Quantitative Monoidal Algebra: Axiomatising Distance with String Diagrams

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Abstract

String diagrammatic calculi have become increasingly popular in fields such as quantum theory, circuit theory, probabilistic programming, and machine learning, where they enable resource-sensitive and compositional algebraic analysis. Traditionally, the equations of diagrammatic calculi only axiomatise exact semantic equality. However, reasoning in these domains often involves approximations rather than strict equivalences.

In this work, we develop a quantitative framework for diagrammatic calculi, where one may axiomatise notions of distance between string diagrams. Unlike similar approaches, such as the quantitative theories introduced by Mardare et al., this requires us to work in a monoidal rather than a cartesian setting. We define a suitable notion of monoidal theory, the syntactic category it freely generates, and its models, where the concept of distance is established via enrichment over a quantale. To illustrate the framework, we provide examples from probabilistic and linear systems analysis.

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1 Introduction

Traditionally, formal semantics models equivalence between programs as equality of their interpretation in a certain mathematical domain. A fundamental question is the one of *axiomatisation*: finding a set of equations between programs that hold precisely when they are semantically equivalent. Such an axiomatisation allows reasoning about semantics purely by syntactic manipulation of programs: this offers a structured, scalable approach to designing protocols (such as refinement and optimisation), automatisation, and formal verification.

In the last few decades, increasingly prominent paradigms of computation such as quantum theory, probabilistic programming, and deep learning have challenged formal semantics, as they demand reasoning about systems that are partially defined, approximate, or sensitive to

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perturbations. This has led to a resource-sensitive enhancement of the above picture, along two axes: quantitative semantics and monoidal syntax (string diagrams). In a sense, our work is about reconciling both perspectives.

Quantitative Semantics. The limitations of “exact” semantics are particularly evident when modelling probabilistic computation [42,55,66,67]: rather than asking if probabilistic programs P and Q yield the same outputs with the same probabilities, it is more informative to ask *how far* Ps behaviour is from Qs behaviour, according to a certain metric. Similar considerations apply to other research areas, such as differential privacy [26], and approximate computing [52]. To address this form of analysis, program semantics has embraced quantitative reasoning, leading to advancements in areas like bisimulation metrics [24,68], coeffectful computation [54], program distances [20, 22], and quantitative rewriting [31]. The focus of our contribution is on *axiomatising* quantitative semantics, for which *quantitative algebraic theories* are particularly relevant. In this line of work, initiated by Mardare et al. [49], equations of the form $s = t$ are replaced with judgments of the form $s =_\varepsilon t$, which should be read as: “s is at distance at most ε from t”. Among their examples, the authors show complete axiomatisations of the total variation and Kantorovich–Wasserstein distances [69] between probability distributions. Quantitative algebraic theories have been developed extensively, including a variety theorem [50], sum and tensor of theories [7], higher-order extensions [23], and the development of significant examples such as Markov processes [6]. They were also generalised to the setting of categorical algebra in various ways [2, 36, 51, 60, 61].

Cartesian vs. Monoidal Syntax. A fundamental feature of the aforementioned approaches is that the syntax of programs (or, more generally, computational processes) is represented by terms of a *cartesian* algebraic theory. The terminology is due to the usual categorical perspective on abstract algebra, initiated by Lawvere [43]: the “syntactic” category freely generated by an algebraic theory (Σ, E) is a *cartesian* category, and models are functors preserving the cartesian structure. Whereas abstractly being cartesian just means to have finite products, via Fox’s theorem [27] this is equivalent to each object X of the category having a “copy” and a “discard” map. If we interpret these objects as variables of our programs, or resources of our systems, the assumption of cartesianity means that these entities may be duplicated or eliminated at will. In other words, the theory is *insensitive* to such resources.

These assumptions are unsuitable in many contexts. A notable example is quantum theory, with its “no-cloning” and “no-deleting” theorems [71]. Also probabilistic computation is inherently non-cartesian: duplicating the outcome of a die roll is not the same as rolling that die twice. There are many more instances in computer science where algebraic modelling needs to be attentive to resource consumption, e.g. in concurrency theory [1] and cryptography [14].

These examples motivated the development of *monoidal* algebra. Processes are studied in (symmetric) monoidal categories, which allow for algebraic reasoning but do not assume a cartesian structure, meaning consumption of resources (variables) becomes explicit in the theory. Because the fundamental operations of a monoidal category are *sequential* and *parallel* composition, process syntax is depicted two-dimensionally, as *string diagrams* [58, 63]. The pictorial representation is not just aesthetically pleasing, but allows for a clearer understanding of how information flows and is exchanged within the process components. For these reasons, string diagrams have been applied in quantum theory [18], concurrency [10], probabilistic programming [57], machine learning [21, 70], cyber-physical systems [11, 15], and even areas further removed from computer science such as linguistics [16, 37], epidemiology [9, 45], and chemistry [30, 47]. When it comes to *axiomatising* semantics of string diagrammatic calculi, tools analogous to those of (non-quantitative) cartesian algebra are available, such as a notion of freely generated "syntactic" category [8, 12, 34] and of model [12].

Towards Quantitative Monoidal Algebra. Similarly to cartesian algebra, it has become apparent that monoidal algebra urges for a quantitative extension. There is an increasing body of work developing the theory of probabilistic processes and Bayesian reasoning in symmetric monoidal categories called Markov categories, see e.g. [29, 35, 48]. Similarly, several categorical models for machine learning algorithms are being proposed, in which string diagrams play a major role, see e.g. the surveys [19, 64]. However, quantitative analysis has received very limited attention so far – one such example is [56], which studies notions of mutual information in Markov categories via relative entropy of string diagrams. In quantum theory, the works [13, 39] use distances between string diagrams to express noise tolerance in quantum protocols, and [33] studies distances between (quantum) channels represented within a monoidal theory. What all these instances are missing is an *axiomatic framework* to reason about distance of string diagrams, playing a role analogous to the one served by quantitative algebra for cartesian computation.

Our Contribution. In this work, we lay the mathematical foundations of quantitative monoidal algebra. To capture a wider range of models, we develop our framework not just for real-valued metric spaces, but for the more general notion of spaces with distances valued in a quantale V [44]. Examples include preorders, pseudometric spaces, ultrametric spaces, etc. We introduce the notion of (symmetric) V-quantitative monoidal theory $\mathcal{U}$ as a triple (Σ, E, E_q), where (Σ, E) is a monoidal theory and E_q is a set of V-quantitative equations, for which we use the same notation $=_\varepsilon$ introduced in [49]. We present the construction of the freely generated syntactic category over $\mathcal{U}$, as an enriched monoidal category $\mathcal{S}_{\mathcal{U}}$. Morphisms of $\mathcal{S}_{\mathcal{U}}$ are depicted as string diagrams, composable sequentially and in parallel. Distances between string digrams, induced by E_q, are modelled in the enrichment of $\mathcal{S}_{\mathcal{U}}$. Being a "variable-free" approach, the interaction between the enrichment and the generating rules of string diagrams in the syntactic category poses additional challenges compared to the cartesian setting. The final piece of our foundations is a suitable notion of model, which is defined à la Lawvere, as enriched functors from the syntactic category to "semantic" categories. We are then able to conclude with an analogue of the completeness theorem of equational logic, for the rules of quantitative diagrammatic reasoning. Our last contribution is a more in-depth comparison with related work [12, 49, 59, 60, 62], which clarifies the relationship between cartesian approaches and our monoidal framework.

We provide two basic examples for our framework, related to linear and probabilistic computation respectively. The first is an axiomatisation for matrices over an ordered semiring with entrywise ordering (Section 4.2). The theory of matrices appears ubiquitously in monoidal algebra (see e.g. [8, 10, 11, 18, 72]), and the order enrichment naturally appears in many such research threads. In particular, matrices on the Boolean semiring $\{0, 1\}$ represent relations, and the ordering is set-theoretic inclusion. Our second example (Section 5.3) is an axiomatisation of discrete probabilistic processes with the total variation metric, a distance fundamental in optimisation, learning theory, statistical inference, etc. Such an example may be thought as the "monoidal version" of an analogous result in the cartesian setting [49, Section 8].

Synopsis. Section 2 provides background on quantales and enriched category theory. Section 3 contains our main theoretical contributions. In Section 3.1, we recall monoidal theories and string diagrams. In Section 3.2 we define quantitative monoidal theories and the construction of the freely generated syntactic categories. We prove the latter are monoidal enriched in Section 3.3. In Section 3.4, we define enriched models and give a sufficient condition for a (classical) model to be enriched. Sections 4 and 5 are devoted to the examples outlined above. We compare formally to related work in Section 6, and conclude in Section 7 with future work. The full version [46] contains an appendix with complete proofs for our results.

2 Preliminaries

Quantale-Valued Generalised Metric Spaces. Following Lawvere [44], we allow distances to be valued not just in the positive reals, but in any quantale. We now recall quantales, as well as hemimetric and pseudometric spaces.

▶ **Definition 1.** *A* quantale *is a tuple* $(V, \sqsubseteq, \oplus, k)$*, where* $(V, \sqsubseteq)$ *is a partial order that has all joins and meets (supremums and infimums), i.e. a complete lattice,* $(V, \oplus, k)$ *is a monoid, and* $\oplus$ *is* join-continuous*, that is,* $a \oplus \bigsqcup S = \bigsqcup_{x\in S} a \oplus x$ *for any* $a \in V$ *and subset* $S \subseteq V$*.*

We write $\bigsqcup S$ *or* $\bigsqcup_{x\in S} x$ *for the join of a subset* $S \subseteq V$*. In particular,* V *has a* bottom $(\bot)$ *and a* top $(\top)$ *element that satisfy* $\bot = \bigsqcup \emptyset \sqsubseteq x \sqsubseteq \bigsqcup V = \top$ *for any* $x \in V$*. We call a quantale* integral *if the monoidal unit is the top element of the underlying lattice (i.e.* $k = \top$*).*

Throughout the paper, we will make the assumption that quantales are integral. This is a common requirement when studying quantales. In Section 4, we use the *Boolean quantale* $2_\sqcap$ consisting of two elements $\bot \sqsubseteq \top$ with underlying monoid $(2_\sqcap, \sqcap, \top)$. In Section 5, we use the *Lawvere quantale* $[0, \infty]_+$ where the underlying lattice is the interval $[0, \infty)$ with the reversed order extended with ∞ as a bottom element, and the monoid operation is addition.

▶ Remark 2. Let $(V, \sqsubseteq)$ be a complete lattice. If meets distribute over infinite joins, that is, for any $x \in V$ and family $\{x_i\}_{i\in I}$ in V, $x \sqcap \bigsqcup_{i\in I} x_i = \bigsqcup_{i\in I}(x \sqcap x_i)$, then V is called *infinitely join distributive (IJD)* (see e.g. [25, 32]).

▶ **Definition 3.** *Let* V *be a quantale. A* V-hemimetric space (X, d) *consists of a set* X *and a function* $d \colon X \times X \to V$ *satisfying, for all* $x, y, z \in X$*,* $k \sqsubseteq d(x, x)$ ***(reflexivity)****, and* $d(x, y) \oplus d(y, z) \sqsubseteq d(x, z)$ ***(triangle inequality)****. We call* (X, d) *a* V-pseudometric space *if it additionally satisfies, for all* $x, y \in X$*,* $d(x, y) = d(y, x)$ ***(symmetry)****. A function* $f \colon X \to Y$ *between* V*-hemimetric spaces* (X, d_X) *and* (Y, d_Y) *is called* nonexpansive *if for all* $x, x' \in X$*,* $d_X(x, x') \sqsubseteq d_Y(f(x), f(x'))$*.*

▶ **Example 4.** In order to make better sense of Definition 3, note that when considering $V = [0, \infty]_+$ we get back the standard definition of hemimetric and pseudometric spaces with possibly infinite distances. Over other quantales, we recover well-known structures. Setting $V = 2_\sqcap$, $2_\sqcap$-hemimetrics are preorders, while $2_\sqcap$-pseudometrics are equivalence relations, and nonexpansive maps are order/relation-preserving functions.

▶ **Definition 5.** *We denote the category of* V*-hemimetric spaces and nonexpansive functions with* $V\mathsf{HMet}$*, and its full subcategory of* V*-pseudometric spaces with* $V\mathsf{PMet}$*.*

Our primary goal is to study categories where morphisms have a distance between them. We will model this extra structure on hom-sets with enriched categories, and this requires us to provide a (symmetric) monoidal product of V-hemi/pseudometric spaces. We consider two monoidal products inspired from well-known products of real-valued metric spaces.

▶ **Example 6.** Let $(V, \oplus, k)$ be a commutative quantale (i.e. $\oplus$ is commutative) and (X, d_X), (Y, d_Y) be two V-hemimetric spaces. We define the *sum hemimetric* $d_X \boxtimes_\oplus d_Y$ on the cartesian product $X \times Y$ by $(d_X \boxtimes_\oplus d_Y)((x,y),(x',y')) \coloneqq d_X(x,x') \oplus d_Y(y,y')$. This yields a monoidal product defined by $(X, d_X) \boxtimes_\oplus (Y, d_Y) \coloneqq (X \times Y, d_X \boxtimes_\oplus d_Y)$ whose monoidal unit is $1_\boxtimes \coloneqq (\{\bullet\}, \top)$, where $\top(\bullet,\bullet) = \top$. The symmetries $\sigma_{X,Y} \coloneqq (x,y) \mapsto (y,x)$ are nonexpansive maps $(X,d_X) \boxtimes_\oplus (Y,d_Y) \to (Y,d_Y) \boxtimes_\oplus (X,d_X)$, and they make $V\mathsf{HMet}$ into a symmetric monoidal category. Since $\boxtimes_\oplus$ preserves symmetry, $V\mathsf{PMet}$ is a full symmetric monoidal subcategory of $V\mathsf{HMet}$.

▶ **Example 7.** If $(V, \oplus, k)$ is a quantale and $(V, \sqsubseteq)$ is IJD (see Remark 2), then we can define the *max hemimetric*. Given $(X, d_X), (Y, d_Y) \in V\mathsf{HMet}$, we define $(X, d_X) \boxtimes_\sqcap (Y, d_Y) \coloneqq (X \times Y, d_X \boxtimes_\sqcap d_Y)$, where $(d_X \boxtimes_\sqcap d_Y)((x,y),(x',y')) \coloneqq d_X(x,x') \sqcap d_Y(y,y')$. The monoidal unit is given by $1_\boxtimes \coloneqq (\{\bullet\}, \top)$. Once again, the evident symmetries are nonexpansive, and we get another symmetric monoidal structure on $V\mathsf{HMet}$. This definition also restricts to a symmetric monoidal product on $V\mathsf{PMet}$.

Categories Enriched over VHMet. In this paper, distances between morphisms of monoidal categories will be cast in terms of *enriched categories*. We are only interested in categories enriched over hemi/pseudometric spaces, so we defer to [38] for the general details.

We will work with categories enriched in $(V\mathsf{HMet}, \boxtimes, 1_\boxtimes)$ or $(V\mathsf{PMet}, \boxtimes, 1_\boxtimes)$, where $\boxtimes$ is defined as $\boxtimes_\oplus$ in Example 6 or $\boxtimes_\sqcap$ in Example 7. To be explicit but concise on which base of enrichment we are considering, we will use the notations $V\mathsf{HMet}_\oplus$, $V\mathsf{HMet}_\sqcap$, $V\mathsf{PMet}_\oplus$, and $V\mathsf{PMet}_\sqcap$. Enrichment over these categories boils down to equipping hom-sets with hemi/pseudometrics and requiring a nonexpansiveness property of composition. In other words, any $V\mathsf{HMet}$-enriched category is determined by an underlying category $\mathcal{C}$, where every hom-set $\mathcal{C}(a,b)$ has a V-hemimetric space structure $(\mathcal{C}(a,b), d_{a,b})$ such that for all $f, f' \in \mathcal{C}(a,b)$ and $g, g' \in \mathcal{C}(b,c)$, we have that $(d_{b,c} \boxtimes d_{a,b})((g,g'),(f,f')) \sqsubseteq d_{a,c}(g \circ f, g' \circ f')$.

Furthermore, any $V\mathsf{HMet}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ is determined by a functor between the underlying categories, which is locally nonexpansive, in the sense that the assignment $f \mapsto Ff$ is a nonexpansive map $\mathcal{C}(a,b) \to \mathcal{D}(Fa, Fb)$ for all $a, b \in Ob(\mathcal{C})$. Similarly, an enriched isomorphism $F\colon \mathcal{C} \to \mathcal{D}$ is an isomorphism between the underlying categories, which in addition is locally an isometry, namely, the assignment $f \mapsto Ff$ is an isometry $\mathcal{C}(a,b) \to \mathcal{D}(Fa, Fb)$ for all $a, b \in Ob(\mathcal{C})$. We carefully develop these claims in the full version, and we note that they can also be applied to enrichment over $V\mathsf{PMet}$.

This concrete characterisation of enrichment allows us to give a convenient definition of enriched monoidal categories for our purposes. It instantiates the more general definition that appears in e.g. [53, Definition 2.1], [40, Definition 4.1]. Morally, we define enriched monoidal categories to be monoidal (Set-)categories equipped with hemi/pseudometrics on their hom-sets such that both composition and monoidal product are nonexpansive.

▶ **Definition 8.** *A $V\mathsf{HMet}$-enriched symmetric monoidal category $\mathcal{C}$ is a category that is both symmetric monoidal and $V\mathsf{HMet}$-enriched, and such that the bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a $V\mathsf{HMet}$-functor. It is called* strict *if the underlying monoidal category is strict. A $V\mathsf{HMet}$-enriched symmetric strict monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ is a strict monoidal functor between the underlying monoidal categories which is also $V\mathsf{HMet}$-enriched (as a functor).*

Unrolling this definition according to our discussion above, a $V\mathsf{HMet}$-enriched symmetric strict monoidal category (SMC) is just a $V\mathsf{HMet}$-category whose underlying category $\mathcal{C}$ is equipped with a symmetric strict monoidal product $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that is nonexpansive with respect to $\boxtimes$, in the sense that for all $a, b, c, d \in Ob(\mathcal{C})$, $f, f' \in \mathcal{C}(a,b)$, $g, g' \in \mathcal{C}(c,d)$, we have $(d_{a,b} \boxtimes d_{c,d})((f,f'),(g,g')) \sqsubseteq d_{a\otimes c, b\otimes d}(f \otimes g, f' \otimes g')$. All the above also applies to $V\mathsf{PMet}$.

3 Quantitative Monoidal Algebra

In this section we fix a commutative integral quantale V and introduce the notion of V-quantitative symmetric monoidal theory. We recall ordinary monoidal theories first, following [12]. We omit the adjective “symmetric” for monoidal theories, as it will be assumed.

3.1 Background: Monoidal Theories

▶ **Definition 9.** *A* monoidal signature Σ *is a set of* generators*, each with an arity* $n \in \mathbb{N}$ *and a coarity* $m \in \mathbb{N}$*, which we often indicate simply with a type* $n \to m$*. In preparation to representing* Σ*-terms using string diagrams, we adopt a graphical representation for generators, as boxes with dangling wires on the left and the right to indicate arity and coarity. For instance,* $=\!\boxed{g}\!\equiv$ *has arity* 2 *and coarity* 3*. We write* $\overset{n}{-}\boxed{g}\overset{m}{-}$ *for a generic generator with arity* n *and coarity* m*. We use* $-\boxed{g}-$ *when the type is irrelevant or clear from context.*

The set of Σ-terms *(and their (co)arities) is defined inductively as follows:*

- *all generators* $\overset{n}{-}\boxed{g}\overset{m}{-} \in \Sigma$*,* $—— : 1 \to 1$*,* $\boxed{} : 0 \to 0$*, and* $\times : 2 \to 2$ *are* Σ*-terms;*
- *if* $s\colon n \to m$ *and* $t\colon m \to \ell$ *are* Σ*-terms, then* $s ; t\colon n \to \ell$ *is a* Σ*-term;*
- *if* $t\colon n \to n'$ *and* $s\colon m \to m'$ *are* Σ*-terms, then* $t \otimes s\colon n + m \to n' + m'$ *is a* Σ*-term.*

A monoidal theory (Σ, E) *consists of a signature* Σ *and a set* E *of pairs* (s, t) *of* Σ*-terms of the same type, which we call* equations *and write* $s = t$*.*

We represent Σ-terms graphically using the same conventions introduced for the generators. Given Σ-terms $s\colon n \to m$ and $t\colon m \to \ell$, we write $s ; t$ as $\overset{n}{-}\boxed{s}\overset{m}{-}\boxed{t}\overset{\ell}{-}$. Similarly, given $t\colon n \to n'$ and $s\colon m \to m'$, we write $t \otimes s$ as $\begin{smallmatrix} \overset{n}{-}\boxed{t}\overset{n'}{-} \\ \overset{m}{-}\boxed{s}\overset{m'}{-} \end{smallmatrix}$. Arbitrary identities $\overset{n}{—} : n \to n$ and symmetries ${}^{m}_{n}\!\times\!{}^{n}_{m} : m + n \to n + m$ may be defined as Σ-terms, by pasting together in the expected way copies of the “basic” identity —— and symmetry $\times$.

When organised into a category (Definition 10 below), operations ; and $\otimes$ become associative and obey the so-called “exchange law”, meaning we can paste together diagrams without worrying about priority of application. Σ-terms modulo the axioms of symmetric strict monoidal categories are called *string diagrams*, see e.g. [58, 63].

▶ **Definition 10.** *The symmetric strict monoidal category (SMC)* $\mathcal{S}_{\Sigma,E}$ *freely generated by* (Σ, E)*, called the* syntactic category*, is defined as follows. Its objects are natural numbers. A morphism* $n \to m$ *is a* Σ*-term of arity* n *and coarity* m *modulo the equations in* E *and the axioms of SMCs. Formally, two* Σ*-terms* s *and* t *are equal in* $\mathcal{S}_{\Sigma,E}$ *if and only if they are in the same equivalence class of the smallest congruence (with respect to* ; *and* $\otimes$*) that contains the pairs in* E *and the axioms of SMCs. Monoidal product on objects is given by addition. Regarding morphisms, composition, monoidal product, identities, and symmetries are defined by their counterparts on* Σ*-terms.*

3.2 Quantitative Monoidal Theories

In the envisioned applications of our work (and in Sections 4 and 5), the string diagrams represent processes for which equality is too coarse a relation to be meaningful. To achieve a finer comparison, we reuse a central idea in [49], that is to replace the “exact” equality relation with equality “up to” some quantity ε in V. This new relation is denoted with $=_\varepsilon$, and $s =_\varepsilon t$ means that the processes represented by s and t are at distance at most ε. A quantitative monoidal theory is a monoidal theory with additional axioms of this shape.

▶ **Definition 11.** *A* V-quantitative (symmetric) monoidal theory *is a triple* (Σ, E, E_q)*, where* (Σ, E) *is a monoidal theory, and* E_q *is a set of triples* (s, t, ε) *comprising two* Σ*-terms* s *and* t*, and an element* $\varepsilon \in V$*, which we call* quantitative equations *and denote with* $s =_\varepsilon t$*.*

We want to construct a syntactic category associated to a V-quantitative monoidal theory (Σ, E, E_q). It must have extra structure describing distance between morphisms, thus it will be VHMet-enriched or VPMet-enriched. To this aim, we will start with $\mathcal{S}_{\Sigma,E}$, the SMC freely generated by the underlying monoidal theory, and define a V-hemimetric (or pseudometric) on all the hom-sets of $\mathcal{S}_{\Sigma,E}$ making sequential and parallel compositions nonexpansive.

Just like equality between Σ-terms in $\mathcal{S}_{\Sigma,E}$ was inferred from the equations in E and the axioms of SMCs, the distance between Σ-terms will be inferred from the quantitative equations in E_q and the axioms of enriched SMCs. This process is more involved than building the smallest congruence, but similar in spirit. It is also inspired from quantitative equational logic in [49]. Note the infinitary Join rule that mirrors the **Arch** rule of [49].

▶ **Definition 12.** *Let* (Σ, E, E_q) *be a* V*-quantitative monoidal theory and* $\mathcal{S}_{\Sigma,E}$ *be the SMC generated by* (Σ, E)*. We define the* $V\mathsf{HMet}_\oplus$*-closure of* E_q*, denoted* $E_q^{\mathsf{H}\oplus\oplus}$*, as the smallest set of quantitative equations containing* E_q *and closed under the following inference rules.*

- *For any* Σ*-terms* $f, g, h \colon n \to m$*, and* $\varepsilon, \varepsilon' \in V$*, we have the following rules. They ensure that the distances defined later in Lemma 16 are* V*-hemimetrics on the hom-sets of* $\mathcal{S}_{\Sigma,E}$*.*

$$\frac{f = g \quad \textit{is provable from } E}{f =_\top g}\,\textsc{Refl} \qquad \frac{f =_\varepsilon g \quad g =_{\varepsilon'} h}{f =_{\varepsilon\oplus\varepsilon'} h}\,\textsc{Triang}$$

$$\frac{}{f =_\bot g}\,\textsc{Bot} \qquad \frac{f =_\varepsilon g \quad \varepsilon' \sqsubseteq \varepsilon}{f =_{\varepsilon'} g}\,\textsc{Mon} \qquad \frac{f =_{\varepsilon_i} g \quad \forall i \in I}{f =_{\sqcup_i \varepsilon_i} g}\,\textsc{Join}$$

- *For any two pairs of composable* Σ*-terms* (f_0, g_0) *and* (f_1, g_1) *and any* $\varepsilon, \varepsilon' \in V$*, the rule* Seq$_\oplus$ *ensures that the syntactic category is* $V\mathsf{HMet}_\oplus$*-enriched.*
- *For any two pairs of* Σ*-terms* (f_0, f_1) *and* (g_0, g_1) *with matching arities and any* $\varepsilon, \varepsilon' \in V$*, the rule* Par$_\oplus$ *ensures that the syntactic category is monoidal* $V\mathsf{HMet}_\oplus$*-enriched.*

$$\frac{f_0 =_\varepsilon f_1 \quad g_0 =_{\varepsilon'} g_1}{f_0 ; g_0 =_{\varepsilon\oplus\varepsilon'} f_1 ; g_1}\,\textsc{Seq}_\oplus \qquad \frac{f_0 =_\varepsilon f_1 \quad g_0 =_{\varepsilon'} g_1}{f_0 \otimes g_0 =_{\varepsilon\oplus\varepsilon'} f_1 \otimes g_1}\,\textsc{Par}_\oplus$$

▶ Remark 13. If $(V, \sqsubseteq)$ is IJD (Remark 2), then we can consider enriching $\mathcal{S}_{\Sigma,E}$ over VHMet with the monoidal product $\boxtimes_\sqcap$ from Example 7. This requires defining a different closure of E_q, that we denote with $E_q^{\mathsf{H}\sqcap\sqcap}$. It is the smallest set of quantitative equations containing E_q and closed under the inference rules above, but Seq$_\oplus$ and Par$_\oplus$ are replaced by Seq$_\sqcap$ and Par$_\sqcap$ below. These ensure that the syntactic category is $V\mathsf{HMet}_\sqcap$-enriched monoidal.

$$\frac{f_0 =_\varepsilon f_1 \quad g_0 =_{\varepsilon'} g_1}{f_0 ; g_0 =_{\varepsilon\sqcap\varepsilon'} f_1 ; g_1}\,\textsc{Seq}_\sqcap \qquad \frac{f_0 =_\varepsilon f_1 \quad g_0 =_{\varepsilon'} g_1}{f_0 \otimes g_0 =_{\varepsilon\sqcap\varepsilon'} f_1 \otimes g_1}\,\textsc{Par}_\sqcap$$

▶ Remark 14. There are two additional possible closures of E_q which we can consider, motivated by the example in Section 5. Let $E_q^{\mathsf{H}\sqcap\oplus}$ be the closure of E_q under the same inference rules as $E_q^{\mathsf{H}\oplus\oplus}$ except Seq$_\oplus$ is replaced by Seq$_\sqcap$. Similarly, let $E_q^{\mathsf{H}\oplus\sqcap}$ be the closure of E_q under the same inference rules as $E_q^{\mathsf{H}\oplus\oplus}$ except Par$_\oplus$ is replaced by Par$_\sqcap$.

Since V is integral, $\forall a, b \in V, a \oplus b \sqsubseteq a \sqcap b$, so we can infer that the rules Seq$_\sqcap$ and Par$_\sqcap$ are tighter than Seq$_\oplus$ and Par$_\oplus$ (respectively). Namely, any quantitative equation in $E_q^{\mathsf{H}\oplus\oplus}$ belongs to $E_q^{\mathsf{H}\sqcap\oplus}$ and $E_q^{\mathsf{H}\oplus\sqcap}$, and any quantitative equation in $E_q^{\mathsf{H}\sqcap\oplus}$ or $E_q^{\mathsf{H}\oplus\sqcap}$ also belongs to $E_q^{\mathsf{H}\sqcap\sqcap}$. Consequently, $E_q^{\mathsf{H}\oplus\sqcap}$ can be used to enrich $\mathcal{S}_{\Sigma,E}$ over $V\mathsf{HMet}_\oplus$, where the monoidal product satisfies an additional nonexpansiveness property. We make use of this in Section 5.2. In the case of $E_q^{\mathsf{H}\sqcap\oplus}$, $\mathcal{S}_{\Sigma,E}$ will be enriched over $V\mathsf{HMet}_{\sqcap}$, but monoidal enriched only over $V\mathsf{HMet}_{\oplus}$.

▶ Remark 15. We can also enforce the distance between morphisms to satisfy the symmetry property, so that $\mathcal{S}_{\Sigma,E}$ will be enriched over $V\mathsf{PMet}$. It suffices to add the following rule.

$$\frac{-\boxed{f}- =_\varepsilon -\boxed{g}-}{-\boxed{g}- =_\varepsilon -\boxed{f}-}\ \textsc{Symm}$$

We write $E_q^{\mathsf{P}\oplus\oplus}$, $E_q^{\mathsf{P}\sqcap\oplus}$, $E_q^{\mathsf{P}\oplus\sqcap}$ and $E_q^{\mathsf{P}\sqcap\sqcap}$ the corresponding closures of E_q after adding Symm.

There is a total of eight possible closures depending on the choice of inference rules considered. They are summarised in Table 1. Our logic offers this flexibility to allow axiomatisation of various examples. We use the closures $E_q^{\mathsf{H}\oplus\oplus}$ and $E_q^{\mathsf{P}\oplus\sqcap}$ respectively in Sections 4.2 and 5.3.

Table 1 Different choices of closures for E_q.

	Rules	IJD	Enrichment
$E_q^{\mathsf{H}\oplus\oplus}$	Seq$_\oplus$,Par$_\oplus$	No	$V\mathsf{HMet}_\oplus$
$E_q^{\mathsf{P}\oplus\oplus}$	" + Symm	No	$V\mathsf{PMet}_\oplus$
$E_q^{\mathsf{H}\sqcap\oplus}$	Seq$_\sqcap$,Par$_\oplus$	Yes	$V\mathsf{HMet}_\oplus$
$E_q^{\mathsf{P}\sqcap\oplus}$	" + Symm	Yes	$V\mathsf{PMet}_\sqcap$
$E_q^{\mathsf{H}\oplus\sqcap}$	Seq$_\oplus$,Par$_\sqcap$	Yes	$V\mathsf{HMet}_\oplus$
$E_q^{\mathsf{P}\oplus\sqcap}$	" + Symm	Yes	$V\mathsf{PMet}_\oplus$
$E_q^{\mathsf{H}\sqcap\sqcap}$	Seq$_\sqcap$,Par$_\sqcap$	Yes	$V\mathsf{HMet}_\sqcap$
$E_q^{\mathsf{P}\sqcap\sqcap}$	" + Symm	Yes	$V\mathsf{PMet}_\sqcap$

3.3 Enrichment of the Syntactic Category

In this section, we will show how to use the inference rules in Definition 12 (or more precisely the different closures they induce) to define enrichments of the syntactic category $\mathcal{S}_{\Sigma,E}$. We work with a generic closure $\overline{E_q}$ that can be instantiated with any closure in Table 1. We explicitly mention what inference rules are needed to show each item. First, we equip each hom-set of $\mathcal{S}_{\Sigma,E}$ with a V-hemimetric, essentially mirroring the definition of $d_{\mathcal{U}}$ in [49, Section 5]. Note that Refl, Triang and Symm correspond, respectively, to reflexivity, triangle inequality, and symmetry for $d^{\mathcal{U}}_{n,m}$.

▶ **Lemma 16.** *Let $\mathcal{U} = (\Sigma, E, E_q)$ be a V-quantitative monoidal theory. For any $n, m \in \mathbb{N}$ and Σ-terms $f, g\colon n \to m$, let $d^{\mathcal{U}}_{n,m}(f, g) := \bigsqcup\{\varepsilon \mid f =_\varepsilon g \in \overline{E_q}\}$. This defines a V-hemimetric on $\mathcal{S}_{\Sigma,E}(n, m)$, which is a V-pseudometric if* Symm *was used in the closure $\overline{E_q}$.*

Now that $\mathcal{S}_{\Sigma,E}$ is equipped with V-hemimetrics (resp. pseudometrics) on its hom-sets, we show it is monoidal enriched over $V\mathsf{HMet}$ (resp. $V\mathsf{PMet}$). This relies on two lemmas showing that sequential and parallel composition are nonexpansive.

▶ **Lemma 17.** *Let $f_0, f_1\colon n \to m$ and $g_0, g_1\colon m \to \ell$ be Σ-terms. When $*$ is $\oplus$ and $\overline{E_q}$ is closed under* Seq$_\oplus$*, or when $*$ is $\sqcap$ and $\overline{E_q}$ is closed under* Seq$_\sqcap$*, the following holds: $d^{\mathcal{U}}_{n,m}(f_0, f_1) * d^{\mathcal{U}}_{m,\ell}(g_0, g_1) \sqsubseteq d^{\mathcal{U}}_{n,\ell}(f_0 ; g_0\, ,\ f_1 ; g_1)$.*

▶ **Lemma 18.** *Let $f_0, g_0\colon n \to n'$ and $f_1, g_1\colon m \to m'$ be Σ-terms. When $*$ is $\oplus$ and $\overline{E_q}$ is closed under* Par$_\oplus$*, or when $*$ is $\sqcap$ and $\overline{E_q}$ is closed under* Par$_\sqcap$*, the following holds: $d^{\mathcal{U}}_{n,n'}(f_0, f_1) * d^{\mathcal{U}}_{m,m'}(g_0, g_1) \sqsubseteq d^{\mathcal{U}}_{n+m,n'+m'}(f_0 \otimes g_0, f_1 \otimes g_1)$.*

By the characterisation of $V\mathsf{HMet}$ enrichment in Section 2, we conclude that the category of string diagrams equipped with the distances $d^{\mathcal{U}}_{n,m}$ is an enriched SMC.

▶ **Proposition 19.** *Let $\mathcal{U} = (\Sigma, E, E_q)$ be a V-quantitative monoidal theory, and $\overline{E_q}$ be one of the closures defined above. Then, $\mathcal{S}_{\Sigma,E}$ equipped with the V-hemimetrics defined in Lemma 16 is a $\mathcal{V}$-enriched SMC, where $\mathcal{V}$ is the base of enrichment corresponding to $\overline{E_q}$ in Table 1. We denote this enriched category with $\mathcal{S}_{\mathcal{U}}$ to distinguish it from its underlying category $\mathcal{S}_{\Sigma,E}$.*

The inference rules are central to the definition of the syntactic category, and it will be convenient for us to reify them in other categories through the notion of validity.

▶ **Definition 20.** *Let $\mathcal{C}$ be an SMC equipped with V-hemimetrics $(\mathcal{C}(a,b), d^{\mathcal{C}}_{a,b})$ on each of its hom-sets. The rules* Refl, Bot, Triang, Mon, *and* Join *are* valid *in $\mathcal{C}$.* Symm *is* valid *if all $d^{\mathcal{C}}_{a,b}$ are V-pseudometrics.* $\text{Seq}_{\oplus}$ *is* valid *if* $; \colon \mathcal{C}(a,b) \boxtimes_{\oplus} \mathcal{C}(b,c) \to \mathcal{C}(a,c)$ *(sequential composition) is nonexpansive.* $\text{Par}_{\oplus}$ *is* valid *if* $\otimes \colon \mathcal{C}(a,a') \boxtimes_{\oplus} \mathcal{C}(b,b') \to \mathcal{C}(a \otimes a', b \otimes b')$ *(parallel composition) is nonexpansive.* $\text{Seq}_{\sqcap}$ *is* valid *if* $; \colon \mathcal{C}(a,b) \boxtimes_{\sqcap} \mathcal{C}(b,c) \to \mathcal{C}(a,c)$ *is nonexpansive.* $\text{Par}_{\sqcap}$ *is* valid *if* $\otimes \colon \mathcal{C}(a,a') \boxtimes_{\sqcap} \mathcal{C}(b,b') \to \mathcal{C}(a \otimes a', b \otimes b')$ *is nonexpansive.*

3.4 Models

Defining syntactic categories allows us to study models as functors à la Lawvere [43]. Recall that a model of a monoidal theory (Σ, E) is a symmetric strict monoidal functor from $\mathcal{S}_{\Sigma,E}$ to another SMC $\mathcal{C}$ (*cf.* [12]). Central to this approach is the fact that because models must preserve the structure used to generate $\mathcal{S}_{\Sigma,E}$, they are entirely determined by their action on the generators. Also, one may check that an assignment of the generators from Σ into $\mathcal{C}$ extends to a model simply by verifying that the equations in E are satisfied. Our goal in this section is to define models of quantitative monoidal theories as functors from the syntactic categories, prove that they are determined by their action on generators, and finally give sufficient conditions for when a model of a monoidal theory can be enriched.

Because there are multiple syntactic categories that can be constructed from a quantitative monoidal theory (depending on the inference rules that are invoked), there are different notions of models. We can unify their definition using the notion of validity (Definition 20).

▶ **Definition 21.** *Let $\mathcal{U} = (\Sigma, E, E_q)$ be a V-quantitative monoidal theory, and $\overline{E_q}$ be one of the closures listed in Table 1. An $\overline{E_q}$-model of $\mathcal{U}$ is a $V\mathsf{HMet}_{\oplus}$-enriched SMC $\mathcal{C}$ wherein all the inference rules used to generate $\overline{E_q}$ are valid, along with a strict monoidal $V\mathsf{HMet}_{\oplus}$-functor $M \colon \mathcal{S}_{\mathcal{U}} \to \mathcal{C}$, where $\mathcal{S}_{\mathcal{U}}$ is constructed according to Proposition 19.*

Independently of the choice of inference rules, the underlying category of $\mathcal{S}_{\mathcal{U}}$ is always $\mathcal{S}_{\Sigma,E}$, where (Σ, E) is the underlying monoidal theory of $\mathcal{U}$. Therefore, any model of $\mathcal{U}$ is always built on top of a model of (Σ, E). The enrichment is merely a property on a strict monoidal functor $M \colon \mathcal{S}_{\Sigma,E} \to \mathcal{C}$. In analogy to how assignments on the generators of Σ can be extended to models of (Σ, E) when they satisfy E, we can give a sufficient condition, in terms of the quantitative equations in E_q, for M to be an enriched model.

▶ **Definition 22.** *Let $\mathcal{U} = (\Sigma, E, E_q)$ be a V-quantitative monoidal theory, $\mathcal{S}_{\mathcal{U}}$ be constructed according to Proposition 19 with a closure $\overline{E_q}$, and $\mathcal{C}$ be a $V\mathsf{HMet}_{\oplus}$-enriched SMC wherein all the inference rules used to generate $\overline{E_q}$ are valid. A quantitative equation $f =_{\varepsilon} g$ is* true *in a model of (Σ, E), $M \colon \mathcal{S}_{\Sigma,E} \to \mathcal{C}$, if $\varepsilon \sqsubseteq d^{\mathcal{C}}(Mf, Mg)$, where $d^{\mathcal{C}}$ is the V-hemimetric (or pseudometric) on the hom-sets of $\mathcal{C}$.*

This definition allows us to define models of (Σ, E, E_q) from certain models of (Σ, E).

▶ **Theorem 23.** *Let M be a model of a monoidal theory (Σ, E). If all the quantitative equations in E_q are true in M, then it is an $\overline{E_q}$-model of (Σ, E, E_q). In particular, M is an enriched functor.*

The theorem below is an analogue of the completeness theorem of equational logic adapted to the case of quantitative monoidal reasoning. It intuitively means that the rules of quantitative monoidal reasoning suffice for proving all the quantitative equations that hold generally in the semantics. The proof relies on the canonical model $\mathrm{id}\colon \mathcal{S}_{\mathcal{U}} \to \mathcal{S}_{\mathcal{U}}$.

▶ **Theorem 24.** *Let $\mathcal{U} = (\Sigma, E, E_q)$ be a V-quantitative monoidal theory, $\overline{E_q}$ be a closure from Table 1, and $f, g\colon n \to m \in \mathcal{S}_{\mathcal{U}}$. If $f =_\varepsilon g$ is true in all $\overline{E_q}$-models $M\colon \mathcal{S}_{\mathcal{U}} \to \mathcal{C}$ of the theory $\mathcal{U}$, meaning that $\varepsilon \sqsubseteq d^{\mathcal{C}}(M(f), M(g))$, then $f =_\varepsilon g$ is in the closure $\overline{E_q}$.*

4 Case Study I: Order on Matrices

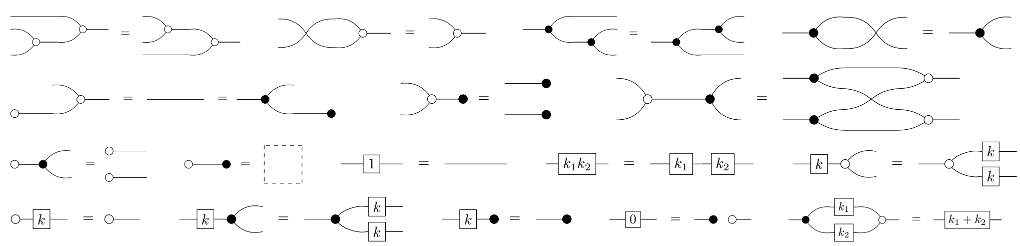
Throughout the section we fix a semiring R (a ring without additive inverses) and write 1_R and 0_R respectively for its multiplicative and additive identities. Also, we write $\underline{\mathbf{n}}$ for $\{0, \dots, n-1\}$. We will show that, when R is ordered, the entrywise ordering of R-matrices can be axiomatised using the framework of Section 3. A notable example is the Boolean semiring $R = \{0, 1\}$, in which case R-matrices represent relations, and the order we axiomatise is set-theoretic inclusion.

4.1 Background: (non-Quantitative) Axiomatisation of Mat_R

Before considering the quantitative theory, we recall the category of R-matrices and the monoidal theory axiomatising it. The axiomatisation result seems to be folklore, see e.g. [34,41] for the Boolean case. We follow the presentation of [72, Section 3.2], which is for a generic ring, but is applicable for semirings as well.

▶ **Definition 25.** *The SMC Mat_R has objects $\mathbb{N}$, and morphisms $n \to m$ the $m \times n$ matrices with entries in R. When n or m is 0, there is a unique empty $m \times n$ matrix $[]$. Composition is by matrix multiplication and the monoidal product is by direct sum: $A \oplus A' = \left[\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{0} & A' \end{smallmatrix}\right]$.*

The monoidal theory HA_R (Hopf Algebras) has generators —• $: 1 \to 0$, —•< $: 1 \to 2$, >○— $: 2 \to 1$, ○— $: 0 \to 1$, and —[k]— $: 1 \to 1$, for each scalar $k \in R$, and equations

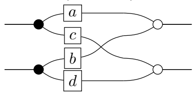


We write $\mathcal{S}_{\mathsf{HA}_R}$ for the SMC freely generated by HA_R, defined according to Definition 10.

▶ **Proposition 26.** *[72, Proposition 3.9] The following assignment of a matrix to each generator of HA_R, $F_R($—•$) = []$, $F_R($—•<$) = \left[\begin{smallmatrix} 1_R \\ 1_R \end{smallmatrix}\right]$, $F_R($>○—$) = [\,1_R\ 1_R\,]$, $F_R($○—$) = []$, $F_R($—[k]—$) = [k]$, yields an identity-on-objects freely generated symmetric monoidal functor $F_R\colon \mathcal{S}_{\mathsf{HA}_R} \to \mathsf{Mat}_R$, which is furthermore an isomorphism of SMCs.*

Showing that F_R is faithful relies on a decomposition result for morphisms of $\mathcal{S}_{\mathsf{HA}_R}$ [72, Lemma 3.10]. We prove a variant of this result that will be convenient to use later.

▶ **Lemma 27.** *Given $n, m \in \mathbb{N}$, there are two morphisms $b^n_m \colon n \to nm$ and $w^n_m \colon nm \to m$ in $\mathcal{S}_{\mathsf{HA}_R}$ such that for any morphism $f \colon n \to m$ in $\mathcal{S}_{\mathsf{HA}_R}$, there are scalars $\{f_{ij} \in R\}_{i\in\underline{\mathbf{m}},j\in\underline{\mathbf{n}}}$ such that $f = b^n_m ; (\bigotimes_{i\in\underline{\mathbf{m}},j\in\underline{\mathbf{n}}} f_{ij}) ; w^n_m$ and the (i,j)-entry of the matrix $F(f)$ is f_{ij}.*

This decomposition is related to the "matrix canonical form" of [72, Lemma 3.10] because it can be shown that b^n_m is represented by a string diagram containing only the generators —● and —●<, while w^n_m only contains ○—, >○—, and ><. For example, the morphism f satisfying $F_R(f) = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ decomposes as  .

4.2 Axiomatising the Preorder Relation for Matrices

When R is an ordered semiring, there is a simple preorder on matrices of the same size defined by entrywise comparisons: $A \leq B$ if and only if, $A_{ij} \leq B_{ij}$ for all i, j.

In this section we consider the task of axiomatising this preorder. There are two main steps. First, formulate the preorder as an enrichment on Mat_R. Second, identify a quantitative extension of the theory HA_R and show it axiomatises the enriched version of Mat_R. A key property for the enrichment is not just the existence of an order on the semiring elements, but also compatibility of this order with matrix multiplication.

▶ **Assumption 28.** *Throughout this subsection we assume R to be an ordered semiring such that, for each $a, a', b, b' \in R$, $a \leq a'$ and $b \leq b'$ implies $a + b \leq a' + b'$ and $ab \leq a'b'$.*

For example, the Boolean semiring $\{0m1\}$ and the semiring $[0, \infty)$ of nonnegative reals satisfies this assumption, whereas $\mathbb{R}$ does not. As seen in Example 4, $2_\sqcap$-hemimetrics are preorders and their nonexpansive maps are order-preserving functions. Thus, we seek to enrich Mat_R in $2_\sqcap\mathsf{HMet}$. Since $\oplus = \sqcap$ in $2_\sqcap$, the two monoidal products provided in Examples 6 and 7 coincide, and are defined as: $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$. Thanks to Assumption 28, we can show that matrix multiplication and direct sum preserve this order, and we obtain the following.

▶ **Theorem 29.** *The category Mat_R equipped with the $2_\sqcap$-hemimetrics corresponding to entrywise comparison on its hom-sets, denoted with $\mathsf{Mat}^{\leq}_R$, is a $2_\sqcap\mathsf{HMet}$-enriched SMC.*

Next, we introduce a $2_\sqcap$-quantitative monoidal theory and show it axiomatises $\mathsf{Mat}^{\leq}_R$.

▶ **Definition 30.** *The $2_\sqcap$-quantitative monoidal theory POHA_R is defined as the monoidal theory HA_R extended with the following family of quantitative equations: $\forall k_1 \leq k_2 \in R$,*

$$-\boxed{k_1}- \;=_\top\; -\boxed{k_2}- \;. \qquad (1)$$

As a reason for (1), recall that, in a $2_\sqcap$-hemimetric space, two objects x, y having distance $\top$ corresponds to $x \leq y$ when seeing the space as a preorder.We may now form the syntactic category $\mathcal{S}_{\mathsf{POHA}_R}$ on POHA_R (Proposition 19) using the closure $E_q^{\mathsf{H}\oplus\oplus}$, where E_q contains the quantitative equations in (1). We may equivalently use any closure in Table 1 since $\oplus = \sqcap$ in $2_\sqcap$. Both $\mathcal{S}_{\mathsf{POHA}_R}$ and $\mathsf{Mat}^{\leq}_R$ are $2_\sqcap\mathsf{HMet}$-enriched monoidal, and we now prove they are isomorphic as enriched SMCs.

▶ **Theorem 31.** *The isomorphism $F_R \colon \mathcal{S}_{\mathsf{HA}_R} \to \mathsf{Mat}_R$ from Proposition 26 induces an isomorphism of $2_\sqcap\mathsf{HMet}$-enriched SMCs $F_R \colon \mathcal{S}_{\mathsf{POHA}_R} \to \mathsf{Mat}^{\leq}_R$.*

Proof. It suffices to prove F_R is locally an isometry, which for $2_{\sqcap}$-hemimetrics means that for any morphisms f and g in $\mathcal{S}_{\mathsf{HA}_R}$, $f \leq g$ if and only if $F_R(f) \leq F_R(g)$.

The forward implication says that F_R is enriched, and by Theorem 23, we can prove this by checking the quantitative equations of POHA_R are true in F_R. The latter are of the form (1) when $k_1 \leq k_2$, and those are clearly true in F_R because $F_R(-\boxed{k}-) = [k]$.

It remains to prove the converse implication. By Lemma 27, we can decompose f and g as $b^n_m; (\bigotimes_{i\in\underline{\mathbf{m}}, j\in\underline{\mathbf{n}}} f_{ij}); w^n_m$ and $b^n_m; (\bigotimes_{i\in\underline{\mathbf{m}}, j\in\underline{\mathbf{n}}} g_{ij}); w^n_m$. Now, $F_R(f) \leq F_R(g)$ means that each entry of $F_R(f)$ is less or equal than each corresponding entry of $F_R(g)$, so for any i and j, $F_R(f_{ij}) \leq F_R(g_{ij})$, hence $-\boxed{f_{ij}}- \leq -\boxed{g_{ij}}-$ by (1). Having established this relation between scalars appearing in f and g, and exploiting the decompositions, we can conclude that $f \leq g$ holds in $\mathcal{S}_{\mathsf{POHA}_R}$ by repeated application of the inference rules $\textsc{Par}_{\sqcap}$ and $\textsc{Seq}_{\sqcap}$. ◀

5 Case Study II: Total Variation Distance

The total variation distance is one of the most widely studied metrics on probability distributions. It appears ubiquitously in various fields of applied mathematics, a prominent example being optimal transport theory [69]. In [49], the authors axiomatise the total variation distance on probability distributions as a quantitative (cartesian) algebraic theory.

In this section, we achieve a similar characterisation result, but in the language of quantitative monoidal theories. Rather than just discrete probability distributions, we focus more generally on stochastic matrices. These form a category FStoch, in which distributions are the $1 \to n$ morphisms.

Our first step is to recall FStoch and the (non-quantitative) monoidal theory axiomatising it (Section 5.1). Second, we introduce the total variation distance and show that FStoch is enriched over metric spaces, so that we can study total variation between its morphisms meaningfully (Section 5.2). Thirdly, we expand the monoidal theory of Section 5.1 to a quantitative monoidal theory, and show that it axiomatises FStoch as an enriched SMC. Effectively, this means that two matrices in FStoch are at total variation distance ε if and only if the corresponding string diagrams can be proven to be at distance ε in the theory.

This can be understood as an axiomatisation of the "metric theory" in [33, Example 3.2.7].

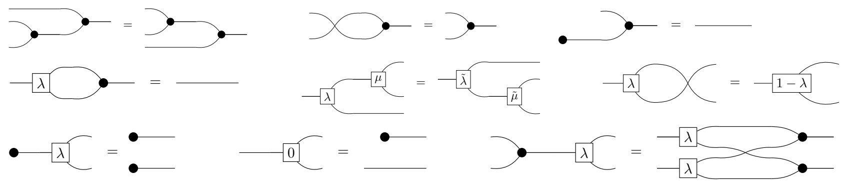
5.1 Background: (non-Quantitative) Axiomatisation of FStoch

Here, we recall the axiomatisation result of FStoch, only focusing on exact equality.

▶ **Definition 32.** *The SMC* FStoch *is the subcategory of* $\mathsf{Mat}_{[0,1]}$ *whose morphisms* $n \to m$ *are the* stochastic matrices*, i.e.* $m \times n$ *matrices with entries in the interval* $[0,1]$*, such that the sum of the entries in a column always equals* 1*.*

In the sequel, we will often (implicitly) see columns of a stochastic matrix as probability distributions and vice-versa. Recall that, given a set X, a *(probability) distribution* on X is a function $\varphi\colon X \to [0,1]$ satisfying $\sum_{x\in X} \varphi(x) = 1$. There is a monad mapping X to the set $\mathcal{D}(X)$ of finitely supported probability distributions on X, and one may regard FStoch as a full subcategory of the Kleisli category of such monad. The following axioms were originally studied in [65], but we follow the more recent [28], casting it in the setting of monoidal categories.

▶ **Definition 33.** *The monoidal theory* Conv *(standing for* convex algebras*) has generators* $\bullet\!-\!: 0 \to 1$*,* $\rangle\!\!\bullet\!-\!: 2 \to 1$*, and* $-\boxed{\lambda}\!\langle\, : 1 \to 2$ *for each* $\lambda \in [0,1]$*, and equations as below (*$\tilde{\lambda}$ *stands for* $\lambda\mu$ *and* $\tilde{\mu}$ *for* $\frac{\lambda-\lambda\mu}{1-\lambda\mu}$ *(with* $\frac{0}{0} = 1$*)). We write* $\mathcal{S}_{\mathsf{Conv}}$ *for the SMC freely generated by* Conv *(recall Definition 10).*



▶ **Proposition 34** ([28, Theorem 3.14]). *The following assignment of a stochastic matrix to each generator of* Conv*,* $F(\bullet\!-) = []$*,* $F(\supset\!\!\bullet\!-) = [\,1\;1\,]$*,* $F(-\boxed{\lambda}\!\subset) = \left[\begin{smallmatrix}\lambda\\1-\lambda\end{smallmatrix}\right]$*, yields an identitity-on-objects freely generated symmetric monoidal functor* $F\colon \mathcal{S}_{\mathsf{Conv}} \to \mathsf{FStoch}$*, which is furthermore an isomorphism of SMCs.*

▶ Remark 35. A few caveats when comparing our presentation with the one of [28]: the category FStoch is called `FinStoMap`; the author uses different graphical conventions for reading sequential and parallel composition of diagrams (top-to-bottom instead of left-to-right); the author has the symmetric structure as explicit part of the presentation, rather than as something generated freely by the syntactic category of string diagrams. Furthermore, note that there are other ways to present FStoch axiomatically, e.g. [12, Example 6.2(c)].

Showing that F is faithful relies on a decomposition for morphisms of $\mathcal{S}_{\mathsf{Conv}}$ recalled below.

▶ **Lemma 36** ([28, Propositions 3.12 and 3.13]). *Given* $n, m \in \mathbb{N}$*, there is a morphism* $p^n_m\colon nm \to m$ *such that* $\forall f\colon n \to m \in \mathcal{S}_{\mathsf{Conv}}$*, there are morphisms* $\{f_i\colon 1 \to m\}_{i\in\underline{\mathbf{n}}}$ *such that* $f = (f_1 \otimes \cdots \otimes f_n); p^n_m$ *and* $F(f_i)$ *is the ith column of* $F(f)$*.*

5.2 Enrichment of FStoch with the Total Variation Distance

In this section we define an enrichment on FStoch based on the *total variation* distance $\mathsf{tv}\colon \mathcal{D}X \times \mathcal{D}X \to [0,1]$, which is defined by $\mathsf{tv}(\varphi,\psi) \coloneqq \max_{S\subseteq X} |\sum_{x\in S}\varphi(x) - \sum_{x\in S}\psi(x)|$.

In the context of this paper, tv is a metric that can be defined on the set of morphisms $\mathsf{FStoch}(1,m)$ for any positive $m \in \mathbb{N}$. Now, in order to define an enrichment of FStoch over $[0,\infty]_+\mathsf{PMet}$, we still need pseudometrics on the other hom-sets. The following definition is somewhat natural (see Remark 38): for every $n, m \in \mathbb{N}$, the metric $\mathsf{tv}_\times$ on $\mathsf{FStoch}(n,m)$ is

$$\mathsf{tv}_\times(A,B) = \max_{i\in\underline{\mathbf{n}}} \mathsf{tv}(A_i, B_i), \tag{2}$$

where A_i is the ith column of A, and $\mathsf{tv}(A_i,B_i)$ is the total variation distance between the corresponding distributions.

We write $\mathsf{FStoch}_{\mathsf{tv}}$ for FStoch equipped with the metric $\mathsf{tv}_\times$ on its hom-sets. After showing that sequential and parallel composition are nonexpansive relative to the monoidal product $\boxtimes_\oplus$ (from Example 6), we conclude the following.

▶ **Theorem 37.** $\mathsf{FStoch}_{\mathsf{tv}}$ *is a* $[0,\infty]_+\mathsf{PMet}_\otimes$*-enriched SMC.*

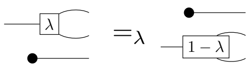
▶ Remark 38. The definition of $\mathsf{FStoch}_{\mathsf{tv}}$ via $\mathsf{tv}_\times$ (2) and the fact that it is enriched can be obtained more abstractly. First, we can show $\mathcal{D}$ lifts to an enriched relative monad on $[0,\infty]_+\mathsf{PMet}_\otimes$ with the total variation metric by adapting [4, Example 5.10] for the theory of LIB algebras in [49, Section 8]. Then, we unroll the construction of the enriched relative Kleisli category in [3, Proposition 8.21] to get the category (opposite to) $\mathsf{FStoch}_{\mathsf{tv}}$.

▶ Remark 39. The category $\mathsf{FStoch}_{\mathsf{tv}}$ is not enriched over $[0,\infty]_+\mathsf{PMet}$ with the monoidal product $\boxtimes_\sqcap$ from Example 7, because sequential composition is not nonexpansive for this monoidal product: with the matrices $A = \left[\begin{smallmatrix}1 & 0.5\\0 & 0.5\end{smallmatrix}\right]$, $B = A$, $C = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, and $C' = \left[\begin{smallmatrix}0.5\\0.5\end{smallmatrix}\right]$, we have $AC = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $BC' = \left[\begin{smallmatrix}0.75\\0.25\end{smallmatrix}\right]$, thus $\mathsf{tv}_\times(C;A,C';B) = \frac{3}{4} > \frac{1}{2} = \max\{\mathsf{tv}_\times(C,C'), \mathsf{tv}_\times(A,B)\}$.

5.3 Quantitative Axiomatisation of $\mathsf{FStoch}_{\mathsf{tv}}$

We introduce the quantitative monoidal theory that axiomatises $\mathsf{FStoch}_{\mathsf{tv}}$.

▶ **Definition 40.** *The* $[0,\infty]$*-quantitative monoidal theory* Lib *is defined as the tuple* (Σ, E, E_q) *where* $(\Sigma, E) = \mathsf{Conv}$ *(Definition 33) and* E_q *contains, for each* $\lambda \in [0,1]$*,*


(TV)

We write $\mathcal{S}_{\mathsf{Lib}}$ *for the* $[0,\infty]_+\mathsf{PMet}_\otimes$*-enriched SMC generated by* Lib *using the inference rules* $\textsc{Seq}_\oplus$*,* $\textsc{Par}_\sqcap$*, and* $\textsc{Symm}$*, constructed according to Proposition 19.*

Note that (TV) is adapted from the quantitative equations **LI** used in [49, Definition 8.1]. Both sides of (TV) will give weight $1-\lambda$ to the second output, thus the parts where they differ will have weight at most λ. Hence, the distance between the results is at most λ.

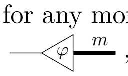
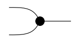
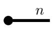
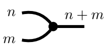
▶ Remark 41. The choice of inference rules used to generate $\mathcal{S}_{\mathsf{Lib}}$ is motivated by our goal to construct an enriched isomorphism $\mathcal{S}_{\mathsf{Lib}} \to \mathsf{FStoch}_{\mathsf{tv}}$. Indeed, by Remark 39 the rule $\textsc{Seq}_\sqcap$ would not be valid in $\mathsf{FStoch}_{\mathsf{tv}}$, and if we took the less strict $\textsc{Par}_\oplus$ over $\textsc{Par}_\sqcap$, nothing would guarantee validity of the latter in $\mathcal{S}_{\mathcal{U}}$. But $\textsc{Par}_\sqcap$ is valid in $\mathsf{FStoch}_{\mathsf{tv}}$.

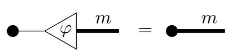
Our axiomatisation result now amounts to showing that the functor F introduced in Proposition 34 is an isomorphism of enriched categories between $\mathcal{S}_{\mathsf{Lib}}$ and $\mathsf{FStoch}_{\mathsf{tv}}$. The following lemma, along with Theorem 23, implies F is enriched.

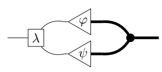
▶ **Lemma 42.** *The quantitative equations* (TV) *are true in* F*.*

To conclude that F is actually an enriched isomorphism, it is enough to show it is locally an isometry. We first focus on the case of morphisms $1 \to m$, that is, probability distributions, and we recall a lemma used in the axiomatisation of Mardare et al.

▶ **Lemma 43** ([5, Lemma 10.12])**.** *For any two distributions* $\varphi, \psi \in \mathcal{D}X$ *with* $\lambda = \mathsf{tv}(\varphi,\psi)$*, there exist three distributions* $\varphi', \psi', \tau \in \mathcal{D}X$ *such that* $\varphi = \varphi' +_\lambda \tau$ *and* $\psi = \psi' +_\lambda \tau$*.*

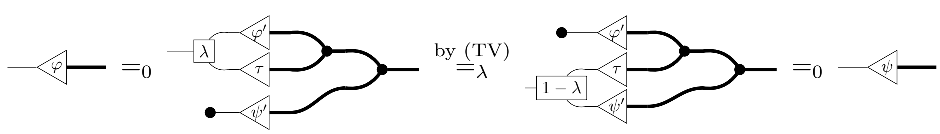
We also introduce thick wires that represent the tensor of multiple wires. For example, for any morphism $f\colon 1 \to m$ in $\mathcal{S}_{\mathsf{Lib}}$, we can choose a representative diagram that we draw as , where φ is the distribution corresponding to $F(f)$ (we often omit the number on top of the thick wire). Moreover, there are also thick versions of and  drawn as  and  respectively, which obey the same equations as their thin counterparts. In particular, we can show the following equation is in $\mathcal{S}_{\mathsf{Lib}}$ by induction.


(3)

Now, a convex combination $\varphi +_\lambda \psi$ can be represented diagrammatically with , which facilitates a diagrammatic proof of the following result.

▶ **Lemma 44.** *The function* $f \mapsto F(f)$ *is a bijective isometry* $\mathcal{S}_{\mathsf{Lib}}(1,m) \to \mathsf{FStoch}_{\mathsf{tv}}(1,m)$*.*

Proof sketch. Given two morphisms $f, g\colon 1 \to m$, let their corresponding distributions be $\varphi, \psi \in \mathcal{D}\underline{\mathbf{m}}$, and let φ', ψ', and τ be given by Lemma 43. We provide a derivation in Lib of $f =_\lambda g$, with $\lambda = \mathsf{tv}(\varphi,\psi) = \mathsf{tv}_\times(F(f), F(g))$.



Both $=_0$ steps follow from hypotheses $\varphi = \varphi' +_\lambda \tau$, $\psi = \tau +_{1-\lambda} \psi'$, and (3). This shows $d^{\mathsf{Lib}}(f, g) \leq \mathsf{tv}(\varphi, \psi)$. The converse inequality holds because F is an enriched functor (Lemma 42), hence F is an isometry on the hom-sets with domain 1. It is bijective by Proposition 34. ◀

Lemma 36 allows us to extend our argument to arbitrary morphisms of $\mathsf{FStoch}_{\mathsf{tv}}$.

▶ **Theorem 45.** *The functor $F \colon \mathcal{S}_{\mathsf{Lib}} \to \mathsf{FStoch}_{\mathsf{tv}}$ is an isomorphism of enriched categories.*

As we mentioned, our axiomatisation of total variation distance between stochastic matrices was inspired by Mardare et al.'s for distributions. We discuss the link between our work and quantitative algebraic theories in the following section.

6 Comparison with Related Work: Cartesian vs Monoidal

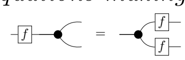
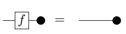
The work [12] relates monoidal theories and (cartesian) algebraic theories, showing that terms of an algebraic theory $\mathcal{U}$ correspond to string diagrams in a monoidal theory $\mathcal{U}'$, where $\mathcal{U}'$ only adds a natural commutative monoid structure to $\mathcal{U}$. This follows by an isomorphism between the Lawvere category generated by $\mathcal{U}$ and the SMC freely generated by $\mathcal{U}'$ [12, Theorem 6.1]. In this section, we establish an analogous link between the unconditional quantitative algebraic theories of [49] and our quantitative monoidal theories (Definition 11), via the discrete enriched Lawvere theories of [59].

Recasting [49] (and [62], which generalises [49] to quantales), an *unconditional V-quantitative algebraic theory* $\mathcal{U}$ is a triple (Σ, E, E_q), where Σ is a signature of operations with coarity 1, E is a set of equations between cartesian terms (the standard terms from universal algebra), and E_q is a set of V-quantitative equations between cartesian terms. Elements of E_q correspond to quantitative equations of [49] with no premises ($\emptyset \vdash s =_\varepsilon t$), and they are called unconditional in *loc. cit.* Any such theory generates a discrete enriched Lawvere theory [59, Definition 4] as follows.

▶ **Definition 46.** *The discrete $V\mathsf{HMet}_\sqcap$-Lawvere theory generated by $\mathcal{U}$ is the $V\mathsf{HMet}$-category $\mathcal{L}_\mathcal{U}$, where objects are natural numbers, and morphisms $n \to m$ are n-tuples of cartesian terms with at most m variables, e.g. $\langle f(x_1, x_3), x_3 \rangle \colon 2 \to 3$, considered modulo the equations between terms derived in quantitative equational logic from the axioms in E and E_q (see [46, Figure 2]). Composition of morphisms is by substitution. The distance between morphisms is computed between terms as the join of derivable distances, and between tuples as the coordinatewise meet. Namely, if $\mathcal{U} \vdash f =_\varepsilon g$ denotes that $f =_\varepsilon g$ is derivable from the axioms in E and E_q, then $d_{\mathcal{L}_\mathcal{U}}(\langle f_i \rangle, \langle g_i \rangle) = \sqcap_i \bigsqcup \{\varepsilon \mid \mathcal{U} \vdash f_i =_\varepsilon g_i\}$.*

▶ Remark 47. Following [60], an equivalent description of $\mathcal{L}_\mathcal{U}$ is as the restriction of the enriched Kleisli category for the free $\mathcal{U}$-algebra monad to the discrete spaces on finite sets.

We will show $\mathcal{L}_\mathcal{U}$ can be freely generated from the theory that combines $\mathcal{U}$ with a natural cocommutative comonoid structure.

▶ **Definition 48.** *The V-quantitative monoidal theory $\mathcal{U}' = (\Sigma', E', E'_q)$ is defined by $\Sigma' := \Sigma \sqcup \{ -\!\bullet, -\!\!\bullet\!\!< \}$, $E' := E \cup E^c \cup E^d$, and $E'_q := E_q$, where E^c contains the equations making $\{ -\!\bullet, -\!\!\bullet\!\!< \}$ into a cocommutative comonoid and E^d the naturality equations*  *and*  *for each in $f \in \Sigma$.*

Our constructions clearly treat the quantitative axioms and the distances separate from the rest. In other words, the underlying categories of $\mathcal{L}_\mathcal{U}$ and $\mathcal{S}_{\mathcal{U}'}$ are, respectively, the Lawvere category generated by (Σ, E) and the prop generated by (Σ', E'). Hence, it readily follows from [12, Theorem 6.1] that the underlying categories are isomorphic. It remains to show this isomorphism is an isometry. At this point, it is important to note the choice of inference rules used to generate $\mathcal{S}_{\mathcal{U}'}$: we take $\textsc{Seq}_{\oplus}$ and $\textsc{Par}_{\sqcap}$. We also need to assume that V is IJD at a technical point in the proof.

▶ **Theorem 49.** *Let $\mathcal{U}$ be an unconditional V-quantitative algebraic theory and $\mathcal{U}'$ a V-quantitative monoidal theory constructed as in Definition 48. There is an isomorphism of $V\mathsf{HMet}$-enriched categories between $\mathcal{L}_{\mathcal{U}}$ and $\mathcal{S}_{\mathcal{U}'}$.*

Proof sketch. We see the isomorphism between the underlying categories as a model of (Σ', E') valued in $\mathcal{L}_{\mathcal{U}}$. We apply Theorem 23 to show it is an enriched model. It remains to prove that the distance between cartesian terms in $\mathcal{L}_{\mathcal{U}}$ is smaller than the distance between the corresponding diagrams in $\mathcal{S}_{\mathcal{U}'}$. We do this by simulating all the rules in quantitative equational logic with the rules used to build the closure ${E'_q}^{\mathsf{H}\oplus\sqcap}$. ◀

In words, Theorem 49 shows that we can always extract the linear part of an unconditional quantitative algebraic theory just as we can extract the linear part of an algebraic theory.

▶ **Example 50.** The theory of LIB algebras in [49, Definition 8.1] is an unconditional quantitative algebraic theory, call it $\mathcal{U}$. Unrolling Remarks 38 and 47, we find that the enriched Lawvere theory generated by $\mathcal{U}$ is the opposite of $\mathsf{FStoch}_{\mathsf{tv}}$. Thus Theorem 49 provides a quantitative monoidal theory $\mathcal{U}'$ and an enriched isomorphism $\mathcal{S}_{\mathcal{U}'} \cong \mathsf{FStoch}_{\mathsf{tv}}^{\mathrm{op}}$.

At first sight, this seems like another axiomatisation of the total variation distance complementary to Theorem 45. Further investigation shows that $\mathcal{U}'$ and Lib are morally the same. This situation exactly mirrors the differences between the axiomatisations of FStoch in [12, Example 6.2(c)] and [28]. Namely, the latter avoids redundant equations.

7 Conclusions

Our work provides mathematical foundations to enhance monoidal algebra with quantitative equations. We are motivated by the increasing relevance of string diagrammatic calculi in areas such as quantum theory, machine learning, probabilistic programming, and circuit theory, in which quantitative reasoning plays a fundamental role. Our basic examples in Sections 4 and 5 are intended merely as a proof-of-concept for our framework. More sophisticated examples, building on diagrammatic calculi for quantum [17, 18, 37], probability theory [29, 35, 56, 57], and machine learning [21, 70], deserve a separate development, which we will explore in future work.

A notable aspect of this work is the flexibility we provide to generate a syntactic category from a monoidal theory, where the different inference rules depend on which quantale operations we pick. This is due to how monoidal terms are formed differently from cartesian terms, and is motivated by the examples we developed. For instance, in Lib, sequential and parallel composition are nonexpansive with respect to the sum and max metric respectively, so the rules $\textsc{Seq}_{\oplus}$ and $\textsc{Par}_{\sqcap}$ are used to generate $\mathcal{S}_{\mathsf{Lib}}$. To encompass more examples, one could devise other rules corresponding to enrichment over other monoidal products. One may also consider a logic whose judgments are inference rules (or implications) rather than quantitative equations, so that $\textsc{Seq}_{\oplus}$ and $\textsc{Par}_{\sqcap}$ become part of the theory Lib.

Other questions concern the relation between cartesian and monoidal theories. Can Theorem 49 be obtained more abstractly via distributive laws like the non-quantitative result in [12]? Also, in [49] and subsequent works, distances between complex terms depend on the distances between variables used in those terms. For example, the construction of $\mathsf{FStoch}_{\mathsf{tv}}$ in

Remark 38 relies on the Kantorovich lifting of the distribution monad relative to the inclusion **FinSet** $\hookrightarrow$ **Met**. If we use **FinMet** $\hookrightarrow$ **Met** instead, the distance between distributions depends on the finite metric space considered. Axiomatising this category would require the diagrammatic syntax to incorporate some quantitative information on the inputs and outputs: it is an open question how to represent it in monoidal algebra.

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