Right-Linear Lattices

An Algebraic Theory of ω-Regular Languages, with Fixed Points

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Abstract

Alternating parity automata (APAs) provide a robust formalism for modelling infinite behaviours and play a central role in formal verification. Despite their widespread use, the algebraic theory underlying APAs has remained largely unexplored. In recent work [10], a notation for non-deterministic finite automata (NFAs) was introduced, along with a sound and complete axiomatisation of their equational theory via *right-linear algebras*. In this paper, we extend that line of work to the setting of infinite words. In particular, we present a dualised syntax, yielding a notation for APAs based on *right-linear lattice* expressions, and provide a natural axiomatisation of their equational theory with respect to the standard language model of ω-regular languages. The design of this axiomatisation is guided by the theory of fixed point logics; in fact, the completeness factors cleanly through the completeness of the linear-time μ-calculus.

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1 Introduction

1.1 A half century of ω-automata theory

ω-automata, *i.e.* finite state machines running on infinite inputs, are useful for modelling behaviour of systems that are not expected to terminate, such as hardware, operating systems and control systems. The prototypical ω-automaton model, *Büchi automaton*, is widely used in model checking [30, 15, 14, 17].

The theory of ω-regular languages, *i.e.* languages accepted by ω-automata, have been studied for more than half a century. Büchi's famous complementation theorem [7] for his automata is the engine underlying his proof of the decidability of monadic second-order logic (MSOL) over infinite words. Its extension to infinite trees, *Rabin's Tree Theorem* [24], is often referred to as the "mother of all decidability results".

McNaughton [23] showed that, while Büchi automata could not be determinised per se, a naturally larger class of acceptance conditions (Muller or parity) allowed such determinisation, a highly technical result later improved by Safra [26]. A later relaxation was the symmetrisa-

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tion of the transition relation itself: instead of only allowing non-deterministic states, allow co-nondeterministic ones too. This has led to beautiful accounts of ω-regular language theory via the theory of positional and finite memory games. The resulting computational model, *alternating parity automaton* (APA), is now the go-to model in textbook presentations, e.g. [16]. Indeed, their features more closely mimic those of logical settings where such symmetries abound, e.g. linear-time μ-calculus [29] and MSOL over infinite words.

1.2 An algebraic approach

In the world of finite words, the theory of regular languages have been axiomatised as *Kleene Algebras* (KAs). In fact, KAs are part of a bigger cohort of *regular algebras* and they have been studied for several decades and completeness proofs for different variants have been obtained [27, 21, 20, 3, 4]. KAs and various extensions have found applications in specification and verification of programs and networks [1].

Note that KAs and other regular algebras axiomatise the equational theory of *regular expressions*. An alternative (and arguably more faithful) notation for NFAs may be given by identifying them with *right-linear grammars*. Recall that a right-linear grammar is a CFG where each production has RHS either aX or ε. They may also be written as *right-linear expressions*, by choosing an order for resolving non-terminals. Formally, **right-linear expressions** (aka **RLA expressions**), written $e, f, \ldots$, are generated by:

$$e, f, \ldots \quad ::= \quad 1 \quad | \quad X \quad | \quad e + f \quad | \quad a \cdot e \quad | \quad \mu X e$$

for $a \in \mathcal{A}$, a finite **alphabet** and $X \in \mathcal{V}$, a countable set of **variables**. Indeed [10] takes this viewpoint seriously and proposed an alternative algebraic foundation of regular language theory, via *right-linear algebras* (RLAs). Notably, RLAs are strictly more general than KAs, as they lack any multiplicative structure. In particular, this means that ω-languages naturally form a model of them (unlike KAs). This is the starting point of the current work.

In this work, we investigate the algebraic structures induced by the theory of APAs. To do so, we dualise the (1-free)[1] syntax of RLA expressions to obtain *right-linear lattice* (RLL) expressions, formally generated by:

$$e, f, \ldots \quad ::= \quad X \quad | \quad a \cdot e \quad | \quad e + f \quad | \quad e \cap f \quad | \quad \mu X e(X) \quad | \quad \nu X e(X)$$

Compared to RLA expressions, RLL expressions enjoy more symmetric relationships to games and consequently, are a notation for APAs. Our main contribution is a sound and complete axiomatisation $\mathsf{RLL}_{\mathcal{L}}$ of the theory of RLL expressions for the language model.

1.3 Related work

Two kinds of variations of KAs are relevant to this work. Firstly, the generalisation of regular algebras to ω-regular algebras [31, 8, 22, 9], by axiomatising the theory of ω-regular expressions, a generalisation of regular expressions admitting terms of the form e^ω. Secondly, following the idea of dualisation from KAs leads to *action lattices*, an extension with meet (dual to the sum), and residuals (adjoint to the product). Since RLA does not have products we do not consider residuals, – so perhaps *Kleene lattices* [6, 12], the extension of KAs with meet is the closest cousin of our proposed right-linear lattices.

[1] This restriction imposed to so that the intended interpretation is over ω-words only, not $\leq \omega$-words.

1.4 Roadmap

In Section 2, we recall right-linear algebras and define RLL expressions, a notation for APAs. We identify several principles governing their behaviour in the standard model $\mathcal{L}$ of ω-languages; namely, their interpretations satisfy a theory of bounded distributive lattices, certain lattice homomorphisms and least and greatest fixed points (of definable operators). In Section 3 recover a syntactic form of *complements*, and we present our ultimate axiomatisation in Section 4. In Section 5, we prove the completeness of the axiomatisation by reducing it to the completeness of linear time μ-calculus. We conclude with some remarks on the axiomatisation and comparison with existing literature in Section 6. Further examples and detailed proofs can be found in an accompanying preprint [11].

2 Right-linear lattice expressions for ω-regular languages

Let us fix a finite set $\mathcal{A}$ (the **alphabet**) of **letters**, written a, b, etc., and a countable set $\mathcal{V}$ of **variables**, written X, Y, etc.

2.1 RLL expressions and ω-regular languages

RLL expressions, written $e, f, \dots$, are generated by:

$$e, f, \dots \quad ::= \quad X \quad | \quad a \cdot e \quad | \quad e + f \quad | \quad e \cap f \quad | \quad \mu X e(X) \quad | \quad \nu X e(X)$$

for $a \in \mathcal{A}$ and $X \in \mathcal{V}$. We usually just write ae instead of $a \cdot e$. A variable X is said to occur **freely** in an expression e if it not under the scope of any binder μX or νX. An expression is said to be **closed** if it has no occurrences of free variables.

▶ Remark 1 (0). The original presentation of right-linear expressions includes a symbol 0 that was always interpreted as a unit for + in structures over this syntax. Here we shall more simply just write $0 := \mu X X$, and remark on the consequences of this choice as we go.

The intended interpretation of an RLL expression is a language of ω-words over $\mathcal{A}$.

▶ **Definition 2** (Interpretation). *Let us temporarily expand the syntax of RLL expressions to include each language $A \subseteq \mathcal{A}^\omega$ as a constant symbol. We interpret each closed expression (of this expanded language) as a subset of $\mathcal{A}^\omega$ as follows:*

- $\mathcal{L}(A) := A$
- $\mathcal{L}(e + f) := \mathcal{L}(e) \cup \mathcal{L}(f)$
- $\mathcal{L}(e \cap f) := \mathcal{L}(e) \cap \mathcal{L}(f)$
- $\mathcal{L}(ae) := \{a\sigma \mid \sigma \in \mathcal{L}(e)\}$
- $\mathcal{L}(\mu X e(X)) := \bigcap\{A \mid A \supseteq \mathcal{L}(e(A))\}$
- $\mathcal{L}(\nu X e(X)) := \bigcup\{A \mid A \subseteq \mathcal{L}(e(A))\}$

Note that Remark 1 is justified by this interpretation: indeed $\mathcal{L}(\mu X X)$ is just the empty language. Furthermore:

▶ Remark 3 ($\top$). Dual to $0 := \mu X X$, we define $\top := \nu X X$, that denotes the universal language in $\mathcal{L}$.

To justify that μ and ν are indeed interpreted as *fixed point* operators, we will first recall some terminology. For $(S, \leqslant_S)$ a complete lattice, $x \in S$ is a **prefixed** (**postfixed** respectively) point of a map $f : S \to S$ if $f(x) \leqslant_S x$ ($x \leqslant_S f(x)$ respectively). If x is both a pre and postfixed point, it is called a **fixed point** of f.

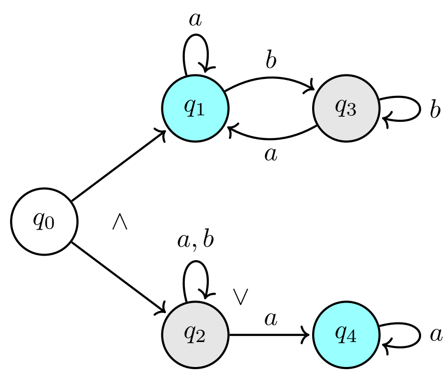

Figure 1 An alternating parity automaton. Blue and grey nodes are of parity 0 and 1 respectively.

▶ **Theorem 4** (Knaster-Tarski theorem [19, 28]). *Let* $(S, \leqslant_S)$ *be a complete lattice and* $f : S \to S$ *be monotone. The set of fixed points of* f *forms a complete lattice under* $\leqslant_S$*.*

Let us now point out that $\mathcal{P}(\mathcal{A}^\omega)$ indeed forms a complete lattice under $\subseteq$, and closed under concatenation with letters on the left. Since all the operations are monotone, $\mathcal{L}(\mu X e(X))$ and $\mathcal{L}(\nu X e(X))$ are indeed the least and greatest fixed points, respectively, of the operation $A \mapsto \mathcal{L}(e(A))$, by the Knaster-Tarski theorem.

▶ **Example 5.** Let us consider some examples of RLL expressions and the languages they compute in $\mathcal{L}$, over the alphabet $\{a, b\}$:

- $i_a := \nu X \mu Y (aX + bY)$ computes the language I_a of words with infinitely many as:
  - First note that, for any language A, we have that $\mu Y (A + bY)$ computes $b^* A$.
  - Now let us show that I_a is a postfixed point of $X \mapsto \mu Y (aX + bY)$. By the above point, it suffices to show that $I_a \subseteq b^* a I_a$, which holds since every word w with infinitely many as can be written $w = b^* a w'$, where w' has infinitely many as too.
  - Now suppose B is another postfixed point, i.e. that $B \subseteq b^* a B$. Then we have $B \subseteq b^* a B \subseteq b^* a b^* a B \subseteq \cdots \subseteq (b^* a)^\omega = I_a$.
- $f_b := \mu X (bX + aX + a \nu Y a Y)$ computes the language F_b of words with finitely many bs:
  - First note that, $\nu Y a Y$ computes a^ω.
  - By a similar argument as above, F_b is a prefixed point of $X \mapsto bX + aX + aa^\omega$, contained in any other prefixed point.
- $i_a \cap f_b$ computes the language $I_a \cap F_b$ of words with infinitely many as and at most finitely many bs.

As the reader might have expected, the range of $\mathcal{L}(\cdot)$ is just the ω-regular languages.

▶ **Proposition 6.** *A language* $L \subseteq \mathcal{A}^\omega$ *is* ω*-regular if and only if there is a closed RLL expression* e *such that* $\mathcal{L}(e) = L$*.*

One direction, exhaustion of all ω-regular languages, follows swiftly from the inductive definition of the set of all ω-regular languages and was established in previous work [10], without making use of $\cap$. One way to prove the converse is to introduce a game-theoretic mechanism for deciding word membership in $\mathcal{L}(e)$. This was introduced in [10] without $\cap$ and can be straightforwardly lifted to our setting. We will illustrate with an example.

▶ **Example 7.** Consider $i_a \cap f_b$ as defined in Example 5. We will consider each variable as a state. So, $i_a = \nu X_1 \mu X_3 (aX_1 + bX_3)$ gives us two states q_1 and q_3 such that $q_1 \xrightarrow{a} q_1$, $q_1 \xrightarrow{b} q_3$, $q_3 \xrightarrow{a} q_1$, and $q_3 \xrightarrow{b} q_3$. To model the priority, we assign colours to states such that $\chi(q_1)$ is even, $\chi(q_3)$ is odd, and $\chi(q_1) < \chi(q_3)$. So, let $\chi(q_1) = 0$ and $\chi(q_3) = 1$.

Similarly, $f_b = \mu X_2(bX_2 + aX_2 + a\nu X_4 aX_4)$ gives us two states q_2 and q_4 such that $q_2 \xrightarrow{b} q_2$, $q_2 \xrightarrow{a} q_2 \vee q_4$, and $q_4 \xrightarrow{a} q_4$ with $\chi(q_2) = 1$ and $\chi(q_4) = 0$.

Finally, we model the meet with an initial state q_0 with arbitrary colour and $q_0 \xrightarrow{\varepsilon} q_1 \wedge q_2$. This gives us an alternating parity automaton (see Figure 1) computing exactly the set of words with finitely many as and infinitely many bs.

▶ Remark 8 (ε). Note that we have allowed ε-transitions in our APAs in order to mimic the RLL syntax as closely as possible. This choice does not affect the class of ω-languages represented by APAs (see, e.g., [5] for a similar exposition).

2.2 Some properties of the intended model

Let us take a moment to remark upon some principles valid in the intended interpretation $\mathcal{L}$ of RLL expressions, in order to motivate the axiomatisation we introduce later. As usual we write $e \leq f$ for $e + f = f$, equivalently $e = e \cap f$ (so in $\mathcal{L}$, $\leq$ just means $\subseteq$). First:

- $(0, \top, +, \cap)$ forms a bounded distributive lattice:[2]

$$\begin{array}{rclcrcl} e+0 &=& e & \qquad & e\cap\top &=& e \\ e+(f+g) &=& (e+f)+g & & e\cap(f\cap g) &=& (e\cap f)\cap g \\ e+f &=& f+e & & e\cap f &=& f\cap e \\ e+e &=& e & & e\cap e &=& e \\ e+(e\cap f) &=& e & & e\cap(e+f) &=& e \\ e+(f\cap g) &=& (e+f)\cap(e+g) & & e\cap(f+g) &=& (e\cap f)+(e\cap g) \end{array} \tag{1}$$

- Each $a \in \mathcal{A}$ is a (lower) semibounded lattice homomorphism:

$$\begin{array}{rcl} a0 &=& 0 \\ a(e+f) &=& ae+af \\ a(e\cap f) &=& ae\cap af \end{array} \tag{2}$$

In particular, of course $\mathcal{L} \not\models a\top = \top$, so in this sense 0 and $\top$ do not behave dually in $\mathcal{L}$. Instead we have a variant of this principle, indicating that the homomorphisms freely factor the structure:

- The ranges of $a \in \mathcal{A}$ partition the domain:

$$\begin{array}{rcll} ae\cap bf &=& 0 & \qquad \text{whenever } a\neq b \\ \top &=& \sum\limits_{a\in\mathcal{A}} a\top & \end{array} \tag{3}$$

Finally, $\mathcal{L}$ is a complete lattice and so interprets the least and greatest fixed points as such. Being a complete lattice is a *second-order* property, but we have the following first order (even quasi-equational) consequences:

- $\mu X e(X)$ is a least prefixed point of $X \mapsto e(X)$:

$$\begin{array}{ll} \text{(Prefix)} & e(\mu X e(X)) \leq \mu X e(X) \\ \text{(Induction)} & e(f) \leq f \implies \mu X e(X) \leq f \end{array} \tag{4}$$

- $\nu X e(X)$ is a greatest postfixed point of $X \mapsto e(X)$:

$$\begin{array}{ll} \text{(Postfix)} & \nu X e(X) \leq e(\nu X e(X)) \\ \text{(Coinduction)} & f \leq e(f) \implies f \leq \nu X e(X) \end{array} \tag{5}$$

[2] Some of these axioms are redundant, but we include them all to facilitate the exposition.

Note that Induction and Coinduction are axiom *schemas*, as one might expect (e.g. as in Peano Arithmetic).

▶ **Example 9** (0)**.** Recall $0 := \mu XX$ and $\top := \nu XX$. Indeed $0 \leq e$ (i.e. $0 + e = e$) is a consequence of the axioms (4) above: it follows by Induction from $e \leq e$. Dually $e \leq \top$ follows from (5).

The principles above also suffice to derive some basic properties of the operators defined by RLL expressions:

▶ **Proposition 10** (Functoriality)**.** *Equations* (1)–(5) $\vdash f \leq g \implies e(f) \leq e(g)$*.*

As an immediate corollary of functoriality, we have:

▶ **Example 11** (Fixed points are fixed points)**.** By a standard argument mimicking the proof of the Knaster-Tarski theorem, $\mathsf{RLL}_{\mathcal{L}} \vdash \mu X e(X) \leq e(\mu X e(X))$ and dually, $\mathsf{RLL}_{\mathcal{L}} \vdash e(\nu X e(X)) \leq \nu X e(X)$. We will show the first one. By Induction it suffices to show that $e(\mu X e(X))$ is a prefixed point, i.e. $e(e(\mu X e(X))) \leq e(\mu X e(X))$. Now, by the functors of Proposition 10 above it suffices to show $e(\mu X e(X)) \leq \mu X e(X)$, which is just the Prefix axiom.

Recall that RLA expressions are notation for NFAs and thus can be duly interpreted as regular languages over *finite* words. In previous work [10] we showed soundness and completeness of a subset of the above mentioned axioms for RLA expressions with respect to the language interpretation (also written $\mathcal{L}$ hedging the risk of confusion). Writing RLA for the subset of axioms from Equations (1)–(5) not involving $\cap, \top, \nu$, we have:

▶ **Theorem 12** ([10])**.** *For RLA expressions e, f,* $\mathsf{RLA} \vdash e = f \iff \mathcal{L}(e) = \mathcal{L}(f)$*.*

The goal of the present work is to establish a similar sort of result for RLL expressions, in the ω-regular world rather than the (finitely) regular world.

3 Boolean subalgebra of RLL expressions

As the ω-regular languages are closed under complementation, we actually have that the initial term submodel of RLL expressions in $\mathcal{L}$ forms a Boolean algebra. In this section, we shall inline this structure axiomatically.

3.1 Complements

We can define complements of the RLL expressions, wrt $\mathcal{L}$, quite simply, thanks to closure of the syntax under duality:

▶ **Definition 13** (Complement)**.** *Define e^c by induction on an expression e:*

- $(ae)^c := ae^c + \sum_{b \neq a} b\top$
- $X^c := X$
- $(e + f)^c := e^c \cap f^c$
- $(e \cap f)^c := e^c + f^c$
- $(\mu X e)^c := \nu X e^c$
- $(\nu X e)^c := \mu X e^c$

▶ **Proposition 14.** *e and e^c are complementary in $\mathcal{L}$, i.e. $\mathcal{L}(e^c) = \mathcal{A}^\omega \setminus \mathcal{L}(e)$ for any closed expression e.*

Thus the set of RLL expressions denote a Boolean subalgebra of $\mathcal{L}$, a fact subsumed by adequacy for ω-regular languages, Proposition 6. Of course duality of $+, \cap$ hold in any bounded distributive lattice. The homomorphism axioms also guarantee that our definition of $(ae)^c$ is well-behaved:

▶ **Example 15.** Let $\mathfrak{L}$ be a bounded distributive lattice (i.e. a model of (1)) satisfying Equations (2) and (3), and suppose A has a complement A^c in $\mathfrak{L}$.[3] Then aA has complement $(aA)^c = aA^c + \sum_{b \neq a} b\top$:

$$\begin{aligned}
0 = A \cap A^c &\implies 0 = aA \cap aA^c && \text{by (2)}\\
&\implies 0 = (aA \cap aA^c) + \sum_{b\neq a} (aA \cap b\top) && \text{by (3)}\\
&\implies 0 = aA \cap (aA^c + \sum_{b\neq a} b\top) && \text{by distributivity}\\
&\implies 0 = aA \cap (aA)^c && \text{by definition}
\end{aligned}$$

Similarly, one can show $\top = A + A^c \implies \top = aA + (aA)^c$.

However, the issue with the principles thusfar, Equations (1)–(5), is that they do not guarantee such duality of μ and ν. Let us address this issue now.

3.2 Incompleteness strikes!

Not all models of Equations (1)–(5) interpret e and e^c as complements. Indeed it is well known that there are even completely distributive lattices, let alone models of Equations (1)–(5), that are not even Heyting algebras, let alone Boolean algebras. Still, this does not quite yet give unprovability of the complementary laws for closed expressions (which carve out a substructure of a model). Indeed in even complete distributive lattices μ and ν are at least dual, in the sense that they *preserve* complements. Let us develop an appropriate counterexample, exploiting the incompleteness of the lattice structure:

▶ **Example 16** (Incompleteness)**.** Consider the Cantor topology $\mathcal{C}$ on $\mathcal{A}^\omega$: $A \subseteq \mathcal{A}^\omega$ is *open* if it is a (possibly infinite) union of sets of form $a_1 \cdots a_n \mathcal{A}^\omega$. $\mathcal{C}$ is closed under finite meets and infinite joins, as it is a topology, so it forms a (bounded) join-complete lattice. So we have:

- $\mathcal{C}$ satisfies (1), under the usual set-theoretic union and intersection; and,
- We can interpret least and greatest fixed points in $\mathcal{C}$ by setting, for monotone open operators F:
  - $\mathcal{C}(\mu F) := \bigcup_{\alpha \in \mathsf{Ord}} F^\alpha(\varnothing)$; and,
  - $\mathcal{C}(\nu F) := \bigcup_{A \subseteq F(A)} A$.

  where $F^\alpha(X)$ is defined by transfinite induction on α as follows:
  - $F^0(X) := X$;
  - $F^{\alpha+1}(X) := F(F^\alpha(X))$; and,
  - $F^\lambda(X) := \bigcup_{\beta < \lambda} F^\beta(X)$ for limit ordinals λ.

[3] Recall that complements are unique in distributive lattices.

It is not difficult to see that these interpretations of μF and νF are always least/greatest pre/post fixed points, respectively, in $\mathcal{C}$, as long as F is monotone. Thus $\mathcal{C}$ furthermore satisfies Equations (4) and (5).

Now define the homomorphisms $a \in \mathcal{A}$ in $\mathcal{C}$ just as in $\mathcal{L}$: $aA := \{aw : w \in A\}$. Clearly this is an open map and, under this interpretation, $\mathcal{C}$ satisfies Equations (2) and (3) as it is a substructure of $\mathcal{L}$.

However the denotation of greatest fixed points in $\mathcal{C}$ may be smaller than in $\mathcal{L}$, as its definition as a union of postfixed points ranges over only open sets, not all languages. Indeed we have:

- $\mathcal{C}(\nu X(aX)) = \varnothing$. For this, reasoning in $\mathcal{C}$, note that surely $\nu X(aX) \leq \top$ by boundedness, and so $\nu X(aX) \leq a^n\top$ for all $n \in \mathbb{N}$, by monotonticity and since $\nu X(aX)$ is a fixed point of $X \mapsto aX$. The only nonempty subset of $\mathcal{A}^\omega$ satisfying this property is $\{a^\omega\}$, but this is not open and so does not belong to $\mathcal{C}$. On the other hand, evidently $a\varnothing = \varnothing$.
- $\mathcal{C}(\nu X(aX))^c \neq \mathcal{A}^\omega$. Reasoning in $\mathcal{C}$, we have that $(\nu X(aX))^c = \mu X(aX + \sum\limits_{b \neq a} b\top)$, which (necessarily) has the same denotation in $\mathcal{C}$ as in $\mathcal{L}$: the set of words with at least one letter $b \neq a$.

Thus $\nu X(aX)$ and $(\nu X(aX))^c$ are not complementary in $\mathcal{C}$. Since $\mathcal{C}$ is a model of Equations (1)–(5), it is immediate that this set of axioms is incomplete for $\mathcal{L}$: it does not prove $\top = \nu X(aX) + (\nu X(aX))^c$.

The issue for Equations (1)–(5), towards completeness for $\mathcal{L}$, is that, in the absence of completeness of the lattice, it is not immediately clear that μ and ν are dual. Duality is derivable for $+$ and $\cap$ from Equation (1), but the infinitary nature of the fixed points means that it does not follow as a consequence of Equations (1)–(5).

▶ Remark 17. In [10] it was asked whether an axiomatisation νRLA for $\cap$-free expressions is complete over $\mathcal{L}$. νRLA is essentially the $\cap$-free part of Equations (1)–(5) (see [10] for a formal definition), and so the structure $\mathcal{C}$ in Example 16 above also models νRLA. Since the equation $\top = \nu X(aX) + (\nu X(aX))^c$ is $\cap$-free, we thus have that νRLA is in fact *incomplete* with respect to $\mathcal{L}$.

4 An axiomatisation

In this section, we will develop an axiomatisation $\mathsf{RLL}_{\mathcal{L}}$ for equations over RLL expressions that are valid in $\mathcal{L}$. Towards a definition of our ultimate axiomatisation, let us give a final property in $\mathcal{L}$:

- μ and ν are dual:

$$\begin{array}{l} \forall X, Y(\top \leq X + Y \implies \top \leq e(X) + f(Y)) \implies \top \leq \mu X e(X) + \nu Y f(Y) \\ \forall X, Y(X \cap Y \leq 0 \implies e(X) \cap f(Y) \leq 0) \implies \mu X e(X) \cap \nu Y f(Y) \leq 0 \end{array} \tag{6}$$

It is not difficult to see that the above principles hold in any completely distributive lattice, not just in $\mathcal{L}$, by induction on the closure ordinals of fixed points. However, unlike completeness, the principle above is first-order, not second-order. Note also that the principle above does not state the *existence* of complements, just that μ and ν behave well wrt complements in the same way that $+$ and $\cap$ do. For all these reasons it is quite natural to include (6) natively within any "right linear lattice axiomatisation" for $\mathcal{L}$. We are now ready to axiomatise the right-linear lattice theory of $\mathcal{L}$.

▶ **Definition 18.** *Write* $\mathsf{RLL}_{\mathcal{L}}$ *for the theory axiomatised by Equations* (1)–(6).

Our main result is that this axiomatisation is indeed sound and complete for the RLL theory of $\mathcal{L}$:

▶ **Theorem 19** (Soundness and completeness of $\mathsf{RLL}_{\mathcal{L}}$). $\mathcal{L} \models e = f \iff \mathsf{RLL}_{\mathcal{L}} \vdash e = f$.

Let us point out that the soundness direction, $\Longleftarrow$, follows from the discussion of each of the axioms Equations (1)–(5) in Section 2.2 and Equation (6) above. For the completeness direction, $\Longrightarrow$, we shall reduce to the completeness result for the fixed point logic $\mu\mathsf{LTL}$. Section 5 is dedicated to proving this formally.

For completeness a key result is the provable correctness of the syntactic notion of complementation we introduced at the beginning of this section:

▶ **Proposition 20** (Complementation). $\mathsf{RLL}_{\mathcal{L}}$ *proves the following, for all closed e:*

$$\begin{aligned} \top &\leq e + e^c \\ e \cap e^c &\leq 0 \end{aligned} \tag{7}$$

To prove this we need a more general intermediate result, establishing "complement functoriality" (*cf.* Proposition 10 earlier):

▶ **Lemma 21.** $\mathsf{RLL}_{\mathcal{L}}$ *proves*

$$\begin{aligned} &\forall \vec{X}, \vec{Y} (\textstyle\bigwedge_i \top \leq X_i + Y_i \implies \top \leq e(\vec{X}) + e^c(\vec{Y})) \\ &\forall \vec{X}, \vec{Y} (\textstyle\bigwedge_i X_i \cap Y_i \leq 0 \implies e(\vec{X}) \cap e^c(\vec{Y}) \leq 0) \end{aligned} \tag{8}$$

Note that Proposition 20 follows immediately by setting $\vec{X}$ and $\vec{Y}$ to be empty in Lemma 21 above.

Proof sketch of Lemma 21. By induction on $e(\cdot)$. When the outermost connective of e is a $+$ or $\cap$ we appeal to the induction hypothesis by duality of $+$ and $\cap$ more generally in bounded distributive lattices. The case when e has form af is handled similarly to Example 15, only with the presence of free variables. It remains to check the fixed point cases.

Suppose $e(\vec{X})$ has form $\mu X f(X, \vec{X})$. Reasoning in $\mathsf{RLL}_{\mathcal{L}}$, suppose $\top \leq X_i + Y_i$ and $X_i \cap Y_i \leq 0$ for all i. We have:

$$\begin{array}{lll} & \forall X, Y (\top \leq X + Y \implies \top \leq f(X, \vec{X}) + f^c(Y, \vec{Y})) & \text{by IH} \\ \therefore & \top \leq \mu X f(X, \vec{X}) + \nu X f^c(X, \vec{Y}) & \text{by (6)} \end{array}$$

$$\begin{array}{lll} & \forall X, Y (X \cap Y \leq 0 \implies f(X, \vec{X}) \cap f^c(Y, \vec{Y}) \leq 0) & \text{by IH} \\ \therefore & \mu X f(X, \vec{X}) \cap \nu X f^c(X, \vec{X}) \leq 0 & \text{by (6)} \end{array}$$

The argument for the case when $e(\vec{X})$ has form $\nu X f(X, \vec{X})$ is symmetric. ◀

We end this section with some remarks on models of $\mathsf{RLL}_{\mathcal{L}}$.

In Section 3 we defined a complement expression e^c of each RLL expression e, and Proposition 20 showed that e and e^c are provable complementary in $\mathsf{RLL}_{\mathcal{L}}$. This means that any model of $\mathsf{RLL}_{\mathcal{L}}$ has a substructure, namely the denotations of RLL expressions, that forms a Boolean algebra. The same holds for Kleene Algebras, as each regular expression can also be associated with one computing its complement, with respect to the regular language model. Just like KA, this does not mean that all models of $\mathsf{RLL}_{\mathcal{L}}$ are Boolean algebras themselves.

▶ **Example 22** ($\mathsf{RLL}_{\mathcal{L}}$ model without general complements). Fix the alphabet $\{0, 1\}$. Consider the substructure $\mathcal{K}$ of $\mathcal{L}$ that is the smallest $\bigcup$-complete lattice containing every ω-regular language and $Q := (0, 1) \cap \mathbb{Q}$. First, note that indeed $\mathcal{K} \models \mathsf{RLL}_{\mathcal{L}}$:

- Equations (1)–(3) hold as $\mathcal{K} \leq \mathcal{L}$.
- For (4), we define $(\mu X e(X))^{\mathcal{K}} := \bigcup_{\alpha \in \mathsf{Ord}} e^{\alpha}(\varnothing)$. This is well defined and coincides with $\mathcal{L}(\mu X e(X))$ by $\bigcup$-completeness and the approximant definition of the latter.
- For (5), we define $(\nu X e(X))^{\mathcal{K}} := \bigcup\{A \subseteq e(A)\}$. Since, in particular, $\mathcal{L}(\nu X e(X))$ is a postfixed point and an ω-regular language, it must coincide with $(\nu X e(X))^{\mathcal{K}}$.

However it is not hard to see that Q does not have a complement in $\mathcal{K}$, i.e. that $(0,1) \setminus \mathbb{Q}$ does not belong to $\mathcal{K}$. For this note that, as powerset lattices are completely distributive (and therefore so are their (semi)complete sublattices), we can write any element A of $\mathcal{K}$ as an infinite union of finite intersections of ω-regular languages and Q, i.e. of the form $\bigcup_{i \in I} A_{i1} \cap \cdots \cap A_{in_i}$, where each A_{ij} is ω-regular or Q. Now, if $A \neq \varnothing$, then also some $A_i := A_{i1} \cap \cdots \cap A_{in_i} \neq \varnothing$ as well. However, since ω-regular languages are closed under intersection, A_i must contain the rational part of some nonempty ω-regular language. Since any non-empty ω-regular language must contain some ultimately periodic word, this means that $A \cap \mathbb{Q} \supseteq A_i \cap \mathbb{Q} \neq \varnothing$, and so A cannot be a complement of Q in $\mathcal{K}$.

5 Completeness via μLTL

In this section, we will prove the completeness of $\mathsf{RLL}_{\mathcal{L}}$. Our completeness proof relies on the completeness of an axiomatisation of the linear-time μ-calculus called μLTL. We show several syntactic and semantic simulations between $\mathsf{RLL}_{\mathcal{L}}$ and μLTL. For the sake of brevity, we only give the directions necessary to recover completeness of $\mathsf{RLL}_{\mathcal{L}}$ wrt. $\mathcal{L}$. Key to this reduction is our recovery of syntactic complements in $\mathsf{RLL}_{\mathcal{L}}$, Proposition 20, which allows $\mathsf{RLL}_{\mathcal{L}}$ to simulate the Boolean reasoning available in classical modal logics like μLTL.

5.1 A (very quick) recap of μLTL

Linear temporal logic (LTL) is a modal logic with modalities referring to time. In LTL, one can encode formulas about the future of *paths*. In particular, we have formulas of the form $\bigcirc\varphi$ and $\varphi \mathbf{U} \psi$ that are (informally) interpreted as "at the next time step φ holds" and "φ holds until ψ holds." Naturally they are interpreted over discrete linear orders.

We can construe $\varphi \mathbf{U} \psi$ as a fixed point $\nu X(\psi \vee (\varphi \wedge \bigcirc X))$. Generalising this idea leads to μLTL, the extension of LTL with arbitrary fixed points.

We shall write $P, Q, \dots$ for propositional symbols. μLTL **formulas**, written $\varphi, \psi, \dots$, are generated by:

$$\varphi, \psi, \dots \quad ::= \quad \bot \mid \top \mid P \mid \bar{P} \mid X \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \bigcirc\varphi \mid \mu X \varphi \mid \nu X \varphi$$

Formulas are interpreted over infinite words. To this end, we shall assume that the propositional symbols $P, Q, \dots$ are from some finite set $\mathbf{P}$, and henceforth fix an alphabet $\mathcal{A} = \mathcal{P}(\mathbf{P})$.

Axioms	All propositional tautologies
	$\bigcirc(\varphi \vee \psi) \leftrightarrow \bigcirc\varphi \vee \bigcirc\psi \qquad \bigcirc(\varphi \wedge \psi) \leftrightarrow \bigcirc\varphi \wedge \bigcirc\psi$
	$\varphi(\mu X\varphi(X)) \to \mu X\varphi(X) \qquad \nu X\varphi(X) \to \varphi(\nu X\varphi(X))$
Rules	$MP\ \dfrac{\varphi \quad \varphi \to \psi}{\psi} \qquad \bigcirc\ \dfrac{\varphi}{\bigcirc\varphi}$
	$\mu\ \dfrac{\varphi(\psi) \to \psi}{\mu X\varphi(X) \to \psi} \qquad \nu\ \dfrac{\psi \to \varphi(\psi)}{\psi \to \nu X\varphi(X)}$

Figure 2 A Hilbert-style axiomatisation of μLTL.

▶ **Definition 23** (Semantics of μLTL). *Let us temporarily expand the syntax of formulas by a constant symbol α for each subset $\alpha \subseteq \omega$. For ω-words $\sigma \in \mathcal{A}^\omega$ (i.e. $\sigma \in \mathcal{P}(\mathbf{P})^\omega$) and formulas φ, we define $\varphi^\sigma \subseteq \omega$ by:*

$$\begin{aligned} &\bot^\sigma := \varnothing && \top^\sigma := \omega \\ &P^\sigma := \{n \in \omega : P \in \sigma_n\} && \bar{P}^\sigma := \{n \in \omega : P \notin \sigma_n\} \\ &\alpha^\sigma := \alpha && (\bigcirc\varphi)^\sigma := \{n \in \omega : n+1 \in \varphi^\sigma\} \\ &(\varphi \wedge \psi)^\sigma := \varphi^\sigma \cap \psi^\sigma && (\varphi \vee \psi)^\sigma := \varphi^\sigma \cup \psi^\sigma \\ &(\mu X\varphi(X))^\sigma := \bigcap\{\alpha \supseteq \varphi(\alpha)^\sigma\} && (\nu X\varphi(X))^\sigma := \bigcup\{\alpha \subseteq \varphi(\alpha)^\sigma\} \end{aligned}$$

*Write $\sigma \models \varphi$ if $0 \in \varphi^\sigma$. We say φ is **valid**, written $\models \varphi$, if for all $\sigma \in \mathcal{A}^\omega$ we have $\sigma \models \varphi$.*

μLTL enjoys a sound and complete axiomatisation, due to Kaivola [18] (see also [13]). To recast this axiomatisation in the current logical basis, let us point out that we can extend negation to all μLTL formulas by defining $\bar{\varphi}$ exploiting De Morgan duality of $\bot, \top$ and $\vee, \wedge$ and μ, ν, and finally self-duality of $\bigcirc$: $\overline{\bigcirc\varphi} := \bigcirc\bar{\varphi}$. Therefore, we may freely use other propositional connectives such as $\neg, \to, \leftrightarrow$ as suitable macros. The following axiomatisation is equivalent to that of [18], only adapted to our negation normal syntax.

▶ **Definition 24** (Hilbert-style axiomatisation of μLTL). *μLTL[4] is the logic defined by the axioms and inference rules in Figure 2.*

▶ **Theorem 25** ([18]). *μLTL is sound and complete i.e. $\mu\mathsf{LTL} \vdash \varphi \iff \models \varphi$.*

Note that the self-duality of $\bigcirc$, semantically, is reflected in the axioms too, with $\bigcirc$ distributing over both $\wedge$ and $\vee$.

▶ **Example 26.** Write $\varphi\mathbf{U}\psi := \nu X(\psi \vee (\varphi \wedge \bigcirc X))$. We will prove the LTL tautology $\bigcirc(\varphi\mathbf{U}\psi) \to \bigcirc\varphi\mathbf{U}\bigcirc\psi$. First note that the following modal rule is derivable,

$$(\star)\ \frac{\varphi \to \psi}{\bigcirc\varphi \to \bigcirc\psi}$$

as $\varphi \to \psi := \bar{\varphi} \vee \psi$. Thus we have:

$$\begin{aligned} \bigcirc(\varphi\mathbf{U}\psi) &\to \bigcirc(\psi \vee (\varphi \wedge \bigcirc(\varphi\mathbf{U}\psi))) && \text{by } (\star) \text{ and } \nu\text{-unfolding} \\ &\to \bigcirc\psi \vee \bigcirc(\varphi \wedge \bigcirc(\varphi\mathbf{U}\psi)) && \text{by distributing } \bigcirc \text{ over } \vee \\ &\to \bigcirc\psi \vee (\bigcirc\varphi \wedge \bigcirc\bigcirc(\varphi\mathbf{U}\psi)) && \text{by distributing } \bigcirc \text{ over } \wedge \end{aligned}$$

Applying the ν rule, we are done.

[4] By abuse of notation, we refer to both the language and the axiomatisation as μLTL.

5.2 Interpreting $\mathsf{RLL}_\mathcal{L}$ in $\mu\mathsf{LTL}$, semantically

Our aim is to reduce the completeness of $\mathsf{RLL}_\mathcal{L}$ to that of $\mu\mathsf{LTL}$. For this reason we need to embed $\mathsf{RLL}_\mathcal{L}$ into $\mu\mathsf{LTL}$.

▶ **Definition 27.** *For (possibly open) RLL expressions e we define a $\mu\mathsf{LTL}$ formula e° by induction on the structure of e as follows:*

- $X^\circ := X$
- $(ae)^\circ := \bigwedge\limits_{P \in a} P \wedge \bigwedge\limits_{P \notin a} \bar{P} \wedge \bigcirc e^\circ$
- $e + f^\circ := e^\circ \vee f^\circ$
- $e \cap f^\circ := e^\circ \wedge f^\circ$
- $(\mu X e)^\circ := \mu X e^\circ$
- $(\nu X e)^\circ := \nu X e^\circ$

We need to show that the translation above is faithful wrt. the two semantics we have presented, for RLL expressions and for $\mu\mathsf{LTL}$ formulas. Writing $\mathcal{L}(\varphi) := \{\sigma \models \varphi\}$ for closed $\mu\mathsf{LTL}$ formulas φ, we have:

▶ **Proposition 28** (Semantic adequacy)**.** *$\mathcal{L}(e) \subseteq \mathcal{L}(e^\circ)$, for closed expressions e.*

To prove this, we must first address the fact that our two semantics interpret syntax as different types of sets, and duly have different types of constant symbols. To this end, let us temporarily introduce into the language of $\mu\mathsf{LTL}$ a constant symbol A for each language $A \subseteq \mathcal{A}^\omega$. We extend the definition of $-^\circ$ by the clause $A^\circ := A$ and duly extend the definition of $-^\sigma$ by the clause $A^\sigma := \{n \in \omega : \sigma^n \in A\}$ where σ^n is the n^{th} tail of σ, i.e. we set $\sigma^0 := \sigma$, and σ^{n+1} to be the tail of σ^n. Now we can establish a sort of substitution lemma that relates our two semantics:

▶ **Lemma 29** (Mixed substitution)**.** *$\varphi(\mathcal{L}(\chi))^\sigma \subseteq \varphi(\chi)^\sigma$, for $\mu\mathsf{LTL}$ formulas $\chi, \varphi(X)$ (all free variables indicated).*

Proof sketch. By Induction on the size of $\varphi(X)$, i.e. its number of symbols. We will only exhibit the fixed point case. Suppose $\varphi(X) = \mu Y \psi(X, Y)$.

$$\begin{array}{rlll} & \psi(\mathcal{L}(\chi), (\mu Y \psi(\chi, Y))^\sigma)^\sigma & \subseteq \psi(\chi, (\mu Y \psi(\chi, Y))^\sigma)^\sigma & \text{by Induction hypothesis} \\ & & \subseteq \psi(\chi, \mu Y \psi(\chi, Y))^\sigma & \text{by substitution property of } -^\sigma \\ & & \subseteq (\mu Y \psi(\chi, Y))^\sigma & \text{since } \mu^\sigma \text{ is a prefixed point} \\ \therefore & (\mu Y \psi(\mathcal{L}(\chi), Y))^\sigma & \subseteq (\mu Y \psi(\chi, Y)^\sigma & \text{by } \mu^\sigma\text{-induction} \end{array}$$

The case for $\varphi(X) = \nu Y \psi(X, Y)$ is symmetric. ◀

Now, semantic adequacy is readily proved:

Proof sketch of Proposition 28. We proceed by induction on the size of e. Again, we will only exhibit the fixed point cases. If e is $\mu X f(X)$ then:

$$\begin{array}{rlll} & \mathcal{L}(f(\mathcal{L}((\mu X f(X))^\circ))) & \subseteq \mathcal{L}(f(\mathcal{L}((\mu X f(X))^\circ)^\circ) & \text{by Induction hypothesis} \\ \therefore & \mathcal{L}(f(\mathcal{L}(\mu X f^\circ(X)))) & \subseteq \mathcal{L}(f^\circ(\mathcal{L}(\mu X f^\circ(X))) & \text{by definition of } -^\circ \\ & & \subseteq \mathcal{L}(f^\circ(\mu X f^\circ(X))) & \text{by Lemma 29} \\ & & \subseteq \mathcal{L}(\mu X f^\circ(X)) & \text{since } \mathcal{L}(\mu) \text{ is a prefixed point} \\ \therefore & \mathcal{L}(\mu X f(X)) & \subseteq \mathcal{L}(\mu X f^\circ(X)) & \text{by } \mathcal{L}(\mu)\text{-induction} \end{array}$$

If e is $\nu X f(X)$ then:

$$
\begin{array}{rll}
 & \mathcal{L}(f(\mathcal{L}(\nu X f(X)))) \subseteq \mathcal{L}(f(\mathcal{L}(\nu X f(X)))^\circ) & \text{by Induction hypothesis} \\
 & \subseteq \mathcal{L}(f^\circ(\mathcal{L}(\nu X f(X)))) & \text{by definition of } -^\circ \\
 & \mathcal{L}(\nu X f(X)) \subseteq & \text{since } \mathcal{L}(\nu) \text{ is a postfixed point} \\
\therefore & \mathcal{L}(\nu X f(X))^\sigma \subseteq \mathcal{L}(f^\circ(\mathcal{L}(\nu X f(X))))^\sigma & \text{by monotonicity property of } -^\sigma \\
 & \subseteq f^\circ(\mathcal{L}(\nu X f(X)))^\sigma & \text{by Lemma 29} \\
 & \subseteq f^\circ(\mathcal{L}(\nu X f(X))^\sigma)^\sigma & \text{by substitution property of } -^\sigma \\
\therefore & \mathcal{L}(\nu X f(X))^\sigma \subseteq (\nu X f^\circ(X))^\sigma & \text{by } \nu^\sigma\text{-coinduction} \\
 & \subseteq (\nu X f(X))^{\circ\sigma} & \text{by definition of } -^\circ
\end{array}
$$

So in particular, $\sigma \in \mathcal{L}(\nu X f(X)) \implies 0 \in \mathcal{L}(\nu X f(X))^\sigma \implies 0 \in (\nu X f(X))^{\circ\sigma} \implies \sigma \models (\nu X f(X))^\circ \implies \sigma \in \mathcal{L}(\nu X f(X))$. ◀

5.3 Interpreting μLTL in RLL$_{\mathcal{L}}$, syntactically

In order to leverage the completeness of μLTL within RLL$_{\mathcal{L}}$, we need to simulate its reasoning, for which we must embed μLTL back into RLL$_{\mathcal{L}}$. As mentioned at the beginning of this section, a key component here is our recovery of syntactic complements, Proposition 20, allowing for Boolean reasoning on RLL expressions in RLL$_{\mathcal{L}}$.

▶ **Definition 30.** *For (possibly open) μLTL formulas φ we define an RLL expression $\varphi^\bullet$ by induction on the structure of φ as follows:*

$$
\begin{array}{ll}
\bot^\bullet := 0 & \top^\bullet := \top \\
P^\bullet := \sum_{a \ni P} a\top & \bar{P}^\bullet := \sum_{a \not\ni P} a\top \\
X^\bullet := X & (\bigcirc\varphi)^\bullet := \sum_{a \in \mathcal{A}} a\varphi^\bullet \\
(\varphi \vee \psi)^\bullet := \varphi^\bullet + \psi^\bullet & (\varphi \wedge \psi)^\bullet := \varphi^\bullet \cap \psi^\bullet \\
(\mu X e)^\bullet := \mu X e^\bullet & (\nu X e)^\bullet := \nu X e^\bullet
\end{array}
$$

We can again establish the adequacy of this interpretation, though this time we need a syntactic result rather than a semantic one:

▶ **Theorem 31** (Syntactic adequacy). μLTL $\vdash \varphi \implies$ RLL$_{\mathcal{L}} \vdash \varphi^\bullet = \top$.

Proof. By induction on μLTL proofs.

- All the propositional axioms are handled by the fact that RLL expressions RLL$_{\mathcal{L}}$-provably form a Boolean Algebra (*cf.* Section 3), and since $\bullet$ is defined directly as a homomorphism $(\bot, \top, \vee, \wedge) \to (0, \top, +, \cap)$. We also need duality of $P^\bullet$ and $\bar{P}^\bullet$ in RLL$_{\mathcal{L}}$:

$$
\begin{array}{rl}
P^\bullet + \bar{P}^\bullet & = \sum_{a \ni P} a\top + \sum_{a \not\ni P} a\top \\
 & = \sum_{a \in \mathcal{A}} a\top \\
 & = \top
\end{array}
\qquad
\begin{array}{rl}
P^\bullet \cap \bar{P}^\bullet & = \sum_{a \ni P} a\top \cap \sum_{b \ni P} b\top \\
 & = \sum_{a \ni P} \sum_{b \not\ni P} a\top \cap b\top \\
 & = 0
\end{array}
$$

- For normality of $\bigcirc$ wrt $\vee$, it suffices by Boolean reasoning in RLL$_{\mathcal{L}}$ to derive:

$$
\begin{array}{rll}
(\bigcirc(\varphi \vee \psi))^\bullet & = \sum_{a \in \mathcal{A}} a(\varphi^\bullet + \psi^\bullet) & \text{by definition of } -^\bullet \\
 & = \sum_{a \in \mathcal{A}} (a\varphi^\bullet + a\psi^\bullet) & \because a \text{ is a } +\text{-homomorphism} \\
 & = \sum_{a \in \mathcal{A}} a\varphi^\bullet + \sum_{a \in \mathcal{A}} a\psi^\bullet & \text{by commutativity and associativity of } + \\
 & = (\bigcirc\varphi \vee \bigcirc\psi)^\bullet & \text{by definition of } -^\bullet
\end{array}
$$

Similarly for normality of $\bigcirc$ wrt $\wedge$.

- The simulation of axioms for μ and ν are immediate, by functoriality, as $-^\bullet$ commutes with μ and ν.
- Obtaining the rules is mostly straightforward. Modus ponens reduces to transitivity of $\leq$, under Boolean reasoning. Necessitation is simulated by $\top = \sum_{a\in\mathcal{A}} a\top$. Simulating (co)induction rules are immediate as $-^\bullet$ commutes with μ and ν. ◀

5.4 Compatibility of interpretations and completeness

To complete our reduction of $\mathsf{RLL}_{\mathcal{L}}$ completeness to $\mu\mathsf{LTL}$ completeness, as well as simulating $\mu\mathsf{LTL}$ reasoning, we need compatibility of the two translations.

▶ **Proposition 32** (Compatibility). $\mathsf{RLL}_{\mathcal{L}} \vdash e^{\circ\bullet} = e$

Proof. By induction on the structure of e. Almost all cases are immediate, as $-^{\circ\bullet}$ commutes with $X, +, \cap, \mu, \nu$. For the remaining homomorphism case, we reason in $\mathsf{RLL}_{\mathcal{L}}$:

$$\begin{aligned}
(ae)^{\circ\bullet} &= \left(\bigwedge_{P\in a} P \wedge \bigwedge_{P\notin a} \bar{P} \wedge \bigcirc e^\circ\right)^\bullet && \text{by definition of } -^\circ\\
&= \bigcap_{P\in a}\sum_{b\ni P} b\top \cap \bigcap_{P\notin a}\sum_{b\not\ni P} b\top \cap \sum_{c\in\mathcal{A}} ce^{\circ\bullet} && \text{by definition of } -^\bullet\\
&= a\top \cap \sum_{c\in\mathcal{A}} ce^{\circ\bullet} && \text{by set theoretic reasoning}\\
&= \sum_{c\in\mathcal{A}} (a\top \cap ce^{\circ\bullet}) && \text{by distributivity}\\
&= a\top \cap ae^{\circ\bullet} && \text{since } ae\cap bf = 0 \text{ when } a\neq b\\
&= a(\top \cap e^{\circ\bullet}) && \text{as } a \text{ is a } \cap\text{-homomorphism}\\
&= ae^{\circ\bullet} && \text{as } \top \text{ is a } \cap\text{-unit}\\
&= ae && \text{by induction hypothesis}
\end{aligned}$$

To explain a little further the third line above, note that any $b \neq a$ is distinguished from a by either some $P \in a \setminus b$ or some $P \in b \setminus a$. ◀

We can finally assemble our main completeness result, the $\Longrightarrow$ direction of Theorem 19:

▶ **Theorem 33** (Completeness of $\mathsf{RLL}_{\mathcal{L}}$). $\mathcal{L}(e) = \mathcal{L}(f) \implies \mathsf{RLL}_{\mathcal{L}} \vdash e = f$.

Proof. By Boolean reasoning in $\mathsf{RLL}_{\mathcal{L}}$ it suffices to show that $\mathcal{L}(e) = \mathcal{A}^\omega \implies \mathsf{RLL}_{\mathcal{L}} \vdash e = \top$:

$$\begin{aligned}
\mathcal{L}(e) = \mathcal{A}^\omega &\implies \models e^\circ && \text{by Proposition 28}\\
&\implies \mu\mathsf{LTL} \vdash e^\circ && \text{by Theorem 25}\\
&\implies \mathsf{RLL}_{\mathcal{L}} \vdash e^{\circ\bullet} = \top && \text{by Theorem 31}\\
&\implies \mathsf{RLL}_{\mathcal{L}} \vdash e = \top && \text{by Proposition 32}
\end{aligned}$$
◀

6 Concluding remarks and future work

In this work, we introduced RLL expressions, a notation for APAs and gave a sound and complete axiomatisation for their equational theory. We make some observations about our choice of axioms and compare with existing literature.

6.1 Alternative axiomatisation(s)

Our axiomatisation $\mathsf{RLL}_{\mathcal{L}}$ for $\mathcal{L}$ is first-order, avoiding second-order axioms such as completeness of lattices. Still, stating the duality of μ and ν, Equation (6), requires quantifiers.

Let us point out that the completeness argument for $\mathsf{RLL}_{\mathcal{L}}$ only used the principles (7), an equational consequence of (6) under Equations (1)–(5). In fact, Equations (1)–(5) and (7) axiomatises the same first-order theory as $\mathsf{RLL}_{\mathcal{L}}$.[5]

▶ **Proposition 34.** *Equations* (1)–(5) *and* (7) *proves Equation* (6)*.*

Of course, (7) is an axiom *schema*. For what it is worth, let us also point out that we can present (6) as quantifier-free *rules* rather than an axiom:

$$\frac{\top \leq X + Y \Rightarrow \top \leq e(X) + f(Y)}{\top \leq \mu X e(X) + \nu Y f(Y)}\, X, Y \text{ fresh} \qquad \frac{X \cap Y \leq 0 \Rightarrow e(X) \cap f(Y) \leq 0}{\mu X e(X) \cap \nu Y f(Y) \leq 0}\, X, Y \text{ fresh}$$

Following from the presentation of (6) as sequent rules above, we may consider an alternative but equational rule for duality of μ and ν, now given in sequent style:

$$\frac{\Gamma, X + Y \Rightarrow \Delta, e(X) + f(Y)}{\Gamma \Rightarrow \Delta, \mu X e(X) + \nu Y f(Y)} \tag{9}$$

One can also show that these rules suffice to establish (7) under Equations (1)–(5), and so is also complete for the equational theory of $\mathcal{L}$. One way to distinguish the above mentioned formulations of the RLL theory of $\mathcal{L}$ is to conduct a proof theoretic analysis, investigating which (if any) of the formulations behave well under *cut-elimination*.

Note that it is not clear whether it is even possible to *finitely* quantifier-free axiomatise the RLL theory of $\mathcal{L}$. For comparison, it is known that regular expressions do not have a finite equational axiomatisation [25], though it does have a finite quantifier-free (even quasi-equational) axiomatisation [20, 21].

6.2 Comparison with ω-algebras

Recall that ω-regular expressions are an extension of regular expressions with terms of the form e^ω that are adequate to capture all ω-regular languages. The intended interpretation is $\mathcal{L}(e^\omega) = \{u_0 u_1 u_2 \cdots \mid u_i \in \mathcal{L}(e), \forall i \in \omega\}$. Surprisingly, the algebraic theory of ω-regular expressions has not been explored until recently. Wagner [31] gave a two-sorted axiomatisation that was proved complete in [9]. Cohen [8] proposed an axiomatic theory with ω-regular expressions but not with the intension of proving completeness for $\mathcal{L}$. In fact, it is indeed incomplete for the language model because it cannot prove identities like $e^\omega f = e^\omega$. In [9] Cohen's axiomatic theory was extended to be complete for $\mathcal{L}$.

It would be interesting to compare various ω-regular algebras to $\mathsf{RLL}_{\mathcal{L}}$ presented in this work. For comparison, in the finite world, every "left-handed" Kleene Algebra is an right-linear algebra, but not vice versa [10].

6.3 Axiomatising relational models

KAs admit relational models interpreting product as composition, sum as union, and the Kleene star as reflexive, transitive closure. It is well-known that the class of relational models satisfies the same equational theory of regular expressions as $\mathcal{L}$. Similarly, interpreting each

[5] Nonetheless, note that it is possible that $\mathsf{RLL}_{\mathcal{L}}$ does not have the same first-order theory as its extension by a native negation operator.

$a\cdot$ as pre-composition by some fixed binary relation $a^{\mathcal{R}}$ and μ as the least fixed point, the class of relational models of RLA satisfies the same equational theory over RLA-expressions as $\mathcal{L}$.

However, in Kleene lattices, relational and language models start to differ: $ef \cap 1 = (e \cap 1)(f \cap 1)$ is valid in $\mathcal{L}$ but not in the relational interpretations [2]. Analogously relational structures, when interpreting letters as relations, do not model $\mathsf{RLL}_{\mathcal{L}}$ (in general). The interpretations $a^{\mathcal{R}}$ are not necessarily lattice homomorphisms: we have $a(e \cap f) \leq ae \cap af$ but not the converse. Thus this interpretation refutes Equation (2). At the same time this interpretation does not necessarily satisfy Equation (3) either: for instance $a^{\mathcal{R}}$ and $b^{\mathcal{R}}$ may intersect, even when $a \neq b$. On the other hand, $a^{\mathcal{R}}\top = \top$ as soon as $a^{\mathcal{R}} \neq \varnothing$. It is therefore a natural question if there is a natural restriction of $\mathsf{RLL}_{\mathcal{L}}$ that is complete for the relational interpretation. Conversely, it might be interesting to explore an alternative interpretation of letters, as certain homomorphisms on lattices of relations.

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