c(H)$.*

Proof. To prove Theorem 5.4, we employ our usual strategy of simulating the robber-moves of G on H , and the corresponding cop-moves of H on G . Let $\{U'_1, \dots, U'_p\}$ be a set of p cops where $p = c(H)$. We will show that a set $\{U_1, \dots, U_p\}$ of p cops in G is sufficient to capture a robber in G . Consider the first round of the game. For every cop U'_i occupying a vertex $v \in V(H)$: if $v \in V(G)$, place U_i on v ; else place U_i on a vertex of G neighbouring v . If the robber's initial position in G is on a vertex $x \in V(G)$, then place the robber on the same vertex $x \in V(H)$. Thereafter, every single round played in G can be simulated as two rounds played in H : If the robber in G moves along an edge $(x, y) \in E(G)$, then the robber in H moves along a path (x, f, y) in H , for some vertex $f \in V(H) \setminus V(G)$ (this is possible since $G = H^2[V]$). Similarly, the moves of a cop U'_i can be simulated by U_i in G such that at the end of every two rounds played in H : if U'_i is on a vertex $v \in V(G)$, so is U_i ; else if U'_i is on a vertex $f \notin V(G)$, then U_i is on some vertex of G neighbouring f .

Now consider the final round in H when the robber gets captured by a cop U'_i . Suppose that the robber in G takes an edge (x, y) . As seen before, this corresponds to a path (x, f, y) in H . If the robber is captured on y , then he is captured by U_i in G . If the robber is captured on f , then U_i captures the robber on y as the neighbours of f induce a clique in G . ◀

Theorem 5.4 gives us a simple proof of the fact that full 1-planar graphs have cop-number at most 3 (Corollary 4.7), as all full 1-planar graphs are map graphs on the sphere [19]. Another interesting corollary is that all optimal 2-planar graphs also have cop-number at most 3. This is again due to the fact that all optimal 2-planar graphs are map graphs on the sphere [5].

► **Corollary 5.5.** *If G is a full 1-planar or an optimal 2-planar graph, then $c(G) \leq 3$.*

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