Homomorphism Indistinguishability and Game Comonads for Restricted Conjunction and Requantification

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Abstract

The notion of homomorphism indistinguishability offers a combinatorial framework for characterizing equivalence relations of graphs, in particular equivalences in counting logics within finite model theory. That is, for certain graph classes, two structures agree on all homomorphism counts from the class if and only if they satisfy the same sentences in a corresponding logic. This perspective often reveals connections between the combinatorial properties of graph classes and the syntactic structure of logical fragments. In this work, we extend this perspective to logics with restricted requantification, refining the stratification of logical resources in finite-variable counting logics. Specifically, we generalize Lovász-type theorems for these logics with either restricted conjunction or bounded quantifier-rank and present new combinatorial proofs of existing results. To this end, we introduce novel path and tree decompositions that incorporate the concept of reusability and develop characterizations based on pursuit-evasion games. Leveraging this framework, we establish that classes of bounded pathwidth and treewidth with reusability constraints are homomorphism distinguishing closed. Finally, we develop a comonadic perspective on requantification by constructing new comonads that encapsulate restricted-reusability pebble games. We show a tight correspondence between their coalgebras and path/tree decompositions, yielding categorical characterizations of reusability in graph decompositions. This unifies logical, combinatorial, and categorical perspectives on the notion of reusability.

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1 Introduction

A fundamental result in graph theory due to Lovász [25] states that two graphs G and H are isomorphic if and only if for every graph F it holds $\hom(F, G) = \hom(F, H)$, where $\hom(F, G)$ denotes the number of homomorphisms from F to G. More generally, G and H are said to be *homomorphism indistinguishable* over a graph class $\mathcal{F}$, denoted by $G \equiv_{\mathcal{F}} H$, if $\hom(F, G) = \hom(F, H)$ for all $F \in \mathcal{F}$. In this formulation, the seminal result of Lovász states that the equivalence relation $\equiv_{\mathcal{G}}$ is the same as graph isomorphism, where $\mathcal{G}$ is the class of all graphs. Understanding characterizations of the homomorphism indistinguishability relation $\equiv_{\mathcal{F}}$ not only deepens our understanding of various graph invariants but also informs algorithmic approaches to problems like graph isomorphism [32], subgraph counting [9], and counting

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answers to conjunctive queries [19]. In recent years, the relation $\equiv_{\mathcal{F}}$ was characterized for several graph classes $\mathcal{F}$ as a natural equivalence relation arising from logic and algebra. Notable examples include graphs of bounded treewidth [12, 13], bounded pathwidth [21, 27], bounded tree-depth [20, 21], and planar graphs [26]. Two central questions have emerged in this line of research and attract ongoing interest:

(1) How do structural properties of the class $\mathcal{F}$ relate to the semantics of the relation $\equiv_{\mathcal{F}}$?

This first question often admits elegant characterizations in terms of mathematical logic, particularly when the graph classes are defined via *graph decompositions*. In this context, logical equivalence provides a natural way to express homomorphism indistinguishability. For a logic on graphs L, two graphs G and H are said to be L-*equivalent* if they satisfy exactly the same sentences from L. In [13] Dvořák shows that two graphs are homomorphism indistinguishable over the class of graphs of treewidth at most k if and only if they are equivalent in the *k-variable fragment* C^k of *first-order counting logic* C. The techniques developed in [13] have been refined in [17] to prove that homomorphism indistinguishability over the class $\mathcal{T}_q^k$ of graphs admitting tree-decompositions of *width k and depth q* is the same as equivalence in C_q^k, the fragment of C^k of quantifier-rank at most q. The general technique is to directly translate between formulas in C_q^k and graphs from $\mathcal{T}_q^k$ by induction on the structure of formulas and tree decompositions in both directions.

Beyond the characterization of homomorphism indistinguishability relations, the following natural question is fundamental to understand these relations:

(2) When do different graph classes $\mathcal{F}$ induce the same indistinguishability relation $\equiv_{\mathcal{F}}$?

This second question was approached methodically by Roberson [31] via introducing the notion of *homomorphism distinguishing closedness*. A graph class $\mathcal{F}$ is called *homomorphism distinguishing closed* (also *h.d. closed*) if for every graph $F \notin \mathcal{F}$ there exist graphs G, H with $G \equiv_{\mathcal{F}} H$ and $\hom(F, G) \neq \hom(F, H)$, i.e., if no graphs can be added to $\mathcal{F}$ without changing the relation $\equiv_{\mathcal{F}}$. The significance of this notion is that any two distinct homomorphism distinguishing closed classes must induce distinct homomorphism indistinguishability relations. In turn, equivalence relations on graphs characterized by homomorphism indistinguishability can be separated by separating the underlying graph classes, given they are homomorphism distinguishing closed. In general, it appears to be a hard task to establish that a given class is homomorphism distinguishing closed, leading to only a short list of known examples. These include the class of graphs of bounded degree [31], bounded tree-depth [17], bounded treewidth [29], bounded depth treewidth [5], and essentially profinite classes [35]. Beyond the investigation of specific graph classes, in [35] the relation between closure properties of $\mathcal{F}$ and preservation properties of $\equiv_{\mathcal{F}}$ was studied systematically guided by the two aforementioned main questions. In [31] Roberson conjectures that every graph class which is closed under taking disjoint unions and minors is homomorphism distinguishing closed. For certain such graph classes $\mathcal{F}$, homomorphism distinguishing closedness has been successfully proven when $\equiv_{\mathcal{F}}$ is characterized by a *model-comparison game* and membership in $\mathcal{F}$ is determined by a *pursuit-evasion game*. A key tool in such proofs is the *CFI-construction* [8], which has been instrumental in separating the homomorphism indistinguishability relations of C^k-equivalence and C^{k+1}-equivalence. For a graph G, the construction yields two *CFI-graphs* $X(G)$ and $\widetilde{X}(G)$ for which their distinguishability by C^k depends on the structural complexity of G. Crucial technical challenges arise, particularly in proving the connection between pursuit-evasion and model-comparison games for the CFI-construction (see [29]) and establishing the *monotonicity* of pursuit-evasion games (see [5]). The monotonicity of a game ensures that when searchers have a winning strategy, the reachable positions for the evading player only decreases as the game progresses.

For many logics of interest in finite model theory, like C_q^k, their equivalence can be characterized in terms of model-comparison games such as *Ehrenfeucht–Fraïssé* or *pebble games* (see [15]). This correspondence was utilized in [2] to give a novel approach to logical resources in terms of *game comonads*. The central observation is that model-comparison games induce comonads on categories of relational structures. In this framework, several essential constructions from finite model theory can be given a categorical account, see [3] for a survey. In particular, coalgebras for some game comonads encode combinatorial parameters of structures [4] leading to a uniform approach to homomorphism indistinguishability developed in [11]. There, the first characterization of C_q^k-equivalence by homomorphism counts was shown for graphs admitting *pebble forest covers*. The comonadic approach was recently used to show a categorical characterization of the graph parameter pathwidth and prove that homomorphism indistinguishability over graphs of pathwidth at most k is logical equivalence in the *restricted conjunction logic* $\wedge\mathsf{C}^k$ [27] by building on previous work of Dalmau [10].

The concept of *requantification*, recently introduced in [30], allows for a more refined view on stratification by logical resources in finite variable counting logics. The logic $\mathsf{C}^{(k_1,k_2)}$ is defined as the fragment of C using at most $k_1 + k_2$ distinct variables of which only k_1 may be requantified, i.e. quantified within scopes of their own quantification. To analyze the expressive power of $\mathsf{C}^{(k_1,k_2)}$, the *bijective* (k_1, k_2)*-pebble game* $\mathrm{BP}^{(k_1,k_2)}$ and the *q-round* (k_1, k_2)*-cops-and-robber game* $\mathrm{CR}_q^{(k_1,k_2)}$ were introduced, incorporating reusability into model-comparison and pursuit-evasion games, respectively. In the context of Question (1), it is only natural to ask whether logics with restricted conjunction and requantification admit homomorphism indistinguishability characterizations. Subsequently, Question (2) asks whether the corresponding graph classes are homomorphism distinguishing closed.

Contribution

In this work, we extend the study of homomorphism indistinguishability to graph classes with restricted reusability and thereby provide characterizations by counting logics with restricted requantification. A central contribution of our work is to show that decomposition-based techniques provide flexible tools for characterizing homomorphism indistinguishability relations, giving novel answers to Question (1). Furthermore, we use these techniques for establishing homomorphism distinguishing closedness, giving new answers to Question (2). By embedding these results into the broader framework of game comonads, we provide a unified categorical perspective on requantification in finite variable logics. In the following, we give a more detailed description of our contribution in terms of techniques and results:

Graph decompositions. We answer an open question from [30] by characterizing the class of graphs $\mathcal{T}_q^{(k_1,k_2)}$ where the cops have a winning strategy for $\mathrm{CR}_q^{(k_1,k_2)}$ by various graph decompositions, which adapt the concept of reusability (Theorem 3.8). Furthermore, we introduce the node-searching game $\mathrm{NS}^{(k_1,k_2)}$ where only k_1 of the $k_1 + k_2$ searchers may be reused and characterize the class of searcher-win graphs $\mathcal{P}^{(k_1,k_2)}$ by novel path decompositions (Theorem 3.7). We demonstrate a new effect that differentiates pathwidth from treewidth in the context of reusability. Namely, for bounded pathwidth, non-reusable resources can be employed uniformly for the full decomposition while for bounded treewidth their usage highly depends on intermediate parts of the decomposition (Proposition 3.4). Moreover, we prove that both games $\mathrm{CR}_q^{(k_1,k_2)}$ and $\mathrm{NS}^{(k_1,k_2)}$ are monotone by showing that reusability is compatible with monotonicity of the non-restricted games (Proposition 3.5).

Characterizations by logical equivalence. The newly defined decompositions for $\mathcal{P}^{(k_1,k_2)}$ and $\mathcal{T}_q^{(k_1,k_2)}$ form the basis of our homomorphism indistinguishability results, providing more fine-grained answers to Question (1). By imposing constraints on requantification in the restricted conjunction logic $\wedge\mathsf{C}^k$ we obtain the new fragment $\wedge\mathsf{C}^{(k_1,k_2)}$ and show that $\wedge\mathsf{C}^{(k_1,k_2)}$-equivalence is exactly the same as homomorphism indistinguishability over $\mathcal{P}^{(k_1,k_2)}$ (Theorem 4.5). This extends the Lovász-type theorem for $\mathcal{P}^{(k,0)}$ from [27] to the setting of restricted reusability. Interestingly, this also reproves the previous result in a purely combinatorial manner by adapting the constructive techniques from [13, 17]. We further underline the versatility of this strategy by proving that $\mathsf{C}_q^{(k_1,k_2)}$-equivalence is homomorphism indistinguishability over $\mathcal{T}_q^{(k_1,k_2)}$ (Theorem 4.6). Also here we show how the interplay of requantification and restricted conjunction differentiates the two logics: We prove a normal form result for $\wedge\mathsf{C}_{\infty\omega}^{(k_1,k_2)}$ with respect to requantification (Proposition 4.2) which in stark contrast was ruled out for the logic $\mathsf{C}^{(k_1,k_2)}$ in [30].

Homomorphism distinguishing closedness. We utilize the established framework for counting homomorphisms from the class $\mathcal{P}^{(k_1,k_2)}$ to prove our main technical result: For every graph $G \notin \mathcal{P}^{(k_1,k_2)}$ the CFI-graphs $X(G), \widetilde{X}(G)$ are $\wedge\mathsf{C}^{(k_1,k_2)}$-equivalent (Theorem 4.8 and Lemma 4.9). Using a similar argument for the class $\mathcal{T}_q^{(k_1,k_2)}$, we obtain that the classes $\mathcal{P}^{(k_1,0)}$, the closure of $\mathcal{P}^{(k_1,k_2)}$ under disjoint unions, and $\mathcal{T}_q^{(k_1,k_2)}$ are homomorphism distinguishing closed (Theorems 4.10 and 4.11). This gives new answers to Question (2) and further exemplifies the technique of using games to establish homomorphism distinguishing closedness. In the light of Roberson's conjecture, Theorem 4.10 is particularly interesting as the class of graphs of pathwidth at most k must exclude a fixed forest as a minor [33]. Next, we employ an argument from [29] to give an exact characterization which subgraph counts are recognized by the logics $\wedge\mathsf{C}^{(k_1,k_2)}$ and $\mathsf{C}_q^{(k_1,k_2)}$. For the logic $\mathsf{C}_q^{(k_1,k_2)}$, this characterizes the ability of a reusability-restricted Weisfeiler-Leman variant to detect subgraph counts (Remark 4.14). *Note:* Recently, Lemma 4.9 and a part of its consequence Theorem 4.10 were independently obtained for the case without constraints on reusability in the PhD thesis [36].

A comonadic perspective. Finally, we give a comonadic account of requantification as a logical resource. The *pebbling comonad* [2] and *pebble-relation comonad* [27] were constructed from organizing the respective pebble games as endofunctors on categories of relational structures. We use reusability-restricted variants of these pebble games from [30] and this work to obtain similar constructions, namely the comonads $\mathbb{P}^{(k_1,k_2)}$ and $\mathbb{PR}^{(k_1,k_2)}$. By proving close correspondences between coalgebras of these comonads and our newly defined path and tree decompositions, we obtain categorical characterizations of reusability in graph decompositions (Theorem 5.4). Finally, we show that coKleisli isomorphisms correspond to Duplicator winning strategies in the corresponding pebble games and hence characterize equivalence for the logics $\wedge\mathsf{C}^{(k_1,k_2)}$ and $\mathsf{C}_q^{(k_1,k_2)}$ (Theorem 5.5). We also devise restricted pebble games to capture coKleisli morphisms and thereby characterize preservation in counting-free logics with restricted requantification (Theorem 5.8).

2 Preliminaries

We write $\mathbb{N} = \{0, 1, 2, \dots\}$ for the set of natural numbers, $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ for the set of positive integers, and for $n \in \mathbb{N}_+$ we define $[n] := \{1, \dots, n\}$. Unless stated explicitly otherwise, we let $k_1, k_2 \in \mathbb{N}$ throughout the paper. We fix the *variable sets* (also called *pebble sets*) $[x_{k_1}] := \{x_1, \dots, x_{k_1}\}$, $[y_{k_2}] := \{y_1, \dots, y_{k_2}\}$, and $[x_{k_1}, y_{k_2}] := \{x_1, \dots, x_{k_1}, y_1, \dots, y_{k_2}\}$.

For the following definitions we let V be a set. We write 2^V for the *power set* of V and set $\binom{V}{2} = \{U \in 2^V : |U| = 2\}$. A *partial function* $\alpha\colon [x_{k_1}, y_{k_2}] \rightharpoonup V$ assigns to every variable $z \in [x_{k_1}, y_{k_2}]$ at most one element $\alpha(z) \in V$. If α does not assign an element to z, we write $\alpha(z) = \bot$. Also, we write $\mathrm{im}(\alpha)$ and $\mathrm{dom}(\alpha)$ for the *image* and *domain* of α respectively. We write V^+, V^n and $V^{\leq n}$ for the sets of non-empty finite sequences, sequences of length n, and sequences of length at most n over V respectively. We denote sequences of elements $s_1, \dots, s_n \in V$ by $\overline{s} = [s_1, \dots, s_n] \in V^n$ and for $\overline{s}, \overline{t} \in V^+$ we write $\overline{s} \sqsubseteq \overline{t}$ if $\overline{s}$ is a *prefix* of $\overline{t}$. The *concatenation* of $\overline{s}$ and $\overline{t}$ is denoted by $\overline{s}\overline{t}$. For $i, j \in [n]$ with $i \leq j$ we define $\overline{s}[i,j] \coloneqq [s_i, s_{i+1}, \dots, s_j]$ and $\overline{s}[i] \coloneqq s[i,i]$. Also, we indicate that s_i occurs in $\overline{s}$ by writing $s_i \in \overline{s}$. For a variable $z \in [x_{k_1}, y_{k_2}]$ and $v \in V$ we write $\alpha[z/v]$ for the partial function that is obtained from α by replacing the image $\alpha(z)$ by v. Given a sequence $\overline{s} = [(z_1, v_1), \dots, (z_n, v_n)] \in ([x_{k_1}, y_{k_2}] \times V)^n$ and $z \in [x_{k_1}, y_{k_2}]$ we denote by $\mathrm{last}_z(\overline{s})$ the v_i with the largest index i such that $(z, v_i) \in \overline{s}$. We call the first entry of each element in $\overline{s}$ a *pebble index* or *variable index*. For a proposition P, we use the *Iverson bracket* $[P] \in \{0, 1\}$ to indicate whether P is satisfied.

Finite model theory. We fix a finite signature σ of relation symbols and associate to each $R \in \sigma$ an *arity* $\mathrm{ar}(R) \in \mathbb{N}_+$. A σ*-structure* $\mathcal{A}$ consists of a *universe* of elements $V(\mathcal{A})$ and interpretations $R^{\mathcal{A}} \subseteq V(\mathcal{A})^{\mathrm{ar}(R)}$ for each $R \in \sigma$. For σ-structures $\mathcal{A}$ and $\mathcal{B}$ we say that $\mathcal{B}$ is a *substructure* of $\mathcal{A}$ if $V(\mathcal{B}) \subseteq V(\mathcal{A})$ and $R^{\mathcal{B}} \subseteq R^{\mathcal{A}}$ for each $R \in \sigma$. Every set of elements $A \subseteq V(\mathcal{A})$ induces a substructure of $\mathcal{A}$ with universe A and relations $R^{\mathcal{A}} \cap A^{\mathrm{ar}(R)}$ for $R \in \sigma$. A *homomorphism* between σ-structures $\mathcal{A}$ and $\mathcal{B}$ is a function $h\colon V(\mathcal{A}) \to V(\mathcal{B})$ such that for all $R \in \sigma$ we have that $(v_1, \dots, v_{\mathrm{ar}(R)}) \in R^{\mathcal{A}}$ implies $(h(v_1), \dots, h(v_{\mathrm{ar}(R)})) \in R^{\mathcal{B}}$. The function h is called an *isomorphism* if it is a bijective homomorphism and h^{-1} is a homomorphism. Let L be a logic over the signature σ with variable set $\mathcal{V}$ (for a formal definition see [14]). For a formula $\varphi \in \mathsf{L}$ we write $\varphi(v_1, \dots, v_n)$ to indicate that the set of *free variables* of φ, which we denote by $\mathrm{free}(\varphi)$, is a subset of $\{v_1, \dots, v_n\}$. Given a σ-structure $\mathcal{A}$ and an assignment $\alpha\colon \mathcal{V} \rightharpoonup V(\mathcal{A})$ we write $\mathcal{A}, \alpha \models \varphi$ to indicate that $\mathcal{A}$ satisfies φ with $\mathrm{free}(\varphi)$ interpreted according to α. For a tuple $\overline{a} \in V(\mathcal{A})^n$ we write $\mathcal{A}, \overline{a} \models \varphi$ by assigning $v_i \mapsto a_i$. For σ-structures $\mathcal{A}, \mathcal{B}$ we write $A \equiv_{\mathsf{L}} \mathcal{B}$ if $\mathcal{A}$ and $\mathcal{B}$ satisfy exactly the same sentences from L. We write $A \Rrightarrow_{\mathsf{L}} \mathcal{B}$ if every sentence from L satisfied by $\mathcal{A}$ is also satisfied by $\mathcal{B}$. *First-order counting logic* C extends first-order logic FO by counting quantifiers $\exists^{\geq n}\varphi$ for $\varphi \in \mathsf{C}$. We say that a variable x_i is *requantified* in a logical formula if it either occurs free and bound or if it is quantified within the scope of a quantification over x_i. The logic $\mathsf{C}^{(k_1,k_2)}$ is obtained from C by fixing the variable set $[x_{k_1}, y_{k_2}]$ and requiring that only variables from $[x_{k_1}]$ are requantified. Finally, $\mathsf{C}_q^{(k_1,k_2)}$ is the fragment of $\mathsf{C}^{(k_1,k_2)}$ with quantifier-rank at most q (see [30] for details).

Graphs. A finite *graph* is a pair $G = (V(G), E(G))$ consisting of a finite set $V(G)$ of *vertices* and a set $E(G) \subseteq \binom{V(G)}{2}$ of *edges*. For an edge $\{u, v\} \in E(G)$ we also write $uv \in E(G)$. Given a set $W \subseteq V(G)$ we define the *induced subgraph* $G[W] = (W, \{uv \in E(G) : u, v \in W\})$. For a graph G and $v \in V(G)$ we write $E(v)$ for the set of edges incident to v. We denote the set of vertex sets of *connected components* (i.e. maximal connected subgraphs) of G by $\mathcal{C}_G$. A *rooted tree* is a pair (T, r) such that T is a tree and $r \in V(T)$ is a designated vertex, called the *root* of T. With a rooted tree we associate a partial order $\preceq_T$ on the vertices of T by setting $s \preceq_T t$ exactly if s is on the unique path from r to t. The *height* of a rooted tree is the maximal number of vertices on a path from the root to a leaf. A *rooted forest* is a pair $(F, \overline{r})$ such that if $F_1, \dots, F_p$ are the connected components of F we have $\overline{r} = (r_1, \dots, r_p)$ and

(F_i, r_i) is a rooted tree for each $i \in [p]$. A *labeled graph* is a graph G together with a finite set of *labels* L and a partial labeling function $\nu_G \colon L \rightharpoonup V(G)$. We write $L_G \coloneqq \mathrm{dom}(\nu)$ for the set of labels occurring in G. A labeled graph is called *fully labeled* if the labeling function is surjective. We denote the class of all $[x_{k_1}, y_{k_2}]$-labeled graphs by $\mathcal{G}_{[x_{k_1}, y_{k_2}]}$. The product of two labeled graphs $G_1, G_2 \in \mathcal{G}_{[x_{k_1}, y_{k_2}]}$ is the graph G_1G_2 obtained by taking the disjoint union of G_1 and G_2, identifying vertices with the same label, and suppressing any loops or parallel edges that might be created. Note that for a graph $G \in \mathcal{G}_{[x_{k_1}, y_{k_2}]}$ the labeling ν_G is a partial variable assignment and hence we may write $G \models \varphi$ for $G, \nu_G \models \varphi$ if $\mathrm{free}(\varphi) \subseteq L_G$.

Two central concepts in this work are *tree decompositions* and *path decompositions*, which we briefly introduce next. For a more detailed exposition, we refer the reader to [7].

▶ **Definition 2.1.** *Let G be a graph. A* tree decomposition *of G is a tuple (T, β) such that T is a tree and $\beta \colon V(T) \to 2^{V(G)}$ is a function such that*

- $\bigcup_{t \in V(T)} \beta(t) = V(G)$,
- *for all $uv \in E(G)$ there exists a $t \in V(T)$ with $u, v \in \beta(t)$, and*
- *for all $v \in V(G)$ the set of vertices $\beta^{-1}(\{v\}) = \{t \in V(T) : v \in \beta(t)\}$ is connected in T.*

The width *of (T, β) is $\max_{t \in V(T)} |\beta(t)| - 1$ and the* treewidth *of G is the minimal width of a tree decomposition of G. If T is a rooted tree with root $r \in V(T)$ we call (T, r, β) a* rooted tree decomposition. *A* path decomposition *of G is a tree decomposition (P, β) such that the underlying tree P is a path. If the decomposition is rooted, then we define the root to be an endpoint of P. The* width *of (P, β) is again $\max_{p \in V(P)} |\beta(p)| - 1$ and the* pathwidth *of G is the minimal width of a path decomposition of G.*

The CFI-construction. We use the *CFI-construction* introduced in [8] in the variant presented in [31]. Let G be a connected graph and $U \subseteq V(G)$. For each $v \in V(G)$ we set $\delta_{v,U} \coloneqq |\{v\} \cap U|$. The *CFI-graph* $X_U(G)$ is defined by

$$V(X_U(G)) \coloneqq \{(v, S) : v \in V(G), S \subseteq E(v), |S| \equiv \delta_{v,U} \bmod 2\},$$
$$E(X_U(G)) \coloneqq \{(v, S)(u, T) : uv \in E(G), uv \notin S \triangle T\}.$$

The connected graph G is called the *base graph* of $X_U(G)$. We also define $\rho \colon V(X_U(G)) \to G, (u, S) \mapsto u$. For $v \in V(G)$ we also denote $F_U(v) \coloneqq \{(v, S) : S \subseteq E(v), |S| \equiv \delta_{v,U} \bmod 2\}$ for the vertices in $X_U(G)$ associated with v. These vertices are also referred to as *gadget vertices* of v and $F_U(v)$ as the corresponding *gadget*.

▶ **Lemma 2.2** ([8, Lemma 6.2], [31, Lemma 3.2])**.** *For all sets of base vertices $S, T \subseteq V(G)$, the graphs $X_T(G)$ and $X_S(G)$ are isomorphic if and only if $|S| \equiv |T| \bmod 2$.*

Thus, we set $X(G) \coloneqq X_\emptyset(G)$ and $\widetilde{X}(G) \coloneqq X_{\{v\}}(G)$ for some $v \in V(G)$ as the isomorphism type only depends on the parity of $|U|$. Note that the vertex sets of the graphs $X(G)$ and $\widetilde{X}_u(G)$ only differ in $F_\emptyset(u)$ and $F_u(u)$.

▶ **Lemma 2.3** ([29, Lemma 11])**.** *Let G be a connected graph, $u, v \in V(G)$, and P be a path from u to v in G. Then there exists an isomorphism $\varphi_{u,v} \colon X_{\{u\}}(G) \to X_{\{v\}}(G)$ such that for all $(w, S) \in V(X_{\{u\}}(G))$ it holds that*

1. *$\rho(\varphi_{u,v}(w, S)) = w$, and*
2. *if $w \in V(G) \setminus P$ then $\varphi_{u,v}(w, S) = (w, S)$.*

Homomorphism indistinguishability. We denote the number of homomorphisms from a possibly labeled graph F to a graph G by $\hom(F,G)$. For a class of labeled graphs $\mathcal{F}$ we write $\mathbb{R}\mathcal{F}$ for the class of all formal finite linear combinations with real coefficients of graphs in $\mathcal{F}$. For a linear combination $\mathfrak{F} = \sum_i c_i F_i \in \mathbb{R}\mathcal{F}$ and a labeled graph G we define $\hom(\mathfrak{F}, G) := \sum_i c_i \hom(F_i, G)$ and $L_{\mathfrak{F}}$ to be the set of labels occurring in $\mathfrak{F}$.

For a class $\mathcal{F}$ of graphs we say that two graphs G and H are *homomorphism indistinguishable over* $\mathcal{F}$ if for all $F \in \mathcal{F}$ it holds that $\hom(F,G) = \hom(F,H)$. In this case we write $G \equiv_{\mathcal{F}} H$. The *homomorphism distinguishing closure* of $\mathcal{F}$ is defined as the class

$$\mathrm{cl}(\mathcal{F}) := \{F \in \mathcal{G} : \forall G, H \in \mathcal{G} \quad G \equiv_{\mathcal{F}} H \Rightarrow \hom(F,G) = \hom(F,H)\}.$$

The class $\mathcal{F}$ is called *homomorphism distinguishing closed* if $\mathrm{cl}(\mathcal{F}) = \mathcal{F}$.

▶ **Lemma 2.4** ([16, Proposition 47]). *Let $\mathcal{F}$ be a graph class that is closed under taking disjoint unions and summands (i.e. $F_1 \dot\cup F_2 \in \mathcal{F}$ exactly if $F_1, F_2 \in \mathcal{F}$). If for every connected graph $G \notin \mathcal{F}$ it holds that $X(G) \equiv_{\mathcal{F}} \widetilde{X}(G)$, then $\mathcal{F}$ is homomorphism distinguishing closed.*

Category theory. We assume only very basic background in category theory, see [1] for details. For a category $\mathbf{C}$ we denote its objects by $\mathrm{Obj}(\mathbf{C})$ and its morphisms (or arrows) by $\mathrm{Ar}(\mathbf{C})$. We denote the category of σ-structures with their homomorphisms by $\mathbf{Str}(\sigma)$.

A *comonad (in coKleisli form)* on a category $\mathbb{C}$ is given by:

- an object map $\mathbb{G} \colon \mathrm{Obj}(\mathbf{C}) \to \mathrm{Obj}(\mathbf{C})$,
- a *counit morphism* $\varepsilon_{\mathcal{A}} \colon \mathbb{G}\mathcal{A} \to \mathcal{A}$ for every $\mathcal{A} \in \mathrm{Obj}(\mathbf{C})$,
- and a *coextension operation* $(\cdot)^*$ associating with each morphism $f \colon \mathbb{G}\mathcal{A} \to \mathcal{B}$ another morphism $f^* \colon \mathbb{G}\mathcal{A} \to \mathbb{G}\mathcal{B}$ for $\mathcal{A}, \mathcal{B} \in \mathrm{Obj}(\mathbf{C})$

such that for all morphisms $f \colon \mathbb{G}\mathcal{A} \to \mathcal{B}$, $g \colon \mathbb{G}\mathcal{B} \to \mathcal{C}$ we have $\varepsilon^*_{\mathcal{A}} = \mathrm{id}_{\mathbb{G}\mathcal{A}}$, $\varepsilon_{\mathcal{B}} \circ f^* = f$, and $(g \circ f^*)^* = g^* \circ f^*$. From this, a *comonad in standard form* $(\mathbb{G}, \varepsilon, \delta)$ on the category $\mathbf{C}$ can be obtained by setting $\mathbb{G}f := (f \circ \varepsilon_{\mathcal{A}})^*$ (turning $\mathbb{G}$ into a functor) and $\delta_{\mathcal{A}} := \mathrm{id}^*_{\mathbb{G}\mathcal{A}}$ for $\mathcal{A} \in \mathrm{Obj}(\mathbf{C})$. For a comonad in standard form, a *coalgebra over* $\mathbb{G}$ is a pair $(\mathcal{A}, \alpha)$ where $\mathcal{A} \in \mathrm{Obj}(\mathbf{C})$ and $\mathrm{Ar}(\mathbf{C}) \ni \alpha \colon \mathcal{A} \to \mathbb{G}\mathcal{A}$ such that $\delta_{\mathcal{A}} \circ \alpha = \mathbb{G}\alpha \circ \alpha$ and $\varepsilon_{\mathcal{A}} \circ \alpha = \mathrm{id}_{\mathcal{A}}$.

For a comonad $(\mathbb{G}, \varepsilon, (\cdot)^*)$ in coKleisli form we define the *coKleisli category* $\mathcal{K}(\mathbb{G})$:

- $\mathrm{Obj}(\mathcal{K}(\mathbb{G}))$ is the class of objects $\mathrm{Obj}(\mathbf{C})$.
- $\mathrm{Ar}(\mathcal{K}(\mathbb{G}))$ are all morphisms $f \colon \mathbb{G}\mathcal{A} \to \mathcal{B}$ for $\mathcal{A}, \mathcal{B} \in \mathrm{Obj}(\mathbf{C})$.
- The composition $\circ_{\mathcal{K}}$ is defined by setting $g \circ_{\mathcal{K}} f := g \circ f^*$.
- The identity morphisms are given by the counit morphisms $\varepsilon_{\mathcal{A}}$ for $\mathcal{A} \in \mathrm{Obj}(\mathbf{C})$.

3 Graph decompositions with restricted reusability

In this section, we introduce several graph decompositions that incorporate constraints on reusability of vertices within the parts of each decomposition. Specifically, we define four distinct types of decompositions, each of which comes in two variants corresponding to two underlying structural models: the *path model* and the *(bounded-depth) tree model*. Each decomposition is parameterized by two values: k_1, representing the number of *reusable resources*, and k_2, representing the number of *non-reusable resources*. Our goal is to show that, for each of the two models, the corresponding decompositions define the same graph class with aligned parameters. This equivalence allows us to explore the concept of reusability in graph decompositions from multiple, yet consistent, perspectives. The motivation for each decomposition arises from its significance in homomorphism indistinguishability over the associated graph class, establishing a unifying theme for our study. In Table 1 we summarize the significance of the various decompositions introduced in this section for our results on homomorphism indistinguishability.

Table 1 Utilization of graph decompositions for homomorphism indistinguishability and game comonads.

	graph class	homomorphism indistinguishability	h.d. closedness	game comonads
path	$\mathcal{P}^{(k_1,k_2)}$ defined by path decompositions	$\wedge\mathsf{C}^{(k_1,k_2)}$-equivalence via construction caterpillars $\mathcal{LP}^{(k_1,k_2)}$	only for $\dot\cup\mathcal{P}^{(k_1,k_2)}$ via $\mathrm{NS}^{(k_1,k_2)}$	coalgebras over $\mathbb{PR}_{(k_1,k_2)}$ correspond to linear component forest covers
tree	$\mathcal{T}_q^{(k_1,k_2)}$ defined by bounded depth tree decompositions	$\mathsf{C}_q^{(k_1,k_2)}$-equivalence via construction trees $\mathcal{LT}_q^{(k_1,k_2)}$	true via $\mathrm{CR}_q^{(k_1,k_2)}$	coalgebras over $\mathbb{P}^q_{(k_1,k_2)}$ correspond to forest covers

We begin by introducing the notion of *exception sets* for tree decompositions of bounded depth and for path decompositions. The key idea is to generalize the concept of a tree decomposition of width k_1 by permitting up to k_2 *exceptions* along each branch of the underlying tree. However, there is an important restriction: if a vertex is designated as an exception at some node t in the rooted tree decomposition, then it must remain fixed, i.e., cannot be replaced by a different vertex at any descendant of t. In this sense, the exception status is not reusable along the subtree rooted at t.

▶ **Definition 3.1.** *Let G be a graph, $k_2 \in \mathbb{N}$, $k_1 \in \mathbb{N} \cup \{-[k_2 > 0]\}$, and $k_1 + k_2, q \in \mathbb{N}_+$. A rooted tree decomposition (T, r, β) of G has*

- width (k_1, k_2) *if for each leaf $\ell \in V(T)$ there exists a set of* exceptions $S_\ell \subseteq V(G)$ *with $|S_\ell| \leq k_2$ such that $\max_{t \preceq_T \ell} |\beta(t) \setminus S_\ell| - 1 \leq k_1$,*
- *and* depth $\max_{v \in V(T)} |\bigcup_{t \preceq_T v} \beta(t)|$.

We write $\mathcal{T}^{(k_1+1,k_2)}$ for the class of all graphs admitting a tree decomposition of width (k_1, k_2) and $\mathcal{T}_q^{(k_1+1,k_2)}$ for the subclass admitting such a cover of depth q.

If (P, r, β) is a path decomposition, we define that (P, r, β) has component width (k_1, k_2) *if for each connected component $C \subseteq V(G)$ there is an exception set $S_C \subseteq C$ with $|S_C| \leq k_2$ such that $\max_{p \in V(P)} |\beta(p) \setminus \bigcup_{C \in \mathcal{C}_G} S_C| - 1 \leq k_1$.*

We write $\mathcal{P}^{(k_1+1,k_2)}$ for the class of graphs admitting a path decomposition of width (k_1, k_2).

For $k_2 = 0$ we recover the class $\mathcal{T}^{(k_1+1,0)}$ of graphs of treewidth at most k_1 but for $k_2 > 0$ we allow the technical nuisance of having $k_1 = -1$ in order to avoid a case distinction.

Note that for the *width* of a path decomposition we only require the existence of one single set S of exceptions. Thus, for fixed k_1 the class of graphs admitting a path decomposition of width (k_1, k_2) can be seen as an approximation of the class of graphs of pathwidth at most k_1 up to deleting k_2 vertices.

Next, we extend the notion of *pebble forest covers* to incorporate the concept of reusability. Originally introduced in [2] as *k-traversals*, these structures provide a combinatorial characterization of coalgebras over the pebbling comonad $\mathbb{P}^k$, finally demonstrating how the comonadic structure of $\mathbb{P}^k$ can be used to characterize treewidth. Our aim is to follow a similar approach to characterize reusability (specifically, width (k_1, k_2)) within tree decompositions of bounded depth via comonadic methods. To this end, we use *non-reusable pebbles* $y_1, \ldots, y_{k_2}$, which mark fixed positions in a forest cover that cannot be reassigned, thereby encoding the non-reusability constraint directly into the structure.

▶ **Definition 3.2.** *Let G be a graph and $k_1 + k_2 \in \mathbb{N}_+$. A* (k_1, k_2)-pebble forest cover *of G is a tuple $(F, \overline{r}, p)$ where $(F, \overline{r})$ is a rooted forest with $V(F) = V(G)$ and $p\colon V(G) \to [x_{k_1}, y_{k_2}]$ is a pebbling function such that*
1. *if $uv \in E(G)$, then $u \preceq_F v$ or $v \preceq_F u$,*
2. *if $uv \in E(G)$ and $u \prec_F v$, then for all $w \in V(G)$ with $u \prec_F w \preceq_F v$ it holds that $p(u) \neq p(w)$, and*
3. *if $u \in V(G)$ and $p(u) \in [y_{k_2}]$, then for all $w \in V(G)$ with $u \prec_F w$ it holds that $p(u) \neq p(w)$.*

The forest cover $(F, \overline{r}, p)$ has depth *$q \in \mathbb{N}_+$ if $(F, \overline{r})$ has height q.*

We call $(F, \overline{r}, p)$ a linear forest cover *if it additionally holds that*
1. *every connected component of F is a path, and*
2. *if $u \in V(G)$ and $p(u) \in [y_{k_2}]$, then for every $w \in V(G) \setminus \{u\}$ it holds that $p(u) \neq p(w)$.*

If we relax Item 2 such that for $u \in V(G)$ with $p(u) \in [y_{k_2}]$ only for every w in the same path of F as u it must hold $p(u) \neq p(w)$, we say that $(F, \overline{r}, p)$ is a linear component forest cover.

The *tree-depth* of a graph G is the minimum q such that G has a forest cover of depth at most q (see [28]). The class of graphs of tree-depth at most q is denoted by $\mathcal{T}_q$.

The classes $\mathcal{P}^{(k,0)}$, $\mathcal{T}^{(k,0)}$, and $\mathcal{T}_q$ admit characterizations in terms of pursuit-evasion games by [37], [6, 22], and [18] respectively. The characterization of $\mathcal{T}^{(k,0)}$ was refined to $\mathcal{T}_q^{(k,0)}$ in [17]. In [30] a cops-and-robber game with constraints on the reusability of cops and the number of rounds was introduced. We recall its definition and also modify it to match the game-theoretic characterization of $\mathcal{P}^{(k,0)}$.

▶ **Definition 3.3.** *Let G be a graph and let $k_1 + k_2, q \in \mathbb{N}_+$. The* cops-and-robber game *$\mathrm{CR}_q^{(k_1,k_2)}(G)$ is defined as follows: The game is played between a group of $k_1 + k_2$* cops *denoted by the elements in $[x_{k_1}, y_{k_2}]$ and one* robber. *The position of the cops is given by a function $\gamma\colon [x_{k_1}, y_{k_2}] \rightharpoonup V(G)$ and the position of the robber is a vertex $v \in V(G)$. We denote the connected component of v in the graph $G - \mathrm{im}(\gamma)$ by C_v^γ. In one round of the game, the following steps are performed:*
1. *The cops choose one cop $z \in [x_{k_1}] \cup \{y \in [y_{k_2}] : \gamma(y) = \bot\}$ and declare a new destination $w \in V(G) \cup \{\bot\}$.*
2. *The robber chooses a vertex v' in $C_v^{\gamma[z/\bot]}$.*
3. *If $v' \in \mathrm{im}(\gamma[z/w])$ the cops win. Otherwise, the game continues from the new position $(\gamma[z/w], v')$.*

Initially, neither the cops nor the robber are placed on the graph. The robber wins if the cops do not win after q rounds. The game variant $\mathrm{CR}^{(k_1,k_2)}$ is defined in the same way with the modification that the robber wins if the cops never win a round.

The node searching game *$\mathrm{NS}^{(k_1,k_2)}(G)$ is defined as the variant of $\mathrm{CR}^{(k_1,k_2)}$ in which the robber is* invisible *to the cops. That is, the choice of the assignment γ_i can only depend on γ_{i-1}, but not on v. Here the cops are called* searchers *instead and the robber is called* fugitive. *In this formulation the only difference to $\mathrm{CR}^{(k_1,k_2)}$ is that searchers do not know the position v, but the fugitive knows the position γ.*

The strategy of the cops or searchers is called monotone *if in each round it holds that $C_v^\gamma \supseteq C_v^{\gamma[z/\bot]}$. We say that a game variant is* monotone *if the existence of a winning strategy implies the existence of a monotone winning strategy.*

A winning strategy of the searchers in the game $\mathrm{NS}^{(k_1,k_2)}(G)$ can be specified as a sequence of positions $\gamma_1, \ldots, \gamma_m\colon [x_{k_1}, y_{k_2}] \rightharpoonup V(G)$ while in the game $\mathrm{CR}^{(k_1,k_2)}$ the cop strategy depends on the moves of the robber and therefore each position γ_i additionally depends on the robber position v_i in round i. The proof of [30, Theorem 15] hinges on this fact and shows that in the game $\mathrm{CR}^{(k_1,k_2)}$ the use of non-reusable cops is restricted to a pattern involving arbitrarily long sequences of reusing all reusable cops before utilizing a new non-reusable cop. We show that the situation is entirely different for $\mathrm{NS}^{(k_1,k_2)}$ due to the invisibility of the fugitive.

▶ **Proposition 3.4.** *The searchers have a winning strategy in* $\mathrm{NS}^{(k_1,k_2)}(G)$ *if and only if they have a winning strategy in* $\mathrm{NS}^{(k_1,0)}(G)$ *with* k_2 *initial fixed placements of non-reusable searchers.*

To prove the characterization of a graph parameter by a pursuit-evasion game an important step often is to establish that the respective game is monotone. Alongside this, in some cases proving the monotonicity of a game is the crucial step in establishing homomorphism distinguishing closedness of the associated graph class, see [5, 17] for a discussion. Our goal is to show monotonicity of the games introduced here in order to utilize this property to prove game-theoretic characterizations of $\mathcal{P}^{(k_1,k_2)}$ and $\mathcal{T}_q^{(k_1,k_2)}$ towards homomorphism distinguishing closedness. The monotonicity of $\mathrm{NS}^{(k_1,0)}$ was proven in [23] and recently [5] established the monotonicity of $\mathrm{CR}_q^{(k_1,0)}(G)$. We build on top of these results to show that reusability is compatible with monotonicity by replacing parts of winning strategies by monotone strategies. The notion of monotonicity we use here is usually referred to as *robber-monotonicity* in the literature.

▶ **Proposition 3.5.** *Let* $k_1 + k_2, q \in \mathbb{N}_+$ *and* G *be a graph. Then both games* $\mathrm{NS}^{(k_1,k_2)}(G)$ *and* $\mathrm{CR}_q^{(k_1,k_2)}(G)$ *are monotone.*

To characterize homomorphism indistinguishability over the class $\mathcal{T}_q^{(k,0)}$, in [17] the notion of *construction trees* was introduced, building on techniques from [13]. We extend this concept to accommodate the classes $\mathcal{T}_q^{(k_1,k_2)}$ and $\mathcal{P}^{(k_1,k_2)}$. This generalized definition forms the basis of our framework for analyzing homomorphism indistinguishability over these broader classes.

▶ **Definition 3.6.** *Let* $k_1+k_2 \in \mathbb{N}_+$ *and* G *be a* $[x_{k_1}, y_{k_2}]$*-labeled graph. A* (k_1, k_2)-construction tree *for* G *is a tuple* (T, λ, r)*, where* (T, r) *is a rooted tree and* $\lambda \colon V(T) \to \mathcal{G}_{[x_{k_1}, y_{k_2}]}$ *is a function assigning* $[x_{k_1}, y_{k_2}]$*-labeled graphs to the nodes of* T *such that*

- $\lambda(r) = G$*,*
- *all leaves* $\ell \in V(T)$ *are assigned fully labeled graphs,*
- *all internal nodes* $t \in V(T)$ *with exactly one child* t' *are* elimination nodes*, that is* $\lambda(t)$ *is obtained from* $\lambda(t')$ *by deleting a label,*
- *all internal nodes* $t \in V(T)$ *with more than one child are* product nodes*, that is* $\lambda(t)$ *is the product of its children,*
- *if* $t \in V(T)$ *is an elimination node deleting a label* $y \in [y_{k_2}]$*, then for all* $s \prec_T t$ *it holds that* $y \notin L_{\lambda(s)}$*.*

The elimination depth *of* (T, λ, r) *is the maximum number of elimination nodes on any path from the root* r *to a leaf. If* G *has a* (k_1, k_2)*-construction tree of elimination depth* q *we say that* G *is* (k_1, k_2, q)-constructible*. We write* $\mathcal{LT}_q^{(k_1,k_2)}$ *for the class of* (k_1, k_2, q)*-constructible labeled graphs. If* G *has a* (k_1, k_2)*-construction tree* T *such that each product node* $v \in V(T)$ *has at most one child which is not a leaf, we say that* T *is a* construction caterpillar *and* G *is* linearly (k_1, k_2)-constructible*. We write* $\mathcal{LP}^{(k_1,k_2)}$ *for the class of linearly* (k_1, k_2)*-constructible labeled graphs.*

We obtain the following characterizations for the classes $\mathcal{T}_q^{(k_1,k_2)}$ and $\mathcal{P}^{(k_1,k_2)}$, showing that all previous definitions are equivalent for unlabeled graphs.

▶ **Theorem 3.7.** *For a graph G and $k_1 + k_2 \in \mathbb{N}_+$ the following are equivalent:*
1. $G \in \mathcal{P}^{(k_1,k_2)}$*, i.e., G has a path decomposition of width $(k_1 - 1, k_2)$.*
2. *G admits a linear (k_1, k_2)-pebble forest cover.*
3. *The searchers have a winning strategy for the game $\mathrm{NS}^{(k_1,k_2)}(G)$.*
4. $G \in \mathcal{LP}^{(k_1,k_2)}$*, i.e., G is linearly (k_1, k_2)-constructible.*

▶ **Theorem 3.8.** *For a graph G and $k_1 + k_2, q \in \mathbb{N}$ the following are equivalent:*
1. $G \in \mathcal{T}_q^{(k_1,k_2)}$*, i.e., G has a tree decomposition of width $(k_1 - 1, k_2)$ and depth q.*
2. *G admits a (k_1, k_2)-pebble forest cover of depth q.*
3. *The cops have a winning strategy for the game $\mathrm{CR}_q^{(k_1,k_2)}(G)$.*
4. $G \in \mathcal{LT}_q^{(k_1,k_2)}$*, i.e., G is (k_1, k_2, q)-constructible.*

▶ Remark 3.9. Regarding inclusions between classes $\mathcal{T}^{(k_1,k_2)}$ (or $\mathcal{P}^{(k_1,k_2)}$) for varying parameters k_1 and k_2 we observe that the proof of [30, Theorem 13] can be used to show a complete classification of all inclusions, which in particular separates $\mathcal{T}^{(k_1,k_2)}$ from $\mathcal{T}^{(k_1',k_2')}$ for $(k_1, k_2) \neq (k_1', k_2')$ (and likewise for $\mathcal{P}^{(k_1,k_2)}$, except that $\mathcal{P}^{(1,k_2)} \not\subseteq \mathcal{P}^{(0,k_2')}$ for all k_2' by Proposition 3.4).

We observe that classes the $\mathcal{T}_q^{(k_1,k_2)}$ and $\mathcal{P}^{(k_1,k_2)}$ admit usual closure properties, except that $\mathcal{P}^{(k_1,k_2)}$ is not closed under taking disjoint unions.

▶ **Proposition 3.10.** *Let $k_1 + k_2 \in \mathbb{N}_+$. The class $\mathcal{T}_q^{(k_1,k_2)}$ is closed under taking disjoint unions, summands, and minors. The class $\mathcal{P}^{(k_1,k_2)}$ is closed under taking summands and minors but not under taking disjoint unions for $k_2 \geq 1$.*

The class $\mathcal{P}^{(k_1,k_2)}$ formalizes the notion of reusability for path decompositions in an appropriate sense as exemplified by the characterization through $\mathrm{NS}^{(k_1,k_2)}$. However, to overcome the obstacle that this class is not closed under disjoint unions we define the class $\dot{\cup}\mathcal{P}^{(k_1,k_2)}$ as the closure of $\mathcal{P}^{(k_1,k_2)}$ under disjoint unions. This class formalizes the notion of restricted reusability componentwise on a graph. The next proposition, which follows directly from Theorem 3.7, makes this explicit.

▶ **Proposition 3.11.** *For a graph G and $k_1 + k_2 \in \mathbb{N}_+$ the following are equivalent:*
1. $G \in \dot{\cup}\mathcal{P}^{(k_1,k_2)}$*, i.e., G has a path decomposition of component width $(k_1 - 1, k_2)$.*
2. *G admits a linear (k_1, k_2)-pebble component forest cover.*
3. *For each component $C \in \mathcal{C}_G$ the searchers have a winning strategy for $\mathrm{NS}^{(k_1,k_2)}(G[C])$.*

4 Homomorphism indistinguishability and logical equivalence

In this section, we characterize homomorphism indistinguishability over the classes $\mathcal{P}^{(k_1,k_2)}$ and $\mathcal{T}_q^{(k_1,k_2)}$ by logics with restricted requantification. We start by giving the definition of finite variable counting logic with restricted conjunction and requantification, extending a definition from [27].

▶ **Definition 4.1.** *We define the set of logical formulas $\wedge\mathsf{C}_{\infty\omega}^{(k_1,k_2)}$ over the variable sets $[x_{k_1}, y_{k_2}]$ and $\mathcal{W} = \{w_1, w_2, \dots\}$. The* non-counting formulas *of the logic are given by*

$$\varphi ::= z_i = z_j \mid R(\overline{z}) \mid \neg p \mid \bigvee_{i \in I} \psi_i \mid \bigwedge_{j \in J} \psi_j \mid \exists z_i (z_i = w_\ell \wedge \psi(\overline{z}, \overline{w}))$$

for $z_i, z_j \in [x_{k_1}, y_{k_2}]$, $\overline{z} \in [x_{k_1}, y_{k_2}]^n$, p atomic, I and J countable index sets, $\bigwedge_{j \in J} \psi_j$ a restricted conjunction, and a non-counting formula $\psi(\overline{z}, \overline{w})$ with $\overline{w} \in \mathcal{W}^m$, $w_\ell \notin \overline{w}$. Here, restricted conjunction *means that at most one formula* ψ_j *containing a quantifier is not a sentence. Furthermore, the logic contains the formulas*

$$\varphi ::= \exists^n (w_{\ell_1}, \dots, w_{\ell_m}) \psi(w_{\ell_1}, \dots, w_{\ell_m}) \mid \psi_1 \vee \psi_2$$

for $n, m \in \mathbb{N}$*,* $w_{\ell_1}, \dots, w_{\ell_m} \in W$*, a non-counting formula* ψ*, and* $\psi_1, \psi_2 \in \wedge\mathsf{C}_{\infty\omega}^{(k_1,k_2)}$*. We additionally require that only variables from* $[x_{k_1}]$ *are requantified. The fragment* $\wedge\mathsf{C}^{(k_1,k_2)}$ *is defined by additionally requiring that all conjunctions and disjunctions are finite.*

We call a non-counting formula primitive *if it contains no disjunction and every restricted conjunction does not contain a sentence. A formula* $\varphi \in \wedge\mathsf{C}_{\infty\omega}$ *is called* primitive *if it is of the form* $\varphi = \exists^n \overline{w}\ \psi(\overline{w})$ *for a primitive non-counting formula* ψ*.*

We first prove that there is a normal form for restricted conjunction counting logic with respect to requantification and primitivity, enabling a more direct correspondence between the syntax of formulas and construction caterpillars. The idea is to translate the scheme from Proposition 3.4 into the language of logic: It suffices to first fix all non-requantifiable variables followed by a well-behaved employment of requantifiable variables.

► **Proposition 4.2.** *Every sentence* $\varphi \in \wedge\mathsf{C}_{\infty\omega}^{(k_1,k_2)}$ *is logically equivalent to disjunction of sentences of the form*

$$\exists^n \overline{w} \exists y_1 \dots \exists y_{k_2} \bigwedge_{i \in [k_2]} y_i = w_{\ell_i} \wedge \chi$$

for a primitive non-counting formula χ *only containing quantification over variables from* $[x_{k_1}]$*.*

In [27] it was shown that equivalence in $\wedge\mathsf{C}_{\infty\omega}^{(k,0)}$ is the same as homomorphism indistinguishability over $\mathcal{P}^{(k,0)}$. This result was proven by evoking a categorical meta-theorem from [11]. We utilize the constructive nature of the proofs in [13, 17] to give a new combinatorial proof of the result that also generalizes to the setting with restricted requantification.

As a first step, we show that homomorphism counts from graphs in $\mathcal{LP}^{(k_1,k_2)}$ are $\wedge\mathsf{C}^{(k_1,k_2)}$-definable by inductively building up the formula along a construction caterpillar.

► **Lemma 4.3.** *Let* $F \in \mathcal{LP}^{(k_1,k_2)}$ *and* $m \in \mathbb{N}$*. There exists a formula* $\varphi_m^F \in \wedge\mathsf{C}^{(k_1,k_2)}$ *with* $\mathrm{free}(\varphi) = L_F$ *such that for each* $[x_{k_1}, y_{k_2}]$*-labeled graph* G *with* $L_F \subseteq L_G$ *it holds that*

$$G \models \varphi_m^F \textit{ if and only if } \hom(F, G) = m.$$

Next, we aim to prove that also every property definable in $\wedge\mathsf{C}^{(k_1,k_2)}$ can be modeled by counting homomorphisms from $\mathcal{LP}^{(k_1,k_2)}$. In fact, the number of solutions to a non-counting formula in a graph can be expressed by counting homomorphisms from linear combinations:

► **Lemma 4.4.** *Let* $k_1 + k_2 \in \mathbb{N}_+$ *and* $\psi(w_{\ell_1}, \dots, w_{\ell_m}, \overline{z}) \in \wedge\mathsf{C}^{(k_1,k_2)}$ *be a non-counting formula. Then there exists a linear combination* $\mathfrak{F}_\psi \in \mathbb{R}\mathcal{LP}^{(k_1,k_2)}$ *such that for all* $G \in \mathcal{G}_{[x_{k_1}, y_{k_2}]}$ *we have*

$$\hom(\mathfrak{F}_\psi, G) = \begin{cases} |\{\overline{v} \in V(G)^m : G, \overline{v} \models \psi\}| & \textit{if } \mathrm{free}(\psi) \cap \mathcal{W} \neq \emptyset \\ [G \models \psi] & \textit{otherwise} \end{cases}$$

Combining the two previous results allows us to prove that homomorphism indistinguishability over the class $\mathcal{P}^{(k_1,k_2)}$ is the same as logical equivalence with restricted conjunction and requantification.

▶ **Theorem 4.5.** *For $k_1 + k_2 \in \mathbb{N}_+$ and graphs G, H the following are equivalent:*

- *G and H are homomorphism indistinguishable over the class $\mathcal{P}^{(k_1,k_2)}$.*
- *G and H are $\wedge\mathsf{C}^{(k_1,k_2)}$-equivalent.*

Following the proof from [17], this strategy can be used to make all constructions for the logic C_q^k with unrestricted conjunction also preserve requantification.

▶ **Theorem 4.6.** *For $k_1 + k_2, q \in \mathbb{N}_+$ and graphs G, H the following are equivalent:*

- *G and H are $\mathsf{C}_q^{(k_1,k_2)}$-equivalent.*
- *G and H are homomorphism indistinguishable over the class $\mathcal{T}_q^{(k_1,k_2)}$.*

4.1 Homomorphism distinguishing closedness

We follow the approach from [29] to combine pursuit-evasion and model-comparison games to prove homomorphism distinguishing closedness for the classes $\mathcal{T}_q^{(k_1,k_2)}$ and $\dot\cup\mathcal{P}^{(k_1,k_2)}$.

First, we introduce a reusability-restricted variant of the all-in-one bijective pebble game from [27] to characterize $\wedge\mathsf{C}_{\infty\omega}^{(k_1,k_2)}$-equivalence.

▶ **Definition 4.7.** *Let $\mathcal{A}, \mathcal{B}$ be σ-structures and $k_1 + k_2 \in \mathbb{N}_+$. The* all-in-one bijective (k_1, k_2)-pebble game $\mathrm{ABP}^{(k_1,k_2)}(\mathcal{A}, \mathcal{B})$ *is defined as follows:*

The game is played by the two players Spoiler *and* Duplicator *on the structures $\mathcal{A}$ and $\mathcal{B}$. During the first and only round of the game, the following steps are performed:*

1. *Spoiler chooses a sequence of pebbles $\overline{z} = [z_1, \dots, z_n] \in [x_{k_1}, y_{k_2}]^n$ such that each $y_j \in [y_{k_2}]$ occurs at most once in $\overline{z}$.*
2. *Duplicator chooses a bijection $h_{\overline{z}} \colon V(\mathcal{A})^n \to V(\mathcal{B})^n$.*
3. *Spoiler chooses $\overline{v} \in V(\mathcal{A})^n$ and defines the sequence $\overline{s} := [(z_i, v_i)]_{i\in[n]}$.*
4. *Duplicator responds with the sequence $\overline{d} := [(z_i, h_{\overline{z}}(\overline{v})[i])]_{i\in[n]}$.*

Duplicator wins if for all $i \in [n]$ the function η_i defined by setting $\eta_i(\mathrm{last}_z(\overline{s}[1,i])) := \mathrm{last}_z(\overline{d}[1,i])$ for each $z \in [x_{k_1}, y_{k_2}]$ is a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

We extend the proof of [27, Theorem 5.9.] to the setting with restricted requantification and obtain the following:

▶ **Theorem 4.8.** *Let $\mathcal{A}, \mathcal{B}$ be finite σ-structures and $k_1 + k_2 \in \mathbb{N}_+$. Then the following assertions are equivalent:*

1. *$\mathcal{A}$ and $\mathcal{B}$ are $\wedge\mathsf{C}_{\infty\omega}^{(k_1,k_2)}$-equivalent.*
2. *$\mathcal{A}$ and $\mathcal{B}$ are $\wedge\mathsf{C}^{(k_1,k_2)}$-equivalent.*
3. *Duplicator has a winning strategy for $\mathrm{ABP}^{(k_1,k_2)}(\mathcal{A}, \mathcal{B})$.*

The proof shows that the theorem also holds for infinite structures if we omit Item 2.

The next lemma is the key technical ingredient to establish our first homomorphism distinguishing closedness result. But more generally, it shows that the capability of G to be decomposed in a path-like fashion with reusability constraints provides a lower bound for the distinguishability of the CFI graphs $X(G), \widetilde{X}(G)$ by $\wedge\mathsf{C}^{(k_1,k_2)}$.

▶ **Lemma 4.9.** *Let $k_1 + k_2 \in \mathbb{N}_+$ and G be a connected graph. If the fugitive has a winning strategy for the game $\mathrm{NS}^{(k_1,k_2)}(G)$, then Duplicator has a winning strategy for $\mathrm{ABP}^{(k_1,k_2)}(X(G), \widetilde{X}(G))$.*

The proof idea is that the position of the fugitive in $\mathrm{NS}^{(k_1,k_2)}(G)$ corresponds to the position where the difference of the CFI graphs is moved to in $\mathrm{ABP}^{(k_1,k_2)}(X(G), \widetilde{X}(G))$ by Lemma 2.3. Thus, the invisibility of the fugitive corresponds to the fact that Spoiler has to fix their entire strategy at once. We conjecture that using similar techniques also the reverse implication of this lemma can be shown, which would in particular yield that the pathwidth of a graph G is exactly the minimum k such that $\wedge\mathsf{C}^{k+1}$ distinguishes $X(G)$ and $\widetilde{X}(G)$. However, to prove our next theorem the direction form Lemma 4.9 suffices.

▶ **Theorem 4.10.** *The classes $\mathcal{P}^{(k_1,0)}$ and $\dot\cup\mathcal{P}^{(k_1,k_2)}$ are homomorphism distinguishing closed.*

Proof. For $\mathcal{F} \in \{\mathcal{P}^{(k_1,0)}, \dot\cup\mathcal{P}^{(k_1,k_2)}\}$ the class $\mathcal{F}$ is closed under taking disjoint unions and summands. For every connected graph $G \notin \mathcal{F}$ the fugitive has a winning strategy for $\mathrm{NS}^{(k_1,k_2)}(G)$ by Theorem 3.7. By Theorem 4.5 and Lemma 4.9 this in turn yields $X(G) \equiv_{\mathcal{F}} \widetilde{X}(G)$. Finally, by Lemma 2.4 it follows that $\mathcal{F}$ is h.d. closed. ◀

Note that the class $\mathcal{P}^{(k_1,k_2)}$ is not homomorphism distinguishing closed since the relations $\equiv_{\dot\cup\mathcal{P}^{(k_1,k_2)}}$ and $\equiv_{\mathcal{P}^{(k_1,k_2)}}$ are identical, but $\mathcal{P}^{(k_1,k_2)} \subsetneq \dot\cup\mathcal{P}^{(k_1,k_2)}$.

After establishing the monotonicity of $\mathrm{CR}_q^{(k_1,k_2)}$ in Proposition 3.5, we can use the correspondence to the bijective pebble game [30, Lemma 8] to recast the proof for $\mathcal{T}_q^{(k_1,k_2)}$.

▶ **Theorem 4.11.** *The class $\mathcal{T}_q^{(k_1,k_2)}$ is homomorphism distinguishing closed.*

▶ Remark 4.12. In the language of [31] the proofs of Theorem 4.10 and Theorem 4.11 show that the respective classes are *closed under weak oddomorphisms* by [31, Theorem 3.13].

4.2 Invariance of subgraph counts

As an application of the previous results of homomorphism distinguishing closedness and its relation to logic, we provide characterizations of the logical invariance of subgraph counts with respect to hereditary graph structure as in [29].

For graphs G, F we denote by $\mathrm{sub}(F, G)$ the number of subgraphs of G which are isomorphic to F. We write $\mathrm{spasm}(F)$ for the set of homomorphic images of a graph F containing exactly one representative from each isomorphism class. For a logic L and a graph F we say that the function $\mathrm{sub}(F, \cdot)$ is L-*invariant* if for all graphs G and H the implication $G \equiv_{\mathsf{L}} H \implies \mathrm{sub}(F, G) = \mathrm{sub}(F, H)$ holds.

▶ **Theorem 4.13.** *Let $k_1 + k_2, q \geq 1$ and F be a graph. Then the following assertions hold:*
- *The function $\mathrm{sub}(F, \cdot)$ is $\wedge\mathsf{C}^{(k_1,k_2)}$-invariant if and only if $\mathrm{spasm}(F) \subseteq \mathcal{P}^{(k_1,k_2)}$.*
- *The function $\mathrm{sub}(F, \cdot)$ is $\mathsf{C}_q^{(k_1,k_2)}$-invariant if and only if $\mathrm{spasm}(F) \subseteq \mathcal{T}_q^{(k_1,k_2)}$.*

▶ Remark 4.14. By [30, Theorem 6] the second assertion of this theorem characterizes which subgraph counts are detected after q iterations of the (k_1, k_2)-dimensional oblivious Weisfeiler-Leman algorithm.

5 A comonadic account of requantification

In this section, we present variations of the pebble-relation comonad $\mathbb{PR}_k$ introduced in [27] and the pebbling comonad $\mathbb{P}_k$ from [2], with the aim of capturing requantification as a logical resource from a categorical perspective. We emphasize that, in order to achieve this, it suffices to adapt the definitions of the associated universes $\mathbb{PR}_{(k_1,k_2)}\mathcal{A}$ and $\mathbb{P}_{(k_1,k_2)}\mathcal{A}$. This approach highlights the versatility of the pebble-relation and pebbling comonads, enabling concise proofs building on previous work despite the significant differences among the corresponding graph classes.

For a sequence $\overline{s} = [(z_1, a_1), \ldots, (z_n, a_n)] \in ([x_{k_1}, y_{k_2}] \times V(\mathcal{A}))^n$ and $i \in [n]$ define $\pi_{\mathcal{A}}(\overline{s}, i) = z_i$. When no index i is given, we set $\pi_{\mathcal{A}}(\overline{s}) \coloneqq z_n$.

▶ **Definition 5.1.** *For a σ-structure $\mathcal{A}$ we define the σ-structure $\mathbb{PR}_{(k_1,k_2)}\mathcal{A}$ as follows:*

- *The universe of $\mathbb{PR}_{(k_1,k_2)}\mathcal{A}$ consists of all pairs $(\overline{s}, i) = ([(z_1, a_1), \ldots, (z_n, a_n)], i)$ for $n \in \mathbb{N}$, $i \in [n]$, and $(z_j, a_j) \in [x_{k_1}, y_{k_2}] \times V(\mathcal{A})$ for $j \in [n]$ such that every $y_j \in [y_{k_2}]$ appears at most once as pebble index in $\overline{s}$.*
- *The counit morphism $\varepsilon_\mathcal{A} \colon \mathbb{PR}_{(k_1,k_2)}\mathcal{A} \to \mathcal{A}$ is defined by $\varepsilon_\mathcal{A}([(z_1, a_1), \ldots, (z_n, a_n)], i) := a_i$*
- *For $R \in \sigma$ it holds $R^{\mathbb{PR}_{(k_1,k_2)}\mathcal{A}}((\overline{s}_1, i_1), \ldots, (\overline{s}_m, i_m))$ exactly if there exists $\overline{s}$ such that*
  - *for all $j \in [m]$ it holds $\overline{s}_j = \overline{s}$,* *(equality)*
  - *$\pi_\mathcal{A}(\overline{s}, i_j)$ does not appear in $\overline{s}[i_j + 1, i]$ for $i = \max\{i_1, \ldots, i_m\}$, and* *(active pebble)*
  - *$R^\mathcal{A}(\varepsilon_\mathcal{A}(\overline{s}_1, i_1), \ldots, \varepsilon_\mathcal{A}(\overline{s}_m, i_m))$.* *(compatibility)*

For a σ-structure $\mathcal{B}$ and a homomorphism $f \colon \mathbb{PR}_{(k_1,k_2)}\mathcal{A} \to \mathcal{B}$ we define the coextension *$f^* \colon \mathbb{PR}_{(k_1,k_2)}\mathcal{A} \to \mathbb{PR}_{(k_1,k_2)}\mathcal{B}$ of f by setting $b_i = f([(z_1, a_1), \ldots, (z_m, a_m)], i)$ for $i \in [m]$ and $f^*([(z_1, a_1), \ldots, (z_m, a_m)], i) := ([(z_1, b_1), \ldots, (z_m, b_m)], i)$.*

▶ **Definition 5.2.** *For a σ-structure $\mathcal{A}$ we define the σ-structure $\mathbb{P}_{(k_1,k_2)}\mathcal{A}$ as follows:*

- *The universe of $\mathbb{P}_{(k_1,k_2)}\mathcal{A}$ consists of all sequences $\overline{s} \in ([x_{k_1}, y_{k_2}] \times V(\mathcal{A}))^+$ such that every pebble $y_j \in [y_{k_2}]$ appears at most once as pebble index in $\overline{s}$.*
- *The counit morphism $\varepsilon_\mathcal{A} \colon \mathbb{P}_{(k_1,k_2)}\mathcal{A} \to \mathcal{A}$ is defined by $\varepsilon_\mathcal{A}([(z_1, a_1), \ldots, (z_m, a_m)]) := a_m$*
- *For $R \in \sigma$ it holds $R^{\mathbb{P}_{(k_1,k_2)}\mathcal{A}}(\overline{s}_1, \ldots, \overline{s}_m)$ exactly if*
  - *$R^\mathcal{A}(\varepsilon_\mathcal{A}(\overline{s}_1), \ldots, \varepsilon_\mathcal{A}(\overline{s}_m))$,* *(compatibility)*
  - *for $i, j \in [m]$ we have $\overline{s}_i \sqsubseteq \overline{s}_j$ or $\overline{s}_j \sqsubseteq \overline{s}_i$, and* *(comparability)*
  - *for $i, j \in [m]$ if $\overline{s}_i \sqsubseteq \overline{s}_j$ then $\pi_\mathcal{A}(\overline{s}_i)$ does not occur as a first coordinate in $\overline{s}_+$ for $\overline{s}_i = \overline{s}_j\overline{s}_+$.* *(active pebble)*

For a σ-structure $\mathcal{B}$ and a homomorphism $f \colon \mathbb{P}_{(k_1,k_2)}\mathcal{A} \to \mathcal{B}$ the coextension *f^* is defined by $f^*([(z_1, a_1), \ldots, (z_m, a_m)]) := [(z_1, b_1), \ldots, (z_m, b_m)]$ where $b_i = f([(z_1, b_1), \ldots, (z_i, a_i)])$ for $i \in [m]$. For $q \in \mathbb{N}_+$ the structure $\mathbb{P}^q_{(k_1,k_2)}\mathcal{A}$ is defined as the substructure of $\mathbb{P}_{(k_1,k_2)}\mathcal{A}$ over the universe $([x_{k_1}, y_{k_2}] \times V(\mathcal{A}))^{\leq q}$.*

To show that $\mathbb{PR}_{(k_1,k_2)}$ and $\mathbb{P}^q_{(k_1,k_2)}$ are again comonads it suffices to observe that the definitions of $R^{\mathbb{PR}_{(k_1,k_2)}\mathcal{A}}$, $R^{\mathbb{P}^q_{(k_1,k_2)}\mathcal{A}}$, and the coextensions are invariant under reusability constraints. Thus, from [27, Proposition 3.1] and [2, Theorem 4] we obtain the following:

▶ **Proposition 5.3.** *The triples $(\mathbb{P}^q_{(k_1,k_2)}, \varepsilon, (\cdot)^*)$ and $(\mathbb{PR}_{(k_1,k_2)}, \varepsilon, (\cdot)^*)$ are comonads in coKleisli form on $\mathbf{Str}(\sigma)$.*

Utilizing the characterization of the classes $\mathcal{T}_q^{(k_1,k_2)}$ and $\mathcal{P}^{(k_1,k_2)}$ in terms of pebble forest covers, we now provide a categorical account of reusability in path- and bounded depth tree decompositions. Similar results are called *coalgebra characterization theorems* in the literature of game comonads [4]. We have defined our decompositions in terms of graphs rather than relational structures. The definition of forest covers can be adapted for structures, which yields the same as considering forest covers of the *Gaifman graph* $G(\mathcal{A})$ of $\mathcal{A}$.

▶ **Theorem 5.4.** *For every finite σ-structure $\mathcal{A}$ there is a bijective correspondence between*

1. *(k_1, k_2)-pebble linear component forest covers of $G(\mathcal{A})$ and coalgebras $\alpha \colon \mathcal{A} \to \mathbb{PR}_{(k_1,k_2)}\mathcal{A}$.*
2. *(k_1, k_2)-pebble forest covers of depth q of $G(\mathcal{A})$ and coalgebras $\alpha \colon \mathcal{A} \to \mathbb{P}^q_{(k_1,k_2)}\mathcal{A}$.*

Further employing the theory of game comonads, we give a categorical formulation of equivalence in the counting logics $\wedge\mathsf{C}^{(k_1,k_2)}_{\infty\omega}$ and $\mathsf{C}^{(k_1,k_2)}_q$ as isomorphism in the coKleisli category. Accordingly, similar theorems are also called *isomorphism power theorems* in the literature. As in [27], we use the extended signature $\sigma^+ = \sigma \cup \{I\}$ to define the functor $J \colon \mathbf{Str}(\sigma) \to \mathbf{Str}(\sigma^+)$ which extends each σ-structure $\mathcal{A}$ by the identity relation $I^{J\mathcal{A}} = \{(a, a) : a \in V(\mathcal{A})\}$ and gives rise to a *relative comonad on J* (see [27]).

▶ **Theorem 5.5.** *For all σ-structures $\mathcal{A}$ and $\mathcal{B}$ the following hold:*

1. *There exists a coKleisli isomorphism $\mathbb{PR}_{(k_1,k_2)}J\mathcal{A} \to J\mathcal{B}$ if and only if $\mathcal{A} \equiv_{\wedge\mathsf{C}^{(k_1,k_2)}_{\infty\omega}} \mathcal{B}$.*
2. *There exists a coKleisli isomorphism $\mathbb{P}^q_{(k_1,k_2)}\mathcal{A} \to \mathcal{B}$ if and only if $\mathcal{A} \equiv_{\mathsf{C}^{(k_1,k_2)}_q} \mathcal{B}$.*

One of the contributions of game comonads is to provide a unified language for various relations from finite model theory. Specifically, in the remainder of this section we show that morphisms in the coKleisli category characterize preservation and winning strategies for logics and games without counting. We first introduce the notion of reusability to the *all-in-one* pebble game from [27] and the well-known *existential k-pebble game* from [24], allowing for a more fine grained analysis.

▶ **Definition 5.6.** *Let $\mathcal{A}, \mathcal{B}$ be σ-structures and $k_1 + k_2 \in \mathbb{N}_+$. The* all-in-one (k_1, k_2)-pebble game $\mathrm{AP}^{(k_1,k_2)}(\mathcal{A}, \mathcal{B})$ *and* $\exists$-(k_1, k_2)-pebble game *are defined as follows: Both games are played by the two players* Spoiler *and* Duplicator *on the structures $\mathcal{A}$ and $\mathcal{B}$ and start from a (possibly empty) position $\overline{s}_0 \in ([x_{k_1}, y_{k_2}] \times V(\mathcal{A}))^m, \overline{d}_0 \in ([x_{k_1}, y_{k_2}] \times V(\mathcal{B}))^m$ with the same pebble sequence in which each pebble pair occurs at most once.*

In each round $n \in \mathbb{N}_+$ of the $\exists$-(k_1, k_2)-pebble game, the following steps are performed:

1. *Spoiler picks a pebble $z_n \in [x_{k_1}, y_{k_2}]$ such that z_n is not yet placed or $z_n \in [x_{k_1}]$ and places it on an element $a_n \in V(\mathcal{A})$.*
2. *Duplicator places the pebble z_n on an element $b_n \in V(\mathcal{B})$.*

This induces sequences $\overline{s} = [(z_1, a_1), \dots, (z_n, a_n)]$ and $\overline{d} \coloneqq [(z_1, b_1), \dots, (z_n, b_n)]$ of placements after round n.

In the single round of the game $\mathrm{AP}^{(k_1,k_2)}(\mathcal{A}, \mathcal{B})$, the following steps are performed:

1. *Spoiler chooses a sequence $\overline{s} = [(z_1, a_1), \dots, (z_n, a_n)] \in ([x_{k_1}, y_{k_2}] \times V(\mathcal{A}))^n$ such that each $y_j \in [y_{k_2}]$ occurs at most once in $\overline{z} = (z_1, \dots, z_n)$ and not in $\overline{s}_0$.*
2. *Duplicator responds with a sequence $\overline{d} \coloneqq [(z_1, b_1), \dots, (z_n, b_n)] \in ([x_{k_1}, y_{k_2}] \times V(\mathcal{B}))^n$.*

The winning condition for both games is the following: Duplicator wins the game if for all $i \in [m + n]$ the function η_i defined by setting $\eta_i(\mathrm{last}_z(\overline{s}_0\overline{s}[1, i])) \coloneqq \mathrm{last}_z(\overline{d}_0\overline{d}[1, i])$ for each $z \in [x_{k_1}, y_{k_2}]$ is a partial homomorphism between $\mathcal{A}$ and $\mathcal{B}$.

The logic $\exists^+\mathsf{L}^{(k_1,k_2)}$ is defined as the fragment of existential positive first-order logic (i.e. no universal quantification and negation) over the variable set $[x_{k_1}, y_{k_2}]$ such that only variables from $[x_{k_1}]$ are requantified. We obtain $\exists^+\wedge\mathsf{L}^{(k_1,k_2)}$ by additionally requiring that every conjunction is restricted and $\exists^+\mathsf{L}^{(k_1,k_2)}_q$ by bounding the quantifier-rank by q.

▶ **Proposition 5.7.** *For all σ-structures $\mathcal{A}$ and $\mathcal{B}$ the following hold:*

1. *Duplicator wins $\mathrm{AP}^{(k_1,k_2)}(\mathcal{A}, \mathcal{B})$ if and only if $\mathcal{A} \Rrightarrow_{\exists^+\wedge\mathsf{L}^{(k_1,k_2)}} \mathcal{B}$.*
2. *Duplicator wins q rounds of the $\exists$-(k_1, k_2)-pebble game if and only if $\mathcal{A} \Rrightarrow_{\exists^+\mathsf{L}^{(k_1,k_2)}_q} \mathcal{B}$.*

Finally, we also obtain what is called a *morphism power theorem* for game comonads with restricted reusability.

▶ **Theorem 5.8.** *For all σ-structures $\mathcal{A}$ and $\mathcal{B}$ the following hold:*

1. *There exists a coKleisli morphism $f \colon \mathbb{PR}_{(k_1,k_2)}\mathcal{A} \to \mathcal{B}$ if and only if $\mathcal{A} \Rrightarrow_{\exists^+\wedge\mathsf{L}^{(k_1,k_2)}} \mathcal{B}$.*
2. *There exists a coKleisli morphism $f \colon \mathbb{P}^q_{(k_1,k_2)}\mathcal{A} \to \mathcal{B}$ if and only if $\mathcal{A} \Rrightarrow_{\exists^+\mathsf{L}^{(k_1,k_2)}_q} \mathcal{B}$.*

6 Conclusion

In this work, we extended the analysis of homomorphism indistinguishability to graph classes characterized by graph decompositions with restricted reusability. We demonstrate how decomposition-based approaches offer robust and adaptable techniques for characterizing indistinguishability relations as well as for establishing homomorphism distinguishing closedness. Moreover, by integrating these results within the broader framework of game comonads, we present a unified categorical perspective on the role of requantification in finite variable counting logics. We list some open questions for future work:

- The homomorphism indistinguishability relation $\equiv_{\mathcal{T}_q^{(k_1,k_2)}}$ can be decided by a more space-efficient variant of the (k_1+k_2)-dimensional Weisfeiler-Leman algorithm [30]. It would be interesting to develop a *linearized variant* to efficiently decide the relation $\equiv_{\mathcal{P}^{(k_1,k_2)}}$. The logic $\exists^+\mathsf{L}^k$ is closely related to the *k-consistency algorithm* for solving *constraint satisfaction problems*, so it might be fruitful to explore whether restricting reusability yields improved algorithmic techniques.
- Motivated by the constructive nature of our results, it is natural to ask for more connections between model-comparison and pursuit-evasion games. Specifically, identifying broader classes of games that align with logical equivalences could yield more results on characterizations and h.d. closedness.
- While we have established h.d. closedness for specific graph classes, a comonadic treatment of this property could provide a deeper understanding of its categorical structure. In particular, investigating whether h.d. closedness can be characterized via coalgebraic properties of game comonads might reveal fundamental principles governing homomorphism indistinguishability of relational structures as in [11].

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