

# Enumerating Minimal Dominating Sets and Variants in Chordal Bipartite Graphs

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## Abstract

Enumerating minimal dominating sets with polynomial delay in bipartite graphs is a long-standing open problem. To date, even the subcase of chordal bipartite graphs is open, with the best known algorithm due to Golovach, Heggenes, Kanté, Kratsch, Sæther, and Villanger running in incremental-polynomial time. We improve on this result by providing a polynomial delay and space algorithm enumerating minimal dominating sets in chordal bipartite graphs. Additionally, we show that the total and connected variants admit polynomial and incremental-polynomial delay algorithms, respectively, within the same class. This provides an alternative proof of a result by Golovach et al. for total dominating sets, and answers an open question for the connected variant. Finally, we give evidence that the techniques used in this paper cannot be generalized to bipartite graphs for (total) minimal dominating sets, unless  $P = NP$ , and show that enumerating minimal connected dominating sets in bipartite graphs is harder than enumerating minimal transversals in general hypergraphs.

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## 1 Introduction

Hypergraphs are encountered in countless areas of computer science or discrete mathematics. Formally, they consist of a pair  $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$  where  $V(\mathcal{H})$  is a set of elements called *vertices*, and  $E(\mathcal{H})$  is a family of subsets of  $V(\mathcal{H})$  called hyperedges (or simply edges). Hypergraphs are known as *graphs* when their edges are of size precisely two. A *transversal* in a hypergraph  $\mathcal{H}$  is a set  $T \subseteq V(\mathcal{H})$  which intersects every edge of  $\mathcal{H}$ . It is called *minimal* if it is inclusion-wise minimal, i.e., if *none* of its proper subsets is a transversal. The problem of enumerating the minimal transversals of a hypergraph, usually denoted by TRANS-ENUM, is one of the most important problems in algorithmic enumeration, for it has far-reaching



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implications in various areas of computer science such as logic, database theory, data profiling, or artificial intelligence [5, 16, 30, 37]; we refer the reader to [16] and the surveys [18, 22] for a comprehensive overview on the fundamental and practical aspects of this problem.

In a typical enumeration problem such as TRANS-ENUM, a first observation to be made is that the number of solutions can be exponential in both  $n$ , the number of vertices, and  $m$ , the number of edges. Thus, asking for running times which are polynomial in  $n + m$  is meaningless. Rather, we are interested in algorithms performing in output-polynomial time, i.e., that run in a time which is polynomial in both the input size, and the output size. More strict notions of tractability can be required, such as incremental-polynomial times, and polynomial delay. We say that an algorithm runs in *incremental-polynomial* time if it produces the  $i^{\text{th}}$  solution in a time which is polynomial in the input size plus  $i$ . An algorithm is said to run with *polynomial delay* if the times before the first output, after the last output, and in between two consecutive outputs are bounded by a polynomial in the input size only. To date, the best-known algorithm solving TRANS-ENUM is due to Fredman and Khachiyan [21] and runs in quasi-polynomial incremental time. Since then, output-polynomial time algorithms were obtained for several important classes, including those generalizing that of bounded degree [17], or bounded edge size [38].

Dominating sets are among the most studied objects in graphs. A *dominating set* in a graph  $G$  is a set  $D \subseteq V(G)$  such that every vertex is either in  $D$ , or adjacent to a vertex of  $D$ . Again, it is called *minimal* if it is inclusion-wise minimal. The problem of enumerating minimal dominating sets in graphs is denoted by DOM-ENUM. Note that being a dominating set of  $G$  can be rephrased as being a transversal of each closed neighborhood of  $G$ , so, DOM-ENUM is a special case of TRANS-ENUM with a linear number of hyperedges.

The problem DOM-ENUM has gained lots of interest since it has been shown, perhaps surprisingly, to be polynomially equivalent to TRANS-ENUM, even when restricted to cobipartite graphs [33]. In particular, this work has initiated a fruitful line of research of characterizing which graph classes admit output-polynomial algorithms, in the hope of better understanding the underlying structure that makes TRANS-ENUM tractable. For example, polynomial-delay algorithms were obtained for various graph classes including chordal graphs [34] (with linear delay on some subclasses [13, 33]), graphs of bounded degeneracy [17], line graphs [35], graphs of bounded clique-width [11], or graph classes related to permutation graphs and their generalizations [8, 23, 32]. Incremental-polynomial time algorithms were obtained for graphs of bounded conformality [38],  $C_{\leq 6}$ -free graphs [27], chordal bipartite graphs [24],  $(C_4, C_5, \text{claw})$ -free graphs [32], or graphs arising in geometrical contexts [20, 23] including unit-disk and unit-square graphs. Output-polynomial time algorithms are known for log  $n$ -degenerate graphs [17], and graphs of bounded clique number [7].

In [33], the authors also include two natural variants of domination in their study: total and connected dominating sets. A *total dominating set* is a set  $D$  such that every vertex of  $G$  has a neighbor in  $D$ . In other words, elements of  $D$  cannot “dominate” themselves and need a neighbor in the set. A *connected dominating set* is a dominating set that induces a connected subgraph. Note that any connected dominating set is a total dominating set, as long as it has more than one element, making the intersection of these two notions irrelevant to study. Perhaps not surprisingly, the authors in [33] show that the problems of enumerating minimal total and connected dominating sets, denoted TDOM-ENUM and CDOM-ENUM, respectively, are equivalent to TRANS-ENUM when restricted to split graphs. These equivalences on a relatively restricted graph class suggest that these variants of DOM-ENUM are challenging, and may explain why they have received less attention in the literature.

Regarding total dominating sets, a first observation is that they are transversals of open neighborhoods. This implies an incremental quasi-polynomial time algorithm for the problem using the results of Fredman and Khachiyan [21]. However, few output-polynomial time algorithms are known for specific instances of TDOM-ENUM. Among them, we cite the class of chordal bipartite graphs, which admits a polynomial-delay algorithm for the problem [24].

Concerning connected dominating sets, it is not known whether the problem reduces to TRANS-ENUM. It has in fact been conjectured that this is not the case in [6]. Rather, it has been shown in [33] that the problem reduces to the enumeration of minimal transversals of an implicit hypergraph given by the minimal separators of the graph. However, the number of such separators may be exponential, and are intractable to compute [9]. The only work known to us dealing with connected dominating sets enumeration, apart from [33], falls under the input-sensitive approach [1, 25, 26] where one aims at small exponential times. Still, we note that some essential questions concerning output-sensitive enumeration were addressed in [1] such as the NP-hardness of the associated extension problem, i.e., deciding if a given set  $U \subseteq V(G)$  is contained in some minimal connected dominating set.

Note that, in the literature, particular attention has been put into characterizing the status of these problems on bipartite, split, and cobipartite graphs. This is because such instances contain the incidence graph of any hypergraph, which is fundamentally challenging and makes interesting links with TRANS-ENUM. However, and in contrast to the split and cobipartite cases, the case of bipartite graphs has been particularly challenging with the question of the (in)tractability of these problems raised in various works [24, 36]. Only recently has an output-polynomial time algorithm been provided for DOM-ENUM in a generalization of bipartite graphs [7]. Still the problem is not known to admit a polynomial-delay algorithm. Concerning TDOM-ENUM and CDOM-ENUM, it is also open, to the best of our knowledge, whether they can be solved with polynomial delay, or even in output-polynomial time, in bipartite graphs. Among natural subclasses of bipartite graphs, the case of chordal bipartite graphs has been studied as it admits nice structural properties analogous to chordal graphs, such as perfect elimination orderings and linearly many minimal separators. We note, however, that even in this particular class, the existence of a polynomial-delay algorithm for DOM-ENUM seems open, and has been left open in [24].

The following two natural questions arise from the literature.

► **Question 1.** *Can the minimal dominating sets of a chordal bipartite graph be enumerated with polynomial delay?*

► **Question 2.** *Can the minimal connected dominating sets of a chordal bipartite graph be enumerated in output-polynomial time?*

Only the case of minimal total dominating sets has been settled for chordal bipartite graphs, with the following result.

► **Theorem 1 ([24]).** *The minimal total dominating sets of a chordal bipartite graph can be enumerated with polynomial delay.*

**Our results and techniques.** This paper is devoted to providing a positive answer to Questions 1 and 2, almost completing the picture for chordal bipartite graphs. Collaterally, we provide an alternative proof of Theorem 1 involving small modifications of our algorithm for minimal dominating sets; we note that the obtained algorithm has similar time bounds as the one in [24].

Our algorithms for DOM-ENUM (Theorem 18) and TDOM-ENUM (Theorem 1) are based on the *ordered generation* framework (also known as *sequential method* in various works) which has been introduced in [17] as a generalization of the algorithm in [45] (see also [31] for a concise description). This approach has since then proved to be fruitful in various contexts related to domination and minimal transversals [7, 10, 14]. Roughly, the idea is to decompose the instance with respect to an ordering of its elements, and, given the partial solutions of the instance induced by the  $i$  first elements, to extend them to the next element of the instance. To achieve polynomial delay, a parent-child relation on partial solutions is defined. Then, the resulting tree is traversed in a DFS manner, and only the leaves (which correspond to actual solutions) are output. Overall, the technique allows to reduce the general enumeration to the enumeration of the children of a partial solution, and is particularly suitable for instances admitting good elimination orderings. In our case, we use the weak-elimination ordering of chordal bipartite graphs (see Section 2) to solve children generation in polynomial time, yielding a polynomial-delay algorithm for DOM-ENUM and TDOM-ENUM in that class.

Concerning CDOM-ENUM, we show that the minimal separators of a chordal bipartite graph define a hypergraph of bounded conformality (Lemma 20), a parameter introduced by Berge [3] and successfully used in the design of tractable algorithms for domination [32] and minimal transversals enumeration [38]. This, combined with the fact that chordal bipartite graphs have polynomially many minimal separators, which can be listed in polynomial time (see Section 2), allows us to obtain an incremental-polynomial time algorithm for CDOM-ENUM in that class (Theorem 21).

Additionally, we give evidence that the sequential method cannot be directly used to generalize our results to bipartite graphs. More specifically, we show that the problem of generating the children of a given partial solution is intractable for DOM-ENUM (Lemma 22) and TDOM-ENUM. For CDOM-ENUM we make a stronger statement (Theorem 24): the problem is at least as hard as TRANS-ENUM, which was not known to the best of our knowledge, despite the reduction being straightforward.

**Organization of the paper.** We introduce preliminary notions and properties dealing with chordal bipartite graphs and minimal separators in Section 2. We describe the sequential approach in Section 3, and provide our algorithms for DOM-ENUM and TDOM-ENUM in Section 4. The arguments for an incremental-polynomial time algorithm for CDOM-ENUM in chordal bipartite graphs are given in Section 5. Limitations of the sequential method in bipartite graphs are discussed in Section 6, and the reduction from TRANS-ENUM to CDOM-ENUM in bipartite graphs is given in Section 7. We discuss open questions in Section 8.

## 2 Preliminary notions

We assume familiarity with standard terminology from graph and hypergraph theory, and refer to [3, 15] for notations not included below.

**Graphs.** All graphs considered in this paper are simple and finite. Given  $X \subseteq V(G)$ ,  $G[X] := (X, \{uv \in E(G) \mid u, v \in X\})$  is the subgraph of  $G$  induced by  $X$ . Given  $v \in V(G)$ , we denote by  $N(v)$  its open neighborhood, by  $N[v]$  its closed neighborhood, and define its *distance-2 neighborhood* as  $N^2(v) := N[N[v]] \setminus N[v]$ . The first two notions are naturally extended to subsets by  $N[S] := \bigcup_{v \in S} N[v]$  and  $N(S) := (\bigcup_{v \in S} N[v]) \setminus S$ . An *independent set* is a set of pairwise non-adjacent vertices. A graph is *bipartite* if its vertex set  $V(G)$  can be partitioned into two independent sets  $X, Y$ . It is called *bipartite chain* if, in addition, there

exist orderings  $x_1, \dots, x_p$  and  $y_1, \dots, y_q$  of  $X$  and  $Y$ , respectively, such that  $N(x_i) \subseteq N(x_j)$  for all  $1 \leq i \leq j \leq p$ , and  $N(y_i) \supseteq N(y_j)$  for all  $1 \leq i \leq j \leq q$ . Note that bipartite chain graphs are closed under taking induced subgraphs.

**Hypergraphs.** Let  $\mathcal{H}$  be a hypergraph. Given a vertex  $v \in V(\mathcal{H})$ , the set of edges that contain  $v$ , also called edges *incident* to  $v$ , is denoted by  $\text{inc}(v)$ . The hypergraph *induced* by a set  $X \subseteq V(\mathcal{H})$  of vertices is defined as  $\mathcal{H}[X] = (X, \{E \in E(\mathcal{H}) : E \subseteq X\})$ . A *transversal* of  $\mathcal{H}$  is a subset  $T \subseteq V(\mathcal{H})$  such that  $E \cap T \neq \emptyset$  for all  $E \in E(\mathcal{H})$ , i.e., every edge is incident to (or *hit* by) a vertex of  $T$ . It is *minimal* if it is minimal by inclusion, or equivalently, if  $T \setminus \{v\}$  is not a transversal for any  $v \in T$ . The set of minimal transversals of  $\mathcal{H}$  is denoted by  $\text{Tr}(\mathcal{H})$ . The *private edges* of an element  $v$  in a transversal  $T$  are those edges of  $\mathcal{H}$  which are only hit by  $v$  in  $T$ ; we denote the set of such edges by  $\text{priv}(T, v)$ . It is well-known (see e.g., [3]) that a set  $T \subseteq V(\mathcal{H})$  is a minimal transversal of  $\mathcal{H}$  if and only if  $T$  is a transversal and every  $v$  in  $T$  has a private edge. We formally state the problem of enumerating minimal transversals of a hypergraph as follows.

MINIMAL TRANSVERSALS ENUMERATION (TRANS-ENUM)

**Input:** A hypergraph  $\mathcal{H}$ .

**Output:** The set  $\text{Tr}(\mathcal{H})$ .

Note that in the problem above,  $\mathcal{H}$  can be considered inclusion-wise minimal (or *Sperner*) as this does not change its set of minimal transversals, and removing non-minimal hyperedges can be performed in polynomial time [18].

**Domination.** We recall the different notions of domination given in the introduction. A *dominating set* (or DS for short) in a graph  $G$  is a set  $D \subseteq V(G)$  such that  $V(G) = N[D]$ . It is called *connected* if the graph  $G[D]$  is connected. A *total dominating set* in  $G$  is a set  $D \subseteq V(G)$  such that  $V(G) = \bigcup_{v \in D} N(v)$ . A (resp. connected, total) dominating set is said to be *minimal* if it is inclusion-wise minimal. We denote by  $\text{MDS}(G)$  (resp.  $\text{MCDS}(G)$ ,  $\text{MTDS}(G)$ ) the families of minimal such sets in a graph  $G$ . The problems of enumerating these families are our main interest in this paper.

We recall that the precise complexity status of each of the problems above is widely open, the first one admitting a quasi-polynomial time algorithm [21, 33], while the third is conjectured to be intractable [6]. Their study has been initiated in [33], where it is noted that each of the problems is a particular instance of TRANS-ENUM: using the family of (closed and open) neighborhoods for the first two, and a more involved hypergraph for the third one, as formalized in Propositions 2–4. See also [3].

► **Proposition 2.** *For every graph  $G$ ,  $\text{MDS}(G) = \text{Tr}(\{N[v] : v \in V(G)\})$ .*

► **Proposition 3.** *For every graph  $G$ ,  $\text{MTDS}(G) = \text{Tr}(\{N(v) : v \in V(G)\})$ .*

**Minimal separators.** A *separator* in a graph  $G$  is a subset  $S \subseteq V(G)$  such that  $G - S$  is disconnected. In the following, let  $\mathcal{S}(G) := \{S \subseteq V : G - S \text{ is not connected and } \forall S' \subset S, G - S' \text{ is connected}\}$  denote the set of (inclusion-wise) minimal separators of  $G$ . Then we have the following.

► **Proposition 4** ([33]). *For every graph  $G$ ,  $\text{MCDS}(G) = \text{Tr}(\mathcal{S}(G))$ .*

Minimal separators are not to be confused with the slightly different notion of *minimal  $a$ - $b$  separators* which are minimal sets  $S$  such that  $a$  and  $b$  lie in different components of  $G - S$ . Indeed, while every minimal separator is a minimal  $a$ - $b$  separator, the converse is not always true: minimal  $a$ - $b$  separators may contain another  $c$ - $d$  separator as a subset; see [28] for an early mention of this observation, and [9] for a thorough discussion on this matter. A simple example is the cycle on four vertices with a pendant edge [9]. Moreover, while enumerating  $a$ - $b$  minimal separators can be done with polynomial delay [4, 40, 44], enumerating minimal separators is a hard task [9]. However, and as is noted in [33], CDOM-ENUM reduces to TRANS-ENUM for graph classes admitting polynomially many  $a$ - $b$  separators.<sup>1</sup> Let  $\mathcal{G}$  be the hypergraph where each hyperedge is an  $a$ - $b$  minimal separator of  $G$ ; using the above-cited algorithms for enumerating  $a$ - $b$  minimal separators, we can construct  $\mathcal{G}$  in polynomial time on  $|\mathcal{G}| + |V(G)| + |E(G)|$ ; moreover, note that  $\mathcal{S}(G) = \text{Min}_{\subseteq} \mathcal{G}$ . Indeed, recall that  $\text{Tr}(\mathcal{H}) = \text{Tr}(\mathcal{G})$  for any pair of hypergraphs  $\mathcal{H}, \mathcal{G}$  such that for any  $E \in E(\mathcal{G})$  there exists  $E' \in E(\mathcal{H})$  with  $E' \subseteq E$ . In other words,  $\text{Tr}(\mathcal{H}) = \text{Tr}(\text{Min}_{\subseteq} \mathcal{H})$  for any hypergraph  $\mathcal{H}$  [3].

**Chordal bipartite graphs.** A bipartite graph is *chordal bipartite* if it does not contain induced cycles of length greater than four [28]. Note that chordal bipartite graphs are *not* chordal, as they may contain the induced cycle on four vertices. This terminology should be regarded as an analogue to bipartite graphs of chordality, as bicliques are to cliques.

A vertex  $v$  in a graph  $G$  is called *weak-simplicial* if  $N(v)$  is an independent set, and for any two  $x, y \in N(v)$ , we have  $N(x) \subseteq N(y)$  or  $N(y) \subseteq N(x)$ ; two such vertices  $x, y$  are said to be *comparable*. Note that this is precisely to say that the graph  $G[N(v) \cup N^2(v)]$  induced by the vertices at distance at most 2 from  $v$  is a bipartite chain; see [43, Lemma 3]. A *weak-simplicial elimination ordering* of  $G$  is an ordering  $v_1, \dots, v_n$  of its vertices such that, for all  $1 \leq i \leq n$ ,  $v_i$  is weak-simplicial in  $G_i = G[\{v_1, \dots, v_i\}]$ . The following characterization is due to [43] and may be regarded as a simplification of an earlier characterization and recognition algorithm by Uehara [46].

► **Proposition 5** ([43, 46]). *A graph is chordal bipartite if and only if it admits a weak-simplicial elimination ordering, which can be computed in polynomial time.*

Another characterization of chordal bipartite graphs based on separators is given in [29], where it is proved that a bipartite graph is chordal bipartite if and only if every minimal *edge* separator induces a complete bipartite subgraph. In [41], the following analogue of the forward implication is given for minimal  $a$ - $b$  separators (hence for minimal separators).

► **Proposition 6** ([41]). *If  $G$  is chordal bipartite and  $S$  is an  $a$ - $b$  minimal separator then  $G[S]$  is complete bipartite.*

In the same paper, the authors show that chordal bipartite graphs have at most  $O(n + m)$  minimal  $a$ - $b$  separators [41]. Since they can be listed with polynomial delay [40], we derive a polynomial time procedure listing  $a$ - $b$  minimal separators and filtering out the non-minimal ones. We formalize this in the next statement that will be of use in Section 5.

► **Proposition 7.** *Let  $G$  be an  $n$ -vertex chordal bipartite graph. Then its set  $\mathcal{S}(G)$  of minimal separators can be enumerated in polynomial time in  $n$ .*

<sup>1</sup> It should be noted that the statement [33, Corollary 33] presents an inaccuracy where by *minimal separators* it should be understood *minimal  $a$ - $b$  separators*. Indeed, listing minimal separators is hard [9] while enumerating minimal  $a$ - $b$  separators is tractable [40]. The present inaccuracy comes from an ambiguity and common confusion in the literature around this terminology [41, 44].

Finally, we will need two extra properties of minimal separators in general and in chordal bipartite graphs. Let  $S$  be a separator of  $G$  and  $C$  be a component of  $G - S$ . We say that  $C$  is *close* to  $S$  if every vertex in  $S$  has at least one neighbor in  $C$ . It is folklore that minimal  $a$ - $b$  separators  $S$  are characterized by the existence of two close components in  $G - S$ ; see [28, 41, 44] or [39] for a proof. We derive one implication for minimal separators since, in particular, they are  $a$ - $b$  minimal separators for some specific pairs  $a, b$  of vertices. Note however that the converse is not true.

► **Proposition 8.** *Let  $G$  be a graph and  $S$  be a minimal separator of  $G$ . Then there exist two components  $C_1, C_2$  in  $G - S$  that are close to  $S$ .*

Finally, the following will be of use in Section 5. We state it for minimal separators but this more generally holds for minimal  $a$ - $b$  separators [41].

► **Proposition 9** ([41]). *Let  $G$  be chordal bipartite with bipartition  $(A, B)$ . Let  $S$  be a minimal separator of  $G$  and  $C$  be a component of  $G - S$  that is close to  $S$ . If  $S \cap A \neq \emptyset$  then there exists a vertex  $x$  in  $C$  with  $N(x) \cap S = S \cap A$ .*

### 3 The sequential method

We describe the *sequential method*. As by Propositions 2–4, the problems DOM-ENUM, TDOM-ENUM, and CDOM-ENUM can be seen as particular instances of TRANS-ENUM we choose to follow the presentation of [2], and refer the reader to [7, 17, 19] for more details on this approach.

Let  $\mathcal{H} \subseteq 2^V$  be a hypergraph on vertex set  $V = \{v_1, \dots, v_n\}$ . For  $1 \leq i \leq n$  we denote by  $V_i := \{v_1, \dots, v_i\}$  the set of its first  $i$  vertices, by  $\mathcal{H}_i := \mathcal{H}[V_i]$  the hypergraph induced by  $V_i$ , and set  $V_0 := \mathcal{H}_0 := \text{Tr}(\mathcal{H}_0) := \emptyset$ . In addition, for  $S \subseteq V_i$  and  $v \in S$ , we note  $\text{priv}_i(S, v)$  the set of private edges of  $v$  with respect to  $S$  in  $\mathcal{H}_i$ . Given a minimal transversal  $T \in \text{Tr}(\mathcal{H}_{i+1})$  for  $i < n$ , we denote by  $\text{parent}(T, i+1)$  the *parent of  $T$  with respect to  $i+1$*  obtained after repeatedly removing from  $T$  the vertex of smallest index  $v$  such that  $\text{priv}_i(T, v) = \emptyset$ . By definition, every  $T$  has a unique parent. Lastly, we denote by  $\text{children}(T^*, i)$ , for  $T^* \in \text{Tr}(\mathcal{H}_i)$ , the family of  $T \in \text{Tr}(\mathcal{H}_{i+1})$  such that  $T^* = \text{parent}(T, i+1)$ .

The next two lemmas are central to the description of the algorithm: they define a tree structure where nodes are pairs  $(T, i)$  with  $T \in \text{Tr}(\mathcal{H}_i)$  and that will be traversed in a DFS manner in order to produce every leaf  $T \in \text{Tr}(\mathcal{H}) = \text{Tr}(\mathcal{H}_n)$  as a solution. We refer to [2] for the proofs of these statements, which follow from the definition of the *parent* relation.

► **Lemma 10.** *Let  $0 \leq i \leq n-1$  and  $T \in \text{Tr}(\mathcal{H}_{i+1})$ , then  $\text{parent}(T, i+1) \in \text{Tr}(\mathcal{H}_i)$ .*

► **Lemma 11.** *Let  $0 \leq i \leq n-1$  and  $T^* \in \text{Tr}(\mathcal{H}_i)$ , then either:*

- $T^* \in \text{Tr}(\mathcal{H}_{i+1})$  and consequently  $\text{parent}(T^*, i+1) = T^*$ ; or
- $T^* \cup \{v_{i+1}\} \in \text{Tr}(\mathcal{H}_{i+1})$  and  $\text{parent}(T^* \cup \{v_{i+1}\}, i+1) = T^*$ .

Finally, the whole framework culminates in the following statement that reduces TRANS-ENUM to the enumeration of the children of a given pair  $(T, i)$  with  $T \in \text{Tr}(\mathcal{H}_i)$ ,  $i < n$ , while preserving polynomial delay and space.

► **Theorem 12.** *Let  $f, s : \mathbb{N}^2 \rightarrow \mathbb{Z}_+$  be two functions. If there is an algorithm that, given a hypergraph  $\mathcal{H}$  on vertex set  $V = \{v_1, \dots, v_n\}$  and  $m$  edges, an integer  $i \in \{1, \dots, n-1\}$ , and  $T^* \in \text{Tr}(\mathcal{H}_i)$ , enumerates  $\text{children}(T^*, i)$  with delay  $f(n, m)$  and using  $s(n, m)$  space, then there is an algorithm that enumerates  $\text{Tr}(\mathcal{H})$  with  $O(nf(n, m))$  delay and  $O(ns(n, m))$  space.*



In the following, given  $T^*$  and  $i$  as above, let

$$\Delta_{i+1} := \{E \in \mathcal{H}_{i+1} : E \cap T^* = \emptyset\} \subseteq \text{inc}_{i+1}(v_{i+1})$$

be the family of hyperedges of  $\mathcal{H}_{i+1}$  that are not hit by  $T^*$ . The following shows that children may be found by considering minimal extensions to  $\mathcal{H}_{i+1}$  of minimal transversals of  $\mathcal{H}_i$ .

► **Lemma 13.** *If  $T \in \text{children}(T^*, i)$  then  $T = T^* \cup X$  for some  $X \in \text{Tr}(\Delta_{i+1})$ .*

**Proof.** Let  $T \in \text{children}(T^*, i)$ . Then  $T^*$  is obtained from  $T$  by greedily removing elements that have no private edge in  $\mathcal{H}_i$ . Let  $X$  be the set of removed elements. Note that since  $T^*$  is transversal of  $\mathcal{H}_i$ ,  $X$  may only have private edges among those  $E \in \mathcal{H}_{i+1}$  that are not intersected by  $T^*$ , i.e., among  $\Delta_{i+1}$ . Then, either  $\Delta_{i+1} = \emptyset$  in which case  $T = T^*$  by Lemma 11, or  $X$  is non empty and intersects each edge in  $\Delta_{i+1}$ . By minimality of  $T$ , we derive that  $X$  is minimal with this property, as desired. ◀

As a corollary, in order to enumerate  $\text{children}(T^*, i)$  for some  $i \in \{1, \dots, n-1\}$  and  $T^* \in \text{Tr}(\mathcal{H}_i)$ , one can simply enumerate all the minimal transversals of  $\Delta_{i+1}$  and filter out those that do not provide children of  $T^*$ . In the following, we call such sets *minimal extensions* of  $(T^*, i)$ . Note, however that the number of minimal transversal of  $\Delta_{i+1}$  may be greater (typically exponential) than the number of actual children, hence that this approach fails to efficiently produce children in general. Moreover, this approach is intractable in general, with instances admitting non-trivial children if and only if NP-complete problems admit a positive answer; see [2] or Section 6. We nevertheless have the following which says that the approach is still tractable if we can bound the number of minimal extension by a polynomial in the input size, and which is in essence what is done in [17].

► **Corollary 14.** *If there is an algorithm that, given  $T^* \in \text{Tr}(\mathcal{H}_i)$ ,  $i < n$ , enumerates all  $X \in \text{Tr}(\Delta_{i+1})$  in time and space which are polynomial in  $n + m$ , then there is one that lists  $\text{Tr}(\mathcal{H})$  with polynomial delay and space.*

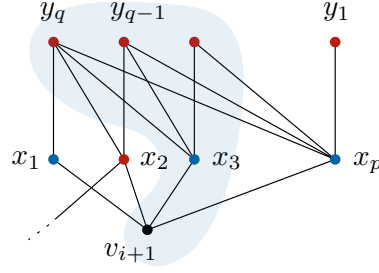
Note that, if  $\mathcal{H}$  is  $d$ -degenerate and  $v_1, \dots, v_n$  is an optimal degeneracy ordering, then minimal extensions have size at most  $d$ . In [17], the authors use this fact to provide a polynomial delay algorithm that lists minimal transversals in  $O(1)$ -degenerate hypergraphs relying on Corollary 14. See [2, 17] for the definition of these notions. In this paper, we combine this corollary together with the elimination ordering of chordal bipartite graphs provided by Proposition 5 to show that DOM-ENUM and TDOM-ENUM can be solved with polynomial delay as well in this class.

#### 4 Minimal extension for (total) dominating sets

In this section, we apply the sequential method to TRANS-ENUM restricted to hypergraphs of closed neighborhoods of chordal bipartite graphs to show that Question 1 admits a positive answer. More specifically, we use the weak-simplicial elimination ordering of chordal bipartite graphs to restrict the instance of minimal extensions enumeration to a variant of domination in bipartite chain graphs, which is tractable. By Corollary 14, this allows us to obtain Theorem 18.

In the following, let  $G$  be a chordal bipartite graph on  $n$  vertices and  $v_1, \dots, v_n$  be a weak-simplicial ordering of its vertices as provided by Proposition 5. Let  $\mathcal{H}$  be its hypergraph of closed neighborhoods,  $i \in \{1, \dots, n-1\}$  be an integer and  $T$  be a minimal transversal of  $\mathcal{H}_i$  as in the statement of Corollary 14. We will show how to enumerate the minimal transversals of  $\Delta_{i+1}$  in polynomial time and space, to derive the desired result.





■ **Figure 1** The bipartite chain instance induced by  $X$  and  $Y$  with  $v_{i+1}$  depicted as well. Blue vertices are those whose closed neighborhoods are incident to  $v_{i+1}$  and included in  $V_{i+1}$  as a subset. Red vertices in  $X$  have neighbors out of  $V_i$ .

First note that since the edges of  $\Delta_{i+1}$  belong to  $\mathcal{H}_{i+1}$  and are incident to  $v_{i+1}$ , they may be equal to  $N[v_{i+1}]$  if  $N[v_{i+1}] \subseteq V_{i+1}$ , or to  $N[u]$  for some neighbor  $u$  of  $v_{i+1}$  if  $N[u] \subseteq V_{i+1}$ . Note that in the later case, the set  $N[u]$  may contain vertices at distance 2 from  $v_{i+1}$ . From now on, let  $B$  be the set of vertices  $u \in N(v_{i+1})$  such that  $N[u] \in \Delta_{i+1}$ , and let

$$R := \left( \bigcup \{E : E \in \Delta_{i+1}\} \right) \setminus B$$

be the remaining vertices that lie in some edge of  $\Delta_{i+1}$ . Consequently  $B \subseteq N(v_{i+1})$  and  $R \subseteq N(v_{i+1}) \cup N^2(v_{i+1})$ . Thus the graph  $H = G_i[R \cup B]$  is an induced subgraph of  $G_i[N(v_{i+1}) \cup N^2(v_{i+1})]$  which, by Proposition 5, is a bipartite chain. Hence  $H$  is a bipartite chain. We denote by  $(X, Y)$  the bipartition of  $H$  where  $B \subseteq X \subseteq N(v_{i+1})$ , and by  $x_1, \dots, x_p$  and  $y_1, \dots, y_q$  the respective orderings of  $X$  and  $Y$  satisfying  $N(x_j) \cap Y \subseteq N(x_\ell) \cap Y$  and  $N(y_\ell) \cap X \subseteq N(y_j) \cap X$  for all  $j < \ell$ ; see Section 2 and Figure 1 for an illustration of the situation.

In what remains we shall call *blue* the vertices in  $B$ , *red* those in  $R$ , and characterize minimal transversals of  $\Delta_{i+1}$  as particular sets analogous to red-blue dominating sets where red and blue vertices may be selected in order to dominate blue vertices.

► **Proposition 15.** *Let  $Z \in \text{Tr}(\Delta_{i+1})$ . Then*

- $|Z \cap R \cap N(v_{i+1})| \leq 1$ ; and
- $|Z \cap N^2(v_{i+1})| \leq 1$ .

**Proof.** First, note that if  $x$  belongs to  $Z \cap R \cap N(v_{i+1})$  then it is incident to at most one edge in  $\Delta_{i+1}$  (as  $H$  is bipartite), namely  $N[v_{i+1}]$ , in which case  $N[v_{i+1}]$  is a private edge of  $x$ . Hence, no two distinct vertices  $x, y$  may lie in  $Z \cap R \cap N(v_{i+1})$  by minimality of  $Z$ , yielding the first inequality.

Now, recall that  $N^2(v_{i+1}) \subseteq Y$  hence that one of  $N(y) \cap X \subseteq N(y') \cap X$  or  $N(y) \cap X \supseteq N(y') \cap X$  holds for any pair  $y, y' \in N^2(v_{i+1})$ . Since edges of  $\Delta_{i+1}$  are of the form  $N[x]$  for some  $x \in X$ , no two distinct  $y, y'$  can have private edges in  $\Delta_{i+1}$  simultaneously. Hence,  $|Z \cap N^2(v_{i+1})| \leq 1$  as claimed. ◀

► **Lemma 16.** *Let  $Z \in \text{Tr}(\Delta_{i+1})$ . Then exactly one of the following holds:*

1.  $Z = \{v_{i+1}\}$ ;
2.  $Z \subseteq B$  in which case  $Z = B$ ;
3.  $Z \subseteq R$  in which case  $|Z| \leq 2$ ; or
4.  $Z = \{r\} \cup (B \setminus N(r))$  for some  $r \in N^2(v_{i+1})$ .

**Proof.** First, note that all the cases are mutually distinct. Hence, we may consider some  $Z \in \text{Tr}(\Delta_{i+1})$ , assume that  $Z \neq \{v_{i+1}\}$ , and prove the conclusion for each of Items 2–4. Note that each  $x \in X$  hits a single edge  $N[x]$  in  $\Delta_{i+1}$ , as  $X$  is an independent set. Hence, if  $Z \subseteq B$  then  $Z = B$  concluding Item 2. If  $Z \subseteq R$  then we have that  $|Z| \leq 2$  by Proposition 15 deriving Item 3. We are thus left with proving Item 4. Suppose that  $Z \cap B \neq \emptyset$  and  $Z \cap R \neq \emptyset$ . Note that if  $x \in X \cap R$ , then it may be incident to at most one edge in  $\Delta_{i+1}$ , namely  $N[v_{i+1}]$ . However, since  $Z \cap B \neq \emptyset$  by assumption of this last case, and since each element in  $B$  is adjacent to  $v_{i+1}$ , the edge  $N[v_{i+1}]$  may not be the private edge of such an  $x$  in  $Z$ . Hence  $Z \cap X \cap R = \emptyset$ , that is,  $Z \cap R \subseteq Y$ . By Proposition 15 there is exactly one vertex, call it  $y$ , belonging to  $Z \cap Y$ . Finally, note that in addition to  $N[v_{i+1}]$ , each  $b \in B$  hits a single edge  $N[b] \in \Delta_{i+1}$  that only  $b$  can hit in  $X$ . Hence  $B \setminus N(r) \subseteq Z$  and Item 4 follows. ◀

Note that the conditions of the items in Lemma 16 can be checked in polynomial time, and provide an upper bound of  $O(n^2)$  on the number of minimal transversals of  $\Delta_{i+1}$ . Clearly these minimal transversals can be constructed within the same time and using polynomial space. We formalize this as follows.

► **Corollary 17.** *The set  $\text{Tr}(\Delta_{i+1})$  can be listed in polynomial time and space.*

We conclude the following by combining Corollaries 14 and 17. This answers positively Question 1.

► **Theorem 18.** *There is a polynomial delay and space algorithm enumerating minimal dominating sets in chordal bipartite graphs.*

Let us now argue that with small modifications we can get a polynomial-delay algorithm for TDOM-ENUM on the same class. Note that the minimal extension problem is almost identical, with the caveat that the hyperedges we must hit are now of the form  $N(x)$  for  $x = v_{i+1}$  or  $x \in X$ , i.e., we are now dealing with open instead of closed neighborhoods. A similar analysis as conducted in the proof of Lemma 16 shows that minimal extensions are red subsets dominating blue elements plus  $v_{i+1}$  if  $N(v_{i+1}) \subseteq V_{i+1}$ . We note that this becomes an instance of proper red-blue domination (where only red elements may be picked) which is precisely what is solved in [24] for chordal bipartite graphs. However in our case, due to the instance being a bipartite chain graph, the problem is trivial. Indeed, it may be seen that solutions have size at most two, intersecting  $X$  and  $Y$  on at most one vertex. As such, we have an alternative proof of Theorem 1, which was already known by [24].

► **Theorem 1 ([24]).** *The minimal total dominating sets of a chordal bipartite graph can be enumerated with polynomial delay.*

## 5 Minimal connected dominating sets

Recall that CDOM-ENUM amounts to list the minimal transversals of the (implicit) hypergraph of minimal separators of a given graph  $G$ , i.e., Proposition 4. By Proposition 7, this hypergraph can be constructed in polynomial time if  $G$  is chordal bipartite. In this section, we moreover show that this hypergraph has bounded conformality in that case. As a corollary, we can use the algorithm of Khachiyan et al. in [38] to list minimal connected dominating sets in incremental-polynomial time in chordal bipartite graphs, providing a positive answer to Question 2.

Let us define the notion of conformality introduced by Berge in [3]. We say that a hypergraph  $\mathcal{H}$  is of *conformality*  $c$  if the following property holds for every subset  $X \subseteq V(\mathcal{H})$ :  $X$  is contained (as a subset) in a hyperedge of  $\mathcal{H}$  whenever each subset of  $X$  of cardinality

at most  $c$  is contained (as a subset) in a hyperedge of  $\mathcal{H}$ . See also [3, 38] for other equivalent characterizations of bounded conformality. We recall that a hypergraph  $\mathcal{H}$  is called Sperner if  $E_1 \not\subseteq E_2$  for any two distinct hyperedges  $E_1, E_2$  in  $\mathcal{H}$ .

Khachiyan, Boros, Elbassioni, and Gurvich proved the following.

► **Theorem 19** ([38]). *The minimal transversals of a Sperner hypergraph of bounded conformality can be enumerated in incremental-polynomial time.*

Using the properties of minimal separators of chordal bipartite graphs given in Section 2, we obtain the following, whose proof is moved to appendix, and derive the desired algorithm as a corollary of Theorem 19 and Lemma 20.

► **Lemma 20.** *If  $G$  is chordal bipartite then  $\mathcal{S}(G)$  has conformality 5.*

► **Theorem 21.** *The minimal connected dominating sets of a chordal bipartite graph can be listed in incremental polynomial time.*

## 6 Limitations of the sequential method

It is already known that the sequential method may not directly provide output-polynomial time algorithms for TRANS-ENUM in general; see e.g. [2] for explicit statements. We adapt the construction in [2] to show that this still holds for hypergraphs of (open or closed) neighborhoods of bipartite graphs, i.e., that we may not expect to get tractable algorithms for TDOM-ENUM and DOM-ENUM using this technique.

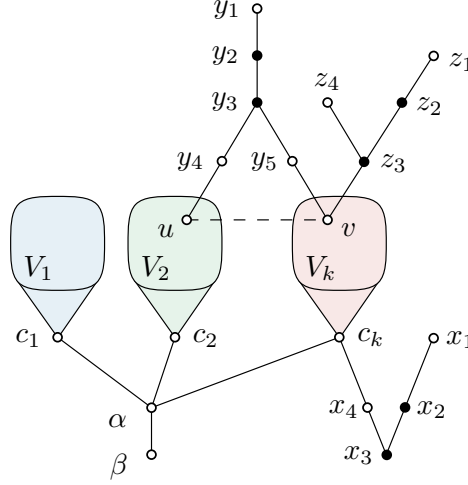
We begin with DOM-ENUM and later detail the modifications to be performed for the statement to hold for TDOM-ENUM. In the following by  $\mathcal{N}(G)$  we mean the hypergraph of closed neighborhoods of  $G$ , whose minimal transversals are precisely the minimal dominating sets of  $G$ ; see Section 2.

Recall that in MULTICOLORED INDEPENDENT SET problem, we are given a graph  $G$ , an integer  $k$ , and a partition  $\mathcal{V} = (V_1, V_2, \dots, V_k)$  of  $V(G)$  such that  $V_i$  is an independent set for every  $1 \leq i \leq k$ . Sets  $V_1, \dots, V_k$  are usually referred to as color classes. The goal is then to decide whether there exists an independent set in  $G$  containing precisely one vertex from each  $V_i$ , i.e., that is multicolored. See [12] for more details on this problem.

Due to space constraints, we omit the formal description of the reduction and only include the statement we derive. We refer the reader to Figure 2 for an idea of the construction, and to the appendix for its formal description and proof. The idea is to show that a minimal transversal  $T^*$  of  $\mathcal{H}$  induced by all but two of its vertices can be extended into a non-trivial child if and only if an instance  $G$  of multicolored independent set admits a solution.

► **Lemma 22.** *Let  $(G, \mathcal{V})$  be an instance of MULTICOLORED INDEPENDENT SET. Then there exists a bipartite graph  $H$ , an ordering  $v_1, \dots, v_n$  of its vertices, and a minimal transversal  $T^*$  of  $\mathcal{N}(H)_{n-2}$ , such that the sets in  $\text{children}(T^*, n-2)$  are all multicolored independent sets of  $G$  except for one.*

A consequence of Lemma 22 is that the sequential approach may not be directly applied to solve DOM-ENUM, even in bipartite graphs. Moreover, in our construction, no vertex  $v$  of  $T^*$  is made self-private, i.e., no  $v$  has its closed neighborhood as a private edge. Furthermore, the elements in the color classes are dominated by  $T^*$ , and their role is thus limited to dominate the  $c_i$ 's in an extension. Hence, by keeping the same construction, all the arguments of the proof hold for the hypergraph of *open* neighborhoods of  $G$ . We derive that the sequential approach may not be directly be used for TDOM-ENUM neither.



■ **Figure 2** The construction of  $H$ , where vertices are ordered arbitrarily except for  $\alpha$  and  $\beta$  which are put last, in this order. Color classes  $V_1, \dots, V_k$  induce independent sets in  $H$ . The dashed edge denote an edge of  $G$  that is not part of  $H$ . Black vertices represent  $T^*$ . Note that there is one copy of the depicted gadgets for each color class, vertex, and edge of  $G$ .

## 7 Trans-Enum and bipartite graphs<sup>2</sup>

Our final result is a proof that TRANS-ENUM reduces to CDM-ENUM on bipartite graphs, which we are not aware of being previously known or directly implied by other results. In fact, by Proposition 4, if we can construct a graph where (almost) every minimal separator is in a one to one correspondence to our hyperedges, we are done.

Let  $\mathcal{H}$  be the input hypergraph to TRANS-ENUM with  $E(\mathcal{H}) = \{E_1, \dots, E_m\}$  and  $V(\mathcal{H}) = \{u_1, \dots, u_n\}$  and assume w.l.o.g. that  $\mathcal{H}$  is a Sperner hypergraph. To obtain  $G$ , the input to CDM-ENUM, we start with the incidence bipartite graph of  $\mathcal{H}$ , that is,  $\{v_1, \dots, v_n\}, \{e_1, \dots, e_m\} \subset V(G)$ , where  $v_i$  corresponds to  $u_i \in V(\mathcal{H})$  and  $e_j$  corresponds to  $E_j \in E(\mathcal{H})$ , and  $v_i e_j \in E(G)$  if and only if  $u_i \in E_j$ . To conclude the construction of  $G$ , we add two vertices  $v'$  and  $v^*$ , the edge  $v'v^*$ , and the edges  $v^*v_i$  for all  $1 \leq i \leq n$ . Due to space constraints the proofs of the following are omitted as well.

► **Lemma 23.** *The minimal separators of  $G$  are  $N(e_1), \dots, N(e_m)$  and  $\{v^*\}$ .*

► **Theorem 24.** *CDM-ENUM on bipartite graphs is at least as hard as TRANS-ENUM.*

## 8 Discussion

In this paper, we investigated the complexity of DOM-ENUM and its two variants TDOM-ENUM and CDM-ENUM in chordal bipartite graphs. We obtained polynomial-delay algorithms for the first two problems, and an incremental-polynomial time algorithm for the last problem. Then we gave evidence that the techniques used for DOM-ENUM and TDOM-ENUM may not be pushed directly to bipartite graphs. As of CDM-ENUM, we proved a stronger statement that CDM-ENUM is as hard as TRANS-ENUM in bipartite graphs, with the later problem being open since more than forty years [16].

<sup>2</sup> After the submission of this work, it has come to our attention that there is a recent stronger result on the subject [42]. We nevertheless leave the proof as is since it is simple and for self-containment.

We leave open the existence of polynomial-delay algorithms for DOM-ENUM and TDOM-ENUM in bipartite graphs, and for CDOM-ENUM in chordal bipartite graphs. Concerning this last case, we note that obtaining polynomial delay is open for TRANS-ENUM on hypergraph of bounded dimension, which define a proper subclass of hypergraphs of bounded conformality. Hence, in order to improve Theorem 21, one either has to find another strategy, or should aim at improving the results in [16, 38] first.

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