A Proof of Shur's Conjecture on the Growth of Power-Free Languages over Large Alphabets

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Abstract

We settle a conjecture of Shur on an estimation of the exponential growth rates of the languages of $\left(\frac{n}{n-1}\right)$-free words and $\left(\frac{n}{n-1}\right)^+$-free words over large alphabets of size k with a correction of order $O\left(\frac{1}{k^2}\right)$.

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1 Introduction

A *square* is a word of the form uu where u is a nonempty word. A *square-free word* is a word that does not contain a square as a factor. For instance, `hotshots` is a square, and `minimize` is a square-free word. In the seminal work [16], which is widely regarded as the starting point of combinatorics on words, Thue showed that there are infinite square-free words over the ternary alphabet as well as infinite cube-free words over the binary alphabet. This notion of powers and its generalizations have received a lot of attention.

One such generalization is the notion of *fractional power*. A word of the form $w = xx\dots xy$ where x is nonempty and y is a prefix of x is a *power of exponent* $\frac{|w|}{|x|}$ and of *period* $|x|$ (we also say that w is a $\left(\frac{|w|}{|x|}\right)$*-power*). A square is a 2-power, and a cube is a 3-power. For any real $\beta > 1$, we say that the word w is β*-free* (resp. β^+*-free*) if it contains no α-power as a factor with $\alpha \geq \beta$ (resp. $\alpha > \beta$). These notions were introduced by Dejean who conjectured in 1972 that for any $k \geq 5$, there exists a $\left(\frac{k}{k-1}\right)^+$-free infinite word over the k-ary alphabet, and proved that there exists no $\left(\frac{k}{k-1}\right)$-free infinite word over the same alphabet [5]. For $k = 3$, she proved that there exists a $\left(\frac{7}{4}\right)^+$ free infinite word over the ternary alphabet [5], and also conjectured that there exists a $\left(\frac{7}{5}\right)^+$ free infinite word over the 4-ary alphabet, which was later proven by Pansiot [10]. Different authors solved Dejean's conjecture for the cases $k \leq 11$ [9], $k \leq 14$ [7], $k \geq 33$ [1], $k \geq 30$ [2], until the remaining cases were finally proven 40 years after the initial conjecture [3, 11]. This led to another conjecture about the number of $\left(\frac{k}{k-1}\right)^+$-free words over the k-ary alphabet. It is conjectured that for all $k \geq 3$, this language grows exponentially [8], and that as k goes to infinity, the growth rate converges to a constant whose first digits are 1.242 [3, 15]. For all but six cases, it has been proven that these languages grow exponentially [6, 17, 4]. The second part of the conjecture remains completely open.

Let $\mathcal{L}(k,p)$ denote the languages of p-free words over an alphabet of size k. The growth rate $\alpha(k,p) = \lim_{n\to\infty} |\mathcal{L}_n(k,p)|^{1/n}$, where $\mathcal{L}_n(k,p)$ is the set of words with length n in $\mathcal{L}(k,p)$, is usually hard to estimate and requires a large amount of computation. The previously mentioned conjecture asserts that $\lim_{k\to\infty} \alpha\left(k, \left(\frac{k}{k-1}\right)^+\right) \approx 1.24$. Shur suggests an

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interesting problem: study the asymptotic order of $\alpha(k,p)$ when $k \to \infty$ for fixed $p > 1$. For an overview of progress on the problem, we refer the reader to the intensive survey of Shur [14]. In particular, Shur [13] settled the problem when the degree of the power is at least 2 in Theorem 1 below.

We extend the strict total order $<$ over the reals to the numbers of the form x^+ in such a way that for all $x < y$, we have $x < x^+ < y$. That is, x^+ is right after x in this order.

▶ **Theorem 1** (Shur, 2010). *If $p \geq 2$ is an integer and β is in $[p^+, p+1]$, then*

$$\alpha(k,\beta) = \begin{cases} k - \frac{1}{k^{p-1}} + \frac{1}{k^p} - \frac{1}{k^{2p-2}} + O(\frac{1}{k^{2p-1}}) & \text{if } \beta \in \left[p^+, p+\frac{1}{2}\right]; \\ k - \frac{1}{k^{p-1}} + \frac{1}{k^p} + O(\frac{1}{k^{2p-1}}) & \text{if } \beta \in \left[\left(p+\frac{1}{2}\right)^+, p+1\right]. \end{cases}$$

However, the problem for powers with degree less than 2 is still open and Shur [13] suggested the following conjecture.

▶ **Conjecture 2** (Shur, 2010). *For every integer $n \geq 2$, we have*

$$\alpha\left(k, \left(\frac{n}{n-1}\right)^+\right) = k + 2 - n - \frac{n-1}{k} + O\left(\frac{1}{k^2}\right),$$

$$\alpha\left(k, \frac{n}{n-1}\right) = k + 1 - n - \frac{n-1}{k} + O\left(\frac{1}{k^2}\right).$$

Note that for all k and for all $x, y \in \mathbb{R} \cup \{x^+ : x \in \mathbb{R}\}$, if $x \leq y$ then $\mathcal{L}(k,x) \subseteq \mathcal{L}(k,y)$. This implies that $\alpha(k,x)$ is a non-decreasing function of x. Moreover, if Shur's conjecture holds then for every integer n and k, we have

$$\alpha\left(k, \frac{n}{n-1}\right) - \alpha\left(k, \left(\frac{n+1}{n}\right)^+\right) = \frac{1}{k} + O\left(\frac{1}{k^2}\right), \tag{1}$$

$$\alpha\left(k, \left(\frac{n}{n-1}\right)^+\right) - \alpha\left(k, \frac{n}{n-1}\right) = 1 + O\left(\frac{1}{k^2}\right). \tag{2}$$

Shur's conjecture implies that most of the jump between $\alpha\left(k, \frac{n}{n-1}\right)$ and $\alpha\left(k, \frac{n+1}{n}\right)$ is located between $\alpha\left(k, \frac{n}{n-1}\right)$ and $\alpha\left(k, \left(\frac{n}{n-1}\right)^+\right)$. It provides precise bounds on the asymptotic behavior of $\alpha(k,\beta)$ tight up to $\frac{1}{k}$ for every $\beta < 2$. In particular, if $p \in \mathbb{R} \cup \{x^+ : x \in \mathbb{R}\}$ is such that $\frac{n+1}{n} < p < \frac{n}{n-1}$, then $\alpha\left(k, \left(\frac{n+1}{n}\right)^+\right) \leq \alpha(k,p) \leq \alpha\left(k, \frac{n}{n-1}\right)$, which implies

$$\left|\alpha(k,p) - \left(k + 1 - n - \frac{n-1/2}{k}\right)\right| \leq \frac{1}{2k} + O\left(\frac{1}{k^2}\right). \tag{3}$$

That is, this conjecture provides for all p a good estimate of the asymptotic behavior of $\alpha(k,p)$ as k goes to infinity. This conjecture implies other similar empirical facts that also hold for $\beta > 2$ and illustrate the particular behavior of $\alpha(k,\beta)$ ((1) and (2) are respectively called *small variation* and *big jump* in [13]).

Using a counting argument, the second author has established the lower bound in [12].

▶ **Theorem 3** (Rosenfeld, 2021). *Let $n \geq 2$ be an integer, then the following holds*

$$\alpha\left(k, \left(\frac{n}{n-1}\right)^+\right) \geq k + 2 - n - \frac{n-1}{k} + O\left(\frac{1}{k^2}\right),$$

$$\alpha\left(k, \frac{n}{n-1}\right) \geq k + 1 - n - \frac{n-1}{k} + O\left(\frac{1}{k^2}\right).$$

Shur has actually confirmed the other direction of the inequality for $n \leq 9$ by a computer-assisted proof which settles the conjecture for these cases [13]. In this article, we settle Shur's conjecture, by proving the upper bound for all $n \geq 2$.

▶ **Theorem 4.** *For every integer $n \geq 2$, we have*

$$\alpha\left(k, \left(\frac{n}{n-1}\right)^+\right) \leq k + 2 - n - \frac{n-1}{k} + O\left(\frac{1}{k^2}\right), \tag{4}$$

$$\alpha\left(k, \frac{n}{n-1}\right) \leq k + 1 - n - \frac{n-1}{k} + O\left(\frac{1}{k^2}\right). \tag{5}$$

Shur's proof for $n \leq 9$ considers a regular language that contains $\mathcal{L}\left(k, \frac{n}{n-1}\right)$ (resp. $\mathcal{L}\left(k, \left(\frac{n}{n-1}\right)^+\right)$) and uses standard tools from automata theory to upper bound the growth of this language. His proof technique allows him to work with all k, for one fixed n, by considering the automaton where states are taken up to isomorphism (that is, up to renaming of the letters). However, this technique needs to explicitly construct the automaton for each specific value n, which requires the use of computers. Proving the result for all n asks for a different approach. Moreover, Shur's approach is to explicitly construct the automata that avoid powers up to a given length. The states of these automata correspond to suffixes of certain lengths, hence the size of the automata grows exponentially. The main idea behind our proof is also to construct a sequence of automata for all n that approximate the true automata. However, they must be simple enough to allow estimating the growth rates manually, and include all the words in the languages (to ensure that we obtain a proper upper bound), but do not include relatively too many forbidden words (to ensure that the upper bound on the growth rate is sharp).

2 Proof of the upper bounds

In this section, we settle Inequality (5) for $\left(\frac{n}{n-1}\right)$-free words. Inequality (4) for $\left(\frac{n}{n-1}\right)^+$-free words can be proved similarly.

For any two integers $a \leq b$, we write $[a \mathinner{.\,.} b]$ for the set of integers $\{a, \ldots, b\}$, and we write $a \ldots b$ for the word $a \cdot (a+1) \cdot (a+2) \ldots (b-1) \cdot b$.

Fix $n, k \geq 2$. We denote by $\mathcal{L}$ the language of words over k letters that avoid p-powers isomorphic to $1 \ldots m1$ and $12 \ldots m12$ for any $p \geq \left(\frac{n}{n-1}\right)$ and any m. Since $\mathcal{L}$ contains all $\left(\frac{n}{n-1}\right)$-free words, the following theorem directly implies Inequality (5) of Shur's conjecture. This section is devoted to the proof of this theorem.

▶ **Theorem 5.** *The growth rate of the language $\mathcal{L}$ is at most*

$$k + 1 - n - \frac{n-1}{k} + O\left(\frac{1}{k^2}\right).$$

The following fact simply illustrates that two occurrences of the same letter cannot be too close, as it would cause the existence of a p-power isomorphic to $1 \ldots m1$ with $p \geq \frac{n}{n-1}$.

▶ **Observation 6.** *Any factor from $\mathcal{L}$ of length n contains n different letters.*

It directly implies that the growth of $\mathcal{L}$ is at most $k + 1 - n$. The term $-\frac{n-1}{k}$ requires considering the powers isomorphic to $12 \ldots m12$.

The idea is to construct an automaton whose language is a superset of $\mathcal{L}$. Shur noted in [13] that the size of the automaton recognizing $\mathcal{L}$ is exponential in n and is really difficult to analyze. But by considering a slightly larger language, we can construct an automaton with a much smaller number of states that we will be able to analyze, but whose growth rate is really close to the original language (up to the $O\left(\frac{1}{k^2}\right)$ term).

For this, we define the following states for any long enough word w (we require $|w| \geq 2n-1$):

- $\boxed{m}$ for $m \in [n \mathinner{.\,.} 2n-2]$: if a suffix of w is isomorphic to $m1\ldots m$.
- $\widehat{m}$ for $m \in [n \mathinner{.\,.} 2n-2]$: if a suffix of w is isomorphic to $x1\ldots m$ with x satisfying

$$\begin{cases} x < m & \text{if } m < 2n-2, \\ x \neq m & \text{if } m = 2n-2. \end{cases}$$

We provide some intuition about this definition. The number m in the states $\widehat{m}$, $\boxed{m}$ stands for the length of the longest suffix containing only distinct letters, hence the letter in front of that suffix should appear in the suffix (the only exception[1] being $\widehat{2n-2}$ as it actually covers all the lengths from $2n-2$ up to k). The states $\boxed{m}$ and $\widehat{m}$ differ based on whether this reoccurring letter is the last letter of the word. The states $\boxed{m}$ are distinguished from the other state, since when $m < 2n-2$ the word ends with $m1\ldots m$ and we know that the next letter cannot be 1 (since this would create a power of the form $m1\ldots m1$), that is, we know that at least one more letter is forbidden in the next position. The length of this power actually suggests the choice of the seemingly-arbitrary number $2n-2$.

▶ Remark 7. By definition, the words in state $\hat{n}$ contain a forbidden factor. It is fine to consider them anyway because we only want an upper bound on the number of words, and introducing this “artificial” state creates some symmetries that simplify the statements and the proofs.

From here on, we assume

$$k \geq 2n-2.$$

▶ **Observation 8.** *Every word $u \in \mathcal{L}$ with $|u| \geq 2n-1$ is in exactly one of the previously defined states.*

We now describe how appending a letter at the end of a word from $\mathcal{L}$ can alter the state. In particular, we need $T(s_1, s_2)$ such that for every states s_1, s_2 and for any $w \in \mathcal{L}$ in state s_1,

$$T(s_1, s_2) \geq |\{\alpha \in \mathcal{A} : w\alpha \in \mathcal{L} \text{ and is in state } s_2\}|.$$

That is, there are at most $T(s_1, s_2)$ letters $\alpha \in [1 \mathinner{.\,.} k]$ such that $w\alpha$ is in $\mathcal{L}$ and is in state s_2.

In the following lemmas, we provide such a T. For convenience, we use the following notation:

$$\begin{aligned} \operatorname{succ}(\widehat{m}) &= \widehat{m+1} \qquad \text{if } m < 2n-2, \\ \operatorname{succ}(\widehat{2n-2}) &= \widehat{2n-2}. \end{aligned}$$

▶ **Lemma 9.** *For $m \in [n \mathinner{.\,.} 2n-2]$, we can set*

[1] It might be more natural to write this state as $\widehat{\geq 2n-2}$, but for the sake of notation it is more convenient to keep it as $\widehat{2n-2}$.

- $T(\widehat{m}, s) = 1$, *for each state* $s \in \{\boxed{n}, \ldots, \boxed{m}\}$,
- $T(\widehat{m}, \mathrm{succ}(\widehat{m})) = k - m$,
- *for the other states* s, $T(\widehat{m}, s) = 0$.

Proof. If $w \in \mathcal{L}$ is in state $\widehat{m}$, then up to renaming of the letters, $x1 \ldots m$ is a suffix of w for some $x < m$. By Proposition 6, the next letter needs to be different from the last $n-1$ letters, so it belongs to $[1\mathinner{.\,.} k] \setminus [m-n+2 \mathinner{.\,.} m]$. If the next letter is in $[1 \mathinner{.\,.} m-n+1]$, then the next state is $\boxed{m}, \boxed{m-1}, \ldots, \boxed{n}$, respectively. If the next letter is in $[m+1 \mathinner{.\,.} k]$, then the next state is always $\mathrm{succ}(\widehat{m})$. ◀

The following lemma and its proof are almost identical to the ones above. The difference is due to the missing loop around the state $\boxed{m}$.

▶ **Lemma 10.** *For* $m \in [n \mathinner{.\,.} 2n-2]$*, we can set*
- $T\left(\boxed{m}, s\right) = 1$, *for each state* $s \in \{\boxed{n}, \ldots, \boxed{m-1}\}$,
- $T\left(\boxed{m}, \mathrm{succ}(\widehat{m})\right) = k - m$,
- *for the other states* s, $T\left(\boxed{m}, s\right) = 0$.

Proof. If $w \in \mathcal{L}$ is in state $\boxed{m}$, then up to renaming of the letters, $m1 \ldots m$ is a suffix of w. By Proposition 6, if $w\alpha$ is in $\mathcal{L}$, then α is different from the last $n-1$ letters of w, so $\alpha \in [1 \mathinner{.\,.} k] \setminus [m-n+2 \mathinner{.\,.} m]$. This time, we also have $\alpha \neq 1$ since $m1 \ldots m1$ is forbidden in $\mathcal{L}$. In particular, if the next letter is $2, \ldots, m-n+1$, then the next state is $\boxed{m-1}, \ldots, \boxed{n}$, respectively. If the next letter is in $[m+1 \mathinner{.\,.} k]$, then the next state is $\mathrm{succ}(\widehat{m})$. ◀

In the remainder of this section, we treat T as a square matrix indexed over the states. We abuse the notation and write T_{s_1,s_2} for $T(s_1, s_2)$. By the definition of T, for every $w \in \mathcal{L}$ in state s_1 we have

$$(T^m)_{s_1,s_2} \geq \{u \in \mathcal{A}^m : wu \in \mathcal{L} \text{ and is in state } s_2\}\,| \,.$$

Since every long enough word from $\mathcal{L}$ is in one of the states, we have the following upper bound on the growth of $\mathcal{L}$.

▶ **Proposition 11.** *The growth rate of the language* $\mathcal{L}$ *is at most the spectral radius of the matrix* T.

Theorem 4 can then be reduced to the following theorem.

▶ **Theorem 12.** *For any* n*, there exists a constant* C *such that for all large enough* k*, the spectral radius of* T *is at most*

$$\lambda = k - (n-1) - \frac{n-1}{k} + C/k^2.$$

Proof. For the proof we fix n, and we let $\lambda = k - (n-1) - \frac{n-1}{k} + C/k^2$, with C large enough as a function of n. We denote the coordinates of a vector v by $v_{\widehat{n}}, \ldots, v_{\widehat{2n-2}}, v_{\boxed{n}}, \ldots, v_{\boxed{2n-2}}$. We consider the vector x with

$$x_{\widehat{m}} = 1 - \frac{m(4n-m-3)}{2k^2}, \tag{6}$$

$$x_{\boxed{m}} = x_{\widehat{m}} - \frac{1}{k}. \tag{7}$$

We first verify that x is positive when k is large enough. Indeed, for all values of n, k, we can prove that for every m,

$$x_{\widehat{m}} = 1 - \frac{m(4n-m-3)}{2k^2} \geq 1 - \frac{(2n-2)(2n-1)}{2(2n-2)^2} = \frac{4n-4-2n+1}{4n-4} = \frac{1}{4} \cdot \frac{2n-3}{n-1} \geq \frac{1}{4}$$

when $n \geq 2$. We also need $x_{\boxed{m}} > 0$ for any m, which by (7) holds for k large enough.

We will also use the following direct consequences of (6) and (7):

$$\begin{aligned} x_{\mathrm{succ}(\widehat{m})} &= x_{\widehat{m}} - \frac{2n-m-2}{k^2}, \\ 1 - \frac{1}{k} &\geq x_{\boxed{m}}. \end{aligned}$$

The nonnegativity of T and Perron-Frobenius Theorem imply that an eigenvector v associated to the dominant eigenvalue $\rho(T)$ is nonnegative. Since x is positive, it is non-orthogonal with v, and we have $x = cv + r$ with c a positive real and r a nonnegative vector. This and the nonnegativity of all the entries implies that $\|T^n x\| \geq \|T^n(cv)\| \geq \Theta(\rho^n)$. It follows that $\rho(T) = \lim_{n\to\infty} \|T^n x\|^{1/n}$. Therefore, in order to deduce $\rho(T) \leq \lambda$, it suffices to show that

$$Tx \leq \lambda x \,,$$

where the inequality is coordinate-wise.

We now verify the inequality $Tx \leq \lambda x$ for each coordinate as follows.

For every $m \in [n \mathrel{..} 2n-2]$,

$$\begin{aligned} (Tx)_{\widehat{m}} =& (k-m)x_{\mathrm{succ}(\widehat{m})} + x_{\boxed{n}} + \cdots + x_{\boxed{m}} \\ \leq& (k-m)\left(x_{\widehat{m}} - \frac{2n-m-2}{k^2}\right) + (m+1-n)\left(1 - \frac{1}{k}\right) \\ =& (k-m)x_{\widehat{m}} + (m+1-n) - \frac{2n-m-2}{k} - \frac{m+1-n}{k} + \frac{m(2n-m-2)}{k^2} \\ =& (k-m)x_{\widehat{m}} + (m+1-n) - \frac{n-1}{k} + \frac{m(2n-m-2)}{k^2} \\ =& \left(k+1-n-\frac{n-1}{k}+\frac{C}{k^2}\right)x_{\widehat{m}} + (1-x_{\widehat{m}})\left(m+1-n-\frac{n-1}{k}\right) \\ & \quad - x_{\widehat{m}}\frac{C}{k^2} + \frac{m(2n-m-2)}{k^2} \\ \leq& \left(k+1-n-\frac{n-1}{k}+\frac{C}{k^2}\right)x_{\widehat{m}} + \frac{m(4n-m-3)}{2k^2}(m+1-n) \\ & \quad - \frac{1}{4}\frac{C}{k^2} + \frac{m(2n-m-2)}{k^2} \\ \leq& \lambda x_{\widehat{m}} \end{aligned}$$

for some large enough C that only depends on n.

We now take care of the coordinates $\boxed{m}$. By Lemma 9 and Lemma 10, we have for all $m \in [n \mathrel{..} 2n-2]$,

$$(Tx)_{\boxed{m}} = (Tx)_{\widehat{m}} - x_{\boxed{m}} \,.$$

We have already proven that $(Tx)_{\widehat{m}} \leq \lambda x_{\widehat{m}}$, for all $m \in [n \mathinner{..} 2n-2]$, and the rest of the computation follows:

$$\begin{aligned}
(Tx)_{\boxed{m}} \leq & \lambda x_{\widehat{m}} - x_{\boxed{m}} \\
= & \lambda \left(x_{\boxed{m}} + \frac{1}{k} \right) - \left(1 - \frac{1}{k} - \frac{m(4n-m-3)}{2k^2} \right) \\
= & \lambda x_{\boxed{m}} + \left(k - n + 1 - \frac{n-1}{k} + \frac{C}{k^2} \right) \frac{1}{k} - \left(1 - \frac{1}{k} \right) + \frac{m(4n-m-3)}{2k^2} \\
= & \lambda x_{\boxed{m}} - \frac{n-2}{k} + \frac{C}{k^3} + \frac{m(4n-m-3) - 2n + 2}{2k^2} \\
\leq & \lambda x_{\boxed{m}}
\end{aligned}$$

where the last inequality holds for large enough k. This concludes our proof. ◀

▶ Remark 13. Although we do not really care about how large C and k should be in Theorem 4, one easily verifies that $C = 20n^3$ and $k = 5(n+6)$ work.

3 Conclusion

As mentioned in the introduction (see Equation (3)), our result provides for all p a good estimate for $\alpha(k,p)$ up to an error term $1/k + O(1/k^2)$. When we only care about powers isomorphic to $1 \ldots \ell 1$ and $12 \ldots \ell 12$, the difference between p-powers for $p \geq \frac{n}{n-1}$ and for $p > \frac{n}{n-1}$ lies only in the range of ℓ. By varying the length ℓ, we might allow ourselves to address p-powers for a fraction p that may be of more complicated nature than $\frac{n}{n-1}$. So applying our approach for all p instead of only p of the form $\frac{n}{n-1}$ or $\left(\frac{n}{n-1}\right)^+$ might be enough to replace this error term by $O(1/k^2)$.

On the other hand, as conjectured by Shur, and later by the second author, the behavior of the growth rate up to the $O(1/k^2)$ term is controlled by powers whose tail is of length 1 or 2. It seems natural to wonder whether increasing the precision up to the $O(1/k^3)$ term requires precisely the considerations of powers with tails of length 3. Shall the degree of the order of precision grow linearly with respect to the length of the tails?

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