From Bisimulation to Traces: The Impact of Parallel Composition on Finite Bases

Rowin Versteeg
Eindhoven University of Technology, The Netherlands

Valentina Castiglioni
Eindhoven University of Technology, The Netherlands

Bas Luttik
Eindhoven University of Technology, The Netherlands

Abstract

We consider process algebras with inaction, action prefix, non-deterministic choice and interleaving parallel composition modulo the behavioural equivalences in van Glabbeek's linear time-branching time spectrum, and study the existence of finite bases (i.e., finite sound and complete axiomatisations) for these algebras. We prove that if the alphabet of actions is infinite and the behavioural equivalence is either simulation equivalence or trace equivalence, then a finite basis exists and is obtained by extending the known ground-complete axiomatisations for these behavioural equivalences. We prove that if the alphabet of actions is finite, then a finite basis does not exist for these equivalences. We also prove for all behavioural equivalences between ready simulation and completed traces there cannot exist a finite basis irrespective of the cardinality of the alphabet of actions (provided that it is non-empty). Finally, we prove that these results are maintained if the process algebra is extended with a constant for successful termination.

2012 ACM Subject Classification Theory of computation → Equational logic and rewriting

Keywords and phrases Equational basis, Parallel composition, Preorders, Equivalences, Linear time - branching time spectrum

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2025.35

1 Introduction

Behavioural semantics have been introduced as simple and elegant tools for an explicit comparison of the semantics of processes. These are preorders and equivalence relations allowing one to establish whether two processes have the same *observable behaviour*: different notions of observability correspond to different levels of abstraction from the information on process execution, which can either be considered irrelevant in an application context, or be unavailable to an external observer. A taxonomy of behavioural relations based on their distinguishing power, known as the *linear time - branching time spectrum* (henceforth referred to as *the spectrum*), was proposed by van Glabbeek in [20].

Semantic properties of processes can also be defined, implicitly, by a set of *equational axioms*, i.e., syntactic equations over terms in a considered process algebra. Informally, if a term t is proved equal to a term u by means of the axioms, then we can say that t and u describe the same behaviour. An axiomatisation $\mathcal{E}$ is *ground-complete* for a behavioural semantics S if all variable-free terms (i.e., processes) that are related by S can be proved equal from $\mathcal{E}$. An axiomatisation $\mathcal{E}$ is *complete* for S if also terms with variables that are related by S can be proved equal from $\mathcal{E}$. We call a finite, sound and complete axiomatisation a *finite basis*. Obtaining an axiomatisation of a behavioural semantics is a key problem in concurrency theory, as it allows us to characterise the semantics of a process algebra in a purely syntactic fashion, making thus such a characterisation independent of the details of the definition of the behavioural semantics of interest. Moreover, an axiomatisation underlines

© Rowin Versteeg, Valentina Castiglioni, and Bas Luttik;
licensed under Creative Commons License CC-BY 4.0

Table 1 Existence of finite bases for the behavioural semantics over $\mathrm{BCCSP}_{\|}$. The rows are clustered to indicate the dependencies of the proofs.

	$\lvert A\rvert = 1$	$2 \leq \lvert A\rvert < \infty$	$\lvert A\rvert = \infty$
Bisimulation	-	-	-
2-Nested Simulation	?	-	-
Possible Futures	?	-	-
Ready Simulation	-	-	-
Possible Worlds	-	-	-
Ready Traces	-	-	-
Failure Traces	-	-	-
Readies	-	-	-
Failures	-	-	-
Completed Simulation	-	-	-
Completed Traces	-	-	-
Simulation	-	-	+
Traces	-	-	+

the differences between the various semantics via a collection of revealing axioms, and, due to its syntactic nature, it can be applied in verification tools based on theorem proving or rewriting.

As part of the analysis of the behavioural semantics in the spectrum, van Glabbeek proposed finite, ground-complete axiomatisations for them over the language BCCSP, which consists of the basic operators of CCS [26] and CSP [24], i.e., inaction, action prefix, and nondeterministic choice. However, the majority of process algebras include some form of *parallel composition operator* to model the concurrent interaction between processes. Hence, the paper [2] proposed a systematic study of the axiomatisability of the CCS interleaving parallel composition operator "$\|$" modulo the behavioural semantics in the spectrum. The study was carried out over the language $\mathrm{BCCSP}_{\|}$, i.e., BCCSP extended with $\|$. Specifically, it was shown that all behavioural semantics that are at least as coarse as ready simulation admit a finite ground-complete axiomatisation over $\mathrm{BCCSP}_{\|}$. Conversely, for all behavioural semantics that are at most as coarse as possible futures there exists no finite ground-complete axiomatisation, and, thus, no finite basis, provided the set of actions A over which the terms are built contains more than one element. (It is still an open problem to determine whether a finite ground-complete axiomatisation exists for possible futures and the n-nested trace and n-nested simulation semantics when $\lvert A\rvert = 1$.)

Our Contribution. In this paper, we move the focus from processes to terms with variables, and we investigate the existence of finite bases for the behavioural semantics in the spectrum over $\mathrm{BCCSP}_{\|}$. Table 1 reports an overview of our findings. We remark that the non-existence of finite bases for possible futures, nested simulations, and bisimulation semantics follows directly from the non-existence of finite ground-complete axiomatisations over $\mathrm{BCCSP}_{\|}$ for them [2, 7, 27]. Interestingly, for the other semantics, we show that almost all finite ground-completeness results from [2] turn into negative results. The only exception is constituted by trace and simulation equivalences over $\mathrm{BCCSP}_{\|}$ terms constructed over an infinite set of actions A. In this case, we provide, for each equivalence, an ω-complete axiomatisation that combined with the ground-completeness results from [2] give us the desired finite bases (Section 4.3). Here, by finite we mean *finitely representable*, namely, the axiomatisation includes only finitely many axiom schemas (that can generate infinitely many equations). The ω-completeness result is obtained by making use of Groote's technique of inverted substitutions [21] and two additional axioms for $\|$ with respect to those in [2] (one of which is original with respect to the related literature). As soon as $|A| < \infty$, this positive results for trace and simulation semantics turn into negative ones. The interesting bit is that those negative results require different proof techniques, according to the cardinality of A. In the case $|A| = 1$ we use the transformation technique from [4] to establish a relation between $\text{BCCSP}_{\|}$, modulo the two semantics, and the max-plus algebra over natural numbers [5]. We then use this relation and the non-finite axiomatisability result for the latter algebra [5] to conclude that the same negative result holds for the former algebra (Section 4.1). In the case $2 \leq |A| \leq \infty$, we prove the non-existence of a finite axiomatisation modulo trace and simulation semantics over $\text{BCCSP}_{\|}$ by applying Moller's proof-theoretic technique [27] (Section 4.2). We make use of the same technique to prove the non-existence of finite basis modulo all semantics between ready simulation and completed traces over $\text{BCCSP}_{\|}$, regardless of the cardinality of A (Section 3). Notably, we prove all negative results for both versions of each semantics: preorder and equivalence. Finally, we also discuss the lifting of our results to $\text{BCCSP}_{\|}$ extended with successful termination (Section 5).

Due to space limitations, we present only a high-level description of our results and the proof techniques applied to obtain them. We refer the interested reader to [30] for the detailed technical development.

Related Work. The axiomatisations of BCCSP have been studied extensively, and we give here only a brief summary of the results. Van Glabbeek presented finite, sound and ground-complete axiomatisations for most equivalences in the spectrum [20]. However, some of these axiomatisations contained conditional axioms. In response Blom, Fokkink and Nain gave finite, sound and ground-complete axiomatisations for ready traces, ready simulation and failure traces without conditions [11]. They also proved that there exists no finite, sound and ground-complete axiomatisation for ready traces if the number of actions is infinite. Aceto et al. proved that there are no finite, sound and ground-complete axiomatisations for 2-nested-simulation and possible futures [7].

Moller proved the existence of a finite basis for bisimulation [27], Groote did this for completed traces and traces [21] and Fokkink and Nain for failures [19]. Conversely, Chen, Fokkink and Nain proved non-finite axiomatisability results for ready simulation and completed simulation [15], Chen and Fokkink for simulation [12], Fokkink and Nain for ready traces, readies and possible worlds [18] and Chen et al. for failure traces [13].

For what concerns preorders, Aceto, Fokkink and Ingólfsdóttir presented the transformation technique for BCCSP to transform finite, sound and (ground-)complete axiomatisations for a behavioural preorder to a (ground-)complete axiomatisation for the respective behavioural equivalence [6]. This method was applied only to the behavioural semantics between ready simulation and traces. The transformation technique can be used also to lift results across process algebras. Aceto, Ésik and Ingólfsdóttir proved there is no finite basis for the max-plus algebra on natural numbers, then used that fact to prove that trace equivalence with $|A| = 1$ has no finite basis over the process algebra BPA [4], which is similar to BCCSP.

Regarding $\text{BCCSP}_{\|}$, Hennessy and Milner [22, 23], proposed a ground-complete axiomatisation over it modulo strong bisimilarity and weak bisimilarity. This axiomatisation contains the expansion law, an axiom schema which expresses equationally the semantics of the parallel composition operator. However, the expansion law generates an infinite number of axioms, which makes any axiomatisation containing it infinite. Moller then proved that no

Table 2 Operational semantics of $\text{BCCSP}_\parallel$.

$$\frac{}{a.t \xrightarrow{a} t} \qquad \frac{t \xrightarrow{a} t'}{t+u \xrightarrow{a} t'} \qquad \frac{u \xrightarrow{a} u'}{t+u \xrightarrow{a} u'} \qquad \frac{t \xrightarrow{a} t'}{t \parallel u \xrightarrow{a} t' \parallel u} \qquad \frac{u \xrightarrow{a} u'}{t \parallel u \xrightarrow{a} t \parallel u'}$$

finite, sound and ground-complete axiomatisation exists for bisimulation [28]. As discussed above, Aceto et al. recently studied the existence of ground-complete axiomatisations for the other behavioural equivalences in the spectrum [2].

2 Preliminaries

The Process Algebra $\text{BCCSP}_\parallel$. The language $\text{BCCSP}_\parallel$ extends BCCSP with the interleaving parallel composition operator, and it is defined by the following grammar:

$$t \ ::= \ \mathbf{0} \ | \ x \ | \ a.t \ | \ t+t \ | \ t \parallel t$$

where a ranges over a non-empty set of actions A, and x ranges over a countably infinite set of variables V. We shall use the meta-variables $t, u, \dots$ to range over the set $\mathcal{O}$ of *open* $\text{BCCSP}_\parallel$ terms, and write $var(t)$ for the collection of variables occurring in the term t. We also adopt the standard convention that prefixing binds strongest and $+$ binds weakest. We use a *summation* $\sum_{i=1}^{n} t_i$ to denote the term $t = t_1 + \dots + t_n$, where terms $t_1, \dots, t_n$ do not have $+$ as head operator and are called *summands* of t. The empty sum represents $\mathbf{0}$. We let $t^n = t \parallel t^{n-1}$, with $t^1 = t$. A term is *closed* if it does not contain any variables. Closed terms are usually called *processes*, and we denote the set of $\text{BCCSP}_\parallel$ closed terms by $\mathcal{P}$. A *substitution* $\sigma : V \to \mathcal{O}$ is a mapping from variables to terms. A *closed substitution* $\rho : V \to \mathcal{P}$ is a mapping from variables to closed terms.

We use the SOS framework [29] to equip processes with an operational semantics. A (*closed*) *literal* is an expression of the form $t \xrightarrow{a} t'$ for some (closed) terms t, t' and action $a \in A$. The inference rules for *prefixing* $a.\_$, *nondeterministic choice* $+$ and *interleaving parallel composition* $\parallel$ are reported in Table 2. These rules induce the *labelled transition system* (LTS) [25] $(\mathcal{P}, A, \to)$ whose transition relation $\to \subseteq \mathcal{P} \times A \times \mathcal{P}$ contains exactly the closed literals that can be derived using the rules in Table 2. As usual, we write $p \xrightarrow{a} p'$ in place of $(p, a, p') \in \to$, and $p \xrightarrow{a}$ if there exists some p' such that $p \xrightarrow{a} p'$. Let the set of *initial actions* of p be defined as $I(p) = \{a \in A \mid p \xrightarrow{a}\}$. Then, we write $p \to$ if $I(p) \neq \emptyset$, and $p \not\to$ otherwise. Given a sequence of actions $s = a_1 \cdots a_n \in A^*$, we write $p \xrightarrow{s} p'$ if and only if there exists a sequence of transitions $p = p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} p_n = p'$. If $p \xrightarrow{s} p'$ holds for some process p', then s is a *trace* of p. Moreover, we say that s is a *completed trace* of p if $I(p') = \emptyset$. We let ε denote the empty trace, and $|s|$ denote the length of trace s. We use $s[i]$ to denote the action occurring at position $1 \leq i \leq |s|$. The prefix of s up to, and including, position i is denoted by $s[..i]$. Similarly, the suffix from position i onward is denoted by $s[i..]$. We write s^n for an n repetition of s, where $s^0 = \varepsilon$ and $s^n = ss^{n-1}$. For ease of notation, given $s \in A^*$ and a term t, we let $s.t = s[1].s[2..].t$, with $\varepsilon.t = t$.

Behavioural Semantics. To compare the behaviour of two processes, we use *behavioural semantics* (BS). These are preorders (reflexive and transitive relations) and equivalences over the states of a LTS allowing us to establish whether two processes have the same *observable* semantics. A *preorder* $p \precsim_{\mathsf{BS}} q$ specifies that the semantics of p is included in that of q, according to the notion of observability specified by BS. We have that p and q are BS *equivalent*, denoted $p \simeq_{\mathsf{BS}} q$, if and only if $p \precsim_{\mathsf{BS}} q$ and $q \precsim_{\mathsf{BS}} p$. Van Glabbeek presented the *linear time–branching time spectrum* [20], henceforth *the spectrum*, which is a taxonomy of BS based on their distinguishing power. Henceforth, we write $\precsim$ (respectively, $\simeq$) to denote an arbitrary preorder (respectively, equivalence) in the spectrum. Below, we report the formal definitions of the BS that we will use in the technical development of our results.

▶ **Definition 1** (Trace and completed trace preorders). *We denote the set of traces of p by $\mathsf{T}(p) = \{s \in A^* \mid \exists p'.p \xrightarrow{s} p'\}$. Then $p \precsim_{\mathsf{T}} q$ if and only if $\mathsf{T}(p) \subseteq \mathsf{T}(q)$. Likewise, the set of completed traces of p is denoted by $\mathsf{CT}(p) = \{s \in A^* \mid \exists p'.p \xrightarrow{s} p' \wedge p' \not\rightarrow\}$. Then, $p \precsim_{\mathsf{CT}} q$ if and only if $\mathsf{T}(p) \subseteq \mathsf{T}(q)$ and $\mathsf{CT}(p) \subseteq \mathsf{CT}(q)$.*

We remark that, since all $\text{BCCSP}_{\|}$ processes have traces of finite length, $\mathsf{CT}(p) \subseteq \mathsf{CT}(q)$ implies $\mathsf{T}(p) \subseteq \mathsf{T}(q)$ for all p, q. Hence, we have $p \precsim_{\mathsf{CT}} q$ if and only if $\mathsf{CT}(p) \subseteq \mathsf{CT}(q)$.

▶ **Definition 2** ((Bi)simulation relations). *A binary relation $\mathcal{R}$ over processes is a simulation if $p\,\mathcal{R}\,q$ and $p \xrightarrow{a} p'$ imply there is some q' such that $q \xrightarrow{a} q'$ and $p'\,\mathcal{R}\,q'$. Then $p \precsim_{\mathsf{S}} q$ if and only if there exists a simulation $\mathcal{R}$ such that $p\,\mathcal{R}\,q$.*
A simulation $\mathcal{R}$ is a ready simulation if $p\,\mathcal{R}\,q$ implies $I(p) = I(q)$. Then $p \precsim_{\mathsf{RS}} q$ if and only if there exists a ready simulation $\mathcal{R}$ such that $p\,\mathcal{R}\,q$.
A simulation $\mathcal{R}$ is a bisimulation if it is symmetric. Then $p \simeq_{\mathsf{B}} q$ if and only if there exists a bisimulation $\mathcal{R}$ such that $p\,\mathcal{R}\,q$.

Using literals and inference rules, we can consider transitions over terms containing variables, e.g. $a.x \xrightarrow{a} x$ and $x \not\rightarrow$, and extend the BS over terms as follows: for $t, u \in \mathcal{O}$, we let $t \precsim u$ if and only if $\rho(t) \precsim \rho(u)$ for all closed substitutions ρ. The following property is straightforward:

▶ **Lemma 3.** *Whenever $t \precsim u$, then $var(t) \subseteq var(u)$.*

Additionally, since $\text{BCCSP}_{\|}$ operators are defined by inference rules in the de Simone format [17], all the preorders in the spectrum are precongruences with respect to them [8]. This means that $t_1 \precsim u_1$ and $t_2 \precsim u_2$ imply $a.t_1 \precsim a.u_1$ for any $a \in A$, $t_1 + t_2 \precsim u_1 + u_2$ and $t_1 \| t_2 \precsim u_1 \| u_2$. Likewise, any equivalence is a *congruence* over $\text{BCCSP}_{\|}$ [8].

We define the *depth* of a term t, notation $|t|$, as the length of the longest trace in $\mathsf{CT}(t)$, formally defined as $|t| = \max\{|s| \mid s \in \mathsf{CT}(t)\}$. Notice that $|x| = 0$ for all $x \in V$. Then, for all behavioural preorders $\precsim$, the following property holds:

▶ **Lemma 4.** *For all terms t, u, if $t \precsim u$, then $|t| \leq |u|$.*

Moreover, a substitution cannot decrease the depth of a term.

▶ **Lemma 5.** *For all terms t and substitutions σ, $|t| \leq |\sigma(t)|$.*

Equational logic. In this paragraph we will use only equivalences to introduce notation and concepts. However, those can be extended to preorders. An *axiom system*, or *axiomatisation*, $\mathcal{E}$ is a collection of *equations* $t \approx u$ over the considered language, thus $\text{BCCSP}_{\|}$ in this paper. An equation $t \approx u$ is *derivable* from an axiom system $\mathcal{E}$, notation $\mathcal{E} \vdash t \approx u$, if there is an equational proof for it from $\mathcal{E}$, namely if $t \approx u$ can be inferred from the axioms in $\mathcal{E}$ using the rules of equational logic. An axiom $t \approx u$ is *sound* modulo $\simeq$ if $t \simeq u$ holds. An axiomatisation $\mathcal{E}$ is sound modulo $\simeq$, if all of its axioms are sound. Consequently, if $\simeq$ is a congruence and $\mathcal{E}$ is sound modulo $\simeq$ then $\mathcal{E} \vdash t \approx u$ implies $t \simeq u$. Then, an axiom system

Table 3 The axiomatisation $\mathcal{E}_1$, which is sound modulo $\simeq_B$.

A0	$x + \mathbf{0} \approx x$	P0	$x \parallel \mathbf{0} \approx x$
A1	$x + y \approx y + x$	P1	$x \parallel y \approx y \parallel x$
A2	$(x + y) + z \approx x + (y + z)$		
A3	$x + x \approx x$		
EL1	$a.x \parallel b.y \approx a.(x \parallel b.y) + b.(a.x \parallel y)$		

$\mathcal{E}$ is *ground-complete* modulo $\simeq$ if $p \simeq q$ implies $\mathcal{E} \vdash p \approx q$ for all closed terms p and q. $\mathcal{E}$ is ω*-complete* modulo $\simeq$ if $\mathcal{E} \vdash \rho(t) \approx \rho(u)$ for all closed substitutions ρ implies $\mathcal{E} \vdash t \approx u$. Lastly, $\mathcal{E}$ is *complete* modulo $\simeq$ if $t \simeq u$ implies $\mathcal{E} \vdash t \approx u$ for all terms t and u.

▶ **Lemma 6.** *Any ground- and ω-complete axiomatisation modulo $\simeq$ is complete modulo $\simeq$.*

Hence, any complete axiomatisation is also ground-complete. An axiomatisation is *finite* if it contains a finite number of axioms. We say that there exists a *finite basis* modulo $\simeq$ if there exists a finite, sound and complete axiomatisation modulo $\simeq$. Consider, for instance axiomatisation $\mathcal{E}_1$ in Table 3. $\mathcal{E}_1$ is sound modulo $\simeq_{\mathsf{B}}$ [2], and, thus, modulo any equivalence in the spectrum. If an axiom contains an action, e.g. axiom EL1, then this may be instantiated with any action in A. Hence, if $|A| = \infty$, this axiom actually represents an infinite number of axioms. However, in this case we still consider the axiomatisation to be finite, since we interpret occurrences of actions in axioms as variables.

We present another axiom, P2, modelling associativity of interleaving:

P2 $\quad (x \parallel y) \parallel z \approx x \parallel (y \parallel z)$

We can show that P2 is sound modulo all the equivalences in the spectrum, and it will be used in the technical results for every finite basis presented in this work.

▶ **Lemma 7.** *Axiom* P2 *is sound modulo* $\simeq_{\mathsf{B}}$.

Henceforth, we write all terms modulo A0-A3 and P0-P2, to improve readability.

Breadth. We conclude this section by introducing the novel notion of *breadth* of a term, that is the largest nested parallel composition of variables in the term. Formally, we define the breadth of t, notation $[\![t]\!]$, inductively as follows:

$$[\![\mathbf{0}]\!] = 0 \qquad [\![x]\!] = 1 \qquad [\![a.t']\!] = [\![t']\!]$$
$$[\![t_1 + t_2]\!] = \max([\![t_1]\!], [\![t_2]\!]) \qquad [\![t_1 \parallel t_2]\!] = [\![t_1]\!] + [\![t_2]\!]$$

Notice that $[\![t]\!] = 0$ if and only if $var(t) = \emptyset$. Breadth satisfies the following property:

▶ **Lemma 8.** *For all terms t, u, if $t \precsim u$, then $[\![t]\!] \leq [\![u]\!]$.*

3 From Ready Simulation to Completed Traces

In this section we show that there exists no finite basis modulo any $\precsim_{\mathsf{RS}} \subseteq \precsim \subseteq \precsim_{\mathsf{CT}}$ or any $\simeq_{\mathsf{RS}} \subseteq \simeq \subseteq \simeq_{\mathsf{CT}}$, as soon as the set of actions A is not empty. To this end, we use the *proof-theoretic technique* [27]: This consists in providing a property of terms that is invariant under provability from finite axiom systems that are sound modulo $\simeq$ ($\precsim$). Then one can identify an infinite family of (in)equations, all sound modulo $\simeq$ ($\precsim$) in which the property is not preserved, i.e., one side of each (in)equation satisfies it, but the other side does not. This implies that there are infinitely many (in)equations that cannot be derived from any finite, sound, axiom system, and thus that there is no finite basis modulo $\simeq$ ($\precsim$).

3.1 Negative Results for the Preorders

Our main objective is to prove the following theorem.

▶ **Theorem 9.** *No precongruence $\precsim$ s.t. $\precsim_{\mathsf{RS}} \subseteq \precsim \subseteq \precsim_{\mathsf{CT}}$ admits a finite basis over BCCSP$_{\|}$.*

Following the strategy above, we introduce an infinite family of sound inequations for which there exist no finite, sound, axiomatisation from which we can derive all of them:

$$\{x \preceq x + x^n \mid n \geq 2\} \tag{1}$$

▶ **Lemma 10.** *The infinite family of inequations in (1) is sound modulo $\precsim_{\mathsf{RS}}$.*

Intuitively, if we have a closed substitution ρ then $\rho(x)$ is simulated by $\rho(x + x^n)$ and $I(\rho(x)) = I(\rho(x + x^n))$. Moreover, Lemma 10 implies that all the inequations in (1) are also sound modulo $\precsim$ for all $\precsim_{\mathsf{RS}}\subseteq\precsim\subseteq\precsim_{\mathsf{CT}}$.

Next, we identify a property of terms that is invariant under provability from a finite axiom system $\mathcal{E}$ that is sound modulo $\precsim_{\mathsf{CT}}$. To this end, notice that, since $\mathcal{E}$ is finite, it can only contain terms with a breadth less than some $n \geq 2$. Consider then a derivation $\mathcal{E} \vdash t \preceq u$ where $u \precsim_{\mathsf{CT}} x + x^n$ and $t \simeq_{\mathsf{CT}} x$. We will argue that no derivation step can increase the breadth of the term, showing that $u \simeq_{\mathsf{CT}} x$ must hold as well. The following proposition shows that this property is an invariant under provability from finite, sound, axiom systems.

▶ **Proposition 11.** *Let $\mathcal{E}$ be a finite set of inequations over* BCCSP$_{\|}$ *that is sound modulo $\precsim_{\mathsf{CT}}$. Let $n \geq 2$ be greater than the breadth of any term in $\mathcal{E}$. Assume that: $\mathcal{E} \vdash t \preceq u$; $u \precsim_{\mathsf{CT}} x + x^n$; and $t \simeq_{\mathsf{CT}} x$. Then $u \simeq_{\mathsf{CT}} x$.*

The following properties are straightforward:

1. $t \precsim_{\mathsf{CT}} u$, by the soundness of $\mathcal{E}$;
2. $|t| = |u| = 0$, by Lemma 4;
3. $[\![t]\!] = 1$, by Lemma 8;
4. $var(t) = var(u) = \{x\}$, by Lemma 3.

The proof of Proposition 11 is by induction on the length of the derivation of $t \preceq u$ from $\mathcal{E}$, with a case distinction on the last rule used in the derivation. We will sketch here the proof of the most interesting case, i.e. the case of the substitution rule. The core of the proof lies in determining the syntactic structure of t and u. We do this by means of the novel notion of breadth: by showing that $[\![u]\!] = 1$ must hold, we can determine that $u \simeq_{\mathsf{CT}} x$. To this end, we present some properties on the depth and breadth of terms as in Proposition 11:

▶ **Lemma 12.** *Let t, u be BCCSP$_{\|}$ terms:*

1. *If $t \precsim_{\mathsf{CT}} x + x^n$ with $n \geq 2$, then $[\![t]\!] = 1$ or $[\![t]\!] = n$.*
2. *If $t \precsim_{\mathsf{CT}} u$ and $|u| = 0$, then $var(u) \subseteq var(t)$.*
3. *If $|t| = [\![t]\!] = 0$, then $t \simeq_{\mathsf{CT}} \mathbf{0}$.*
4. *If $|t| = 0$, $[\![t]\!] = 1$ and $var(t) = \{x\}$, then $t \simeq_{\mathsf{CT}} x$.*

Assume that $\mathcal{E} \vdash t \preceq u$ because $t = \sigma(v)$ and $u = \sigma(w)$ from some $v \preceq w \in \mathcal{E}$ and substitution σ. We have $\sigma(v) \precsim_{\mathsf{CT}} \sigma(w)$, and since $\sigma(w) \precsim_{\mathsf{CT}} x + x^n$, we can use Lemma 12.1 to obtain $[\![\sigma(w)]\!] = 1$ or $[\![\sigma(w)]\!] = n$. We argue that $[\![\sigma(w)]\!] = 1$ must hold, so assume, towards a contradiction, $[\![\sigma(w)]\!] = n$. We have $[\![w]\!] < n$, so there must be some variable $y \in var(w)$ such that $2 \leq [\![\sigma(y)]\!] \leq n$. By the soundness of $\mathcal{E}$ we have $v \precsim_{\mathsf{CT}} w$. Since $\sigma(w) \precsim_{\mathsf{CT}} x + x^n$, by Lemma 4 we get $|\sigma(w)| = 0$, from which, by Lemma 5, we infer $|w| = 0$. Hence, by Lemma 12.2 we obtain $y \in var(v)$. Then we reach a contradiction, since $[\![\sigma(v)]\!] \geq [\![\sigma(y)]\!] > 1$ contradicts $[\![\sigma(v)]\!] = 1$. Consequently, $[\![\sigma(w)]\!] = 1$, so $\sigma(w) \simeq_{\mathsf{CT}} x$ by Lemma 12.4.

Theorem 9 then follows by noticing that no inequation in (1) satisfies the considered property. In fact, while it is clearly the case that $x \simeq_{\mathsf{CT}} x$ and $x + x^n \precsim_{\mathsf{CT}} x + x^n$, it is not true that $x + x^n \simeq_{\mathsf{CT}} x$. In light of Proposition 11, this means that no finite axiomatisation that is sound modulo $\precsim_{\mathsf{CT}}$ can derive all the inequations in (1). Hence, there is not finite basis for $\precsim_{\mathsf{CT}}$ over BCCSP$_{\|}$. Then, recall that the considered inequations are also sound modulo $\precsim_{\mathsf{RS}}$ (Lemma 10). Let $\precsim_{\mathsf{RS}} \subseteq \precsim \subseteq \precsim_{\mathsf{CT}}$ be a precongruence over BCCSP$_{\|}$, and let $\mathcal{E}$ be any finite inequational axiomatisation for BCCSP$_{\|}$ that is sound modulo $\precsim$. Since $\precsim \subseteq \precsim_{\mathsf{CT}}$, we have that $\mathcal{E}$ is sound modulo $\precsim_{\mathsf{CT}}$. Let $n \geq 2$ and larger than the breadth of each term in the inequations in $\mathcal{E}$. Then $x \preceq x + x^n$ cannot be derived from $\mathcal{E}$. Since this inequation is sound modulo $\precsim$, it follows that $\mathcal{E}$ is not complete modulo $\precsim$.

3.2 Negative Results for the Equivalences

We now show how we can extend Theorem 9 from preorders to equivalences. Specifically, the goal is to prove the following theorem.

▶ **Theorem 13.** *No congruence $\simeq$ s.t. $\simeq_{\mathsf{RS}} \subseteq \simeq \subseteq \simeq_{\mathsf{CT}}$ admits a finite basis over BCCSP$_{\|}$.*

From Lemma 10, we infer that the inequation $a.x \preceq a.(x + x^n)$ is sound modulo $\precsim_{\mathsf{RS}}$ for any $a \in A$. Then, we consider the infinite family of equations:

$$\{a.x + a.(x + x^n) \approx a.(x + x^n) \mid n \geq 2\}. \tag{2}$$

▶ **Lemma 14.** *The infinite family of equations in (2) is sound modulo $\simeq_{\mathsf{RS}}$.*

Intuitively, for any closed substitution ρ, $\rho(a.x + a.(x + x^n))$ is simulated by $\rho(a.(x + x^n))$ and vice versa. Additionally, the set of initial actions is $\{a\}$. As a direct consequence of Lemma 14, every equation in (2) is sound modulo $\simeq$ for all $\simeq_{\mathsf{RS}} \subseteq \simeq \subseteq \simeq_{\mathsf{CT}}$.

We can prove that the equations in (2) are not derivable from any finite axiom system $\mathcal{E}$ which is sound modulo $\simeq_{\mathsf{CT}}$ and contains terms up to breadth $n \geq 2$. Informally, consider the derivation $\mathcal{E} \vdash a.x + a.(x + x^n) \approx a.(x + x^n)$. The question is whether we can apply an axiom $v \approx w \in \mathcal{E}$, with $\sigma(v) = a.x + a.(x + x^n)$ and $\sigma(w) = a.(x + x^n)$. Since w has a breadth less than n, there must be some variable $y \in var(v)$ encapsulating some part of x^n. However, equation $a.x + a.(x + x^k \| y) \approx a.(x + x^k \| y)$ with $k \geq 1$ is not sound modulo $\simeq_{\mathsf{CT}}$. Unfortunately, the proof is not this immediate, since the direct application of the axiom does not cover the general case in the considered derivation. Still, notice that $a.x \xrightarrow{a} x$, while $a.(x + x^n) \xrightarrow{a} x + x^n$ is the only transition from $a.(x + x^n)$. Then, we show that having an a-derivative that is completed trace equivalent to a variable is an invariant under provability from finite, sound, axiom systems. Afterwards, we can argue that no equation in (2) satisfies this invariant, from which the non-existence of a finite basis modulo $\simeq_{\mathsf{CT}}$ follows.

▶ **Proposition 15.** *Let $\mathcal{E}$ be a finite set of equations over* BCCSP$_{\|}$ *that is sound modulo $\simeq_{\mathsf{CT}}$. Let $n \geq 2$ be greater than the breadth of any term in $\mathcal{E}$. Assume that: $\mathcal{E} \vdash t \approx u$; $u \precsim_{\mathsf{CT}} a.(x + x^n)$; and $t \xrightarrow{a} t'$ for some $t' \simeq_{\mathsf{CT}} x$. Then $u \xrightarrow{a} u'$ for some $u' \simeq_{\mathsf{CT}} x$.*

Also in this case we have some straightforward properties:

1. $t \simeq_{\mathsf{CT}} u$ by the soundness of $\mathcal{E}$;
2. $|t| = |u| = 1$ by Lemma 4 and the fact that $t \xrightarrow{a}$;
3. $1 \leq [\![t]\!] = [\![u]\!] \leq n$ by Lemma 8 and the fact that $t' \simeq_{\mathsf{CT}} x$; and
4. $var(t) = var(u) = \{x\}$ by Lemma 3 and the fact that $t' \simeq_{\mathsf{CT}} x$.

Moreover, as we consider terms $t \precsim_{\mathsf{CT}} a.(x + x^n)$, the following lemma will be helpful.

▶ **Lemma 16.** *If* $t \precsim_{\mathsf{CT}} a.w$ *and* $t \xrightarrow{a} t'$, *then* $t' \precsim_{\mathsf{CT}} w$.

Proposition 15 is proved by induction on the derivation of $\mathcal{E} \vdash t \approx u$. We sketch only the substitution case. Assume that $\mathcal{E} \vdash t \approx u$ because $t = \sigma(v)$ and $u = \sigma(w)$ for some $v \approx w \in \mathcal{E}$ and substitution σ. We distinguish two cases, according to the derivation of $t \xrightarrow{a} t'$.

Assume first that there exists some v' such that $v \xrightarrow{a} v'$ and $\sigma(v') \simeq_{\mathsf{CT}} x$. Then consider the closed substitution ρ defined as $\rho(y) = \mathbf{0}$ if $[\![\sigma(y)]\!] \leq 1$ and $\rho(y) = a^2.\mathbf{0}$ otherwise. Then, $\rho(v') \not\rightarrow$, so $a \in \mathsf{CT}(\rho(v))$. From the soundness of $\mathcal{E}$, we have $v \simeq_{\mathsf{CT}} w$, so $a \in \mathsf{CT}(\rho(w))$. By construction of ρ, this gives that $w \xrightarrow{a} w'$ for some w' s.t. $\rho(w') \not\rightarrow$. Then, $\sigma(w) \xrightarrow{a} \sigma(w')$. By Lemma 16 we obtain $\sigma(w') \precsim_{\mathsf{CT}} x + x^n$, so by Lemma 1 it holds that either $[\![\sigma(w')]\!] = 1$ or $[\![\sigma(w')]\!] = n$. If $[\![\sigma(w')]\!] = 1$, then $\sigma(w') \simeq_{\mathsf{CT}} x$ by Lemma 4, so the invariant holds. If $[\![\sigma(w')]\!] = n$, then since $[\![w]\!] < n$, there exists some variable $z \in var(w')$ such that $2 \leq [\![\sigma(z)]\!] \leq n$. However, we reach a contradiction, since $\rho(w') \xrightarrow{a}$ contradicts $\rho(w') \not\rightarrow$.

Assume now that there is no v' such that $v \xrightarrow{a} v'$ and $\sigma(v') \simeq_{\mathsf{CT}} x$. We first prove that there is some $y \in var(v)$ s.t. $\sigma(y) \xrightarrow{a} t''$ and either $t' = t''$ or $t' = t'' \| \sigma(v_\|)$ for some $v_\|$. We have $\sigma(v) \xrightarrow{a} t'$, so there must exist some summand v' of v such that $\sigma(v') \xrightarrow{a} t'$. We proceed by a case distinction on the structure of v'. We consider only the case $v' = v_1 \| v_2$. In this case, assume, without loss of generality, that $\sigma(v_1) \xrightarrow{a} t_1$ such that $t' = t_1 \| \sigma(v_2)$. Note that $\sigma(v_2) \not\rightarrow$, since $|\sigma(v)| = 1$. We can then perform a case distinction on the summands of v_1 to prove that there is some $y \in var(v_1)$ such that $\sigma(y) \xrightarrow{a} t''$ and either $t' = t'' \| \sigma(v_2)$ or $t' = t'' \| \sigma(v'_\|) \| \sigma(v_2) = t'' \| \sigma(v'_\| \| v_2)$ for some $v'_\|$. If $t'' = t'$, it is clear that $t'' \simeq_{\mathsf{CT}} x$. If $t' = t'' \| \sigma(v_\|)$, from $t'' \| \sigma(v_\|) \simeq_{\mathsf{CT}} x$ and Lemma 8 it follows that $[\![t'' \| \sigma(v_\|)]\!] = 1$. So either $[\![t'']\!] = 1$ and $[\![\sigma(v_\|)]\!] = 0$ or vice versa. Then $t'' \simeq_{\mathsf{CT}} x$ and $\sigma(v_\|) \simeq_{\mathsf{CT}} \mathbf{0}$ or vice versa, by Lemmas 3, 4. Consider the closed substitution ρ' defined as $\rho'(y) = a^2.\mathbf{0}$, $\rho'(z) = \mathbf{0}$ if $[\![\sigma(z)]\!] \leq 1$ and $\rho'(z) = a^3.\mathbf{0}$ otherwise. Then $a^2 \in \mathsf{CT}(\rho'(v))$ in both forms of t', so, by soundness of $\mathcal{E}$, $a^2 \in \mathsf{CT}(\rho'(w))$. Hence, there is some summand w' of w such that $a^2 \in \mathsf{CT}(\rho'(w'))$. We expand only the case $w' = w_1 \| w_2$. Assume, without loss of generality, that $a^2 \in \mathsf{CT}(\rho'(w_1))$ and $\mathsf{CT}(\rho'(w_2)) = \{\varepsilon\}$ (it is impossible that $a \in \mathsf{CT}(\rho'(w_1))$ and $a \in \mathsf{CT}(\rho'(w_2))$, since $|\sigma(w)| \leq 1$ and by construction of ρ'). From $a^2 \in \mathsf{CT}(\rho'(w_1))$, we can again do a case distinction on the summands of w_1. This process can be repeated a finite number of times, until the nested summand y is reached. Let $w_\|$ be the accumulation of parallel compositions, such that $\mathsf{CT}(\rho'(w_\|)) = \{\varepsilon\}$. We obtain that $\sigma(w) \xrightarrow{a} t'' \| \sigma(w_\|)$. By Lemma 16 we must have that $t'' \| \sigma(w_\|) \precsim_{\mathsf{CT}} x + x^n$. Hence, $[\![t'' \| \sigma(w_\|)]\!] = 1$ or $[\![t'' \| \sigma(w_\|)]\!] = n$ by Lemma 1. In the first case we obtain that $t'' \| \sigma(w_\|) \simeq_{\mathsf{CT}} x$ by Lemma 4, so the invariant holds. In the second case we reason towards a contradiction as above. Hence, $a^2 \in \mathsf{CT}(\rho'(w))$ only holds if $\sigma(w) \xrightarrow{a} u'$ for some $u' \simeq_{\mathsf{CT}} x$.

As discussed above, the equations in (2) do not satisfy the invariant in Proposition 15. Hence Theorem 13 follows for $\simeq_{\mathsf{CT}}$. The same reasoning applied in Section 3.1 then allows us to conclude that Theorem 13 holds for any congruence included between $\simeq_{\mathsf{RS}}$ and $\simeq_{\mathsf{CT}}$.

4 Simulation and Traces

In this section we discuss the axiomatisability modulo trace and simulation semantics over $\mathrm{BCCSP}_\|$. Interestingly, we show that as long as the set of actions is finite, then no finite basis can be given. However, this changes when $|A| = \infty$ since we can provide complete axiomatisations including only finitely many axiom schemas characterising the considered semantics. Moreover, we need to apply various proof techniques to obtain these results. Specifically, if $|A| = 1$ we prove that there exist no finite bases modulo $\simeq_{\mathsf{S}}$ or $\simeq_{\mathsf{T}}$ using the transformation technique from [5]; whereas if $2 \leq |A| < \infty$ we prove that there exist no finite bases modulo $\precsim_{\mathsf{S}}$, $\precsim_{\mathsf{T}}$, $\simeq_{\mathsf{S}}$ or $\simeq_{\mathsf{T}}$ using the proof-theoretic technique described in Section 3. Lastly, for $|A| = \infty$, we present finite bases modulo $\simeq_{\mathsf{S}}$ and $\simeq_{\mathsf{T}}$ using Groote's technique of inverted substitutions [21].

4.1 Negative Results with $|A| = 1$

The main result of this subsection is the following theorem.

▶ **Theorem 17.** *If $|A| = 1$, then no congruence $\simeq$ s.t. $\simeq_{\mathsf{S}} \subseteq \simeq \subseteq \simeq_{\mathsf{T}}$ admits a finite basis over BCCSP$_{\|}$.*

For this section, assume $A = \{a\}$ and that $\simeq$ represents a congruence between $\simeq_{\mathsf{S}}$ and $\simeq_{\mathsf{T}}$. To prove that the algebra of BCCSP$_{\|}$ processes modulo $\simeq$ has no finite basis we apply the transformation technique: we show how a presupposed finite basis for the algebra of BCCSP$_{\|}$ processes modulo $\simeq$ can be transformed into a finite basis for the so-called *max-plus algebra* of natural numbers with max and + as operations. Since it has been shown that the latter algebra is not finitely based [5], it then follows that the former cannot be finitely based either. A similar approach was applied by Aceto et al. to prove that there exists no finite basis modulo $\simeq_{\mathsf{T}}$ if $|A| = 1$ over the process algebra BPA [4].

The following lemma is straightforward to prove.

▶ **Lemma 18.** *If $|A| = 1$, then for all processes p, q, we have $p \simeq q$ if and only if $|p| = |q|$. Moreover, $|\mathbf{0}| = 0$, $|p + q| = \max(|p|, |q|)$ and $|p \parallel q| = |p| + |q|$.*

Note that Lemma 18 establishes an isomorphism from the algebra of closed BCCSP$_{\|}$ expressions modulo $\simeq$ with constant $\mathbf{0}$ and binary operations + and $\|$ and the max-plus algebra with constant 0 and binary operations max and +. Hence, there is a correspondence between the equational theories of these two algebras that we shall now formalise. We need the following inductively defined mapping (with $\mathtt{W}$ the set of variables in the max-plus algebra):

$$\langle \mathbf{0} \rangle = 0 \qquad \langle t_1 + t_2 \rangle = \max(\langle t_1 \rangle, \langle t_2 \rangle)$$
$$\langle x \rangle = \mathtt{x}, \text{ where } \mathtt{x} \in \mathtt{W} \qquad \langle t_1 \parallel t_2 \rangle = \langle t_1 \rangle + \langle t_2 \rangle$$

The following lemma is then the counterpart of Lemma 18 on the max-plus algebra, since the mapping defined above combines the depth of terms with occurrences of variables.

▶ **Lemma 19.** *For all terms t, u without an action prefix, we have $t \simeq u$ if and only if $\langle t \rangle = \langle u \rangle$.*

Now suppose that $\mathcal{E}$ is a finite basis for the algebra BCCSP$_{\|}$ processes modulo $\simeq$. Then, in particular, if $t \simeq u$, then $\mathcal{E} \vdash t \approx u$. Now $\mathcal{E}$ may include axioms with action prefixes, so we cannot readily transform all of $\mathcal{E}$ into a complete set of axioms for the max-plus algebra. The following lemma, however, establishes that whenever two (open) BCCSP$_{\|}$ terms are equivalent modulo $\simeq$, then either both or neither have occurrences of the action prefix.

▶ **Lemma 20.** *If $t \simeq u$, then t contains an action prefix if and only if u contains an action prefix.*

Now, let $\mathcal{E}' \subseteq \mathcal{E}$ be obtained by removing all axioms with occurrences of the action prefix. By Lemma 20 we have, for all BCCSP$_{\|}$-terms t and u without occurrences of action prefixes that $\mathcal{E} \vdash t \approx u$ if and only if $\mathcal{E}' \vdash t \approx u$. Denote by $\langle \mathcal{E}' \rangle$ the set of equations $\langle t \rangle \approx \langle u \rangle$ such that $(t \approx u) \in \mathcal{E}'$. Then, for t, u without occurrences of action prefixes, $\mathcal{E} \vdash t \approx u$ if and only if $\langle \mathcal{E}' \rangle \vdash \langle t \rangle = \langle u \rangle$. It would follow that $\langle \mathcal{E}' \rangle$ is a finite basis for the max-plus algebra, which cannot exist. We conclude that the presupposed finite basis for the algebra of closed $\mathrm{BCCSP}_{\|}$ expressions modulo $\simeq$ can also not exist.

4.2 Negative Results with $2 \leq |A| < \infty$

We use the proof-theoretic technique discussed in Section 3 to prove the following:

▶ **Theorem 21.** *If $2 \leq |A| < \infty$, then no precongruence $\precsim$ s.t. $\precsim_{\mathsf{S}} \subseteq \precsim \subseteq \precsim_{\mathsf{T}}$ admits a finite basis over $BCCSP_{\|}$.*

▶ **Theorem 22.** *If $2 \leq |A| < \infty$, then no congruence $\simeq$ s.t. $\simeq_{\mathsf{S}} \subseteq \simeq \subseteq \simeq_{\mathsf{T}}$ admits a finite basis over $BCCSP_{\|}$.*

Throughout this section consider $A = \{a_1, \dots, a_{|A|}\}$, where $a = a_1, b = a_2$ and $s = a_3 \cdots a_{|A|}$. Note that $s = \varepsilon$ if $|A| = 2$. Let $n \geq 1$, then consider the term

$$h_n = ab^n s.\mathbf{0} + b^n s.x \parallel x + a.(s.x \parallel x^n) + \sum_{i=1}^{|s|} ab^n s[..i-1].(s[i+1..].x \parallel x).$$

Then we introduce the following infinite family of inequations:

$$\{ab^n s.x \preceq h_n \mid n \geq 1\}. \tag{3}$$

Intuitively, for any closed substitution ρ we see that $\rho(ab^n s.x)$ is simulated by one of the summands of $\rho(h_n)$ because of the added parallel composition with x. If $\rho(x) \not\rightarrow$ then it is the first summand, and if $\rho(x) \xrightarrow{a}$, then it is the second summand, etc. Thus, for each of the $|A|+1$ options there is a corresponding summand.

▶ **Lemma 23.** *The infinite family of inequations in (3) is sound modulo $\precsim_{\mathsf{S}}$.*

We argue that not all inequations in the family are derivable using a finite collection of inequations $\mathcal{E}$, which is sound modulo $\precsim_{\mathsf{T}}$ and contains terms up to depth $n \geq 1$. First, we give some intuition why this is the case. Consider term $ab^n s.x$, for which we know that $ab^n s.x \xrightarrow{ab^n s} x$. However, h_n cannot perform trace $ab^n s$ and reach a term with an occurrence of x. We introduce a new notion, named the front of a term, to formally express this.

▶ **Definition 24.** *Let $\mathsf{front}(t) \subseteq var(t)$ be defined inductively as follows:*

$$\mathsf{front}(\mathbf{0}) = \mathsf{front}(a.t') = \emptyset \qquad \mathsf{front}(x) = \{x\}$$
$$\mathsf{front}(t_1 + t_2) = \mathsf{front}(t_1 \parallel t_2) = \mathsf{front}(t_1) \cup \mathsf{front}(t_2)$$

Hence, $\mathsf{front}(t)$ is the set of all variables of t that do not occur after an action prefix in t.

This means that whenever we have a substitution σ such that $\sigma(x) \xrightarrow{a}$, then $x \in \mathsf{front}(t)$ implies $\sigma(t) \xrightarrow{a}$. Additionally, it follows that if $|t| = 0$, then $var(t) = \mathsf{front}(t)$. Using this definition we obtain the following proposition.

▶ **Proposition 25.** *Let $2 \leq |A| < \infty$ and let $\mathcal{E}$ be a finite collection of inequations over* $\mathrm{BCCSP}_{\|}$ *that is sound modulo $\precsim_{\mathsf{T}}$. Let $n \geq 1$ be greater than the depth of any term in $\mathcal{E}$. Assume that: $\mathcal{E} \vdash t \preceq u$; $u \precsim_{\mathsf{T}} h_n$; and $t \xrightarrow{ab^n s} t'$ for some t' with $x \in \mathsf{front}(t')$. Then $u \xrightarrow{ab^n s} u'$ for some u' with $x \in \mathsf{front}(u')$.*

Continuing our intuitive argument, consider $t = ab^n s.x$ and derivation $\mathcal{E} \vdash ab^n s.x \preceq u$ where $u \precsim_{\mathsf{T}} h_n$. Either we directly apply an axiom $v \preceq w \in \mathcal{E}$ on $ab^n s.x$, so $\sigma(v) = ab^n s.x$ and $\sigma(w) = u$, or $\mathcal{E} \vdash b^n s.x \preceq u'$ and $u = a.u'$. To proceed, we need a property of front:

▶ **Lemma 26.** *If $|A| > 2$, $a \in A$, $t \precsim_{\mathsf{T}} u$ and $t \xrightarrow{s'} t'$ for some t' with $x \in \mathsf{front}(t')$ and s' does not contain a, then $u \xrightarrow{s'} u'$ for some u' with $x \in \mathsf{front}(u')$.*

If $\mathcal{E} \vdash b^n s.x \preceq u'$ and $u = a.u'$, we can infer, from $b^n s.x \precsim_{\mathsf{T}} u'$ and Lemma 26, that $u' \xrightarrow{b^n s} u''$ and $x \in \mathsf{front}(u'')$, so that $a.u' \xrightarrow{ab^n s} u''$ and $x \in \mathsf{front}(u'')$. If axiom $v \preceq w \in \mathcal{E}$ is applied such that $\sigma(v) = ab^n s.x$ and $\sigma(w) = u$, then some part of $ab^n s.x$ must be encapsulated in a variable $y \in var(v)$, i.e. $v = (ab^n s)[..i].y$ for some $0 \leq i < n$. To see this, let us first introduce the notion of $\mathbf{0}$-nested summand.

▶ **Definition 27.** *t' is a $\mathbf{0}$-nested-summand of t, denoted $t' \sqsubseteq_{\mathbf{0}} t$, if and only if either:*
1. *t' is a summand of t, or*
2. *t has a summand $t_1 \parallel t_2$, where $t' \sqsubseteq_{\mathbf{0}} t_1$ and $t_2 \simeq_{\mathsf{T}} \mathbf{0}$, or*
3. *t has a summand $t_1 \parallel t_2$, where $t' \sqsubseteq_{\mathbf{0}} t_2$ and $t_1 \simeq_{\mathsf{T}} \mathbf{0}$.*

Using this new notion, we can prove the following lemma.

▶ **Lemma 28.** *If $|A| \geq 2$, $t \precsim_{\mathsf{T}} h_n$ and $t \xrightarrow{(ab^n s)[..i]} t'$ for some t' which has a $\mathbf{0}$-nested-summand of the form $t_1 \parallel t_2$ such that $t_1 \parallel t_2 \xrightarrow{(ab^n s)[i+1..]} t''$ for some t'' with $0 \leq i < n$ and $x \in \mathsf{front}(t'')$, then either $t_1 \simeq_{\mathsf{T}} \mathbf{0}$ or $t_2 \simeq_{\mathsf{T}} \mathbf{0}$.*

Hence, even if v was of the form $v_1 \parallel v_2$, thanks to Lemma 28 we can always eliminate one of the two parallel components, and repeat the reasoning on the remaining component until we reach a term $v' \sqsubseteq_{\mathbf{0}} v$ such that $y \in \mathsf{front}(v')$. Consequently, y is in the front of v after performing trace $(ab^n s)[..i]$ and x is in the front of $\sigma(y)$ after performing the remaining part of the trace, namely $(ab^n s)[i+1..]$. If $|A| \geq 3$, then we can infer from $v \precsim_{\mathsf{T}} w$ and Lemma 26 that y is in the front after performing the same trace in w. Hence, $\sigma(w) \xrightarrow{ab^n s} u'$ for some u' with $x \in \mathsf{front}(u')$. To complete our reasoning, we need a special case of Lemma 26 if $|A| = 2$, covering the case in which variable y is in the front of v after performing trace ab^k for some $k < n$. Note that the following lemma holds for $|A| > 2$ due to Lemma 26.

▶ **Lemma 29.** *If $|A| \geq 2$, $t \precsim_{\mathsf{T}} u$, $1 \leq k < |t|$ and $t \xrightarrow{ab^k} t'$ for some t' with $x \in \mathsf{front}(t')$, then there exist u' and $0 \leq \ell < |u|$ such that $u \xrightarrow{ab^\ell} u'$ and $x \in \mathsf{front}(u')$.*

From Lemma 29 we know that w can perform trace $ab^n[..k]$ for some $0 \leq k < |w|$ for which $y \in \mathsf{front}(w)$. We argue that $k = i$ must hold, because of the constraints of $\sigma(w)$ in regards to the depth and soundness.

Hence, we can conclude that, in all cases, $u = \sigma(w) \xrightarrow{ab^n} u'$ for some term u' with $x \in \mathsf{front}(u')$.

Theorem 21 then follows by applying the same reasoning used to conclude Section 3.

The infinite family of inequations in (3) results in an infinite family of equations as:

$$\{ab^n s.x + h_n \approx h_n \mid n \geq 1\}. \tag{4}$$

▶ **Lemma 30.** *The infinite family of equations in (4) is sound modulo $\simeq_{\mathsf{S}}$.*

Then, following a similar reasoning as used for Proposition 25, we can prove the invariance under equational derivation also in the case of equivalences.

Table 4 Additional axioms for $\simeq_\mathsf{T}$ and $\simeq_\mathsf{S}$.

SP1	$(x+y) \parallel (z+w) \approx x \parallel (z+w) + y \parallel (z+w) + (x+y) \parallel z + (x+y) \parallel w$		
TP	$(x+y) \parallel z \approx x \parallel z + y \parallel z$	SP2	$a.x \parallel (y+z) \approx a.(x \parallel (y+z)) + a.x \parallel y + a.x \parallel z$
T	$a.x + a.y \approx a.(x+y)$	S	$a.(x+y) \approx a.(x+y) + a.x$

▶ **Proposition 31.** *Let $2 \leq |A| < \infty$ and let $\mathcal{E}$ be a finite collection of equations over* BCCSP$_\parallel$ *that is sound modulo $\simeq_\mathsf{T}$. Let $n \geq 1$ be greater than the depth of any term in $\mathcal{E}$. Assume that: $\mathcal{E} \vdash t \approx u$; $u \precsim_\mathsf{T} h_n$; and $t \xrightarrow{ab^n s} t'$ for some t' with $x \in \mathsf{front}(t')$. Then $u \xrightarrow{ab^n s} u'$ for some u' with $x \in \mathsf{front}(u')$.*

Similar arguments to those used to prove Theorem 21 allow us to prove Theorem 22.

4.3 Finite Bases with $|A| = \infty$

In [2] it was proved that the axiomatisation $\mathcal{E}_\mathsf{T} = \mathcal{E}_1 \cup \{\mathrm{T}, \mathrm{TP}\}$, with $\mathcal{E}_1$ given in Table 3 and T, TP from Table 4, is a finite, sound and ground-complete axiomatisation of $\simeq_\mathsf{T}$ over BCCSP$_\parallel$. Similarly, the axiomatisation $\mathcal{E}_\mathsf{S} = \mathcal{E}_1 \cup \{\mathrm{S}, \mathrm{SP1}, \mathrm{SP2}\}$ is finite, sound and ground-complete modulo $\simeq_\mathsf{S}$, with S, SP1 and SP2 given in Table 4. In this section, we extend $\mathcal{E}_\mathsf{T}$ and $\mathcal{E}_\mathsf{S}$ with a finite number of sound axioms that make them ω-complete besides ground-complete, so that the existence of finite bases for the two equivalences follows from Lemma 6.

To this end, consider the following axiom, that, to the best of our knowledge, is original with respect to the related literature:

P3 $\quad a.x \parallel y \approx a.x \parallel y + a.(x \parallel y)$.

▶ **Lemma 32.** *Axiom* P3 *is sound modulo $\simeq_B$.*

Consider $\mathcal{E}'_\mathsf{T} = \mathcal{E}_\mathsf{T} \cup \{\mathrm{P2}, \mathrm{P3}\}$. We use Groote's technique of inverted substitutions to prove that the axiomatisation $\mathcal{E}'_\mathsf{T}$ is ω-complete if the number of actions is infinite. In [21], it is proved that an axiomatization $\mathcal{E}$ is ω-complete if for each equation $t \approx u$, of which all closed instances can be derived from $\mathcal{E}$, we can define a mapping $R : \mathcal{P} \to \mathcal{O}$ such that:

1. We can define a closed substitution $\tilde{\rho}$ such that $\mathcal{E} \vdash R(\tilde{\rho}(t)) \approx t$ and $\mathcal{E} \vdash R(\tilde{\rho}(u)) \approx u$.
2. From $\mathcal{E}$ and $\{p_1 \approx q_1, p_2 \approx q_2, R(p_1) \approx R(q_1), R(p_2) \approx R(q_2)\}$ we can derive $R(a.p_1) \approx R(a.q_1)$ for all $a \in A$, $R(p_1 + p_2) \approx R(q_1 + q_2)$ and $R(p_1 \| p_2) \approx R(q_1 \| q_2)$.
3. For each $v \approx w \in \mathcal{E}$ and each closed substitution ρ, we have $\mathcal{E} \vdash R(\rho(v)) \approx R(\rho(w))$.

In our setting, given an equation $t \approx u$ such that $\mathcal{E}'_\mathsf{T} \vdash \rho(t) \approx \rho(u)$ for all closed substitutions ρ, we consider the following mapping $R : \mathcal{P} \to \mathcal{O}$, where a_x is a unique action for each $x \in V$ and a_x does not appear in either t or u:

$$R(\mathbf{0}) = \mathbf{0} \qquad R(a.p) = a.R(p) \text{ where } a \neq a_x \text{ for all } x \in V$$
$$R(a_x.p) = x \parallel R(p) \qquad R(p+q) = R(p) + R(q) \qquad R(p \parallel q) = R(p) \parallel R(q)$$

We remark that existence, and uniqueness, of action a_x is guaranteed by the fact that we are considering a countably infinite set of actions. Then, we can show, by induction on the structure of terms, that $\mathcal{E}'_\mathsf{T} \vdash R(\tilde{\rho}(t)) \approx t$ and $\mathcal{E}'_\mathsf{T} \vdash R(\tilde{\rho}(u)) \approx u$ for the closed substitution $\tilde{\rho}$ defined by $\tilde{\rho}(x) = a_x.\mathbf{0}$ for all $x \in V$. The other two properties follow by a case analysis on terms and axioms, respectively. In particular, we remark that the new axiom P3 is fundamental in proving the third property for axiom EL1 (Table 3). In fact, we have that if $a \neq a_x$ for all $x \in V$ and $b = b_y$ for some $y \in V$, then, letting $\rho(x) = p$ and $\rho(y) = q$:

$$
\begin{aligned}
& R(a.p \parallel b.q) \\
&= a.R(p) \parallel R(b.q) \overset{\text{P3}}{\approx} a.R(p) \parallel R(b.q) + a.(R(p) \parallel R(b.q)) \\
&= R(a.p) \parallel (y \parallel R(q)) + a.(R(p) \parallel R(b.q)) \overset{\text{P1,P2}}{\approx} y \parallel (R(a.p) \parallel R(q)) + a.(R(p) \parallel R(b.q)) \\
&\overset{\text{A1}}{\approx} a.(R(p) \parallel R(b.q)) + y \parallel (R(a.p) \parallel R(q)) = R(a.(p \parallel b.q) + b.(a.p \parallel q)).
\end{aligned}
$$

The following Theorem is then a direct consequence of [21, Theorem 3.1].

▶ **Theorem 33.** *If $|A| = \infty$, then the axiomatisation $\mathcal{E}'_{\mathsf{T}}$ is ω-complete modulo $\simeq_{\mathsf{T}}$.*

We can then use the same technique, with the same mapping R and closed substitution $\tilde{\rho}$, to show that $\mathcal{E}'_{\mathsf{S}} = \mathcal{E}_{\mathsf{S}} \cup \{\text{P2}, \text{P3}\}$ is ω-complete if the number of actions is infinite.

▶ **Theorem 34.** *If $|A| = \infty$, then the axiomatisation $\mathcal{E}'_{\mathsf{S}}$ is ω-complete modulo $\simeq_{\mathsf{S}}$.*

We remark that axioms T, S, SP2 and P3 are axiom schemas, i.e., they generate an equation for every instantiation of the action prefix. This means that, when $|A| = \infty$, they generate infinitely many axioms. However, as stated in the previous sections, we still consider the axiomatisations $\mathcal{E}_{\mathsf{T}'}$ and $\mathcal{E}_{\mathsf{S}'}$ as finite, since they are finitely representable (they include only finitely many axiom schemas).

5 Extension with Successful Termination

In this section we extend $\text{BCCSP}_{\parallel}$ with successful termination, and we discuss which results from the previous sections can be lifted to the new language.

First of all, we extend the definition of a LTS with successful termination, i.e., a LTS is a tuple $(\mathcal{P}, A, \rightarrow, \downarrow)$ where $\downarrow \subseteq \mathcal{P}$ is a set of successfully terminating processes. We write $p\downarrow$ if and only if $p \in \downarrow$. We extend the initial actions of a process p by successful termination ($\downarrow$), namely we define $\mathsf{menu}(p) = I(p) \cup \{\downarrow\}$ if $p\downarrow$ and $\mathsf{menu}(p) = I(p)$ otherwise. Then we extend the behavioural semantics with requirements on successful termination.

▶ **Definition 35** (Trace and completed trace preorders with termination)**.** *The set of terminating traces of p is denoted by $\mathsf{T}^{\downarrow}(p) = \{s \in A^* \mid \exists p'.p \xrightarrow{s} p' \wedge p'\downarrow\}$. Then $p \precsim^{\downarrow}_{\mathsf{T}} q$ if and only if $\mathsf{T}(p) \subseteq \mathsf{T}(q)$ and $\mathsf{T}^{\downarrow}(p) \subseteq \mathsf{T}^{\downarrow}(q)$. Similarly, $p \precsim^{\downarrow}_{\mathsf{CT}} q$ if and only if $\mathsf{T}(p) \subseteq \mathsf{T}(q)$, $\mathsf{CT}(p) \subseteq \mathsf{CT}(q)$ and $\mathsf{T}^{\downarrow}(p) \subseteq \mathsf{T}^{\downarrow}(q)$.*

▶ **Definition 36** ((Bi)simulations with termination)**.** *A simulation $\mathcal{R}$ is terminating if $p\mathcal{R}q$ and $p\downarrow$ imply $q\downarrow$. Then $p \precsim^{\downarrow}_{\mathsf{S}} q$ if and only if there exists a terminating simulation $\mathcal{R}$ such that $p\mathcal{R}q$.*
A terminating simulation $\mathcal{R}$ is a terminating ready simulation if $p\mathcal{R}q$ implies $\mathsf{menu}(p) = \mathsf{menu}(q)$. Then $p \precsim^{\downarrow}_{\mathsf{RS}} q$ if and only if there exists a terminating ready simulation $\mathcal{R}$ such that $p\mathcal{R}q$.
A terminating simulation $\mathcal{R}$ is a terminating bisimulation relation if it is symmetric. Then $p \simeq^{\downarrow}_{\mathsf{B}} q$ if and only if there exists a terminating bisimulation relation $\mathcal{R}$ such that $p\mathcal{R}q$.

We then extend $\text{BCCSP}_{\parallel}$ with the constant process $\mathbf{1}$, denoting successful termination, obtaining the language $\text{BSP}_{\parallel}$. The operational semantics of $\text{BSP}_{\parallel}$ is obtained by adding the inference rules in Table 5 to those in Table 2. While most axioms of $\mathcal{E}_1$ remain sound, P0 does not. In Table 6, we introduce the axioms P0a, P0b and P0c to replace P0.

▶ **Lemma 37.** *Axiom system $\mathcal{E}_2$, consisting of the axioms in Table 6, is sound modulo $\simeq^{\downarrow}_{\mathsf{B}}$.*

Table 5 Operational semantics of successful termination in $\mathrm{BSP}_{\|}$.

$$\frac{}{\mathbf{1}\downarrow} \qquad \frac{t\downarrow}{t+u\downarrow} \qquad \frac{u\downarrow}{t+u\downarrow} \qquad \frac{t\downarrow \quad u\downarrow}{t \parallel u\downarrow}$$

Table 6 The axiomatisation $\mathcal{E}_2$, which is sound modulo $\simeq^{\downarrow}_{\mathsf{B}}$.

A0	$x + \mathbf{0} \approx x$	P0a	$\mathbf{0} \parallel \mathbf{0} \approx \mathbf{0}$
A1	$x + y \approx y + x$	P0b	$x \parallel \mathbf{1} \approx x$
A2	$(x + y) + z \approx x + (y + z)$	P0c	$a.x \parallel \mathbf{0} \approx a.(x \parallel \mathbf{0})$
A3	$x + x \approx x$	P1	$x \parallel y \approx y \parallel x$
EL1	$a.x \parallel b.y \approx a.(x \parallel b.y) + b.(a.x \parallel y)$		

5.1 Preservation of Negative Results

We present a schema to apply the transformation technique on any finite basis for $\mathrm{BSP}_{\|}$ into a finite basis for $\mathrm{BCCSP}_{\|}$. This schema also works for finite, sound and ground-complete axiomatisations. Consequently, for every behavioural semantics it holds that the non-existence of a finite basis for $\mathrm{BCCSP}_{\|}$ implies the non-existence of a finite basis for $\mathrm{BSP}_{\|}$.

Consider some behavioural semantics in the spectrum, let $\precsim$ and $\precsim^{\downarrow}$ be the corresponding preorders and let $\simeq$ and $\simeq^{\downarrow}$ be the corresponding equivalences. We argue that if $t \precsim u$ holds, then $t \parallel \mathbf{0} \precsim^{\downarrow} u \parallel \mathbf{0}$ must hold. Since from the operational semantics it is clear that for any closed $\mathrm{BSP}_{\|}$ substitution ρ no successfully terminating term can be reached from $\rho(t \parallel \mathbf{0})$ or $\rho(u \parallel \mathbf{0})$, then any closed substitution which uses the successful termination constant has no influence on the behaviour. We define $[t]_{\mathbf{1}\mapsto\mathbf{0}}$ to be $\mathrm{BSP}_{\|}$ term t with every occurrence of the constant $\mathbf{1}$ replaced with the constant $\mathbf{0}$. We then also argue that $t \precsim^{\downarrow} u$ implies $[t]_{\mathbf{1}\mapsto\mathbf{0}} \precsim [u]_{\mathbf{1}\mapsto\mathbf{0}}$, since the successful termination constant has no influence on the old criteria. Using these arguments, consider a finite basis modulo $\simeq^{\downarrow}$, let this be $\mathcal{E}$. We argue that $[\mathcal{E}]_{\mathbf{1}\mapsto\mathbf{0}} \cup \{\mathrm{P0}\}$, where $[\mathcal{E}]_{\mathbf{1}\mapsto\mathbf{0}} = \{[v]_{\mathbf{1}\mapsto\mathbf{0}} \approx [w]_{\mathbf{1}\mapsto\mathbf{0}} \mid v \approx w \in \mathcal{E}\}$, then serves as a finite basis modulo $\simeq$, leading to a contradiction with Theorem 13. Therefore, we can conclude that the following theorem holds.

▶ **Theorem 38.** *No congruence $\simeq^{\downarrow}$ s.t. $\simeq^{\downarrow}_{\mathsf{RS}} \subseteq \simeq^{\downarrow} \subseteq \simeq^{\downarrow}_{\mathsf{CT}}$ admits a finite basis over $BSP_{\|}$. If $|A| < \infty$, then no congruence $\simeq^{\downarrow}$ s.t. $\simeq^{\downarrow}_{\mathsf{S}} \subseteq \simeq^{\downarrow} \subseteq \simeq^{\downarrow}_{\mathsf{T}}$ admits a finite basis over $BSP_{\|}$.*

We can use the same schema also for the preorders, provided we substitute axiom P0 with the inequations P0-l ($x \parallel \mathbf{0} \preceq x$) and P0-r ($x \preceq x \parallel \mathbf{0}$), which are sound modulo $\simeq_{\mathsf{B}}$.

▶ **Theorem 39.** *No precongruence $\precsim^{\downarrow}$ s.t. $\precsim^{\downarrow}_{\mathsf{RS}} \subseteq \precsim^{\downarrow} \subseteq \precsim^{\downarrow}_{\mathsf{CT}}$ admits a finite basis over $BSP_{\|}$. If $|A| < \infty$, then no precongruence $\precsim^{\downarrow}$ s.t. $\precsim^{\downarrow}_{\mathsf{S}} \subseteq \precsim^{\downarrow} \subseteq \precsim^{\downarrow}_{\mathsf{T}}$ admits a finite basis over $BSP_{\|}$.*

5.2 Finite Basis with $|A| = \infty$

Consider the axiomatisations $\mathcal{E}^{\downarrow}_{\mathsf{T}} = \mathcal{E}_2 \cup \{\mathrm{T}, \mathrm{TP}\}$, and $\mathcal{E}^{\downarrow}_{\mathsf{S}} = \mathcal{E}_2 \cup \{\mathrm{P0d}, \mathrm{S}, \mathrm{SP1}, \mathrm{SP2}\}$, where

P0d $\qquad (x + y) \parallel \mathbf{0} \approx x \parallel \mathbf{0} + y \parallel \mathbf{0}$

is a new axiom, sound modulo $\simeq^{\downarrow}_{\mathsf{B}}$. From [10] we know that, when considering BSP terms, the axiom system $\{\mathrm{A0}, \mathrm{A1}, \mathrm{A2}, \mathrm{A3}, \mathrm{T}\}$ is ground-complete modulo $\simeq^{\downarrow}_{\mathsf{T}}$ and $\{\mathrm{A0}, \mathrm{A1}, \mathrm{A2}, \mathrm{A3}, \mathrm{S}\}$ is ground-complete modulo $\simeq_{\mathsf{S}}^{\downarrow}$. Hence, to establish the ground-completeness of $\mathcal{E}_{\mathsf{T}}^{\downarrow}$ and $\mathcal{E}_{\mathsf{S}}^{\downarrow}$, it suffices to show that all occurrences of $\|$ from $\mathrm{BSP}_{\|}$ processes can be eliminated using their respective axioms. In the remainder of this section, let $\mathsf{X} \in \{\mathsf{T}, \mathsf{S}\}$.

▶ **Lemma 40.** *For all* BSP *processes p and q there exists a closed* BSP *term r s.t. $\mathcal{E}_{\mathsf{X}}^{\downarrow} \vdash p\|q \approx r$. Then, for every* $\mathrm{BSP}_{\|}$ *process p there exists a* BSP *process q s.t. $\mathcal{E}_{\mathsf{X}}^{\downarrow} \vdash p \approx q$.*

▶ **Theorem 41.** *The axiomatisation $\mathcal{E}_{\mathsf{X}}^{\downarrow}$ is ground-complete modulo $\simeq_{\mathsf{X}}^{\downarrow}$.*

Consider the axiomatisation $\mathcal{E}_{\mathsf{X}}^{\downarrow'} = \mathcal{E}_{\mathsf{T}}^{\downarrow} \cup \{\mathrm{P2}, \mathrm{P3}\}$. We can use the technique of inverted substitutions to prove the ω-completeness of $\mathcal{E}_{\mathsf{X}}^{\downarrow'}$. To this end, it is enough to extend the mapping R defined in Section 4.3 with $R(\mathbf{1}) = \mathbf{1}$, and consider the closed substitution $\tilde{\rho}(x) = a_x.\mathbf{1}$ in the proof of the first property of the mapping.

▶ **Theorem 42.** *If $|A| = \infty$, then the axiomatisation $\mathcal{E}_{\mathsf{X}}^{\downarrow'}$ is ω-complete modulo $\simeq_{\mathsf{X}}^{\downarrow}$.*

6 Concluding remarks

We have studied the existence of finite bases over $\mathrm{BCCSP}_{\|}$ terms with respect to the behavioural equivalences in van Glabbeek's linear time-branching time spectrum. For the behavioural semantics between ready simulation and completed traces we proved a negative result for both the preorders and equivalences using Moller's proof-theoretic technique. For traces and simulation we obtained a negative result with $|A| = 1$ by applying the transformation technique to the max-plus algebra on natural numbers, for which it is known that no finite basis exists. Additionally, in the case of $2 \leq |A| < \infty$ we proved a negative result for both the preorders and equivalences using the proof-theoretic technique. Conversely, we exploited Groote's technique of inverted substitutions to obtain finite basis for trace and simulation semantics when $|A| = \infty$.

In our investigations, we independently developed the results for the preorders and their corresponding equivalences. While we encountered many similarities, it was not possible to automatically lift the results obtained for one to the other. In [6], it is shown that, under certain conditions, it is possible to lift an axiomatisation for a preorder to the corresponding equivalence over languages without parallel composition. It is an interesting avenue for future research to see whether the lifting technique can be extended to languages including a parallel composition operator and which kind of conditions will be necessary in that case.

The study of (ground-)completeness for possible futures and 2-nested simulation with $|A| = 1$ is a natural venue for future work. Additionally, for the behavioural equivalences between ready simulation and failures, in the case of $|A| = \infty$, it is unknown whether finite, sound and ground-complete axiomatisations exist. Moreover, the axiomatisability of parallel composition modulo possible worlds and impossible futures have not been studied yet [3, 14, 16]. In this paper, we considered the interleaving parallel composition operator, so the next step will be to include communication as well. In [3], it has been proved that the finite, sound and ground-complete axiomatisations over interleaving provided in [2] are preserved if CCS full merge operator is considered. It would be interesting to see whether a similar extension can be obtained for our results regarding finite bases. Furthermore, following [1], CSP-style and ACP-style communication can be studied as well.

Finally, given the amount of negative results that we obtained, it is natural to wonder whether the use of auxiliary operators can help us to find some finite bases. Given its successful application in the case of bisimilarity [9], we will first investigate the role of left merge in obtaining completeness results for all the congruences that we considered in this paper.

References

1 Luca Aceto, Elli Anastasiadi, Valentina Castiglioni, and Anna Ingólfsdóttir. Non-finite axiomatisability results via reductions: CSP parallel composition and CCS restriction. In *A Journey from Process Algebra via Timed Automata to Model Learning*, volume 13560 of *Lecture Notes in Computer Science*, pages 1–26. Springer, 2022. doi:10.1007/978-3-031-15629-8_1.

2 Luca Aceto, Valentina Castiglioni, Anna Ingólfsdóttir, Bas Luttik, and Mathias Ruggaard Pedersen. On the Axiomatisability of Parallel Composition: A Journey in the Spectrum. In *Proceedings of CONCUR 2020*, volume 171 of *LIPIcs*, pages 18:1–18:22. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.CONCUR.2020.18.

3 Luca Aceto, Valentina Castiglioni, Anna Ingólfsdóttir, Bas Luttik, and Mathias Ruggaard Pedersen. On the axiomatisability of parallel composition. *Log. Methods Comput. Sci.*, 18(1), 2022. doi:10.46298/LMCS-18(1:15)2022.

4 Luca Aceto, Zoltán Ésik, and Anna Ingólfsdóttir. On the two-variable fragment of the equational theory of the max-sum algebra of the natural numbers. Technical report, Aarhus University, 1999. doi:10.7146/brics.v6i22.20079.

5 Luca Aceto, Zoltán Ésik, and Anna Ingólfsdóttir. The max-plus algebra of the natural numbers has no finite equational basis. *Theoretical Computer Science*, 293(1):169–188, 2003. doi:10.1016/S0304-3975(02)00236-0.

6 Luca Aceto, Wan Fokkink, and Anna Ingólfsdóttir. Ready to preorder: Get your BCCSP axiomatization for free! In *Algebra and Coalgebra in Computer Science*, pages 65–79. Springer, 2007. doi:10.1007/978-3-540-73859-6_5.

7 Luca Aceto, Wan Fokkink, Rob van Glabbeek, and Anna Ingólfsdóttir. Nested semantics over finite trees are equationally hard. *Information and Computation*, 191(2):203–232, 2004. doi:10.1016/j.ic.2004.02.001.

8 Luca Aceto, Wan Fokkink, and Chris Verhoef. Structural operational semantics. In Jan A. Bergstra, Alban Ponse, and Scott A. Smolka, editors, *Handbook of Process Algebra*, pages 197–292. North-Holland / Elsevier, 2001. doi:10.1016/B978-044482830-9/50021-7.

9 Luca Aceto, Wan J. Fokkink, Anna Ingólfsdóttir, and Bas Luttik. A finite equational base for CCS with left merge and communication merge. *ACM Trans. Comput. Log.*, 10(1):6:1–6:26, 2009. doi:10.1145/1459010.1459016.

10 Josephus C.M. Baeten, Twan Basten, and Michel A. Reniers. *Process algebra: equational theories of communicating processes*, volume 50. Cambridge university press, 2010. doi:10.1017/CBO9781139195003.

11 Stefan Blom, Wan Fokkink, and Sumit Nain. On the axiomatizability of ready traces, ready simulation, and failure traces. In *Automata, Languages and Programming*, pages 109–118. Springer Berlin Heidelberg, 2003. doi:10.1007/3-540-45061-0_10.

12 Taolue Chen and Wan Fokkink. On finite alphabets and infinite bases III: Simulation. In *Proceedings of CONCUR 2006*, pages 421–434. Springer, 2006. doi:10.1007/11817949_28.

13 Taolue Chen, Wan Fokkink, Bas Luttik, and Sumit Nain. On finite alphabets and infinite bases. *Information and Computation*, 206(5):492–519, 2008. doi:10.1016/j.ic.2007.09.003.

14 Taolue Chen and Wan J. Fokkink. On the axiomatizability of impossible futures: Preorder versus equivalence. In *Proceedings of LICS 2008*, pages 156–165. IEEE Computer Society, 2008. doi:10.1109/LICS.2008.13.

15 Taolue Chen, Wan J. Fokkink, and Sumit Nain. On finite alphabets and infinite bases II: completed and ready simulation. In *Proceedings of FOSSACS 2006*, volume 3921 of *Lecture Notes in Computer Science*, pages 1–15. Springer, 2006. doi:10.1007/11690634_1.

16 Taolue Chen, Wan J. Fokkink, and Rob J. van Glabbeek. On the axiomatizability of impossible futures. *Log. Methods Comput. Sci.*, 11(3), 2015. doi:10.2168/LMCS-11(3:17)2015.

17 Robert de Simone. Higher-level synchronising devices in Meije-SCCS. *Theor. Comput. Sci.*, 37:245–267, 1985. doi:10.1016/0304-3975(85)90093-3.

18 Wan J. Fokkink and Sumit Nain. On finite alphabets and infinite bases: From ready pairs to possible worlds. In *Proceedings of FOSSACS 2004*, volume 2987 of *Lecture Notes in Computer Science*, pages 182–194. Springer, 2004. doi:10.1007/978-3-540-24727-2_14.

19 Wan J. Fokkink and Sumit Nain. A finite basis for failure semantics. In *Proceedings of ICALP 2005*, volume 3580 of *Lecture Notes in Computer Science*, pages 755–765. Springer, 2005. doi:10.1007/11523468_61.

20 Rob J. van Glabbeek. The Linear Time - Branching Time Spectrum I. In *Handbook of Process Algebra*, pages 3–99. Elsevier Science, 2001. doi:10.1016/B978-044482830-9/50019-9.

21 Jan Friso Groote. A new strategy for proving ω-completeness applied to process algebra. In *Proceedings of CONCUR 1990*, pages 314–331. Springer, 1990. doi:10.1007/BFb0039068.

22 Matthew Hennessy and Robin Milner. On observing nondeterminism and concurrency. In *Proceedings of ICALP 1980*, volume 85 of *Lecture Notes in Computer Science*, pages 299–309, 1980. doi:10.1007/3-540-10003-2_79.

23 Matthew Hennessy and Robin Milner. Algebraic laws for nondeterminism and concurrency. *J. ACM*, 32(1):137–161, 1985. doi:10.1145/2455.2460.

24 Charles A. R. Hoare. Communicating sequential processes. *Communications of the ACM*, 21(8):666–677, 1978. doi:10.1145/359576.359585.

25 Robert M. Keller. Formal verification of parallel programs. *Commun. ACM*, 19(7):371–384, 1976. doi:10.1145/360248.360251.

26 Robin Milner. *A Calculus of Communicating Systems*. LNCS 92, Springer-Verlag, Berlin, 1980. doi:10.1007/3-540-10235-3.

27 Faron Moller. *Axioms for Concurrency*. PhD thesis, Department of Computer Science, University of Edinburgh, July 1989. Report CST-59-89. Also published as ECS-LFCS-89-84.

28 Faron Moller. The nonexistence of finite axiomatisations for CCS congruences. In *Proceedings of LICS 1990*, pages 142–153. IEEE Computer Society, 1990. doi:10.1109/LICS.1990.113741.

29 Gordon D. Plotkin. A structural approach to operational semantics. Technical report, Aarhus University, 1981.

30 Rowin Versteeg. From Bisimulation to Traces: The Impact of Parallel Composition on Finite Bases. Master's thesis, Eindhoven University of Technology, 2021. URL: https://research.tue.nl/en/studentTheses/from-bisimulation-to-traces.