


Subcoloring of (Unit) Disk Graphs

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Abstract

A subcoloring of a graph is a partition of its vertex set into subsets (called colors), each inducing a disjoint union of cliques. It is a natural generalization of the classical proper coloring, in which each color must instead induce an independent set. Similarly to proper coloring, we define the *subchromatic number* of a graph as the minimum integer k such that it admits a subcoloring with k colors, and the corresponding problem k -SUBCOLORING which asks whether a graph has subchromatic number at most k . In this paper, we initiate the study of the subcoloring of (unit) disk graphs. One motivation stems from the fact that disk graphs can be seen as a dense generalization of planar graphs where, intuitively, each vertex can be blown into a large clique—much like subcoloring generalizes proper coloring. Interestingly, it can be observed that every unit disk graph admits a subcoloring with at most 7 colors. We first prove that the subchromatic number can be 3-approximated in polynomial-time in unit disk graphs. We then present several hardness results for special cases of unit disk graphs which somehow prevents the use of classical approaches for improving this result. We show in particular that 2-SUBCOLORING remains NP-hard in triangle-free unit disk graphs, as well as in unit disk graphs representable within a strip of bounded height. We also solve an open question of Broersma, Fomin, Nešetřil, and Woeginger (2002) by proving that 3-SUBCOLORING remains NP-hard in co-comparability graphs (which contain unit disk graphs representable within a strip of height $\sqrt{3}/2$). Finally, we prove that every n -vertex disk graph admits a subcoloring with at most $O(\log^3(n))$ colors and present a $O(\log^2(n))$ -approximation algorithm for computing the subchromatic number of such graphs. This is achieved by defining a decomposition and a special type of co-comparability disk graph, called Δ -disk graphs, which might be of independent interest.

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1 Introduction

1.1 Definitions and Related Work

One of the most fundamental topics in algorithmics and graph theory is the coloring problem. A *proper coloring* of a graph is an assignment of colors to the vertices of the graph in such a way that no two adjacent vertices receive the same color, or equivalently, such that each color induces an independent set. The *chromatic number* of a graph G , denoted by $\chi(G)$, is the minimum number of colors of a proper coloring of G . In this paper we deal with a natural variant of this number. In 1989, Albertson et al. introduced the notion of *subcoloring* of a graph [1], which is a coloring where each color induces a disjoint union of complete graphs, or equivalently, a graph with no induced path on three vertices. The *subchromatic number* of a graph G , denoted by $\chi_s(G)$, is the minimum number of colors needed to obtain



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a subcoloring of the graph. By definition, any proper coloring is also a subcoloring, and thus $\chi_s(G) \leq \chi(G)$ holds for any graph G . However, the gap between these two parameters may be arbitrarily large, as in the case of complete graphs. The subchromatic number is also upper bounded by two other well-known parameters: the clique cover number $\text{cc}(G)$, which is the chromatic number of the complement graph \overline{G} ; and the co-chromatic number $\text{co}\chi(G)$ which is the minimum number of colors needed to color the graph such that each color class induces either a clique or an independent set.

The k -SUBCOLORING problem, which consists in deciding whether the subchromatic number of a given graph is at most k , received significant attention, both from structural and algorithmic aspects. Its complexity greatly differs from the one of the classical coloring problem. For example, k -SUBCOLORING is NP-hard already for $k = 2$. In addition, 2-SUBCOLORING remains NP-hard in various subclasses of planar graphs, namely triangle-free planar graphs with maximum degree 4 [7, 9], planar graphs with girth 5 [12], and planar comparability graphs with maximum degree 4 [13]. Notice also that any graph with maximum degree 3 is 2-subcolorable, following an argument of Erdős [1, 6]. In 2002, Broersma et al. [4] studied the k -SUBCOLORING problem in several subclasses of perfect graphs, in which the chromatic number is well understood. Among others, they showed that the problem is NP-complete in comparability graphs for $k = 2$, and that in the case of interval graphs, the problem can be solved in time $O(kn^{2k+1})$, where n is the number of vertices. Additionally, they proved that every chordal graph admits a solution with a logarithmic number of colors, which in particular implies that in interval graphs, when k is part of the input, the problem is solvable in quasi-polynomial time. Following this work, it was established that in chordal graphs, the problem is polynomial-time solvable for $k = 2$ [15], but becomes NP-hard for $k = 3$ [14]. Additionally, a 3-approximation algorithm was obtained for computing the subchromatic number of interval graphs [8]. Interestingly, two questions posed by Broersma et al. remain open:

- What is the complexity of k -SUBCOLORING in interval graphs when k is part of the input?
- What is the complexity of k -SUBCOLORING in co-comparability graphs?

In this paper, we initiate the study of the k -SUBCOLORING problem in disk graphs, that is, in intersection graphs of a set of disks drawn on the plane. A celebrated result of Koebe states that every planar graph is also a disk graph [11]. The converse is not true, as disks graph can contain large cliques. However, one might think that the presence of large cliques essentially captures the only difference between these two classes, as triangle-free disk graphs are indeed planar graphs [3]. This observation greatly motivates the study of the subchromatic number of disk graphs since, analogously, this parameter can be seen as a generalization of the chromatic number in which each vertex is blown into a clique. Despite its seemingly nice geometrical properties, disk graphs still possess a complex combinatorial structure, and the complexity of many fundamental problems is still poorly understood in this graph class. For instance, it is still open whether MAXIMUM CLIQUE is polynomial-time solvable in disk graphs [2]. As we will see, the subcoloring problem also suffers from this phenomenon. One of the reason is perhaps that disk graphs also capture interval graphs, containing themselves graphs of arbitrarily large subchromatic number [4]. A natural restriction of disk graphs is the class of *unit* disk graphs, in which all disks in the representation have diameter 1. Although unit disk graph might still contain large cliques, their simpler structure brings them a little closer to planar graphs. For instance, several algorithmic techniques usually designed for planar graphs (and minor-free classes), such as Baker's technique, can sometimes be applied to unit disk graphs. To do so, one generally has to partition the plane into *strips* of constant height, and then solve the considered problem optimally in each strip, by either

using dynamic programming techniques or other structural arguments. For instance, it can be shown that unit disk graphs representable within a strip of height $\sqrt{3}/2$ are co-comparability graphs (which are perfect graphs).

Beyond its theoretical significance, subcoloring of (unit) disk graphs has practical applications, particularly in channel allocation problems for Wi-Fi networks. These practical motivations are discussed further in the extended version of this paper.

1.2 Contribution and organization

The next section presents the technical preliminaries essential for the rest of the paper.

In Section 3, we start by examining the k -SUBCOLORING problem on unit disk graphs. We first establish that the subchromatic number of any unit disk graph is at most 7, which immediately leads to a simple $\frac{7}{2}$ -approximation algorithm. We then refine this result, providing a 3-approximation algorithm.

In Section 4, we demonstrate the challenges in further improving the approximation algorithm by proving several hardness results. Specifically:

► **Theorem 1.** *The k -SUBCOLORING problem is NP-complete on:*

1. *Triangle-free unit disk graphs for $k = 2$.*
2. *Unit disk graphs with bounded height for $k = 2$.*
3. *Co-comparability graphs for any $k \geq 3$.*

The third result resolves an open question posed by Broersma, Fomin, Nešetřil, and Woeginger [4].

In Section 5, we first give some arguments indicating that the methods used to achieve a 3-approximation for interval graphs cannot be directly generalized to disk graphs. To address this, we introduce a new subclass of disk graphs called Δ -disk graphs, which (i) generalize interval graphs, (ii) belong to the class of co-comparability graphs, and (iii) serve as building blocks for decomposing general disk graphs. The structured properties of Δ -disk graphs enable us to derive a logarithmic upper bound and a constant-factor approximation algorithm for their subchromatic number. These results lead to the two main contributions of this section:

► **Theorem 2.** *For any n -vertex disk graph G , the subchromatic number $\chi_s(G)$ satisfies $\chi_s(G) = O(\log^3 n)$.*

► **Theorem 3.** *There exists an $O(\log^2 n)$ -approximation algorithm for computing the subchromatic number of an n -vertex disk graph.*

We believe that Δ -disk graphs and the decomposition technique employed may be of independent interest.

Many proofs (marked with a *) have been omitted here for brevity and are available in the extended version of this paper.

2 Preliminaries

Graph notations. Let G be a simple graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G , respectively. When there is no ambiguity, we denote by n the number of vertices of G , and by m the number of edges of G . An *independent set* of G is a set of pairwise non-adjacent vertices, and we denote by $\alpha(G)$ the *independence number* of G , i.e., the size of a maximum independent set. Similarly, a *clique* in G is a set of

pairwise adjacent vertices, and we denote by $\omega(G)$ the size of a maximum clique. Given a set $R \subseteq V(G)$, we use $G[R]$ to denote the subgraph induced by R , and $G - R$ to denote the graph induced by $V(G) \setminus R$. For a vertex $v \in V(G)$, we denote by $N(v)$ the *open neighborhood* of v , that is, $N(v) = \{u \in V(G) \mid uv \in E(G)\}$, and by $N[v]$ its *closed neighborhood*, defined as $N[v] = N(v) \cup \{v\}$.

Given an integer $t \geq 1$, we denote by K_t the complete graph on t vertices. A graph G is said to be K_t -free if it does not contain K_t as an induced subgraph. In the special case of $t = 3$, we use the term *triangle-free graph* instead of K_3 -free graph. A graph G is said to be a *disjoint union of cliques* if there exists a partition of $V(G)$ into V_1, \dots, V_ℓ such that, for all $1 \leq i \leq \ell$, V_i is a clique, and there are no edge between vertices in different cliques. Equivalently, G is a disjoint union of cliques if it does not contain an induced path on three vertices (i.e., it is P_3 -free).

A graph is a *co-comparability* if it is the intersection graph of curves from a line to a parallel line. Equivalently, a co-comparability graph is a graph that connects pairs of elements that are not comparable to each other in a partial order.

A *hypergraph* $H = (V, \mathcal{E})$ consists of a set of vertices V and a set of hyperedges \mathcal{E} , where each hyperedge is a subset of V . If all hyperedges have exactly k vertices, the hypergraph is called k -uniform. For example, in a 3-uniform hypergraph, every edge connects exactly three vertices. Similarly to graphs, we denote by $V(H)$ the set of vertices of H and by $\mathcal{E}(H)$ its set of hyperedges.

Geometric notations. Given two points $A, B \in \mathbb{R}^2$, we denote by $d(A, B)$ their Euclidean distance. A disk of radius $r \in \mathbb{R}$ and center $A \in \mathbb{R}^2$ is the set $D(A, r) = \{B \in \mathbb{R}^2 \mid d(A, B) \leq r\}$. The diameter of such a disk is $2r$, and if a disk has diameter 1, it is called a *unit disk*.

Note that the intersection of two disks $D(A, r_A)$ and $D(B, r_B)$ is non-empty if and only if $d(A, B) \leq r_A + r_B$. A *(unit) disk graph* is the intersection graph of (unit) disks in the plane. The set of disks $\mathcal{D}_G = \{D_v = D((x_v, y_v), r_v) \mid v \in V(G)\}$ of a given disk graph G is called a *disk representation* of G . For a real number $h > 0$, a unit disk graph G is said to have *height* h if there exists a disk representation of G such that the center of every disk lies within the region $\mathbb{R} \times [0, h]$.

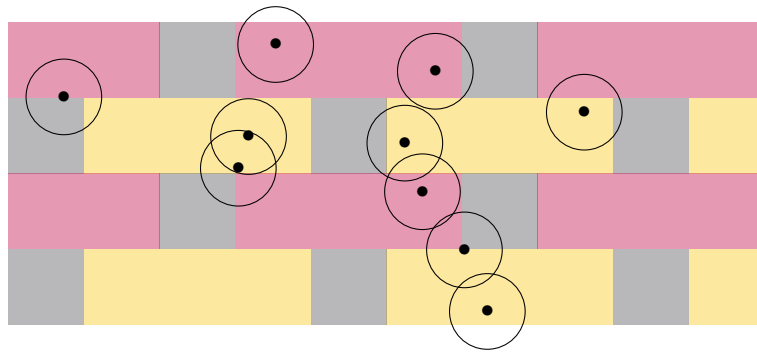
An *interval graph* is the intersection graph of intervals on the real line. The set of intervals $\mathcal{I}_G = \{(\ell_v, r_v)\}_{v \in V(G)}$ is called an *interval representation* of G .

Approximation scheme. A λ -approximation algorithm is an algorithm that runs in polynomial time with respect to the instance size and returns a solution that is guaranteed, in the worst case, to be within a factor of λ from the optimal solution.

3 First results on unit disk graphs

As a warm-up, let us first consider the case of unit disk graphs.

A simple yet important result is that the subchromatic number of a unit disk graph is bounded by a constant. To provide some intuition for this result, imagine partitioning the plane into a grid, where each cell has a diagonal of length 1. All vertices within a cell form a clique, while the cells can be colored using a constant number of colors, ensuring that no two cells within a distance less than 1 share the same color. To optimize this constant, we consider a hexagonal tiling of the plane and the subcoloring induced by this tiling, usually known as *Isbell's coloring*. Notice that in order to turn the following property into a polynomial-time algorithm, we need the unit disk embedding of the input graph.



■ **Figure 1** The 3-partition of the plane to obtain the 3-approximation algorithm. The yellow represents R_0 , the purple R_1 and the gray R_2 .

► **Observation 4 (*)**. *Every unit disk graphs G satisfies $\chi_s(G) \leq 7$.*

An algorithmic consequence of Theorem 4 is the existence of a straightforward polynomial-time $7/2$ -approximation algorithm for computing the subchromatic number, since deciding whether a graph is a disjoint union of cliques (hence deciding if the subchromatic number is 1) can be done in linear time.

► **Corollary 5**. *There exists a $7/2$ -approximation for the subchromatic number on unit disk graphs, running in linear time, given the unit disk representation.*

Here we improve the approximation ratio by presenting a 3-approximation algorithm. A key step involves partitioning the unit disk graph so that each part induces a disjoint union of graphs with a maximum independent set of bounded size (see Figure 1). Then, we conclude by using the fact that in general graphs, 2-SUBCOLORING can be solved in FPT time when parameterized by the total number of connected components of each color [10]. Naturally, this parameter is upper bounded by the independence number of the input graph.

► **Theorem 6 (*)**. *There exists a 3-approximation for the subchromatic number in unit disk graphs running in time $O(n \cdot m)$, given the unit disk representation.*

4 Hardness results

In this section we obtain several negative results concerning the k -SUBCOLORING problem in restricted classes of geometric intersection graphs. As mentioned in the introduction, 2-SUBCOLORING is already known to be NP-complete in various subclasses of planar graphs, such as triangle-free planar graphs. By a celebrated theorem of Koebe, it implies that 2-SUBCOLORING remains NP-complete in triangle-free disk graphs. This result can be extended to k -SUBCOLORING in disk graphs for every $k \geq 3$ by inductively doing the following reduction: take two disjoint copies of a disk graph and add a new disk adjacent to both copies. Observe then that both the subchromatic number and the maximum clique number increase by exactly one. We now first focus on (subclasses of) unit disk graphs, and then on co-comparability graphs.

Unit disk graphs. Here we prove the first and second cases of Theorem 1. Interestingly, they highlight two differences between proper coloring and subcoloring. First, recall that any triangle-free unit disk graph is planar, and thus admits a proper 3-coloring. Second,

observe that k -COLORING can be solved in linear time for every $k \geq 3$ in unit disk graphs of bounded height by first deciding whether the input graph has a clique of size k (in which case the answer is no), and then solving the problem using a dynamic programming over a path decomposition (whose width is thus $O(k)$). We show that these two tractable cases for 3-COLORING remain hard for 2-SUBCOLORING.

► **Theorem 7 (*)**. *2-SUBCOLORING is NP-complete on triangle-free unit disk graphs and in unit disk graphs of bounded height.*

Notice that this NP-hardness cannot be generalized to k -SUBCOLORING, as we saw in the previous section that the subchromatic number of any unit disk graph is at most 7. A direct consequence of the previous result is that k -SUBCOLORING cannot be approximated in unit disk graphs within a ratio of $3/2 - \varepsilon$ for any $\varepsilon > 0$ in polynomial-time, unless $P=NP$.

It remains open whether 2-SUBCOLORING is polynomial-time solvable for unit disk graphs of height at most 1. If so, this would imply a 2-approximation algorithm for SUBCOLORING on unit disk graphs.

k -Subcoloring of co-comparability graphs. Using similar ideas as in the proof of the previous result, we can tackle the case of co-comparability graphs, thus proving the remaining case of Theorem 1. Recall that these graphs contain unit disk graphs of height $\sqrt{3}/2$, and can be defined as the intersection graph of curves between two parallel lines. This result solves an open problem of Broersma, Fomin, Nešetřil, and Woeginger [4]. The proof is a reduction from NO RAINBOW HYPERGRAPH 3-COLORING, which has been shown to be NP-complete [5].

NO RAINBOW HYPERGRAPH 3-COLORING :

Input: A 3-uniform hypergraph $H = (V, \mathcal{E})$ with vertex set V and edge set \mathcal{E} and 3 sets $A_1, A_2, A_3 \subseteq V$.

Question: Is there a 3-coloring $\mu : V \rightarrow \{1, 2, 3\}$ such that no hyperedge $e \in \mathcal{E}$ is “rainbow” (i.e., no hyperedge contains all 3 colors) and such that for any $k \in \{1, 2, 3\}$ and $v \in A_k$, $\mu(v) = k$?

In [16], it was proven that the same problem without fixing some vertices with some color but just requiring that the 3-coloring is surjective, is also NP-complete. We start with a straightforward lemma.

► **Lemma 8**. *In any 3-subcoloring of $K_{4,4,4}$, two vertices of two different parts have different colors, or equivalently, the three parts are all monochromatic with three different colors.*

Proof. Let A , B , and C be the three distinct parts of a $K_{4,4,4}$ graph, and let μ be a 3-subcoloring of this graph. Suppose, for contradiction, that there exist two vertices in different parts with the same color. *W.l.o.g.*, assume $a \in A$ and $b \in B$ are both colored 3. Then, no other vertex $a' \in A \setminus \{a\}$ can have color 3; otherwise, the path aba' would form a monochromatic path of three vertices. Similarly, no other vertex $b' \in B \setminus \{b\}$ can have color 3.

Now, consider part C . If two different vertices $c, c' \in C$ are also colored 3, then the path cac' would also form a monochromatic path of three vertices. Consequently, after removing all vertices colored 3 from the graph, each part still contains at least three vertices. This configuration yields a 2-subcoloring of $K_{3,3,3}$, which cannot exist. ◀

► **Theorem 9**. *3-SUBCOLORING is NP-complete on co-comparability graphs.*

Proof. We reduce from NO RAINBOW HYPERGRAPH 3-COLORING. Let H be a 3-uniform hypergraph with n vertices v_1, \dots, v_n and m hyperedges. Let $A_1, A_2, A_3 \subseteq V(H)$. We construct a co-comparability graph G as follows:

1. For each $j = 1, \dots, m$, create two copies of K_{n+6} , denoted $C_j = \{c_{j,1}, c_{j,2}, \dots, c_{j,n+6}\}$ and $L_j = \{l_{j,1}, l_{j,2}, \dots, l_{j,n+6}\}$.
2. For each $j = 1, \dots, m-1$, add edges to form a matching between C_j and L_j , and similarly between L_j and C_{j+1} . Specifically, add edges $\{(c_{j,i}, l_{j,i}) \mid 1 \leq i \leq n+6\}$ and $\{(l_{j,i}, c_{j+1,i}) \mid 1 \leq i \leq n+6\}$.
3. For each $j = 1, \dots, m$, add a set of n vertices $\{l'_{j,i} \mid 1 \leq i \leq n+6\}$ to L_j , where each $l'_{j,i}$ is a false twin of $l_{j,i} \in L_j$. More precisely, each $l'_{j,i}$ is adjacent to:
 - all vertices in L_j except $l_{j,i}$,
 - the vertices $c_{j,i} \in C_j$ and, if $j < m$, $c_{j+1,i} \in C_{j+1}$.
 Now, L_j denote the set of vertices $\bigcup_{1 \leq i \leq n+6} \{l_{j,i}, l'_{j,i}\}$.
4. For each hyperedge $e_j \in \mathcal{E}(H)$, introduce a new vertex s_j (the “selector” vertex).
 - Connect s_j to all vertices in L_j , i.e., add edges $\{(s_j, u) \mid u \in L_j\}$.
 - Additionally, connect s_j to the vertices $\{c_{j,i} \mid v_i \in e_j\}$ in C_j that correspond to the vertices in the hyperedge e_j .
5. Add a complete multipartite graph $K_{4,4,4}$ with parts B_1, B_2 , and B_3 . For each $k \in \{1, 2, 3\}$ and $v_i \in A_k$, add edges from $c_{1,i}$ to each vertex in B_ℓ for $\ell \in \{1, 2, 3\} \setminus \{k\}$. Additionally, add edges from $\{c_{1,n+1}, c_{1,n+2}\}$ to $B_2 \cup B_3$, from $\{c_{1,n+3}, c_{1,n+4}\}$ to $B_1 \cup B_3$, and from $\{c_{1,n+5}, c_{1,n+6}\}$ to $B_1 \cup B_2$.

Observe that the size of G is polynomial in the size of H , and we also confirm that G is a co-comparability graph.

▷ **Claim 10.** The graph G constructed above is a co-comparability graph.

Proof. In Figure 2, we illustrate a string representation for a simple example of G . Below, we provide a formal proof of the desired property. The class of co-comparability graphs is closed under the addition of false twins. Indeed, in any string representation of such a graph, a string can be replaced by two non-crossing strings following the original one, resulting in the creation of false twins. Therefore, to establish that G is a co-comparability graph, it suffices to show that the graph obtained by contracting each pair of false twins in G is a co-comparability graph.

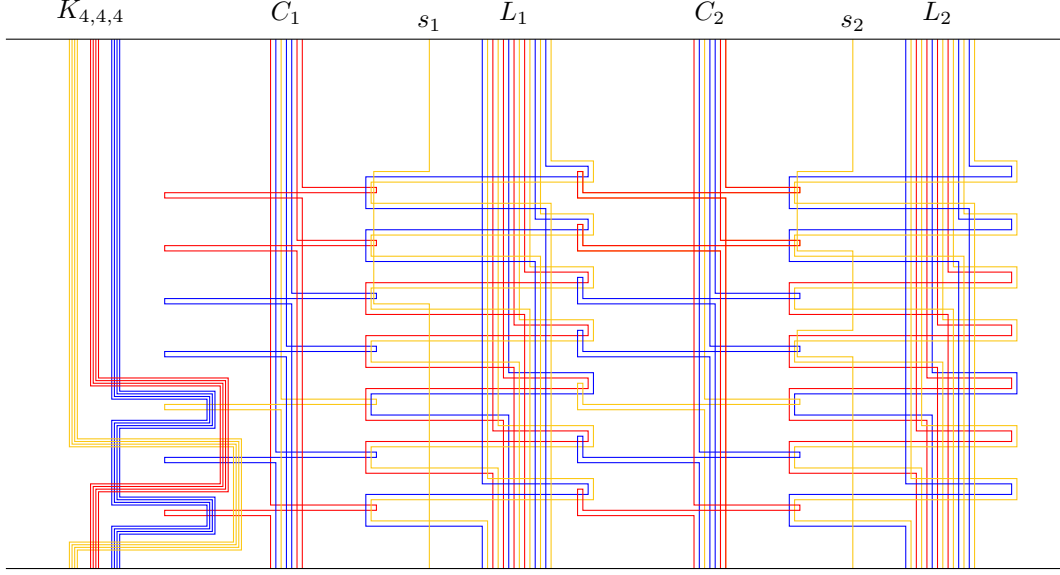
Let G' be the graph obtained by contracting each pair of false twins $\{l_{j,i}, l'_{j,i}\}$ into a single vertex $t_{j,i}$ for all $1 \leq j \leq m$ and $1 \leq i \leq n+6$, and for each $k \in \{1, 2, 3\}$, contracting B_k into a single vertex b_k . Define the $2m+1$ cliques of G' as D_0, \dots, D_{2m} , where $D_0 = \{b_1, b_2, b_3\}$, $D_{2j-1} = C_j$, and $D_{2j} = \{t_{j,i} \mid i = 1, \dots, n+6\} \cup \{s_j\}$ for $1 \leq j \leq m$. Observe that every edge of G' is either within one of these cliques or between two successive cliques.

Now, consider the following order:

$$b_1 \prec b_2 \prec b_3 \prec c_{1,1} \prec \dots \prec c_{1,n+6} \prec s_1 \prec t_{1,1} \prec \dots \prec t_{1,n+6} \prec \dots \\ \prec c_{m,1} \prec \dots \prec c_{m,n+6} \prec s_m \prec t_{m,1} \prec \dots \prec t_{m,n+6}.$$

We will show that this ordering respects the non-edges of G' . Suppose, for some $u, v, w \in V(G')$, that $u \prec v$, $v \prec w$, $uv \notin E(G')$, and $vw \notin E(G')$. Assume, for contradiction, that $uw \in E(G')$. Then, u and w must either belong to the same clique or to two consecutive cliques. If $u, w \in D_j$ for some j , then by the construction of the order, $v \in D_j$, implying $uv \in E(G')$, a contradiction. If $u \in D_j$ and $w \in D_{j+1}$ for some j , then, by the ordering, v

must be in either D_j or D_{j+1} . In the first case, $uv \in E(G')$, and in the second, $vw \in E(G')$, both contradicting the assumption. This confirms that G' respects the co-comparability ordering. \triangleleft



■ **Figure 2** A string representation of G and its 3-subcoloring, when the instance of NO-RAINBOW 3-COLORING is composed of an hypergraph H with 7 vertices $\{v_1, \dots, v_7\}$ and two hyperedges $\{v_1, v_2, v_3\}$ and $\{v_1, v_2, v_4\}$, and $A_1 = \{v_5\}$, $A_2 = \{v_6\}$, $A_3 = \{v_7\}$. For readability, the 6 vertices additional vertices $c_{j,n+1}, \dots, c_{j,n+6}$ are not represented at each layer.

Assume that G has a 3-subcoloring μ . By Lemma 8, the vertex sets B_1 , B_2 , and B_3 must each be monochromatic with three distinct colors. *W.l.o.g.*, assume that for each $k \in \{1, 2, 3\}$ and every $v \in B_k$, we have $\mu(v) = k$. Now, for each $k \in \{1, 2, 3\}$, take $v_i \in A_k$. By construction, the vertex $c_{1,i}$ is adjacent to every vertex in B_ℓ for $\ell \in \{1, 2, 3\} \setminus \{k\}$. Therefore, $\mu(c_{1,i}) = k$, as any other color assignment would form a monochromatic P_3 involving $c_{1,i}$ and vertices from B_ℓ . Applying a similar argument, we find that $\mu(c_{1,n+1}) = \mu(c_{1,n+2}) = 1$, $\mu(c_{1,n+3}) = \mu(c_{1,n+4}) = 2$, and $\mu(c_{1,n+5}) = \mu(c_{1,n+6}) = 3$. In the following claim, we detail how these colors propagate throughout the graph G .

▷ **Claim 11.** For any $1 \leq j \leq m$ and $1 \leq i \leq n$, $\mu(c_{j,i}) = \mu(c_{1,i})$.

Proof. We prove the result by induction on j . The base case holds for $j = 1$.

Assume, for contradiction, that $\mu(c_{j,i}) = \mu(l_{j,i}) = k$, and *w.l.o.g.*, let $k = 1$. This implies that no other vertex in C_j or L_j can have color 1, which leads to a contradiction because, by the induction hypothesis, C_j must contain at least two vertices colored 1, namely $c_{j,n+1}$ and $c_{j,n+2}$. Thus, by symmetry, we conclude that $\mu(c_{j,i}) \neq \mu(l_{j,i})$ and $\mu(c_{j,i}) \neq \mu(l'_{j,i})$.

Now, assume for contradiction that $\mu(l_{j,i}) = \mu(l'_{j,i}) = k'$, and *w.l.o.g.*, let $k' = 2$ (with $k = 1$ as before). This assumption implies that all vertices in $L_j \setminus \{l_{j,i}, l'_{j,i}\}$ cannot have color 2. Consider indices $i_1 \in \{n+1, n+2\} \setminus \{i\}$ and $i_3 \in \{n+5, n+6\} \setminus \{i\}$. By the induction hypothesis, $\mu(c_{j,i_\ell}) = \ell$ for $\ell \in \{1, 3\}$. For both $\ell \in \{1, 3\}$, the vertices l_{j,i_ℓ} and l'_{j,i_ℓ} cannot be colored 2 or ℓ , as that would create a monochromatic P_3 . Consequently, we must have $\mu(l_{j,i_1}) = \mu(l'_{j,i_1}) = 3$ and $\mu(l_{j,i_3}) = \mu(l'_{j,i_3}) = 1$.

Finally, assigning any color to a remaining vertex in $L_j \setminus \{l_{j,i}, l'_{j,i}, l_{j,i_1}, l'_{j,i_1}, l_{j,i_3}, l'_{j,i_3}\}$ would again create a monochromatic P_3 , which contradicts the subcoloring property of μ . Therefore, we conclude that $\mu(c_{j,i}) \neq \mu(l_{j,i}) \neq \mu(l'_{j,i})$.

By symmetry, we also obtain that $\mu(c_{j+1,i}) \neq \mu(l_{j,i})$ and $\mu(c_{j+1,i}) \neq \mu(l'_{j,i})$, which implies $\mu(c_{j,i}) = \mu(c_{j+1,i}) = k$.

Notice that the same proof shows that for any $1 \leq j \leq m$, $\mu(c_{j,i}) \neq \mu(l_{j,i}) \neq \mu(l'_{j,i})$. \triangleleft

Then, we prove that the coloring of the selector vertices ensures that their neighborhood is not a rainbow.

\triangleright **Claim 12.** For any $1 \leq j \leq m$, $\mu(N(s_j) \cap C_j) \neq \{1, 2, 3\}$, i.e., $N(s_j) \cap C_j$ is not a rainbow.

Proof. Suppose, for contradiction, that $\mu(N[s_j] \cap C_j) = \{1, 2, 3\}$. Then, there exists a vertex $v \in N(s_j) \cap C_j$ such that $\mu(s_j) = \mu(v) = k$. For $k = 1$, note that $v \neq c_{j,n+1}$ because $N(s_j) \subseteq C_j \setminus \{c_{j,n+1}, \dots, c_{j,n+6}\}$. Then, the path $s_j v c_{j,n+1}$ induces a monochromatic P_3 of color 1, which is a contradiction. A similar argument applies for $k = 2, 3$. \triangleleft

Let $\nu : V(H) \rightarrow \{1, 2, 3\}$ be defined by $\nu(v_i) = \mu(c_{1,i})$ for $1 \leq i \leq n$, and we show that ν is a no-rainbow 3-coloring of H . First, for any $k \in \{1, 2, 3\}$ and any $v_i \in A_k$, we have $\mu(c_{1,i}) = k$, and thus $\nu(v_i) = k$. Next, let $e_j = \{v_{i_1}, v_{i_2}, v_{i_3}\}$ be a hyperedge of H . By Claim 12, $\mu(N(s_j) \cap C_j) \neq \{1, 2, 3\}$. By construction, $N(s_j) \cap C_j = \{c_{j,i_1}, c_{j,i_2}, c_{j,i_3}\}$, so $\mu(\{c_{j,i_1}, c_{j,i_2}, c_{j,i_3}\}) \neq \{1, 2, 3\}$.

Using Claim 11 and the definition of ν , we have

$$\mu(\{c_{j,i_1}, c_{j,i_2}, c_{j,i_3}\}) = \mu(\{c_{1,i_1}, c_{1,i_2}, c_{1,i_3}\}) = \nu(\{v_{i_1}, v_{i_2}, v_{i_3}\}).$$

Thus, $\nu(e_j) \neq \{1, 2, 3\}$, proving that ν is a no-rainbow 3-coloring of H such that for any $k \in \{1, 2, 3\}$ and $v \in A_k$, $\nu(v) = k$.

Conversely, assume that H has a no-rainbow 3-coloring ν such that for any $k \in \{1, 2, 3\}$ and $v \in A_k$, $\nu(v) = k$. We construct a 3-subcoloring μ of G as follows:

- For each $k \in \{1, 2, 3\}$ and any vertex $b \in B_k$, set $\mu(b) = k$.
- For any $1 \leq j \leq m$ and $1 \leq i \leq n$, set $\mu(c_{j,i}) = \nu(v_i)$. Let k, k' (with $k < k'$) be the two colors different from $\nu(v_i)$; then set $\mu(l_{j,i}) = k$ and $\mu(l'_{j,i}) = k'$.
- For any $1 \leq j \leq m$, set $\mu(c_{j,n+1}) = \mu(c_{j,n+2}) = 1$, $\mu(c_{j,n+3}) = \mu(c_{j,n+4}) = 2$, and $\mu(c_{j,n+5}) = \mu(c_{j,n+6}) = 3$. For each $i \in \{n+1, \dots, n+6\}$, let k, k' (with $k < k'$) be the two colors different from $\mu(c_{j,i})$; then set $\mu(l_{j,i}) = k$ and $\mu(l'_{j,i}) = k'$.
- For each $1 \leq j \leq n$, since ν is a no-rainbow coloring of H , there exists a color $k \in \{1, 2, 3\} \setminus \nu(e_j)$. Set $\mu(s_j) = k$.

We prove that μ is indeed a 3-subcoloring of G . To establish this, we need to show that no vertex in G is the center of a monochromatic P_3 .

First, no vertex in $B_1 \cup B_2 \cup B_3$ has a neighbor of the same color. Assume, for contradiction, that there exists a vertex $b \in B_k$ (for some $k \in \{1, 2, 3\}$) that has a neighbor with the same color. This neighbor must belong to C_1 . However, if any vertex $v_i \in C_1$ is connected to $B_1 \cup B_2 \cup B_3$ and has color k , it cannot be connected to B_k .

Then, for any $1 \leq j \leq m$, notice that any vertex $c_{j,i} \in C_j$ does not have neighbors of the same color outside C_j . The only vertices it neighbors outside C_j are in the sets $\{l_{j-1,i}, l'_{j-1,i}, l_{j,i}, l'_{j,i}, s_j\}$ if $j > 1$ and $\{l_{j,i}, l'_{j,i}, s_j\}$ if $j = 1$. By definition of μ , all these vertices have colors different from $\mu(c_{j,i})$. Additionally, since C_j induces a clique, $c_{j,i}$ cannot be the center of a monochromatic P_3 .

Similarly, for any $l_{i,j} \in L_j$, all its neighbors outside L_j have different colors, except for s_j which can have the same color. Thus, if $l_{i,j}$ is the center of a monochromatic P_3 represented as $ul_{i,j}v$, then $\{u, v\} \subseteq L_j \cup \{s_j\}$. Since u and v are not adjacent, it follows that there exists $1 \leq i' \leq n+6$ such that $\{u, v\} = \{l_{j,i'}, l'_{j,i'}\}$; otherwise, there would be an edge between u and v . However, this leads to the conclusion that $\mu(l_{j,i'}) = \mu(l'_{j,i'})$, which contradicts the definition of μ .

In conclusion, we have shown that μ is indeed a 3-subcoloring of G , completing the proof. \blacktriangleleft

The result can be easily generalized to k -SUBCOLORING for $k \geq 3$ by induction. Indeed, if G is a co-comparability graph, then the graph G' obtained by taking two disjoint copies of G and adding a universal vertex is also a co-comparability graph such that $\chi_s(G') = \chi_s(G) + 1$. Thus, the following holds.

► **Theorem 13.** *k -SUBCOLORING is NP-complete on co-comparability graphs for any $k \geq 3$.*

The only unresolved case from the original question by Broersma et al. concerns the 2-SUBCOLORING problem on co-comparability graphs. It seems plausible that this problem could be solved in polynomial time, given that k -SUBCOLORING is already known to be polynomially solvable for $k = 2$ on chordal graphs [15] but NP-complete for $k \geq 3$. Moreover, co-comparability graphs exhibit a certain lattice structure on their maximal cliques, raising the question of whether the algorithm for chordal graphs can be adapted to handle co-comparability graphs.

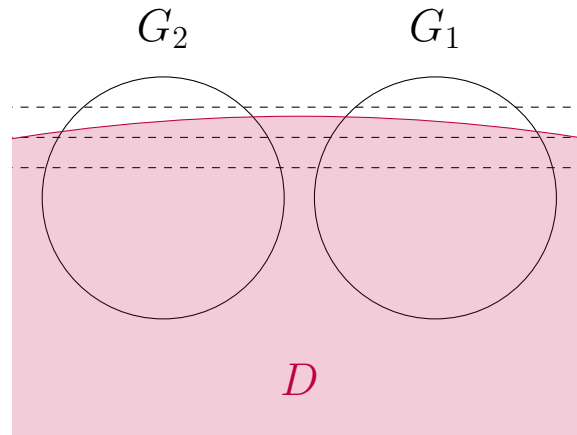
5 The case of disk graphs

Disk graphs are a natural generalization of interval graphs to two dimensions, making it appealing to adapt methods originally developed for interval graphs. As mentioned in the introduction, the best known factor [8] for polynomial-time approximation in interval graphs is 3, and the authors asked whether a similar approach could be used for disk graphs. We first give evidence that this cannot be the case.

To do so, let us give a high-level description of their algorithm. For $k \geq 2$, let $BC(k)$ be the graph constructed inductively by adding a universal vertex to two disjoint copies of $BC(k-1)$, with $BC(1)$ being a single vertex. A straightforward proof shows that $\chi_s(BC(k)) \geq k$ for any $k \geq 1$, and observe that $BC(k)$ are indeed interval graphs, for $k \geq 1$. Then, an interval is said *internal* if it properly contains two independent intervals, and *external* otherwise. Given a graph G , let V_1, \dots, V_k be the partition of the vertex set $V(G)$ obtained iteratively, for $i = 1, \dots, k$, by defining V_i as the set of external vertices of the graph induced by $V(G) \setminus \cup_{j < i} V_j$. Observe that such a partition implies the existence of an induced subgraph isomorphic to $BC(k)$. Additionally, it can be shown that each $G[V_i]$ has a 3-subcoloring for all $1 \leq i \leq k$, leading to a 3-approximation.

A key ingredient of this proof relies on the fact that an interval graph where no interval is internal can be subcolored with a constant number of colors. We show that this no longer holds in disk graphs. We even prove the stronger result that for any $k \geq 1$, $BC(k)$ is a *proper disk graph*, that is an intersection graph of disks such that no disk properly contains another one.

► **Theorem 14 (*)**. *For any $k \geq 1$, the graph $BC(k)$ is a proper disk graph.*



■ **Figure 3** The proper disk representation of $BC(k)$. The dashed lines, from bottom to top, are L_1 , L_2 and L_3 respectively. G_1 and G_2 are two proper disk representations of $BC(k-1)$, where each disk intersects both L_1 and L_3 .

In Section 3, we showed that any unit disk graph can be subcolored with a constant number of colors. The same result cannot be obtained for disk graphs, as there exist disk graphs (and even interval graphs) with subchromatic number logarithmic in the number of vertices [4]. We now prove Theorems 2 and 3, which provide an almost tight upper bound and an approximation algorithm, respectively, in the case of disk graphs.

Our proof relies on a three-steps decomposition, where at each time we find a balanced separator which is itself decomposed, until the pieces we obtain are simple enough to be able to color them with a constant number of colors. In particular, the last step consists of a subclass of disk graphs dubbed Δ -disk graphs which turns out to be included in co-comparability graphs. From an algorithmic perspective, we propose a polynomial-time $O(1)$ -approximation algorithm for the subchromatic number of those Δ -disk graphs.

5.1 Decomposition of disk graphs and Δ -disk graphs

In the following, we say that a subset of vertices $S \subseteq V(G)$ is a *balanced separator* if $V(G) \setminus S$ can be partitioned into two sets A and B containing at most $\lceil |V(G)|/2 \rceil$ vertices each, and such that there is no edge between A and B . In addition, we use the notation $\Delta = \mathbb{R}_+^2$ for the first quadrant of the plane. We now define three subclasses of disk graphs.

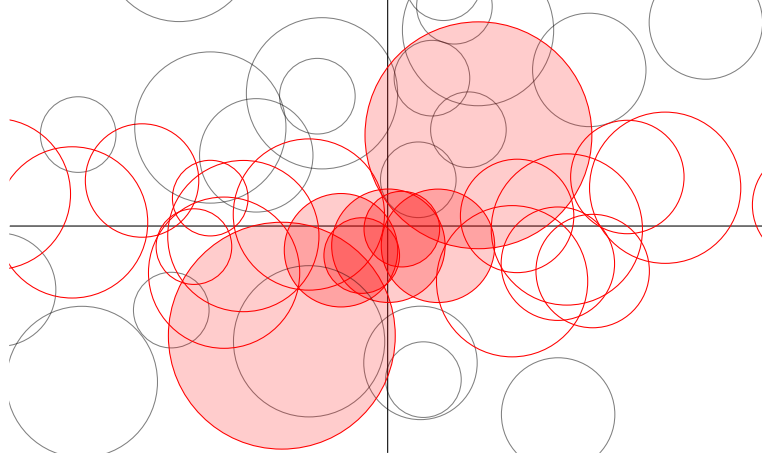
► **Definition 15.** A disk graph is called a *linear disk graph* if there exists a disk representation of G such that a line crosses all disks of G .

A disk graph G is called a Δ -disk graph if there exists a disk representation where each disk intersects both positive axes but does not contain the point $(0,0)$, or more formally such that for any $v \in V(G)$, we have $x_v > 0$, $y_v > 0$ and $\max(x_v, y_v) \leq r_v < \sqrt{x_v^2 + y_v^2}$.

For the remainder of this section, we assume that the representation of any Δ -disk graph satisfies the properties given in the above definition. Additionally, for a given Δ -disk graph G with its associated disk representation and a vertex $v \in V(G)$, we denote by $d_v = \sqrt{x_v^2 + y_v^2}$ the distance of the point (x_v, y_v) from the origin.

► **Theorem 16 (Decomposition of disk graph).** Let G be a disk graph.

1. Any disk graph contains a balanced separator which induces a linear disk graph.
2. Any linear disk graph contains a balanced separator S which can be partitioned into $(V_i)_{1 \leq i \leq 5}$ such that $G[V_i]$ is a Δ -disk graph for $1 \leq i \leq 4$ and $G[V_5]$ is a clique.



■ **Figure 4** A disk graph and its decomposition. The circles with red lines form a line disk graph, which is a separator for the whole graph. In addition, the vertices filled in red can be partitioned into four Δ -disk graphs and one clique.

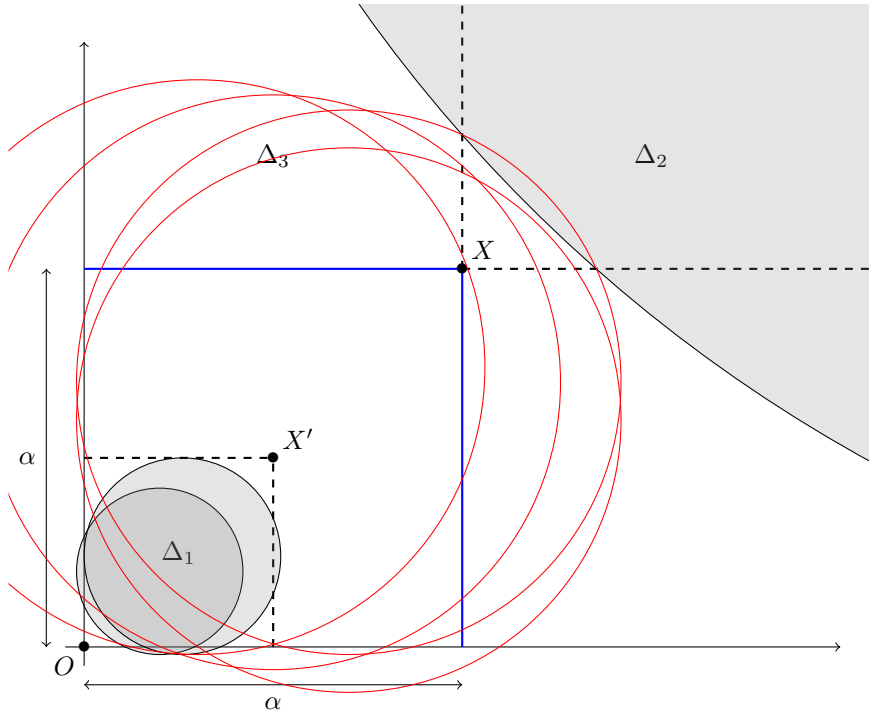
Proof. Proof of 1. Consider a representation of the disk graph G , and let y be the median of all the ordinates of the disks. Consider the line L parallel to the abscissa axis crossing the point $(0, y)$, and let S be all disks that cross L . Observe by construction that all disks of $V \setminus S$ with ordinate less than y do not intersect those with ordinate greater than y , and both sets represent at most $\lceil n/2 \rceil$ disks.

Proof of 2. If G is a linear disk graph, w.l.o.g. we consider that this line is exactly the abscissa axis. Let x be the median of all abscissa of the vertices, and consider the line L' perpendicular to L crossing the point $(x, 0)$. By similar arguments as previously, observe that the set of disks crossing L' is a balanced separator which induces a disk graph where each disk intersect both axis. Let V_5 be the set of vertices such that the corresponding disks intersect the intersection of both lines, and let V_1, V_2, V_3, V_4 be the partition of the remaining vertices corresponding to the division of the plane into four subspaces formed by the two perpendicular lines. By definition, $G[V_i]$ is a Δ -disk graph for $1 \leq i \leq 4$, and since each disk of V_5 intersects the same point, $G[V_5]$ is a clique. ◀

Relation between Δ -disk graphs and other classes. Due to the constraints on their structure, Δ -disk graphs are far less general than classical disk graphs. In particular, they enjoy many structural properties that we can formalize using other known graph classes. For instance, we can show that they strictly generalize interval graphs, and that they are co-comparability graphs.

► **Theorem 17 (*).** *The set of interval graphs is strictly contained in Δ -disk graphs, and Δ -disk graphs are strictly contained in co-comparability graphs. More precisely, any Δ -disk graph G is a co-comparability graph with respect to the following co-comparability order: for any two vertices $u, v \in V(G)$, define $u \prec v$ if and only if $x_u < x_v$, $y_u < y_v$, and $D_u \cap D_v = \emptyset$.*

Properties of Δ -disk graphs. We first establish some basic properties of Δ -disk graphs that will be useful in our approximation algorithm. The following lemma, which will be applied frequently (often implicitly), states that if two disks in a Δ -disk graph intersect, then their intersection must occur within the region $\Delta = \mathbb{R}_+^2$. In other words, there cannot be a “hidden” intersection point located outside Δ , e.g. at a point with a negative coordinate. This result follows directly from the geometry of the disks.



■ **Figure 5** A decomposition of a Δ -disk graph. All the red disks have centers in Δ_3 and thus intersect X' . They gray disks are those with center in Δ_1 (resp. Δ_2) and that do not intersect X' (resp. X). The blue lines represent the separator of Claim 23.

► **Lemma 18 (*)**. Let G be a Δ -disk graph and let $uv \in E(G)$. Then, the disks D_u and D_v intersect within Δ ; that is, the intersection $D_u \cap D_v \cap \Delta$ is non-empty.

► **Lemma 19 (*)**. Let G be a Δ -disk graph, and let $u, v \in V(G)$ such that $uv \notin E(G)$ and $d_u < d_v$. Then, $\min(x_v, y_v) \geq \max(x_u, y_u) + r_u$.

► **Lemma 20 (*)**. Let G be a Δ -disk graph and $v \in V(G)$. If, for some $t \leq \min(x_v, y_v)$, $D_v \cap [0; t]^2$ is not empty, then D_v intersects the point (t, t) .

5.2 Polylogarithmic upper bound

Here we prove that Δ -disk graphs have logarithmic subchromatic number.

► **Lemma 21**. Any Δ -disk graph contains a balanced separator which can be partitioned into two cliques.

Proof. Let G be a Δ -disk graph, and let $\mathcal{D}_G = \{D_v = D((x_v, y_v), r_v) \mid v \in V(G)\}$ be a disk representation of G . Let α be the median of the abscissa of all the centers in the representation \mathcal{D}_G , and let $\Delta = \{(x, y) \mid x > 0 \wedge y > 0\}$ be the set of points with positive coordinates. Consider a partition of Δ using the following sets:

- $\Delta_1 = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < \alpha/2 \wedge 0 < y < \alpha/2\}$
- $\Delta_2 = \{(x, y) \in \mathbb{R}^2 \mid \alpha < x \wedge \alpha < y\}$
- $\Delta_3 = \Delta \setminus (\Delta_1 \cup \Delta_2)$

We can notice that $\Delta = \bigsqcup_{i=1}^3 \Delta_i$. The partition is depicted in Figure 5. In addition, let $X = (\alpha, \alpha)$ and $X' = (\alpha/2, \alpha/2)$.

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▷ **Claim 22.** Any disk of \mathcal{D}_G with center in Δ_3 contains X' .

Proof. Let $v \in V(G)$ such that $C_v = (x_v, y_v) \in \Delta_3$ and $x_v \leq \alpha$. Consider the point A of coordinates $(x_v, \alpha/2)$. We have $d(C_v, X') \leq d(C_v, A) + d(A, X') \leq y_v - \alpha/2 + \alpha/2 = y_v$. Since $r_v \geq y_v$, D_v intersects X' . By symmetry, all the remaining disks with center in Δ_3 intersect X' . ◀

▷ **Claim 23.** Let $v \in V(G)$ such that $C_v = (x_v, y_v) \in \Delta_1$ and $X' \notin D_v$, and let $u \in V(G)$ such that $C_u = (x_u, y_u) \in \Delta_2$ and $X \notin D_u$. Then D_v and D_u do not intersect.

Proof. Let A be a point of coordinates (α, y_A) with $y_A \leq \alpha$ (A is a point below X on the blue line in Figure 5). We show that A intersects neither D_v nor D_u . First, we have

$$\begin{aligned} d(C_u, A)^2 &= (x_u - \alpha)^2 + (y_u - y_A)^2 \\ &\geq (x_u - \alpha)^2 + (y_u - \alpha)^2 \\ &= d(C_u, X)^2 \end{aligned}$$

However, D_u does not contain X and so $d(C_u, X) > r_u$. It follows that D_u does not contain A . Then, notice that $d(C_v, A)^2 \geq (\alpha - x_v)^2 > (\alpha/2)^2$. Now, we want to show that $r_v \leq \alpha/2$. Using $d(C_v, (0, 0)) > r_v$ and $d(C_v, X') > r_v$, we obtain

$$\begin{cases} x_v^2 + y_v^2 > r_v^2 \\ (\frac{\alpha}{2} - x_v)^2 + (\frac{\alpha}{2} - y_v)^2 > r_v^2 \end{cases}$$

By summing the two inequalities and optimizing the quadratic function $z \mapsto z^2 + (\alpha/2 - z)^2$, we obtain that $2r_v^2 < 4(\alpha/4)^2$ and thus $r_v < \frac{\alpha}{2\sqrt{2}} < \frac{\alpha}{2}$. Similar arguments allow to consider any point of coordinate (x_A, α) with $x_A \leq \alpha$. The two semi-lines are separating the plane, and thus D_v and D_u do not intersect. ◀

Let V_1 (resp. V_2) be the vertices with disk center in Δ_1 (resp. Δ_2) that do not contain X' (resp. X). By definition of the median, and since Δ_2 do not contain any center of abscissa α , $|V_1| < \lfloor n/2 \rfloor$ and $|V_2| < \lfloor n/2 \rfloor$. Then, by Claim 23, there is no edge between V_1 and V_2 . Then, consider V_3 to be the set of vertices with disk center in Δ_3 and those with center in Δ_1 which intersect X' . By Claim 22, they all intersect X' and thus $G[V_3]$ is a clique. In a same way, let V_4 be the set of vertices with disk centers in Δ_2 which intersect X . By definition, $G[V_4]$ is a clique. Hence, $V_3 \cup V_4$ is indeed a balanced separator which can be partitioned into two cliques. ◀

By coloring the balanced separator with two colors and inductively coloring the remaining vertices, we obtain the following.

► **Lemma 24.** For any Δ -disk graph G , $\chi_s(G) \leq 2 \log_2(n)$.

Proof. We show the result by induction. Let $C(n)$ be the maximum subchromatic number of a Δ -disk graph with n vertices. Using the previous lemma, we have $C(n) \leq C(\lfloor n/2 \rfloor) + 2$ and by induction $C(n) \leq 2 \log_2(n) + 1$ for any $n \geq 1$. ◀

The proof of Theorem 2 is straightforward with the previous lemma and the decomposition of disk graphs of Theorem 16. Indeed, through classical induction, any linear disk graph G satisfies $\chi_s(G) = O(\log^2(n))$, and finally, any disk graph G satisfies $\chi_s(G) = O(\log^3(n))$. Furthermore, such a subcoloring can be found in time $O(n \log^3(n))$ given the representation, using an algorithm that computes the median in linear time.

5.3 A $O(\log^2(n))$ -approximation algorithm

Given a Δ -disk graph G and two vertices $u, v \in V(G)$, we say that v *contains* u if, and only if $N[u] \subseteq N[v]$. By extension, a vertex v contains a set $U \subseteq V(G)$ if for any $u \in U$, v contains u .

► **Theorem 25.** *There exists a constant-factor approximation algorithm for computing the subchromatic number of a Δ -disk graph.*

The core result for proving this theorem is the following lemma:

► **Lemma 26 (*)**. *Let G be a Δ -disk graph and $S = \{v_1, \dots, v_9\}$ an independent set of size 9 where the v_i 's are sorted by increasing co-comparability order. If a vertex v is adjacent to all vertices of S , then v contains v_5 .*

Proof of Theorem 25. Let G be a Δ -disk graph with a fixed disk representation. We call a vertex *internal* if it contains two non-adjacent vertices, and *external* otherwise. We construct a partition V_1, \dots, V_k such that for any $1 \leq i \leq k$, V_i is exactly the set of external vertices of $G \setminus \bigcup_{j < i} V_j$.

▷ **Claim 27.** $\chi_s(G) \geq k$.

Proof. We show that G contains an induced $BC(k)$. We proceed by induction on i , proving that for any vertex $v \in V_i$, there exists a $BC(i)$ in $N\left[\{v\} \cup \bigcup_{j < i} V_j\right]$. The result is clear for $i = 1$. Suppose it holds for $i - 1 \geq 1$. Let $v \in V_i$, so v contains two independent vertices v_1 and v_2 in V_{i-1} . Let u_1 (resp. u_2) be any vertex contained by v_1 (resp. v_2). Note that u_1 is not adjacent to u_2 , because otherwise $u_2 \in N[u_1]$, implying $u_2 \in N[v_1]$ and reciprocally $v_1 \in N[u_2]$. Since v_2 contains u_2 , $N[u_2] \subseteq N[v_2]$, leading to $v_1 \in N[v_2]$, which is a contradiction. By the induction hypothesis, both v_1 and v_2 contain a $BC(i - 1)$, and both of them are non-adjacent, by the previous argument. Therefore, v contains a $BC(i)$. ◀

Now let us show that $\chi_s(G[V_\ell])$ is bounded by a constant for every $\ell \in \{1, \dots, k\}$. Let $H = G[V_\ell]$, and construct a maximal independent set S of H as follows: start with the vertex of H whose disk has the smallest radius in the representation (breaking ties arbitrarily), add it to S , remove its neighborhood, and repeat recursively. Let v_1, \dots, v_m be the obtained vertices, sorted by increasing radius r_1, \dots, r_m . Observe that this total order on S also corresponds to the corresponding co-comparability order.

For all $1 \leq i \leq m$, let U_i be the set of vertices v in $V(H)$ such that $i = \min\{j \mid vv_j \in E(H)\}$.

▷ **Claim 28.** For all $i \in \{1, \dots, m\}$, $H[U_i]$ can be partitioned into at most 6 cliques.

Proof. Let $t \in U_i \setminus \{v_i\}$. Since t is not adjacent to any of v_1, \dots, v_{i-1} and was not chosen to be part of S , it follows that $r_t \geq r_i$. By a classical geometrical argument, D_t cannot intersect 7 pairwise independent disks of radius at least r_t . It follows that $H[U_i]$ can be partitioned into 6 cliques. ◀

From Lemma 26, note that no vertex $v \in V(H)$ is adjacent to 10 consecutive vertices of S , since otherwise it would contain two independent vertices. Thus, by assigning colors to the cliques of U_i as $(i \bmod 9, j)$ for $1 \leq j \leq 6$, we obtain a subcoloring of H that uses only a constant number of colors (namely, 54).

In summary, we decomposed the vertex set $V(G)$ into k subsets V_1, \dots, V_k such that $\chi_s(G) \geq k$, and each induced subgraph $G[V_\ell]$ ($1 \leq \ell \leq k$) admits a subcoloring with a constant number of colors, c . This decomposition results in a ck -subcoloring of G , yielding a constant-ratio approximation algorithm. ◀

The proof of Theorem 3 follows directly from Theorem 16. Specifically, any n -vertex disk graph can be partitioned into $O(\log^2 n)$ subgraphs, each of which is a disjoint union of Δ -disk graphs. By applying the constant-factor approximation algorithm from Theorem 25 to each of these Δ -disk subgraphs, we obtain an overall $O(\log^2 n)$ -approximation algorithm for SUBCOLORING on disk graphs.

6 Conclusion and Open Questions

In this paper, we explored the subchromatic number of (unit) disk graphs, presenting both positive results and hardness results. Several intriguing questions remain open in this area of research, including two that were initially posed by Broersma et al.:

- What is the complexity of k -SUBCOLORING on interval graphs when k is part of the input?
- What is the complexity of 2-SUBCOLORING on co-comparability graphs?
- Is there a c -approximation algorithm for SUBCOLORING on unit disk graphs with $c < 3$?
- What is the complexity of 3-SUBCOLORING of unit disk graphs?
- Let $f_s(n)$ denote the maximum subchromatic number of an n -vertex disk graph. We have established that $f_s(n) = O(\log^3 n)$ and $f_s(n) = \Omega(\log n)$. Can we narrow this gap?
- Can we develop an $f(\text{OPT})$ -approximation algorithm for SUBCOLORING of disk graphs, where OPT is the subchromatic number of the input disk graph?
- What is the complexity of k -SUBCOLORING on Δ -disk graphs?

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