Improved Approximation Algorithms for Capacitated Vehicle Routing with Fixed Capacity

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Abstract

The Capacitated Vehicle Routing Problem (CVRP) is one of the most extensively studied problems in combinatorial optimization. Based on customer demand, we distinguish three variants of CVRP: unit-demand, splittable, and unsplittable. In this paper, we consider k-CVRP in general metrics and on general graphs, where k is the vehicle capacity. All three versions are APX-hard for any fixed $k \geq 3$. Assume that the approximation ratio of metric TSP is $3/2$. We present a $(5/2 - \Theta(\sqrt{1/k}))$-approximation algorithm for the splittable and unit-demand cases, and a $(5/2 + \ln 2 - \Theta(\sqrt{1/k}))$-approximation algorithm for the unsplittable case. Our approximation ratio is better than the previous results when k is less than a sufficiently large value, approximately 1.7×10^7.

For small values of k, we design independent and elegant algorithms with further improvements. For the splittable and unit-demand cases, we improve the approximation ratio from 1.792 to 1.500 for $k = 3$, and from 1.750 to 1.500 for $k = 4$. For the unsplittable case, we improve the approximation ratio from 1.792 to 1.500 for $k = 3$, from 2.051 to 1.750 for $k = 4$, and from 2.249 to 2.157 for $k = 5$. The approximation ratio for $k = 3$ surprisingly achieves the same value as in the splittable case. Our techniques, such as EX-ITP – an extension of the classic ITP method, have the potential to improve algorithms for other routing problems as well.

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1 Introduction

In the Capacitated Vehicle Routing Problem (CVRP), we are given an undirected complete graph $G = (V \cup \{v_0\}, E)$ with edge weights w satisfying the symmetric and triangle inequality properties. The n nodes in $V = \{v_1, \dots, v_n\}$ represent n customers and each customer v_i has a demand $d_i \in \mathbb{Z}_{\geq 1}$. A vehicle with a capacity of $k \in \mathbb{Z}_{\geq 1}$ is initially located at the depot v_0. A tour is a walk that begins and ends at the depot and the sum of deliveries to all customers in it is at most k. The distance of a tour is the sum of the weights of edges in the tour. In CVRP, we wish to find a set of tours to satisfy the demand of every customer with minimum total distance. We use k-CVRP to denote the problem where the capacity k is a

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fixed integer. In the *unsplittable* version of the problem, the demand of each customer can only be delivered by a single tour. In the *splittable* version, the demand of each customer can be delivered by more than one tour. If the demand of each customer is 1, it is called the *unit-demand* version.

Since CVRP was raised by Dantzig and Ramser [12] in 1959, it has become a very famous problem with numerous applications in combinatorial optimization. It has been widely studied in both theory and application. Readers can refer to a survey [36] for its applications and fast solvers in practice. In theory, it is a rather rich problem in approximation algorithms [8, 7, 30, 31, 11, 14].

When $k = 1$ or $k = 2$, k-CVRP can be solved in polynomial time [4]. However, for each fixed $k \geq 3$, the problem becomes APX-hard, even for the unit-demand case [3]. A classic algorithm known as *Iterated Tour Partitioning (ITP)* was introduced about 40 years ago [19]. ITP is not only efficient in practice but also achieves strong theoretical approximation guarantees. Assuming we are given an α-approximation algorithm for metric TSP, for both splittable k-CVRP and unit-demand k-CVRP, ITP achieves an approximation ratio of $\alpha + 1 - \alpha/k$ [19]. For unsplittable k-CVRP, a variant of ITP called UITP achieves an approximation ratio of $\alpha + 2 - 2\alpha/k$ for even k and $\alpha + 2 - \alpha/k$ for odd k [1].

For metric TSP, there is a well-known 3/2-approximation algorithm [9, 35]. Although a recent breakthrough by Karlin, Klein, and Oveis Gharan [24, 25] has improved the approximation ratio to $3/2-\varepsilon$, where $\varepsilon \approx 10^{-36}$, this improvement is too small to significantly affect the analysis for k-CVRP. Thus, for ease of comparison, we may continue to assume an approximation ratio of $\alpha = 3/2$ for metric TSP in our algorithms.

Due to its simplicity and versatility, the ITP algorithm has been adapted for various other vehicle routing problems [29]. However, for general metrics, there have been few improvements over ITP in approximating k-CVRP.

One interesting improvement was done by Bompadre *et al.* [8] about 20 years ago. For any $\alpha \geq 1$, they improved the approximation ratio by a term of $\frac{1}{3k^3}$ for all three versions of k-CVRP. Specifically, for $\alpha = 3/2$, the improvement was $\frac{1}{4k^2}$ for the splittable and unit-demand cases, and $\frac{1}{3k^2}$ for the unsplittable case. These results are still the best-known approximation ratios for many small values of k. Recently, one significant progress was done by Blauth *et al.* [7]. They improved the approximation ratio to $\alpha+1-\varepsilon$ for the splittable and unit-demand cases, and to $\alpha + 2 - 2\varepsilon$ for the unsplittable case, where ε is a small value related to α, with $\varepsilon > \frac{1}{3000}$ when $\alpha = 3/2$. While this provides a slight improvement in the constant part of the approximation ratio, it does not outperform the approximation ratio in [8] for small values of k. Friggstad *et al.* [15] proposed two further improvements for unsplittable k-CVRP. The first is an $(\alpha + 1.75)$-approximation algorithm using a combinatorial method, while the second is an $(\alpha + \ln 2 + \frac{1}{1-\delta})$-approximation algorithm based on LP rounding, with a running time of $n^{O(1/\delta)}$. They also showed that both approximation ratios can be further improved by a small constant ε' through the method from [7].

For small k, k-CVRP has garnered independent interest [2, 23, 8]. For unit-demand 3-CVRP, ITP achieves an approximation ratio of 2. Based on the 25/33-approximation algorithm for MAX TSP on general graphs [23], Bazgan *et al.* [5] proposed a 1.990-approximation algorithm, marking the first known improvement. Leveraging the best-known 4/5-approximation algorithm for MAX TSP [13], the approximation ratio of their algorithm can be further improved to 1.934. For unit-demand 4-CVRP, ITP achieves an approximation ratio of 2.125. Anily and Bramel [2] proposed an approximation algorithm for *Capacitated TSP with Pickups and Deliveries*, demonstrating that their algorithm can be applied to unit-demand k-CVRP. Their approach achieves an improved approximation ratio only for $k = 4$, with an approximation ratio of 1.750, which remains the best-known result.

Beyond the aforementioned algorithms, another approach for solving k-CVRP with $k = O(1)$ is to reduce it to the more general *minimum weight k-set cover problem* [10, 21, 18]. By treating each feasible tour as a k-set, this reduction applies to both unit-demand k-CVRP and unsplittable k-CVRP. For splittable k-CVRP, it can be shown that it is equivalent to unit-demand k-CVRP when $k = n^{O(1)}$. Since there are $n^{O(k)}$ feasible tours, the reduction is polynomial for $k = O(1)$. Gupta *et al.* [18] improved the approximation ratio for the minimum weight k-set cover problem to $\min\{H_k - \frac{1}{8k},\ H_k - \sum_{i=1}^{k} \frac{\log i}{8ki}\}$, where $H_k = \sum_{i=1}^{k} \frac{1}{i}$ is the k-th harmonic number. For $k = 3$, this achieves an approximation ratio of 1.792, surpassing all previous results for both splittable 3-CVRP and unit-demand 3-CVRP. Furthermore, this algorithm yields the best-known approximation ratio for unsplittable k-CVRP when $k \leq 9$.

In this paper, we focus on approximation algorithms for k-CVRP with bounded k. As highlighted in [6, 8], many practical problems involve small values of k. For example, a logistics company raised the need to transport newly produced cars using a truck with a capacity of at most six cars, where the capacity $k = 6$. In benchmark datasets, the capacities of the instances are also typically in the range of hundreds or thousands [37]. These examples demonstrate the practical relevance and significance of studying k-CVRP with bounded k.

1.1 Our Contributions

We present several algorithms for k-CVRP, improving the best-known approximation ratios for any $k \leq 1.7 \times 10^7$. This threshold is quite large, and to our knowledge, the capacity of all practical and artificial instances does not exceed this value. We summarize our contributions to splittable (including unit-demand) k-CVRP and unsplittable k-CVRP separately below.

For splittable k-CVRP and unit-demand k-CVRP, we have the following contributions.

1. Based on a new concept of *home-edges*, which are edges incident to the depot in an optimal solution, we obtain tighter lower bounds for k-CVRP based on the minimum weight spanning tree and several different kinds of cycle covers (Sect. 3). Our analysis shows that if the lower bound used for the connection part of ITP in [1, 19] is tight, the optimal solution weight (denoted by OPT) will be dominated by home-edges, enabling the computation of a Hamiltonian cycle with cost OPT and a cycle cover with zero cost in polynomial time. Since the upper bound on the cost of the Hamiltonian cycle in [1, 19] is $\frac{3}{2}$OPT, we can obtain significant improvements for this case. Otherwise, since the lower bound for the connection part is not tight, we obtain improvements as well. Note that our lower bounds are also suitable for the unsplittable case.
2. The classic ITP algorithm is based on a given Hamiltonian cycle of the graph. We extend ITP to an algorithm based on any cycle cover, called EX-ITP. One advantage is that an optimal cycle cover is polynomially computable [20]. Based on EX-ITP with our new lower bounds, we can quickly improve the approximation ratio from 1.792 to 1.500 for 3-CVRP and from 1.750 to 1.667 for 4-CVRP. Then, based on a good structural property related to perfect matchings, we can surprisingly improve the approximation ratio to 1.500 for 4-CVRP (Sect. 5). Then, to obtain improvements for larger k, we also consider *mod-k-cycle covers* (the length of each cycle in it is divisible by k). We show that given an α-approximation algorithm for metric TSP, by making a trade-off among two Hamiltonian cycles and a cycle cover with ITP and EX-ITP, we can achieve an approximation ratio of $\alpha + 1 - \alpha/k - \Theta(1/k)$, improving the previous approximation ratio of $\alpha + 1 - \alpha/k - \max\{\Omega(1/k^3), \varepsilon\}$ [7, 8] for small k (Sect. 6.1)[2]. Our approximation ratio achieves $5/2 - 3.5/k$ when $\alpha = 3/2$. If α is smaller, we obtain better results.

[2] With a more careful analysis, the previous results may be further improved slightly. However, the improvement is small and not mentioned in any published paper.

Table 1 Splittable k-CVRP and unit-demand k-CVRP: previous and our approximation ratios for different values of k under $\alpha = 3/2$, where the best approximation ratios are marked in bold.

k	3	4	5	6	7	8	9	10
Previous Results	1.792 [18]	1.750 [2]	2.188 [8]	2.242 [8]	2.280 [8]	2.308 [8]	2.330 [8]	2.348 [8]
Our Results in Sect. 5	**1.500**	**1.500**	-	-	-	-	-	-
Our Results in Sect. 6.2	1.667	1.750	**1.800**	**1.875**	**1.929**	**1.969**	**2.000**	**2.025**

k	29	...	5833	5834	...	1.7×10^7	1.8×10^7
Previous Results	2.44795 [7]	... ...	2.49941 [7]	2.49941 [7]	... ...	2.49967 [7]	**2.49967** [7]
Our Results in Sect. 6.2	**2.22414**	**...**	**2.48140**	**2.48141**	**...**	**2.49966**	2.49967

3. The best approximation ratio for metric TSP is still about 3/2. Under $\alpha = 3/2$, the current best approximation ratio of k-CVRP is approximately $5/2 - 1.5/k - \max\{\Omega(1/k^2), 1.005/3000\}$ [7, 8]. By generalizing the concept of home-edges, we improve two previously-used lower bounds. Based on these bounds and a detailed analysis of ITP, we derive an approximation ratio of $5/2 - \Theta(\sqrt{1/k})$, which improves the previous results for any $k \leq 1.7 \times 10^7$ (Sect. 6.2). The exact approximation ratio is presented in Theorem 24, and some numerical values for specific k are shown in Table 1.

For unsplittable k-CVRP, we achieve similar improvements, though additional techniques are required. A summary of our results is provided in Table 2. Due to limited space, we omit the details for the unsplittable case but outline our main ideas and contributions below.

1. We first propose a refined UITP, improving its approximation ratio for fixed k. Building on this, we extend UITP to EX-UITP, similar to the EX-ITP approach. However, due to the unsplittable constraint, EX-UITP requires that each customer's demand cannot be too large. For cycles involving large-demand customers, we assign a single tour to each, then apply shortcutting to create a simplified cycle before applying EX-UITP. While shortcutting can significantly alter a cycle's structure, we carefully analyze these cycles' local properties and address them separately. Using this approach, along with ideas from algorithms for splittable 3-CVRP and 4-CVRP, we improve the approximation ratios for unsplittable 3-CVRP to 1.500 and 4-CVRP to 1.750.
2. For larger k, we combine the refined UITP with the LP-based technique from [15] to obtain the LP-UITP algorithm. By making a trade-off between two Hamiltonian cycles using LP-UITP[3], we obtain an approximation ratio of $\alpha + 1 + \ln 2 - 2\alpha/k - \Theta(1/k)$, improving the previous approximation ratio of $\alpha + 1 + \ln 2 - 2\alpha/k - \varepsilon'$ [15] for small k.
3. Based on the refined two lower bounds (provided in Sect. 6.2) and using a deep analysis on LP-UITP, we achieve an approximation ratio of $5/2 + \ln 2 - \Theta(\sqrt{1/k})$, improving the previous approximation ratio of $5/2 + \ln 2 + \ln(1 - 1.005/3000) - 3/k$ in [15] for any $k \leq 1.7 \times 10^7$. Additionally, for unsplittable 5-CVRP, we further refine the analysis to improve the approximation ratio to 2.157.

In Tables 1 and 2, the previous results are $\alpha + 1 - \alpha/k - \max\{\Omega(1/k^3), \varepsilon\}$ and $\alpha + 1 + \ln 2 - 2\alpha/k - \varepsilon'$ for splittable and unsplittable k-CVRP, respectively, as well as $\min\{H_k - \frac{1}{8k}, H_k - \sum_{i=1}^{k} \frac{\log i}{8ki}\}$ for both versions. The constant behind Ω is available in [8]. The values ε and ε'

[3] For the unsplittable case, we focus on the trade-off between two algorithms instead of three, as the third algorithm based on cycle cover is not applicable here.

Table 2 Unsplittable k-CVRP: previous and our approximation ratios for different values of k under $\alpha = 3/2$, where the best approximation ratios are marked in bold.

k	3	4	5	6	7	8	9	10
Previous	1.792	2.051	2.249	2.416	2.558	2.684	2.795	2.893
Results	[18]	[18]	[18]	[18]	[18]	[18]	[18]	[15]
Our Results in the full version	**1.500**	**1.750**	**2.157**	**2.163**	**2.343**	**2.337**	**2.471**	**2.448**

k	11720	11721	11722	...	...	1.7×10^7	1.8×10^7
Previous	3.19256	3.19269	3.19256	...	...	3.19282	**3.19282**
Results	[15]	[15]	[15]	...	...	[15]	[15]
Our Results in the full version	**3.17973**	**3.17981**	**3.17973**	**...**	**...**	**3.19281**	3.19282

depending on α and k can be calculated using the results in [7, 15]. We adopt $\alpha = 3/2$. For this case, the values ε and ε' are less than $1.005/3000$ and $-\ln(1 - 1.005/3000)$, respectively. Hence, the approximation ratio of splittable k-CVRP and unit-demand k-CVRP in [7] is at least $5/2 - 1.005/3000 - 1.5/k$ for $k \geq 3$. The approximation ratio of unsplittable k-CVRP in [15] is at least $5/2 + \ln 2 + \ln(1 - 1.005/3000) - 3/k$ for even $k \geq 3$. Note that for unsplittable k-CVRP with odd $k \geq 3$, we need to double the capacity and the demand, resulting in a slightly worse approximation ratio of at least $5/2 + \ln 2 + \ln(1 - 1.005/3000) - 1.5/k$.

Due to limited space, we only present our results for the splittable and unit-demand cases, and that for the unsplittable case are omitted. The proofs of lemmas and theorems marked with "*" are also omitted. The full version is available at `https://arxiv.org/abs/2210.16534`.

2 Definitions, Assumptions, and Notations

A *walk* in a graph is a succession of edges in the graph, where the same edge can appear more than one time. We will use a sequence of vertices to denote a walk: $(v_1, v_2, v_3, \ldots, v_l)$ means a walk with edges (v_1, v_2), (v_2, v_3), and so on. A *path* in a graph is a walk such that no vertex appears twice in the sequence, and a *cycle* is a walk such that only the first and the last vertices are the same. A cycle containing l edges is called an *l-cycle* and the *length* of it is l. Two subgraphs (or two sets of edges) are *vertex-disjoint* if they do not have a common vertex. Given an edge-weighted graph, where the number n of vertices is a multiple of k, a *minimum weight k-cycle cover* is a set of exactly n/k vertex-disjoint k-cycles with the minimum total weight of edges in the k-cycles in the set. A *minimum weight mod-k-cycle cover* (resp., *minimum weight mod-k-tree cover*) is a set of vertex-disjoint cycles (resp., trees) such that the length of each cycle (resp., the number of vertices on each tree) is divisible by k, each vertex of the graph appears in exactly one cycle, and the total weight of edges in the cycles in the set is minimized. A *minimum weight cycle cover* is a set of vertex-disjoint cycles such that the length of each cycle is at least three, each vertex of the graph appears in exactly one cycle, and the total weight of edges in the cycles in the set is minimized.

Problem Definitions. We use $G = (V \cup \{v_0\}, E)$ to denote a complete graph, where the vertex v_0 represents the depot and vertices in V represent customers. There is a non-negative weight function $w : E \to \mathbb{R}_{\geq 0}$ on the edges in E, which denotes the distance between two endpoints of the edge. The weight function w is a metric function, i.e., it satisfies the symmetric and triangle inequality properties. For any weight function $w : X \to \mathbb{R}_{\geq 0}$, we extend it to subsets of X by defining $w(Y) = \sum_{x \in Y} w(x)$ for any $Y \subseteq X$. An *itinerary* is a walk starting and ending at vertex v_0. It can be partitioned into several minimal itineraries containing v_0, each of which is called a *tour*.

The Capacitated Vehicle Routing Problem (CVRP) can be described as follows.

▶ **Definition 1.** *An instance $(G = (V \cup \{v_0\}, E), w, d, k)$ of CVRP consists of:*

- *a complete graph G, where $V = \{v_1, \ldots, v_n\}$ represents the n customers and v_0 represents the depot;*
- *a metric weight function on edges w: $(V \cup \{v_0\}) \times (V \cup \{v_0\}) \to \mathbb{R}_{\geq 0}$, which represents the distances;*
- *the demand of each customer $d = (d_1, \ldots, d_n)$, where $d_i \in \mathbb{Z}_{\geq 1}$ is the demand required by customer $v_i \in V$;*
- *the capacity $k \in \mathbb{Z}_{\geq 1}$ of the vehicle that initially stays at the depot v_0.*

A feasible solution is an itinerary such that

- *each tour delivers at most k of the demand to customers on the tour;*
- *the union of tours meets the demand of every customer.*

The goal is to find such an itinerary I, minimizing the total distances of the succession of edges in the walk, i.e., $w(I) \coloneqq \sum_{e \in I} w(e)$.

We define three variants of the problem. If the demand of each customer must be delivered in one tour, we call it *unsplittable CVRP*. If the demand of a customer can be split into multiple tours, we call it *splittable CVRP*. If the demand of each customer is a unit, we call it *unit-demand CVRP*.

Some Assumptions. We make some assumptions which can be guaranteed by simple observations or polynomial-time reductions (see the full version of this paper).

▶ **Assumption 1.** *In an optimal itinerary, each tour is a cycle, and in each tour, the vehicle delivers an integer amount of the demand to each customer.*

▶ **Assumption 2.** *For splittable k-CVRP, we have $d_i \leq (n-1)(k-1)$ for each $i \in \{1, \ldots, n\}$; for unsplittable k-CVRP, we have $d_i < k$ for each $i \in \{1, \ldots, n\}$.*

For a customer with d_i of the demand in splittable k-CVRP, we take it as d_i customers with unit-demand. Then, we obtain an equivalent instance of unit-demand k-CVRP. By Assumption 2, splittable k-CVRP can be reduced to unit-demand k-CVRP in polynomial time.

▶ **Assumption 3.** *There is an optimal itinerary where every tour delivers exactly k of the demand.*

Assumption 3 can be ensured by adding some dummy customers at the depot. It guarantees the existence of an optimal solution with a good structure. It will be helpful in our analysis. For unit-demand k-CVRP, such an itinerary consists of a set of $(k+1)$-cycles intersecting only at the depot.

Some Important Notations. The following notations are illustrated with the unit-demand case. Most of them will be used to establish some lower bounds for our problems.

- I^*: an optimal solution to our problem;
- Δ: the sum of the weight of the edges from the depot v_0 to the customer, i.e., $\sum_{v_i \in V} w(v_0, v_i)$;
- H^*: a minimum weight Hamiltonian cycle on $V \cup \{v_0\}$;
- H_{CS}: the Hamiltonian cycle on $V \cup \{v_0\}$ obtained by the Christofides-Serdyukov algorithm [9, 35];

- $\mathcal{M}^*$: a minimum weight perfect matching in $G[V]$;
- MST: the total weight of the edges in a minimum weight spanning tree in G;
- $\mathcal{C}^*$: a minimum weight cycle cover in $G[V]$;
- $\mathcal{C}^*_k$: a minimum weight k-cycle cover in $G[V]$;
- $\mathcal{C}^*_{\text{mod } k}$: a minimum weight mod-$k$-cycle cover in $G[V]$.

For an instance $G = (V \cup \{v_0\}, E)$ of splittable or unsplittable CVRP, we construct a corresponding unit-demand instance $G' = (V' \cup \{v_0\}, E')$ by replacing each customer with demand d_i with d_i unit-demand customers. By Assumption 2, this reduction can be done in polynomial time when $k = O(1)$. It is easy to observe that the optimal value in G' is at most the optimal value in G. Thus, for an instance G of splittable or unsplittable CVRP, the lower bound computed in the unit-demand instance G' also constitutes a valid lower bound for G.

Although the above notations are defined for the unit-demand case, they can also be applied to the splittable and unsplittable cases. For an instance $G = (V \cup \{v_0\}, E)$ of splittable or unsplittable CVRP, the above notations, including Δ, H^*, H_{CS}, $\mathcal{M}^*$, MST, $\mathcal{C}^*$, $\mathcal{C}^*_k$, and $\mathcal{C}^*_{\text{mod } k}$, are defined based on the instance G'.

Thus, in the splittable and unsplittable cases, Δ becomes the sum of each customer's demand multiplied by the weight of the edge from v_0 to the customer, i.e., $\sum_{v_i \in V} d_i w(v_0, v_i)$, $\mathcal{C}^*_k$ represents a minimum weight k-cycle cover in $G'[V']$, and so on. Note that, for H^*, H_{CS}, and MST, there is no difference between G and G', so we do not distinguish between them.

We also mention the following. A minimum weight perfect matching in a complete graph can be found in $O(n^3)$ time using classical algorithms [16, 27], although faster and simpler algorithms exist for other graphs; see, for example, Schrijver's book [34]. A minimum weight spanning tree in a complete graph can be computed in $O(n^2)$ time using Prim's algorithm [33], or in optimal time using the algorithm of Pettie and Ramachandran [32]. It is NP-hard to compute a minimum weight k-cycle cover for any $k \geq 3$ [26]. There exists an $O(n^2 \log n)$-time 2-approximation algorithm for the minimum weight mod-k-cycle cover problem based on the primal-dual method [17]. Additionally, a minimum weight cycle cover can be computed via a reduction to the minimum weight perfect matching problem, for which more efficient algorithms are also known [20, 34]. These results will be used in our algorithms.

For an optimal itinerary I^*, the edges incident to v_0 in I^* are called *home-edges*, and the set of home-edges of I^* is denoted by $h(I^*)$. We define χ as the proportion of the weight of home-edges in I^*, i.e., $\chi = \frac{w(h(I^*))}{w(I^*)}$, where we assume $w(I^*) \neq 0$ to exclude the trivial case.

3 Lower Bounds on Unit-Demand CVRP

In this section, we study lower bounds related to Δ, H^*, H_{CS}, $\mathcal{M}^*$, MST, $\mathcal{C}^*_k$, $\mathcal{C}^*_{\text{mod } k}$, and $\mathcal{C}^*$. As previously mentioned, the optimal value for G' is at most the optimal value for G for an instance G of splittable or unsplittable CVRP. Therefore, the lower bounds we establish for the unit-demand case in this section will hold for all three versions of CVRP. We now assume that the problem under consideration is unit-demand CVRP and proceed to prove some lower bounds.

We will use the concept of χ to derive some refined lower bounds, which were not considered in previous work. The first lower bound, related to the minimum weight Hamiltonian cycle H^* on $V \cup \{v_0\}$, has been used in most previous papers.

▶ **Lemma 2** ([19, 1]). *$w(I^*) \geq w(H^*)$.*

▶ **Lemma 3.** *$\frac{k}{2} w(I^*) \geq (\chi + \frac{k-2}{2}) w(I^*) \geq \Delta$.*

Proof. Since $0 \leq \chi \leq 1$, we have $\frac{k}{2} = (1 + \frac{k-2}{2}) \geq (\chi + \frac{k-2}{2})$, which proves the first inequality.

Now, we show the second inequality. By Assumption 3, I^* consists of a set of $(k+1)$-cycles. We consider an arbitrary $(k+1)$-cycle $C = (v_0, v_1, \ldots, v_k, v_0)$ in I^*. Since $k \geq 3$, the triangle inequality implies that $w(C) \geq 2w(v_0, v_i)$ for each $i \in \{2, 3, \ldots, k-1\}$. Thus, we have

$$\sum_{i=1}^{k} w(v_0, v_i) = w(h(C)) + \sum_{i=2}^{k-1} w(v_0, v_i) \leq w(h(C)) + \frac{k-2}{2} w(C).$$

Summing the above inequality over all cycles in I^*, we obtain

$$\begin{aligned}\Delta = \sum_{v_i \in V} w(v_0, v_i) &\leq \sum_{C \in I^*} \left(w(h(C)) + \frac{k-2}{2} w(C) \right) \\ &= \chi w(I^*) + \frac{k-2}{2} w(I^*) = \left(\chi + \frac{k-2}{2} \right) w(I^*),\end{aligned}$$

as desired. ◀

▶ **Lemma 4.** $(1 - \frac{1}{2}\chi) w(I^*) \geq MST$.

Proof. For each tour in I^*, there are exactly two home-edges. We can obtain a spanning tree of the graph from I^* by deleting the longer home-edge in each tour. Since $w(h(I^*)) = \chi w(I^*)$ by definition, the weight of this spanning tree is at least $w(I^*) - \frac{1}{2}\chi w(I^*) = (1 - \frac{1}{2}\chi) w(I^*)$, which is at least the weight of the minimum weight spanning tree. ◀

Recall that H_{CS} is obtained by the Christofides-Serdyukov algorithm.

▶ **Lemma 5** ([9, 35])**.** $MST + \frac{1}{2} w(H^*) \geq w(H_{CS})$.

By Lemmas 2, 4, and 5, we can obtain the following result.

▶ **Lemma 6.** $\frac{3-\chi}{2} w(I^*) \geq w(H_{CS})$.

▶ **Lemma 7.** $\min\{2(1-\chi), 1\} w(I^*) \geq w(\mathcal{C}_k^*) \geq w(\mathcal{C}_{\text{mod } k}^*) \geq w(\mathcal{C}^*)$.

Proof. We first show that there exists a k-cycle cover in $G[V]$ whose weight is at most $\min\{2(1-\chi), 1\} w(I^*)$.

By Assumption 3, I^* consists of a set of $(k+1)$-cycles. Consider an arbitrary $(k+1)$-cycle $C = (v_0, v_1, \ldots, v_k, v_0)$ in I^*. Let $C' = (v_1, \ldots, v_k, v_1)$ be the k-cycle obtained by shortcutting v_0 from C. By the triangle inequality, we have $w(v_1, v_k) \leq \min\{w(v_0, v_1) + w(v_0, v_k), w(C) - w(v_0, v_1) - w(v_0, v_k)\}$. Thus, we have $w(C') = w(C) - w(v_0, v_1) - w(v_0, v_k) + w(v_1, v_k) \leq \min\{w(C), 2w(C) - 2w(v_0, v_1) - 2w(v_0, v_k)\}$. Note that the edges (v_0, v_1) and (v_0, v_k) are home-edges of the cycle C. By summing the above inequality over all cycles in I^*, we obtain the desired result.

Since $\mathcal{C}_k^*$ is the minimum weight k-cycle cover in $G[V]$, we have $w(\mathcal{C}_k^*) \leq \min\{2(1-\chi), 1\} w(I^*)$. Moreover, since any k-cycle cover is a mod-k-cycle cover, and any mod-k-cycle cover is also a cycle cover, we have the relationship $w(\mathcal{C}^*) \leq w(\mathcal{C}_{\text{mod } k}^*) \leq w(\mathcal{C}_k^*)$. ◀

Since H_{CS} is a $\frac{3}{2}$-approximate Hamiltonian cycle [9, 35], some papers [1, 19] used the inequalities $\Delta \leq \frac{k}{2} w(I^*)$ and $w(H_{CS}) \leq \frac{3}{2} w(I^*)$ in the analysis of ITP and UITP. However, if the upper bound of Δ is tight, i.e., $\Delta = \frac{k}{2} w(I^*)$, we have $\chi = 1$ by Lemma 3. In this case, by Lemma 6, we also have $w(H_{CS}) \leq w(I^*)$. Therefore, our new lower bounds show that these two bounds cannot be tight simultaneously, indicating the potential for better approximation ratios. Besides, if $\chi = 1$, by Lemma 7, we have $w(\mathcal{C}_k^*) = w(\mathcal{C}_{\text{mod } k}^*) = w(\mathcal{C}^*) = 0$. So, any $O(1)$-approximate cycle covers may outperform the optimal Hamiltonian cycle.

Before showing how these insights can lead to improved approximation ratios, we first review the ITP algorithms [1, 19] that operate with a Hamiltonian cycle, and then we will propose our EX-ITP algorithm, which works for any cycle cover.

4 The ITP Algorithms

ITP (Iterated Tour Partitioning) is a frequently used technique for unit-demand CVRP. The main idea of ITP is to construct feasible solutions for CVRP based on given Hamiltonian cycles: first split the Hamiltonian cycle into several connected pieces of length at most k and then construct a tour for each piece. There are two versions of the ITP algorithm for CVRP according to the Hamiltonian cycle containing the depot v_0 or not [1, 19].

▶ **Lemma 8** ([1]). *Given a Hamiltonian cycle H on $V \cup \{v_0\}$ as part of the input, for unit-demand k-CVRP with any $k \geq 3$, the AG-ITP algorithm in $O(nk)$ time outputs a solution with weight at most $(2/k)\Delta + (1-1/k)w(H)$.*

▶ **Lemma 9** ([19]). *Given a Hamiltonian cycle H on V as part of the input, for unit-demand k-CVRP with any $k \geq 3$, the HR-ITP algorithm in $O(n^2)$ time outputs a solution with weight at most $(2\lceil n/k\rceil/n)\Delta + (1-\lceil n/k\rceil/n)w(H)$.*

An Extension of ITP. The above two ITP algorithms are based on given Hamiltonian cycles. In fact, the requirement of Hamiltonian cycles is not necessary. We can replace the Hamiltonian cycle with a cycle cover.

Given any cycle cover $\mathcal{C}$ in the graph $G[V]$ or $G[V \cup \{v_0\}]$ (either containing the depot v_0 or not), for each cycle $C \in \mathcal{C}$, we call the HR-ITP algorithm on it if C does not contain the depot v_0, and call the AG-ITP algorithm on it if C contains the depot v_0. By putting all toobtainher, we can obtain a feasible solution for k-CVRP. The quality of the solution is related to the cycle cover. We refer to this algorithm as the EX-ITP algorithm.

▶ **Lemma 10.** *Given a cycle cover $\mathcal{C}$ in $G[V]$ or $G[V \cup \{v_0\}]$ as part of the input, for unit-demand k-CVRP with any $k \geq 3$, the EX-ITP algorithm in $O(n^2)$ time outputs a solution with weight at most $2g\Delta + (1-g)w(\mathcal{C})$, where $g = \max_{C\in\mathcal{C}}\lceil |C|/k\rceil/|C|$.*

Proof. Define $\Delta_C = \sum_{v_i \in C} w(v_0, v_i)$. For the cycle $C \in \mathcal{C}$ satisfying $v_0 \in C$, the AG-ITP algorithm can generate an itinerary on C with weight at most $(2/k)\Delta_C + (1-1/k)w(C)$. For each cycle $C \in \mathcal{C}$ satisfying $v_0 \notin C$, the HR-ITP algorithm can generate an itinerary on C with weight at most $(2\lceil |C|/k\rceil/|C|)\Delta_C + (1-\lceil |C|/k\rceil/|C|)w(C)$.

By the triangle inequality, we have $w(C) \leq 2\Delta_C$. Then, for any $0 \leq x \leq y$, we can obtain $2x\Delta_C + (1-x)w(C) \leq 2y\Delta_C + (1-y)w(C)$. Since $2/k \leq 2\lceil |C|/k\rceil/|C| \leq 2g$, we have

$$(2/k)\Delta_C + (1-1/k)w(C) \leq (2\lceil |C|/k\rceil/|C|)\Delta_C + (1-\lceil |C|/k\rceil/|C|)w(C) \leq 2g\Delta_C + (1-g)w(C).$$

Therefore, the itinerary on $C \in \mathcal{C}$ has a weight of at most $2g\Delta_C + (1-g)w(C)$.

Since $\sum_{C\in\mathcal{C}} \Delta_C = \Delta$, EX-ITP outputs a solution with weight at most

$$\sum_{C\in\mathcal{C}} (2g\Delta_C + (1-g)w(C)) = 2g\Delta + (1-g)w(\mathcal{C}),$$

as desired. ◀

Intuitively, the value $1/g$ in Lemma 10 represents the maximum, over all cycles (excluding the depot), of the average number of customers per tour used to serve that cycle. Thus, a larger value of g indicates that each tour serves fewer customers on average.

If the cycle cover $\mathcal{C}$ used in Lemma 10 satisfies $w(\mathcal{C}) \leq \min\{2(1-\chi), 1\}w(I^*)$, as in Lemma 7, and if $g \leq 1/2$, we have the following corollary.

▶ **Corollary 11 (*).** *Given a cycle cover $\mathcal{C}$ in $G[V]$ or $G[V \cup \{v_0\}]$ as part of the input, where $w(\mathcal{C}) \leq \min\{2(1-\chi), 1\}w(I^*)$ and $g \leq 1/2$, for unit-demand k-CVRP with any $k \geq 3$, EX-ITP achieves an approximation ratio of $1 + g(k-2)$. Moreover, in the worst case, we have $\chi = 1/2$. (If $g = 1/2$, the approximation ratio remains the same for any $1/2 \leq \chi \leq 1$.)*

5 Applications of the EX-ITP Algorithm

In this section, we demonstrate that the EX-ITP algorithm can be used to design improved approximation algorithms for splittable k-CVRP and unit-demand k-CVRP. We show as examples, the approximation ratio can be significantly improved from 1.792 [18] to $3/2 = 1.500$ for 3-CVRP, and from 1.750 [2] to $5/3 < 1.667$ for 4-CVRP. At last, we show that the approximation ratio of 4-CVRP can be further improved to $3/2 = 1.500$.

Splittable 3-CVRP and Unit-demand 3-CVRP. There are little improvements for 3-CVRP in the last 20 years. Only recently an improved result for the general weighted k-set cover problem [18] leads to the current best result for 3-CVRP. We show that we can easily improve the approximation ratio by using EX-ITP. Our algorithm computes a minimum weight cycle cover $\mathcal{C}^*$ in $G[V]$, and then calls EX-ITP on $\mathcal{C}^*$.

▶ **Theorem 12.** *For splittable 3-CVRP and unit-demand 3-CVRP, there is a $\frac{3}{2}$-approximation.*

Proof. Since $|C| \geq 3$ for each cycle $C \in \mathcal{C}^*$, we have

$$g = \max_{C \in \mathcal{C}^*} \lceil |C|/3 \rceil / |C| \leq \max_{|C| \geq 3} \lceil |C|/3 \rceil / |C| = \lceil 4/3 \rceil / 4 = \frac{1}{2}.$$

Note that $w(\mathcal{C}^*) \leq \min\{2(1-\chi), 1\}w(I^*)$ by Lemma 7. Then, by Corollary 11, EX-ITP achieves an approximation ratio of at most $g(k-2) + 1 \leq \frac{3}{2}$, where we have $\chi \geq \frac{1}{2}$ in the worst case. ◀

Splittable 4-CVRP and Unit-demand 4-CVRP. The idea is similar. However, we will construct a good mod-2-cycle cover $\mathcal{C}_{\text{mod } 2}$ on V, instead of using a minimum weight cycle cover $\mathcal{C}^*$ on V, and then call EX-ITP.

We compute the mod-2-cycle cover $\mathcal{C}_{\text{mod } 2}$ in this way: first find a minimum weight perfect matching $\mathcal{M}^*$ in graph $G[V]$, find a minimum weight perfect matching $\mathcal{M}^{**}$ in graph $G[V] \setminus \mathcal{M}^*$, and then let $\mathcal{C}_{\text{mod } 2} = \mathcal{M}^* \cup \mathcal{M}^{**}$.

Note that $\mathcal{C}_{\text{mod } 2}$ is a mod-2-cycle cover without 2-cycles since $\mathcal{M}^*$ and $\mathcal{M}^{**}$ are two edge-disjoint perfect matchings in $G[V]$. Recall that $\mathcal{C}_4^*$ denotes a minimum weight 4-cycle cover in $G[V]$. We have the following result.

▶ **Lemma 13.** *$w(\mathcal{C}_{\text{mod } 2}) \leq w(\mathcal{C}_4^*)$.*

Proof. We first prove the following property.

▷ Claim 14. Given a minimum weight 4-cycle cover $\mathcal{C}_4^*$ and a minimum weight perfect matching $\mathcal{M}^*$, there is a way to color edges in $\mathcal{C}_4^*$ with red and blue such that

(1) the blues (resp., red) edges form a perfect matching $\mathcal{M}_b$ (resp., $\mathcal{M}_r$);
(2) $\mathcal{C}_4^* = \mathcal{M}_b \cup \mathcal{M}_r$;
(3) $\mathcal{M}_b \cup \mathcal{M}^*$ is a mod-2-cycle cover without 2-cycles.

Proof. For each 4-cycle $C = (v_1, v_2, v_3, v_4, v_1) \in \mathcal{C}_4^*$, we color its edges by considering three cases.

Case 1: $|C \cap \mathcal{M}^*| = 0$. We color the edges (v_1, v_2) and (v_3, v_4) with blue and the edges (v_1, v_4) and (v_2, v_3) with red. Now, blue edges (resp., red edges) are vertex-disjoint.

Case 2: $|C \cap \mathcal{M}^*| = 1$. We assume w.l.o.g. that $C \cap \mathcal{M}^* = \{(v_1, v_2)\}$. Then, we color the edges (v_1, v_2) and (v_3, v_4) with red and the edges (v_1, v_4) and (v_2, v_3) with blue.

Case 3: $|C \cap \mathcal{M}^*| = 2$. We assume w.l.o.g. that $C \cap \mathcal{M}^* = \{(v_1, v_2), (v_3, v_4)\}$. Then, we color the edges (v_1, v_2) and (v_3, v_4) with red and the edges (v_1, v_4) and (v_2, v_3) with blue.

It is easy to see that the set of blues edges $\mathcal{M}_b$ and the set of red edges $\mathcal{M}_r$ are two perfect matchings, and $\mathcal{M}_b \cup \mathcal{M}_r = \mathcal{C}_4^*$. Moreover, we know that $\mathcal{M}_b \cap \mathcal{M}^* = \emptyset$, and hence $\mathcal{M}_b \cup \mathcal{M}^*$ is a mod-2-cycle cover without 2-cycles. Thus, the claim holds. ◁

By the claim, we have

$$w(\mathcal{M}_b \cup \mathcal{M}^*) \leq w(\mathcal{M}_b \cup \mathcal{M}_r) = w(\mathcal{C}_4^*).$$

Since the mod-2-cycle cover $\mathcal{C}_{\text{mod } 2} = \mathcal{M}^* \cup \mathcal{M}^{**}$ is the minimum weight mod-2-cycle cover containing the minimum weight perfect matching $\mathcal{M}^*$, we have

$$w(\mathcal{C}_{\text{mod } 2}) = w(\mathcal{M}^* \cup \mathcal{M}^{**}) \leq w(\mathcal{M}_b \cup \mathcal{M}^*).$$

Thus, $w(\mathcal{C}_{\text{mod } 2}) \leq w(\mathcal{C}_4^*)$ and the lemma holds. ◀

If we use the lower bound from Lemma 7, Lemma 13 shows that $\mathcal{C}_{\text{mod } 2}$ performs as well as $\mathcal{C}^*$. By calculations, we obtain $\max_{C \in \mathcal{C}_{\text{mod } 2}} \lceil |C|/4 \rceil / |C| \leq \frac{1}{3}$ and $\max_{C \in \mathcal{C}^*} \lceil |C|/4 \rceil / |C| \leq \frac{2}{5}$. Then, by Corollary 11, it is better to use $\mathcal{C}_{\text{mod } 2}$ when invoking EX-ITP, which achieves an approximation ratio of $g(k-2) + 1 = \frac{5}{3}$. Thus, we have the following result.

▶ **Theorem 15.** *For splittable 4-CVRP and unit-demand 4-CVRP, there is a $\frac{5}{3}$-approximation.*

A Further Improvement on Splittable 4-CVRP and Unit-demand 4-CVRP. Note that we can obtain a solution with weight at most $\frac{1}{2}\Delta + \frac{3}{4}w(\mathcal{C}_4^*)$ for unit-demand 4-CVRP by calling EX-ITP on a minimum weight 4-cycle cover $\mathcal{C}_4^*$. However, it is NP-hard to compute $\mathcal{C}_4^*$ as mentioned before. Surprisingly, we can obtain a solution with the same upper bound in polynomial time without using $\mathcal{C}_4^*$, and hence obtain an improved 3/2-approximation algorithm, which even matches the approximation ratio for unit-demand 3-CVRP.

The main idea is that we can use the minimum weight matching algorithm to optimally compute a minimum weight itinerary containing all edges in $\mathcal{M}^*$: first we construct a multi-graph $G' = G[V]/\mathcal{M}^*$ by contracting every edge in $\mathcal{M}^*$; for each edge $e_i e_j$ in G' where $e_i, e_j \in \mathcal{M}^*$, we set its weight as the minimum weight of a tour in G containing e_i and e_j; then a minimum weight matching in G' corresponds to a minimum weight itinerary in G containing all edges in $\mathcal{M}^*$. Next, through a property of $\mathcal{C}_4^*$, we prove that this solution has a weight of at most $\frac{1}{2}\Delta + \frac{3}{4}w(\mathcal{C}_4^*)$.

▶ **Lemma 16.** *For unit-demand 4-CVRP, there is a polynomial-time algorithm that can generate a solution with weight at most $\frac{1}{2}\Delta + \frac{3}{4}w(\mathcal{C}_4^*)$.*

Proof. First, the claim in Lemma 13 can be strengthened using a similar odd cycle elimination technique in [22].

▷ Claim 17. Given a minimum weight 4-cycle cover $\mathcal{C}_4^*$ and a minimum weight perfect matching $\mathcal{M}^*$, there is a way to color edges in $\mathcal{C}_4^*$ with red and blue such that

(1) the blues (resp., red) edges form a perfect matching $\mathcal{M}_b$ (resp., $\mathcal{M}_r$);
(2) $\mathcal{C}_4^* = \mathcal{M}_b \cup \mathcal{M}_r$;
(3) $\mathcal{M}_b \cup \mathcal{M}^*$ is a mod-4-cycle cover.

Proof. Previously, we have shown that (1) and (2) holds, and $\mathcal{M}_b \cup \mathcal{M}^*$ is a mod-2-cycle cover without 2-cycles.

Then, we modify the cycle cover $\mathcal{M}_b \cup \mathcal{M}^*$ so that (1), (2), and (3) hold. By the previous coloring, we know that each cycle of $\mathcal{C}_4^*$ contains two blue edges and two red edges. Two blue/red edges are called *matched* if they fall on the same cycle of $\mathcal{C}_4^*$.

Consider a cycle $C_i \in \mathcal{M}_b \cup \mathcal{M}^*$ with a minimum length not divisible by 4. Since $\mathcal{M}_b \cup \mathcal{M}^*$ is a cycle cover without 2-cycles and the length of every cycle is divisible by 2, the number of blue edges on C_i is odd. Hence, there must be a blue edge e_1 such that its matched blue edge e_3 falls on a different cycle $C_j \in \mathcal{M}_b \cup \mathcal{M}^*$. Moreover, there are two red edges e_2 and e_3 sharing common vertices with them. See Figure 1 for an illustration.

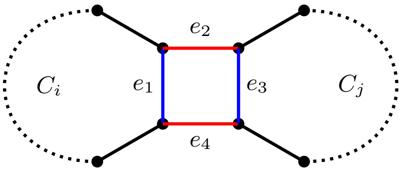

Figure 1 An illustration of the cycles $C_i, C_j \in \mathcal{M}_b \cup \mathcal{M}^*$, the blue edges $\{e_1, e_3\}$, and the red edges $\{e_2, e_4\}$, where $e_1 \in C_i$, $e_3 \in C_j$, and each blue edge is incident to two (black) matching edges.

We can modify the colors of $\{e_1, e_2, e_3, e_4\}$ such that the new color of $\{e_1, e_3\}$ is red, and the new color of $\{e_2, e_4\}$ is blue. We can see that (1) and (2) still holds. Moreover, by repeating this, we obtain a cycle cover $\mathcal{M}_b \cup \mathcal{M}^*$ such that the length of every cycle is divisible by 4. So, (3) also holds. ◁

Now, consider the multi-graph $G' = G[V]/\mathcal{M}^*$, i.e., G' is obtained by contracting edges of $\mathcal{M}^*$ in $G[V]$. There are $n/2$ super-vertices in G' with each corresponding to an edge of $\mathcal{M}^*$. For any two super-vertices $e_i, e_j \in \mathcal{M}^*$, there are four parallel edges between them. Assume that $e_i = (u, u')$ and $e_j = (v, v')$. The augmented weights $\widetilde{w}$ on the four edges are

$$w(v_0, u') + w(e_i) + w(u, v) + w(e_j) + w(v', v_0), w(v_0, u') + w(e_i) + w(u, v') + w(e_j) + w(v, v_0),$$

$$w(v_0, u) + w(e_i) + w(u', v) + w(e_j) + w(v', v_0), w(v_0, u) + w(e_i) + w(u', v') + w(e_j) + w(v, v_0),$$

which measure the weights of tours: (v_0, u', u, v, v', v_0), (v_0, u', u, v', v, v_0), (v_0, u, u', v, v', v_0), (v_0, u, u', v', v, v_0), respectively. Then, a matching $\mathcal{M}$ in G' corresponds to a solution of unit-demand 4-CVRP with an augmented weight of $\widetilde{w}(\mathcal{M})$. The minimum augmented weight matching in G', denoted by $\mathcal{M}^{**}$, has the following property.

▷ Claim 18. $\widetilde{w}(\mathcal{M}^{**}) \leq \frac{1}{2}\Delta + \frac{3}{4}w(\mathcal{C}_4^*)$.

Proof. By Lemma 17, $\mathcal{M}^* \cup \mathcal{M}_b$ is a mod-4-cycle cover. So, the number of blue edges on each cycle $C \in \mathcal{M}^* \cup \mathcal{M}_b$ is even. All blue edges in $\mathcal{M}_b$ can be decomposed into two matchings $\mathcal{M}_1$ and $\mathcal{M}_2$ in $G[V]/\mathcal{M}^*$. Note that $\widetilde{w}(\mathcal{M}_1) + \widetilde{w}(\mathcal{M}_2) = \Delta + 2w(\mathcal{M}^*) + w(\mathcal{M}_b)$. We have $\widetilde{w}(\mathcal{M}^{**}) \leq \frac{1}{2}(\widetilde{w}(\mathcal{M}_1) + \widetilde{w}(\mathcal{M}_2)) = \frac{1}{2}\Delta + w(\mathcal{M}^*) + \frac{1}{2}w(\mathcal{M}_b)$. Recall that $\mathcal{C}_4^* = \mathcal{M}_b \cup \mathcal{M}_r$ and $w(\mathcal{M}^*) \leq \min\{w(\mathcal{M}_b), w(\mathcal{M}_r)\} \leq \frac{1}{2}(w(\mathcal{M}_b) + w(\mathcal{M}_r)) = \frac{1}{2}w(\mathcal{C}_4^*)$. We have $w(\mathcal{M}^*) + \frac{1}{2}w(\mathcal{M}_b) \leq \frac{1}{2}w(\mathcal{M}^*) + \frac{1}{2}w(\mathcal{M}_r) + \frac{1}{2}w(\mathcal{M}_b) \leq \frac{3}{4}w(\mathcal{C}_4^*)$. ◁

Since the minimum augmented weight matching $\mathcal{M}^{**}$ in G' can be found by the minimum weight matching algorithm in $O(n^3)$ time [16, 27], we can find a solution of unit-demand 4-CVRP with weight $\widetilde{w}(\mathcal{M}^{**})$ in polynomial time. ◀

By Lemmas 16, 3, and 7, we have the following result.

▶ **Theorem 19.** *For splittable 4-CVRP and unit-demand 4-CVRP, there is a $\frac{3}{2}$-approximation.*

6 Improvements for the Splittable Case

Given an α-approximate Hamiltonian cycle H in G, by Lemma 8, the ITP algorithm outputs a solution with weight at most $(2/k)\Delta + (1-1/k)w(H) \leq (2/k)\Delta + (1-1/k)\alpha w(H^*)$, where H^* denotes an optimal TSP tour. Recall that $w(H^*) \leq w(I^*)$ by Lemma 2 and $\Delta \leq \frac{k}{2}w(I^*)$ by Lemma 3. Thus, ITP achieves an approximation ratio of $\alpha + 1 - \alpha/k$.

In the following, we apply ITP and EX-ITP to design three algorithms and use the lower bounds in Sect. 3 to obtain initial improvements over the approximation ratio $\alpha + 1 - \alpha/k$. Then, we introduce two refined lower bounds and use them to obtain further improvements by carefully analyzing one of the three algorithms.

6.1 Initial Improvements

Recall that if the upper bound of Δ is tight, we have $\chi = 1$ by Lemma 3. And, we have $w(H_{CS}) \leq w(I^*)$ and $w(\mathcal{C}^*_{\text{mod } k}) = 0$ by Lemmas 6 and 7. Hence, apart from using an α-approximate Hamiltonian cycle H, we also consider the Hamiltonian cycle H_{CS} [9, 35] and a modification of a 2-approximate mod-k-cycle cover [17]. We apply these two Hamiltonian cycles to call the AG-ITP algorithm and the mod-k-cycle cover to call the EX-ITP algorithm. Based on the lower bounds in Sect. 3, the approximation ratios of these three algorithms can be expressed as functions of the parameter χ, and then we can make a trade-off.

Our first algorithm is to call the EX-ITP algorithm on a mod-k-cycle cover $\mathcal{C}_{\text{mod } k}$ on V. Note that the 2-approximate mod-k-cycle cover [17] is obtained by doubling a mod-k-tree cover and then shortcutting. We may find a minimum weight matching on odd-degree vertices in the mod-k-tree cover and then shortcutting like the Christofides-Serdyukov algorithm [9, 35]. We use the following three steps to compute $\mathcal{C}_{\text{mod } k}$.

Step 1. Compute a mod-k-tree cover $\mathcal{T}_{\text{mod } k}$ in $G[V]$ using the algorithm in [17].

Step 2. Find a minimum weight matching $\mathcal{M}$ on odd-degree vertices of $\mathcal{T}_{\text{mod } k}$.

Step 3. Construct a mod-k-cycle cover $\mathcal{C}_{\text{mod } k}$ in $G[V]$ from $\mathcal{T}_{\text{mod } k} \cup \mathcal{M}$ by shortcutting.

The second algorithm is to call the AG-ITP algorithm on any α-approximate Hamiltonian cycle on $V \cup \{v_0\}$. The third algorithm is to call the AG-ITP algorithm on the Hamiltonian cycle H_{CS} on $V \cup \{v_0\}$.

We first analyze some properties of the first algorithm. The mod-k-tree cover computed in the first step has the following property.

▶ **Lemma 20** ([17]). *There is a polynomial-time algorithm that can generate a mod-k-tree cover $\mathcal{T}_{\text{mod } k}$ in $G[V]$ such that $w(\mathcal{T}_{\text{mod } k}) \leq w(\mathcal{C}^*_{\text{mod } k})$.*

In Steps 2 and 3, we generate a mod-k-cycle cover based on the mod-k-tree cover $\mathcal{T}_{\text{mod } k}$ and a minimum weight perfect matching $\mathcal{M}$ on odd-degree vertices of $\mathcal{T}_{\text{mod } k}$ using the Christifides-Serdykov method for TSP. Although it may not guarantee a better approximation ratio for the minimum weight mod-k-cycle cover problem, it is useful for CVRP.

▶ **Lemma 21** (*). *There is a polynomial-time algorithm that generates a mod-k-cycle cover $\mathcal{C}_{\text{mod } k}$ in $G[V]$ such that $w(\mathcal{C}_{\text{mod } k}) \leq w(\mathcal{C}^*_{\text{mod } k}) + \frac{1}{2}w(H^*)$.*

Now, we are ready to analyze the three algorithms.

▶ **Theorem 22 (*).** *Given an α-approximate Hamiltonian cycle on $V \cup \{v_0\}$, there is an approximation algorithm for splittable k-CVRP and unit-demand k-CVRP such that*

- *If $1 \leq \alpha \leq 7/6$ and $k \geq 3$, the approximation ratio is $\alpha + 1 - \alpha/k - (\alpha - 1/2)/k$;*
- *If $7/6 \leq \alpha \leq 3/2$ and $3 \leq k \leq 5$, the approximation ratio is $(13k - 11)/(6k)$;*
- *If $7/6 \leq \alpha \leq 3/2$ and $k \geq 6$, the approximation ratio is $\alpha + 1 - \alpha/k - 4(\alpha - 1)/k$.*

We can obtain an approximation ratio of $\alpha + 1 - \alpha/k - \Theta(1/k)$ for splittable k-CVRP, which improves the previous approximation ratio of $\alpha + 1 - \alpha/k - \max\{\Omega(1/k^3), \varepsilon\}$ [7, 8] for some small k. Specifically, when $\alpha = 3/2$, our approximation ratio achieves $5/2 - 3.5/k$, and in the worst case we have $\chi = 0$. Further details and some comparisons can be found in the full version of this paper.

6.2 Further Improvements

When $\alpha = 3/2$, we can further improve the approximation ratio to $5/2 - \Theta(\sqrt{1/k})$. Our motivation is that the approximation ratio $5/2 - 3.5/k$ is obtained with $\chi = 0$ in the worst case. That is, in the worst case, the total weight of all home-edges in the optimal solution I^* is zero. Recall that each tour of I^* contains two home-edges and the lower bound based on MST in Lemma 4 is obtained by deleting the more weighted home-edge in each tour of I^*. Since $\chi = 0$ in the worst case, we have $w(I^*) \geq \mathrm{MST}$ by Lemma 4. So, by deleting a zero-weighted home-edge, we gain no revenues. We may obtain a refined lower bound based on MST by deleting the highest weighted edge in each tour, which is clearly better than just deleting a zero-weighted home-edge. Based on this, we give two refined lower bounds based on MST as well as Δ to combine them.

Our algorithm is simply to call the classic AG-ITP algorithm on the Hamiltonian cycle H_{CS} on $V \cup \{v_0\}$. By Lemmas 2, 5, and 8, the solution is at most

$$(2/k)\Delta + (1 - 1/k)w(H_{CS}) \leq (2/k)\Delta + (1 - 1/k)\mathrm{MST} + (1/2)(1 - 1/k)w(I^*). \tag{1}$$

To prove an approximation ratio of $\frac{5}{2} - \Theta(\sqrt{1/k})$, we show that $\frac{2}{k}\Delta + (1 - \frac{1}{k})\mathrm{MST} \leq (2 - \Theta(\sqrt{1/k}))w(I^*)$.

Now, we analyze the structure in more details.

By Assumption 3, I^* consists of a set of $(k+1)$-cycles. Consider an arbitrary $(k+1)$-cycle $C = (v_0, v_1, \ldots, v_k, v_0)$. First, let Δ_C denote the sum of the weights of edges between vertices v_0 and v_i for $1 \leq i \leq k$, i.e., $\sum_{i=1}^{k} w(v_0, v_i)$. Note that $\Delta = \sum_{C \in I^*} \Delta_C$. Then, let T_C denote the spanning tree in $G[C]$ obtained by deleting the highest weighted edge in C, i.e., $w(T_C) = w(C) - \max_{0 \leq i \leq k} w(v_i, v_{(i+1) \bmod (k+1)})$. After we delete the highest weighted edge for each cycle $C \in I^*$, the remaining graph is a spanning tree in G. So, $\mathrm{MST} \leq \sum_{C \in I^*} w(T_C)$.

Moreover, since $w(I^*) = \sum_{C \in I^*} w(C)$, to prove $\frac{2}{k}\Delta + (1 - \frac{1}{k})\mathrm{MST} \leq (2 - \Theta(\sqrt{1/k}))w(I^*)$, it is sufficient to prove $\frac{2}{k}\Delta_C + (1 - \frac{1}{k})w(T_C) \leq (2 - \Theta(\sqrt{1/k}))w(C)$.

▶ **Lemma 23.** *When $k \geq 3$, for any cycle $C \in I^*$, we have that $\frac{2}{k}\Delta_C + \left(1 - \frac{1}{k}\right) w(T_C) \leq (2 - \frac{2l^2 + k - 1}{2kl})w(C)$, where $l = \lceil \frac{\sqrt{2k-1}-1}{2} \rceil$.*

Proof Sketch. Our proof is based on generalizing the concept of home-edges.

When k is odd (resp., even), the number of edges in the cycle C is even (resp., odd). Due to different structural properties, these two cases have to be handled separately. We consider the case that k is odd. Another case can be analyzed in a similar way.

We may assume $w(C) \neq 0$; otherwise, Lemma 23 holds trivially. Let $m = \frac{k+1}{2}$. We define

$$a_i = \frac{w(v_{i-1}, v_i) + w(v_{k+1-i}, v_{(k+2-i) \bmod (k+1)})}{w(C)},$$

where $1 \leq i \leq m$. See Figure 2 for an illustration of the definition on a_i.

It is easy to see that $\sum_{i=1}^{m} a_i = 1$ and $a_i \geq 0$ for each $1 \leq i \leq m$. Note that a_1 measures the proportion of weights of home-edges in C, so it can be regarded as a generalization of χ but works on a single cycle.

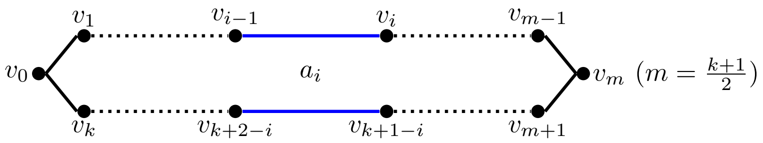

Figure 2 An illustration of the cycle $C = (v_0, v_1, \ldots, v_k, v_0)$ with the case of odd k, where a_i $(1 \leq i \leq m)$ measures the weight proportion of the blue edges compared to the cycle C.

Then, by the definitions of Δ_C and T_C, we have

$$\Delta_C \leq \left(\frac{k+2}{2} - \sum_{i=1}^{m} i a_i\right) w(C) \quad \text{and} \quad w(T_C) \leq \left(1 - \max_{1 \leq i \leq m} \frac{1}{2} a_i\right) w(C).$$

Thus, it holds that

$$\frac{\frac{2}{k}\Delta_C + \left(1 - \frac{1}{k}\right) w(T_C)}{w(C)} \leq \max_{\substack{a_1, a_2, \ldots, a_m \geq 0 \\ a_1 + a_2 + \cdots + a_m = 1}} \left\{\frac{2k+1}{k} - \frac{2}{k}\sum_{i=1}^{m} i a_i - \max_{1 \leq i \leq m} \frac{k-1}{2k} a_i\right\}.$$

In the worst case of the above linear program, we can obtain $a_1 \geq a_2 \geq \cdots \geq a_m$; otherwise, if there exists $a_p < a_q$ for some $p < q$, we can exchange their values, and then since the value $\max_{1 \leq i \leq m} a_i$ does not change, we can obtain a bigger solution since the coefficients of a_p and a_q satisfy $0 > \frac{-2p}{k} > \frac{-2q}{k}$, which causes a contradiction. So, $\max_{1 \leq i \leq m} a_i = a_1$.

Consequently, we have $\frac{\frac{2}{k}\Delta_C + \left(1 - \frac{1}{k}\right) w(T_C)}{w(C)} \leq \frac{2k+1}{k} - \text{LP}$, where LP is the value of the following linear program:

$$\min_{\substack{x_1 \geq x_2 \geq \cdots \geq 0 \\ x_1 + x_2 + \cdots = 1}} \left\{\frac{k+3}{2k} x_1 + \frac{2}{k}\sum_{i=2}^{\infty} i x_i\right\}.$$

In the full version, we prove that the linear program exists an optimal solution, denoted by $\text{SOL}(x_1^*, x_2^*, \ldots)$, satisfying that $x_1^* = x_2^* = \cdots = x_l^* = \frac{1}{l}$ and $x_i^* = 0$ for all $i > l$, where $l = \lceil \frac{\sqrt{2k-1}-1}{2} \rceil$. Therefore, it holds that

$$\text{SOL}(x_1^*, x_2^*, \ldots) = \frac{k+3}{2k} \cdot \frac{1}{l} + \frac{2}{k}\sum_{i=2}^{l} \frac{i}{l} = \frac{2l^2 + 2l + k - 1}{2kl}.$$

Therefore, we have

$$\frac{\frac{2}{k}\Delta_C + \left(1 - \frac{1}{k}\right) w(T_C)}{w(C)} \leq \frac{2k+1}{k} - \frac{2l^2 + 2l + k - 1}{2kl} = 2 - \frac{2l^2 + k - 1}{2kl},$$

where $l = \lceil \frac{\sqrt{2k-1}-1}{2} \rceil$. ◀

Next, we are ready to analyze the AG-ITP algorithm.

▶ **Theorem 24.** *For splittable k-CVRP and unit-demand k-CVRP, the AG-ITP algorithm admits an approximation ratio of $\frac{5}{2} - \frac{2l^2+k+l-1}{2kl} < \frac{5}{2} - \sqrt{2/k}$, where $l = \lceil \frac{\sqrt{2k-1}-1}{2} \rceil$.*

Proof. Using the Hamiltonian cycle H_{CS}, by (1), AG-ITP can output a solution with weight at most $(2/k)\Delta + (1-1/k)\mathrm{MST} + (1/2)(1-1/k)w(I^*)$. Recall that $\Delta = \sum_{C\in I^*}\Delta_C$ and $\mathrm{MST} \leq \sum_{C\in I^*} w(T_C)$. By Lemma 23, we have

$$\begin{aligned}
&\frac{2}{k}\Delta + \left(1-\frac{1}{k}\right)\mathrm{MST} + \frac{1}{2}\left(1-\frac{1}{k}\right)w(I^*)\\
&\leq \sum_{C\in I^*}\left(\frac{2}{k}\Delta_C + \frac{k-1}{k}w(T_C)\right) + \frac{k-1}{2k}w(I^*)\\
&\leq \sum_{C\in I^*}\left(2-\frac{2l^2+k-1}{2kl}\right)w(C) + \frac{k-1}{2k}w(I^*)\\
&= \left(2-\frac{2l^2+k-1}{2kl}\right)\sum_{C\in I^*} w(C) + \frac{k-1}{2k}w(I^*)\\
&= \left(2-\frac{2l^2+k-1}{2kl} + \frac{k-1}{2k}\right)w(I^*) = \left(\frac{5}{2}-\frac{2l^2+k+l-1}{2kl}\right)w(I^*).
\end{aligned}$$

To show the approximation ratio $\frac{5}{2}-\frac{2l^2+k+l-1}{2kl} < \frac{5}{2}-\sqrt{2/k}$ holds for any $k \geq 3$, it is sufficient to prove

$$\frac{2l^2+k+l-1}{2kl} > \sqrt{2/k} \quad\Longleftrightarrow\quad 2l^2-(2\sqrt{2k}-1)l+k-1>0.$$

The latter holds since the discriminant of the quadratic equation satisfies $(2\sqrt{2k}-1)^2 - 8(k-1) = 9-4\sqrt{2k} < 0$ for any $k \geq 3$. ◀

In the initial improvements, when $\alpha = 3/2$ and $k > 5$, the approximation ratio of our algorithm is $5/2 - 3.5/k$. In further improvements, we achieve a result of $5/2 - \Theta(\sqrt{1/k})$, which is strictly better for any $k > 5$. This is also better than $5/2 - 1.005/3000 - 1.5/k$ for any $k \leq 1.7 \times 10^7$ (see Table 1). When $3 \leq k \leq 5$, the approximation ratio is $2 - 1/k$, which is tight for this algorithm, since it has been shown [28] that the approximation ratio of AG-ITP is at least $2 - 1/k$ in general metrics, even using an optimal Hamiltonian cycle.

Our further improvements only consider $\alpha = 3/2$. For other values of α, similar improved results, as in Theorem 22, could potentially be obtained by applying refined lower bounds based on MST and Δ. However, the analysis is sophisticated and achieving significant improvements for metric TSP appears challenging. We leave the exploration of this direction for future research.

7 Concluding Remarks

In this paper, we consider k-CVRP in general metrics and improve the approximation ratio for k less than a sufficiently large value, approximately $k \leq 1.7 \times 10^7$. Although most of our algorithms, such as EX-ITP, are simple or extensions of the classic methods, the analysis is technically involved. We need to carefully analyze the structure of the solutions and most of our analysis is based on pure combinatorial analysis. We think that analyzing better theoretical bounds for simple and classic algorithms is an interesting and important task in algorithm research. We also believe that our methods have potential to be applied to more routing problems.

References

1 Kemal Altinkemer and Bezalel Gavish. Heuristics for unequal weight delivery problems with a fixed error guarantee. *Oper. Res. Lett.*, 6(4):149–158, 1987.

2 Shoshana Anily and Julien Bramel. Approximation algorithms for the capacitated traveling salesman problem with pickups and deliveries. *Naval Research Logistics*, 46(6):654–670, 1999.

3 Tetsuo Asano, Naoki Katoh, Hisao Tamaki, and Takeshi Tokuyama. Covering points in the plane by k-tours: a polynomial approximation scheme for fixed k. *IBM Tokyo Research Laboratory Research Report RT0162*, 1996.

4 Tetsuo Asano, Naoki Katoh, Hisao Tamaki, and Takeshi Tokuyama. Covering points in the plane by k-tours: Towards a polynomial time approximation scheme for general k. In Frank Thomson Leighton and Peter W. Shor, editors, *Proceedings of the Twenty-Ninth Annual ACM Symposium on the Theory of Computing, El Paso, Texas, USA, May 4-6, 1997*, pages 275–283. ACM, 1997. doi:10.1145/258533.258602.

5 Cristina Bazgan, Refael Hassin, and Jérôme Monnot. Approximation algorithms for some vehicle routing problems. *Discret. Appl. Math.*, 146(1):27–42, 2005. doi:10.1016/j.dam.2004.07.003.

6 Walter J Bell, Louis M Dalberto, Marshall L Fisher, Arnold J Greenfield, Ramchandran Jaikumar, Pradeep Kedia, Robert G Mack, and Paul J Prutzman. Improving the distribution of industrial gases with an on-line computerized routing and scheduling optimizer. *Interfaces*, 13(6):4–23, 1983.

7 Jannis Blauth, Vera Traub, and Jens Vygen. Improving the approximation ratio for capacitated vehicle routing. *Math. Program.*, 197(2):451–497, 2023. doi:10.1007/s10107-022-01841-4.

8 Agustín Bompadre, Moshe Dror, and James B. Orlin. Improved bounds for vehicle routing solutions. *Discret. Optim.*, 3(4):299–316, 2006. doi:10.1016/j.disopt.2006.04.002.

9 Nicos Christofides. Worst-case analysis of a new heuristic for the travelling salesman problem. *Oper. Res. Forum*, 3(1), 2022. doi:10.1007/s43069-021-00101-z.

10 Vasek Chvátal. A greedy heuristic for the set-covering problem. *Math. Oper. Res.*, 4(3):233–235, 1979. doi:10.1287/moor.4.3.233.

11 Vincent Cohen-Addad, Arnold Filtser, Philip N. Klein, and Hung Le. On light spanners, low-treewidth embeddings and efficient traversing in minor-free graphs. In Sandy Irani, editor, *61st IEEE Annual Symposium on Foundations of Computer Science, FOCS 2020, Durham, NC, USA, November 16-19, 2020*, pages 589–600. IEEE, 2020. doi:10.1109/FOCS46700.2020.00061.

12 George B Dantzig and John H Ramser. The truck dispatching problem. *Manag. Sci.*, 6(1):80–91, 1959.

13 Szymon Dudycz, Jan Marcinkowski, Katarzyna E. Paluch, and Bartosz Rybicki. A 4/5 - approximation algorithm for the maximum traveling salesman problem. In Friedrich Eisenbrand and Jochen Könemann, editors, *Integer Programming and Combinatorial Optimization - 19th International Conference, IPCO 2017, Waterloo, ON, Canada, June 26-28, 2017, Proceedings*, volume 10328 of *Lecture Notes in Computer Science*, pages 173–185. Springer, 2017. doi:10.1007/978-3-319-59250-3_15.

14 Arnold Filtser and Hung Le. Low treewidth embeddings of planar and minor-free metrics. In *63rd IEEE Annual Symposium on Foundations of Computer Science, FOCS 2022, Denver, CO, USA, October 31 - November 3, 2022*, pages 1081–1092. IEEE, 2022. doi:10.1109/FOCS54457.2022.00105.

15 Zachary Friggstad, Ramin Mousavi, Mirmahdi Rahgoshay, and Mohammad R. Salavatipour. Improved approximations for capacitated vehicle routing with unsplittable client demands. In Karen I. Aardal and Laura Sanità, editors, *Integer Programming and Combinatorial Optimization - 23rd International Conference, IPCO 2022, Eindhoven, The Netherlands, June 27-29, 2022, Proceedings*, volume 13265 of *Lecture Notes in Computer Science*, pages 251–261. Springer, 2022. doi:10.1007/978-3-031-06901-7_19.

16 Harold Neil Gabow. *Implementation of algorithms for maximum matching on nonbipartite graphs*. PhD thesis, Stanford University, 1974.

17 Michel X. Goemans and David P. Williamson. A general approximation technique for constrained forest problems. *SIAM J. Comput.*, 24(2):296–317, 1995. doi:10.1137/S0097539793242618.

18 Anupam Gupta, Euiwoong Lee, and Jason Li. A local search-based approach for set covering. In Telikepalli Kavitha and Kurt Mehlhorn, editors, *2023 Symposium on Simplicity in Algorithms, SOSA 2023, Florence, Italy, January 23-25, 2023*, pages 1–11. SIAM, SIAM, 2023. doi:10.1137/1.9781611977585.ch1.

19 Mordecai Haimovich and Alexander H. G. Rinnooy Kan. Bounds and heuristics for capacitated routing problems. *Math. Oper. Res.*, 10(4):527–542, 1985. doi:10.1287/moor.10.4.527.

20 David Hartvigsen. *Extensions of matching theory*. PhD thesis, Carnegie-Mellon University, 1984.

21 Refael Hassin and Asaf Levin. A better-than-greedy approximation algorithm for the minimum set cover problem. *SIAM J. Comput.*, 35(1):189–200, 2005. doi:10.1137/S0097539704444750.

22 Refael Hassin and Shlomi Rubinstein. An approximation algorithm for maximum packing of 3-edge paths. *Inf. Process. Lett.*, 63(2):63–67, 1997. doi:10.1016/S0020-0190(97)00097-5.

23 Refael Hassin and Shlomi Rubinstein. Better approximations for max TSP. *Inf. Process. Lett.*, 75(4):181–186, 2000. doi:10.1016/S0020-0190(00)00097-1.

24 Anna R. Karlin, Nathan Klein, and Shayan Oveis Gharan. A (slightly) improved approximation algorithm for metric TSP. In Samir Khuller and Virginia Vassilevska Williams, editors, *STOC 2021: 53rd Annual ACM SIGACT Symposium on Theory of Computing, Virtual Event, Italy, June 21-25, 2021*, pages 32–45. ACM, 2021. doi:10.1145/3406325.3451009.

25 Anna R. Karlin, Nathan Klein, and Shayan Oveis Gharan. A deterministic better-than-3/2 approximation algorithm for metric TSP. In Alberto Del Pia and Volker Kaibel, editors, *Integer Programming and Combinatorial Optimization - 24th International Conference, IPCO 2023, Madison, WI, USA, June 21-23, 2023, Proceedings*, volume 13904 of *Lecture Notes in Computer Science*, pages 261–274. Springer, 2023. doi:10.1007/978-3-031-32726-1_19.

26 David G. Kirkpatrick and Pavol Hell. On the completeness of a generalized matching problem. In Richard J. Lipton, Walter A. Burkhard, Walter J. Savitch, Emily P. Friedman, and Alfred V. Aho, editors, *Proceedings of the 10th Annual ACM Symposium on Theory of Computing, May 1-3, 1978, San Diego, California, USA*, pages 240–245. ACM, 1978. doi:10.1145/800133.804353.

27 Eugene Lawler. *Combinatorial Optimization: Networks and Matroids*. Holt, Rinehart and Winston, 1976.

28 Chung-Lun Li and David Simchi-Levi. Worst-case analysis of heuristics for multidepot capacitated vehicle routing problems. *INFORMS J. Comput.*, 2(1):64–73, 1990. doi:10.1287/ijoc.2.1.64.

29 Claire Mathieu and Hang Zhou. Probabilistic analysis of euclidean capacitated vehicle routing. In Hee-Kap Ahn and Kunihiko Sadakane, editors, *32nd International Symposium on Algorithms and Computation, ISAAC 2021, December 6-8, 2021, Fukuoka, Japan*, volume 212 of *LIPIcs*, pages 43:1–43:16. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021. doi:10.4230/LIPIcs.ISAAC.2021.43.

30 Claire Mathieu and Hang Zhou. A PTAS for capacitated vehicle routing on trees. In Mikolaj Bojanczyk, Emanuela Merelli, and David P. Woodruff, editors, *49th International Colloquium on Automata, Languages, and Programming, ICALP 2022, July 4-8, 2022, Paris, France*, volume 229 of *LIPIcs*, pages 95:1–95:20. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPIcs.ICALP.2022.95.

31 Claire Mathieu and Hang Zhou. A tight ($1.5+\varepsilon$)-approximation for unsplittable capacitated vehicle routing on trees. In Kousha Etessami, Uriel Feige, and Gabriele Puppis, editors, *50th International Colloquium on Automata, Languages, and Programming, ICALP 2023, July*

10-14, 2023, Paderborn, Germany, volume 261 of *LIPIcs*, pages 91:1–91:16. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2023. doi:10.4230/LIPIcs.ICALP.2023.91.

32 Seth Pettie and Vijaya Ramachandran. An optimal minimum spanning tree algorithm. *J. ACM*, 49(1):16–34, 2002. doi:10.1145/505241.505243.

33 Robert Clay Prim. Shortest connection networks and some generalizations. *Bell Syst. Tech. J.*, 36(6):1389–1401, 1957.

34 Alexander Schrijver. *Combinatorial optimization: polyhedra and efficiency*, volume 24. Springer, 2003.

35 Anatolii Ivanovich Serdyukov. Some extremal bypasses in graphs. *Upravlyaemye Sistemy*, 17:76–79, 1978.

36 Paolo Toth and Daniele Vigo. *Vehicle routing: problems, methods, and applications*. SIAM, 2014.

37 Eduardo Uchoa, Diego Pecin, Artur Alves Pessoa, Marcus Poggi, Thibaut Vidal, and Anand Subramanian. New benchmark instances for the capacitated vehicle routing problem. *Eur. J. Oper. Res.*, 257(3):845–858, 2017. doi:10.1016/j.ejor.2016.08.012.