An Improved Guillotine Cut for Squares

Parinya Chalermsook
University of Sheffield, England

Axel Kugelmann
PSL Research University, Paris, France

Ly Orgo
Aalto University, Finland

Sumedha Uniyal
Aalto University, Finland

Minoo Zarsav
Aalto University, Finland

Abstract

Given a set of n non-overlapping geometric objects, can we separate a constant fraction of them using straight-line cuts that extend from edge to edge? In 1996, Urrutia posed this question for compact convex objects. Pach and Tardos later refuted it for general line segments by constructing a family where any separable subfamily has size at most $O\left(n^{\log_3 2}\right)$. However, for axis-parallel rectangles, they provided positive evidence, showing that an $\Omega(1/\log n)$-fraction can be separated.

This problem naturally arises in geometric approximation algorithms. In particular, when restricting cuts to only orthogonal straight lines, known as a *guillotine cut sequence*, any bound on the separability ratio directly translates into a clean and simple dynamic programming for computing a maximum independent set of geometric objects.

This paper focuses on the case when the objects are squares. For squares of arbitrary sizes, an $\Omega(1)$-fraction can be separated (Abed et al., APPROX 2015), recently improved to $1/40$ (and $1/160 \approx 0.62\%$ for the weighted case) (Khan and Pittu, APPROX 2020). We further improve this bound, showing that a $9/256 \approx 3.51\%$ can be separated for the weighted case. This result significantly narrows the possible range for squares to $[3.51\%, 50\%]$. The key to our improvement is a refined analysis of the existing framework.

2012 ACM Subject Classification Theory of computation → Approximation algorithms analysis; Theory of computation → Computational geometry

Keywords and phrases Guillotine cuts, Geometric Approximation Algorithms, Rectangles, Squares

Digital Object Identifier 10.4230/LIPIcs.WADS.2025.16

Funding This work was supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 759557).

Acknowledgements In memory of our friend, Sumedha Uniyal.

1 Introduction

In this paper, we investigate the following combinatorial question, posed initially by Urrutia in 1996 [17]: Given a collection of n disjoint compact convex objects, what is the largest subset that a sequence of straight-line cuts can separate? This combinatorial geometric question naturally arises in two distinct scenarios:

- **Cutting Stock Problems:** The cutting stock problem involves cutting a standard-sized large piece of material – such as a roll of paper, a sheet of glass, or a wooden plank – into smaller specified pieces while minimizing waste [5]. This class of problems has inspired numerous challenges in operations research, combinatorics, and approximation algorithms (see, e.g., [9, 13, 15]).

- **Simple Algorithms for Geometric Packing:** In geometric packing, we seek to compute the maximum independent set in the intersection graph of geometric objects, i.e., given a collection of convex polygons, find the largest subset of non-overlapping polygons. These problems have been extensively studied, with a focus on approximation algorithms (due to their NP-hardness even for the simplest objects like disks or squares) [8, 2, 4, 3, 6, 10, 14, 7].

While Urrutia's question was refuted by Pach and Tardos [16] – who constructed a family of line segments where any separable subset has size at most $O(n^{\log_3 2})$ – there remains hope for axis-parallel rectangles:

- For any family of rectangles, there exists a separable set of size $\Omega(n/\log n)$ [16].
- For squares, a constant fraction of any given family is separable, thus affirmatively answering Urrutia's original question for square objects [1].

Both [16] and [1] (and a recent follow-up [12]) rely only on a sequence of orthogonal straight-line cuts (known as *guillotine cuts*) to separate the rectangles. Guillotine cuts are particularly appealing from an approximation algorithms perspective, as they lead to clean and simple combinatorial techniques. Specifically, any non-constructive result that guarantees a guillotine separable γ fraction of objects immediately implies a simple DP-based $(1/\gamma)$-approximation algorithm for geometric packing of the respective object families.

This connection, in fact, holds in a more general setting when the geometric objects have weights, i.e., we are looking for a separable set that retains γ fraction of total weights. For squares, the current best known factor is $1/40$ (and $1/160 \approx 0.62\%$ for the weighted setting) [12], while obtaining a factor better than $1/2$ is not possible [1].

1.1 Our Contributions

Our main contribution is summarized in the following theorem.

▶ **Theorem 1.** *For all families of non-overlapping weighted squares, there exists a separable subset that retains at least* $9/256 \approx 3.51\%$ *of the total weight.*

So our result improves upon the previous result by a factor of 5.6 in the weighted setting.

Broader connections to the DP framework. Our results relate to a dynamic programming framework for geometric packing, initiated by Adamaszek and Wiese [3, 2]. Their framework is parametrized by (α, t), where a geometric object family is called (α, t)-good if it contains a subset of at least an α-fraction of objects that can be separated using polygonal cuts that always create pieces with at most t edges. If a family is (α, t)-good, the framework yields a $(1/\alpha)$-approximation algorithm with running time $n^{O(t)}$. In Adamaszek, Har-Peled, and Wiese's elegant result [2], they show that every family of polygons in 2D is $((1-\epsilon), poly \log n)$-good, implying a quasi-polynomial time approximation scheme (QPTAS) for the independent set of polygons.

Since the existence of a γ-fraction guillotine separable subset implies that the family is $(\gamma, 4)$-good, our results directly contribute to a better understanding of this framework for squares. The recent breakthrough by Mitchell [14] show that rectangles are $(1/10, 5)$-good, and further refinements [11] prove that they are $(\frac{1}{2} - \epsilon, O(1/\epsilon))$-good, leading to the best known $(2 + \epsilon)$-approximation algorithm for the maximum independent set of rectangles.

1.2 Overview of Techniques

This paper builds on the techniques from [1]. There, the guillotine cut sequence is created in two phases. In the first phase, which we call the **grid drawing phase**, the procedure draws hierarchical grid lines so that each square is assigned its own cell roughly the size of its own. Those squares that do not fit this criterion would be removed from the first phase. We call the squares that remain after this phase the surviving squares. Denote by γ_1 the surviving ratio of the first phase. In the second phase, called **conflict removal phase**, we define a conflict graph H where the vertices correspond to the surviving squares and the edges connect the squares that are "difficult" to save simultaneously. The graph H is defined so that every independent set in this graph corresponds to squares that can be guillotine-separated. Therefore, the crux of this approach is to ensure that the conflict graph is sufficiently sparse, so that the graph is k-colorable for some small value of k. This will imply a guillotine-separable set of size γ_1/k.

Using this terminology, the best known factor of 160 [12] ensures that the conflict graph is 5-colorable and loses the factor of 32 in the grid drawing phase [12]; the earlier work by Abed et al. [1] also aims for a simple conflict removal phase, and therefore losing a relatively large factor in the grid drawing phase. Our improvements are obtained by looking at this differently. We ensure that the first phase has a good surviving rate (i.e., our $\gamma_1 = 9/16$), thus naturally complicating the second phase. Our conflict graph is more sophisticated and is no longer 5-colorable. We, nevertheless, show that the graph is 16-colorable, which implies the factor of $9/256$ as claimed.

The paper is written so that the ideas are introduced one by one. We first present the proof where $\gamma_1 = 1/4$ and $k = 15$ (so the factor is $1/60$) and improve k to 9 and 8 respectively (while keeping $\gamma_1 = 1/4$). Finally, we present the main result with $\gamma_1 = 9/16$ and $k = 16$.

2 Preliminaries

Assume we are given a set of non-overlapping axis-parallel squares of arbitrary sizes. We aim to separate as many of them as possible with a sequence of axis-parallel cuts, defined as follows.

Guillotine cuts are a series of horizontal and vertical cuts separating objects on a plane, such that each cut is performed on a previously separated part. For example, if the first vertical cut separates the plane into two, the next two horizontal cuts can separate the two pieces independently.

Given a collection of non-overlapping squares $\mathcal{R}$, $|\mathcal{R}| = n$ on a plane, the problem is to find the largest set $R \subseteq \mathcal{R}$ separable by Guillotine cuts.

2.1 Grid construction

In this section, we give the random hierarchical grid construction used in all the separation algorithms. Let M be the maximum width/height of a square in $\mathcal{R}$. Instead of width/height, we will say the **size** of a square.

We define a hierarchical grid structure similar to Abed et al. [1], which will act as a scaffolding for our instance $\mathcal{R}$. In our cutting strategy, we will perform cuts only along these lines.

We choose two integers $a, b \in [0, M)$ uniformly at random, corresponding to the random *horizontal* and *vertical* shift, respectively. The positive integer r will be a *"scaling factor"*.

Algorithm 1 Grid drawing.

```
foreach i ∈ [1, 2, ...] do
    Draw vertical grid lines at x coordinates a + (rM/2^i) · k for every integer k
    Draw horizontal grid lines at y coordinates b + (rM/2^i) · k for every integer k
```

We say that a grid line is ***level-i*** if it is drawn by Algorithm 1 in iteration i. Note that all level-i grid lines are also level-$(i+1)$ grid lines, etc.

▶ **Definition 2** (Cell levels). *Any two consecutive horizontal and vertical grid lines on level-i form a unique square, called a level-i cell.*

▶ **Definition 3** (Square levels). *We say that $S \in \mathcal{R}$ is a level-i square, if its size is within $\left(\frac{M}{2^{i+1}}, \frac{M}{2^i}\right]$.*

▶ **Fact 4.** *A level-i cell is at least r times (and strictly less than $2r$ times) larger than a level-i square.*

▶ **Proposition 5.** *Grid cells satisfy the following properties:*

- *A level-i cell has a size of $\frac{rM}{2^i}$.*
- *Two cells of the same level do not intersect.*
- *For two cells that intersect, one must contain the other.*

We say that C is a **subcell** of D if D contains C, and $D \neq C$ (they are not the same cell). We use $\mathsf{size}(C)$ to refer to the size of cell C and similarly for squares.

2.2 Surviving set $\mathcal{R}_1$

▶ **Definition 6** (Original cells). *Let S be any square and i be the level of S. If S does not intersect a grid line of level i, we define the original cell of S to be the level-i cell that contains S. We denote it as $\mathcal{OC}(S)$.*

Note that $\mathcal{OC}(S)$ is not defined for every square S. The set of squares for which original cells are defined is called the **surviving** set $\mathcal{R}_1$.

▶ **Theorem 7** (Claim 2.1, [1]). *The horizontal and vertical shifts $a, b \in [0, M)$ can be chosen such that the surviving set $\mathcal{R}_1$ is of size at least $(1 - \frac{1}{r})^2|\mathcal{R}|$.*

The proof can be found in Appendix A.

3 Algorithmic Results

3.1 Overview

We will introduce multiple strategies for separating a set of squares by performing guillotine cuts. All of these strategies will follow the basic structure outlined here.

Algorithm 2 Cutting framework.

Choose r.
Choose horizontal and vertical shifts $a, b \in [0, M)$ such that the surviving set $\mathcal{R}_1$ is at least $(1 - \frac{1}{r})^2 |\mathcal{R}|$.
For the surviving set of squares, define "a conflicting pair".
Based on the definition:
  Construct a conflict graph $G = (\mathcal{R}_1, E)$, where for all squares $S_1, S_2 \in \mathcal{R}_1$, $\{S_1, S_2\} \in E$, if and only if S_1 and S_2 conflict.
  Color G with β colors, such that each color (independent set) contains no conflicting pairs.
  Show that any independent set of G can be the result of a sequence of guillotine cuts.

This strategy leads to a set of size at least $(1 - \frac{1}{r})^2 \cdot \frac{1}{\beta} \cdot |\mathcal{R}|$. The specific ratio depends on r and, most importantly, on how we define a conflict, which will gradually be improved in each section, resulting in increasingly larger sets.

3.2 A direct cutting strategy to get $1/60$-separation

In this subsection, we fix $r = 2$, define the conflict graph G_1 on $\mathcal{R}_1$, and show that it is 15-colorable. Thus, the cutting sequence separating any independent set of G_1 will imply a $1/60$-separation.

We say that two squares $S_1, S_2 \in \mathcal{R}_1$ *conflict* if the original cells of S_1 and S_2 are the same, i.e., $\mathcal{OC}(S_1) = \mathcal{OC}(S_2)$, or we have the case that $\mathcal{OC}(S_1)$ contains $\mathcal{OC}(S_2)$ and S_1 intersects $\mathcal{OC}(S_2)$. Note that S_1 and S_2 do not conflict if $\mathcal{OC}(S_1)$ and $\mathcal{OC}(S_2)$ are of the same level but not the same cells. Figure 1 shows examples of conflicting and non-conflicting squares.

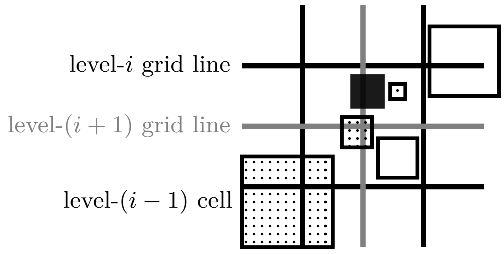


Figure 1 Dot-filled squares conflict with the black square, but the empty squares do not. The original cell of the black square is the black level-i cell.

Now, we define the *conflict graph* G_1, so that the vertices are the squares in $\mathcal{R}$ and two vertices are adjacent if they conflict.

For any cell C, we denote by $\mathsf{Squares}_{\mathcal{OC}}(C)$ the set of all squares $S \in \mathcal{R}_1$, such that $\mathcal{OC}(S) = C$. Note that some higher- or lower-level squares might be drawn inside C but not contained in $\mathsf{Squares}_{\mathcal{OC}}(C)$.

▶ **Lemma 8.** *The conflict graph G_1 is 15-colorable.*

Proof. We prove it by induction on the size of G_1. If G_1 is empty, it is obviously 15-colorable. Otherwise, we take the smallest non-empty cell C_1 such that $\mathsf{Squares}_{\mathcal{OC}}(C_1) \neq \emptyset$, i.e. there exists a square $S \in \mathcal{R}_1$ such that $\mathcal{OC}(S) = C_1$, and for any cell C_2 contained in C_1, we have $\mathsf{Squares}_{\mathcal{OC}}(C_2) = \emptyset$. Let i be the level of cell C_1.

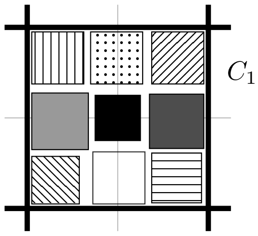


Figure 2 We color nine squares contained in C_1 using different colors.

According to the induction hypothesis, $G_1 \setminus \mathsf{Squares}_{\mathcal{OC}}(C_1)$ is 15-colorable, so let us assume it is already colored. We show that we can color all the squares in $\mathsf{Squares}_{\mathcal{OC}}(C_1)$ to get a 15 coloring for G_1. These squares all have to have a level at most i; otherwise, their original cell would have been smaller, and we wouldn't have chosen C_1. According to Fact 4, the size of level-i squares is strictly bigger than $\frac{1}{2r} = \frac{1}{4}$ of the size of C_1, and therefore at most 3 squares of level i fit into a cell of level i horizontally as well as vertically. Since squares of smaller levels are larger, it follows that at most 9 squares can be inside C_1 (Figure 2). If C_1 contains squares of smaller levels, they will already be colored by the induction hypothesis, and these colors will not be changed. We can color all the squares in $\mathsf{Squares}_{\mathcal{OC}}(C_1)$ with at most nine different colors (see Figure 2), avoiding the already used colors inside C_1.

Now, we bound the number of squares that conflict with any square in $\mathsf{Squares}_{\mathcal{OC}}(C_1)$ but are not contained in C_1. Since C_1 is the smallest non-empty cell, any square $S \in \mathcal{R}_1$ must cross the boundary of C_1. Since C_1 is a level-i cell, then $\mathcal{OC}(S)$ is a level-j cell for $j < i$. We bound the number of squares that can conflict with any square in $\mathsf{Squares}_{\mathcal{OC}}(C_1)$ based on the way they intersect C_1 (see Figure 3a): First, there can be at most four squares containing one of the corners of C_1. Next, let S be a square intersecting the side of C_1 without containing any corner. Using Fact 4, we know that the size of S is strictly bigger than half of the size of C_1. Thus, there can be at most one square in this position for each side of C_1, which adds up to four squares. We can further reduce this number because two sides of any level-i cell are also level-$(i-1)$ grid lines. Then, there cannot be any intersecting square on those sides, implying that at most two squares intersecting only sides of C_1 (see Figure 3b).

(a) Without considering the geometry of the cells.

(b) Such squares shouldn't cross the level-$(i-1)$ grid lines.

Figure 3 The dot-filled squares show the possible positions of squares not contained in cell C (darkened area) and conflict with the black square in C.

We showed that at most six squares can conflict with squares in $\mathsf{Squares}_{\mathcal{OC}}(C_1)$ by intersecting the boundary of C_1 and at most 9 colors can be required to color the squares inside C_1, leading to a valid 15 coloring for G_1. ◀

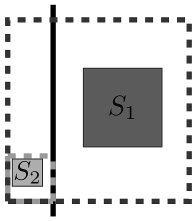


Figure 4 The dashed squares are $\mathcal{OC}(S_1)$ and $\mathcal{OC}(S_2)$, and black-line cut separates S_1 and S_2.

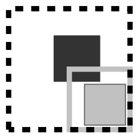


Figure 5 For the black square $\mathcal{FC}(S) = \mathcal{OC}(S)$, whereas for the grey square $\mathcal{FC}(S) \neq \mathcal{OC}(S)$.

▶ **Lemma 9.** *Any independent set of G_1 is guillotine separable.*

Proof. Let $\mathcal{I} \subseteq \mathcal{R}_1$ be an independent set of G_1. First, we can separate the two squares with the "biggest" original cells without cutting through any other squares. Let $S_1, S_2 \in \mathcal{I}$ be the squares such that $\mathsf{size}(\mathcal{OC}(S_1)) \geq \mathsf{size}(\mathcal{OC}(S_2))$ and for any other square $S \in \mathcal{I}$, $\mathsf{size}(\mathcal{OC}(S_2)) \geq \mathsf{size}(\mathcal{OC}(S))$. Note that since S_1 and S_2 cannot conflict, then $\mathcal{OC}(S_1) \neq \mathcal{OC}(S_2)$.

Since the grid lines defining $\mathcal{OC}(S_2)$ cannot go through any cell smaller than itself, we can safely cut along these grid lines and do not touch any other squares in $\mathcal{I}$ except for S_1. Now, we claim that one of the four grid lines defining $\mathcal{OC}(S_2)$ separates S_1 and S_2 (see Figure 4). Since S_1 cannot intersect $\mathcal{OC}(S_2)$, it must be entirely on the left, right, top, or bottom of $\mathcal{OC}(S_2)$. The grid line of the corresponding side then separates S_1 from S_2 without cutting any square in $\mathcal{I}$. We can separate all the squares by recursively and independently applying this process to the two resulting sub-instances. ◀

3.3 Fitting Cells to get $1/44$-separation

In this section, we improve coloring by using the additional concept of fitting cells. The grid's scaling factor remains 2.

Note that the grid structure defined in Section 2 forms a **laminar family** of cells. Let the set of these cells be $\mathcal{L}_1$. For any laminar family of cells, we define the notion of fitting cells as follows.

▶ **Definition 10** (Fitting cells). *Given a laminar family of cells $\mathcal{L}_1$ and a surviving set $\mathcal{R}_1$, the fitting cell $\mathcal{FC}(S)$ of any square $S \in \mathcal{R}_1$ is the smallest cell in $\mathcal{L}_1$ containing S.*

The fitting and original cells of square $S \in \mathcal{R}_1$ can differ when the former is too small to qualify as an original cell because of the scaling factor $r = 2$ (Figure 5).

▶ **Fact 11.** *For a level-i square S, the fitting cell $\mathcal{FC}(S)$ is always of level j, for $i \leq j \leq i+1$.*

Proof. The bound $j \leq i+1$ is a direct result of the scaling factor: for a cell C of level i, $\mathsf{size}(S) > \frac{1}{4}\mathsf{size}(C)$ (Fact 4), therefore for a cell C_1 of level $i+1$, we would have $\mathsf{size}(S) > \frac{1}{2}\mathsf{size}(C_1)$ and for a level-$(i+2)$ cell C_2, $\mathsf{size}(S) > \mathsf{size}(C_2)$. This implies that C_2 cannot be a fitting cell of S, bounding j at most $i+1$. The first part of the inequality, $i \leq j$, is easy to see since every original cell must contain S, so the smallest cell that contains S cannot be larger than $\mathcal{OC}(S)$. ◀

Conflict graph G_2. We define the new conflict graph G_2 based on the definition of conflicts below.

▶ **Definition 12** (Conflicts). *Let S_1 and S_2 be distinct squares in $\mathcal{R}_1$. We say that S_1 and S_2 conflict if one of the following conditions is true: (i) $\mathcal{FC}(S_1) = \mathcal{FC}(S_2)$, or (ii) $\mathsf{size}(\mathcal{FC}(S_1)) > \mathsf{size}(\mathcal{FC}(S_2))$ and S_1 intersects $\mathcal{FC}(S_2)$.*

Figure 6 shows examples of conflicting and non-conflicting squares.

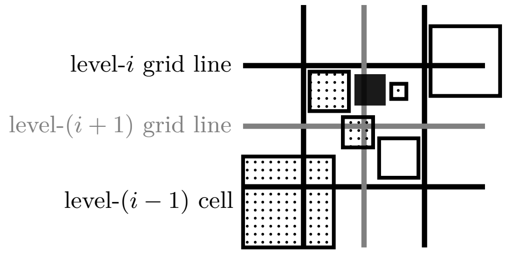


Figure 6 Dot-filled squares conflict with the black square, and empty squares do not. The fitting cell of the black square is the black level-i cell.

Analogously to $\mathsf{Squares}_{\mathcal{OC}}$ for original cells, we define $\mathsf{Squares}_{\mathcal{FC}}$ for fitting cells.

▶ **Definition 13.** *Let C be a cell of $\mathcal{L}_1$. We use $\mathsf{Squares}_{\mathcal{FC}}(C)$ to denote the set of all squares $S \in \mathcal{R}_1$, such that $\mathcal{FC}(S) = C$.*

▶ **Lemma 14.** *The conflict graph G_2 is 11-colorable.*

The proof can be found in Appendix A.

▶ **Lemma 15.** *Any independent set of G_2 is guillotine-separable.*

The proof can be seen by replacing the concept of original cells in Lemma 9 with fitting cells.

4 Further Improvements for cutting Squares

4.1 Twin squares to reach 1/36-separation

In this subsection, we introduce the notion of *twin squares.* The idea is to identify pairs of squares (belonging to the same fitting cell) that can be separated using the next level's vertical or horizontal grid line. We will get a sparser conflict graph G_3 on $\mathcal{R}_1$ by removing conflicts between such squares. We show that the new conflict graph G_3 is 9-colorable, and any independent set of this graph is guillotine separable, implying a 1/36-separation for $\mathcal{R}$. We start with the same set $\mathcal{R}_1$, laminar family $\mathcal{L}_1$, and the fitting cells defined in Section 3.3.

▶ **Definition 16** (Twin squares). *For a laminar family $\mathcal{L}_1$, Let S_1 and S_2 be two squares with the same level-i fitting cell, i.e. $\mathcal{FC}(S_1) = \mathcal{FC}(S_2)$. We say S_1 and S_2 are* twin squares *if they can be separated by the level-$(i+1)$ horizontal or vertical grid line cutting through $\mathcal{FC}(S_1)$ (see Figure 7).*

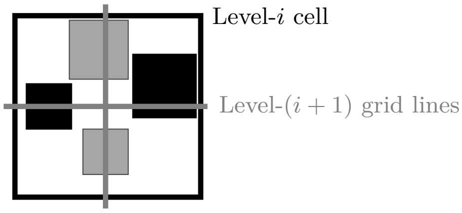


Figure 7 The two gray and two black squares all have a common fitting cell.

Conflict graph G_3. We define the new conflict graph G_3 based on the definition of conflicts below.

▶ **Definition 17** (Conflicts). *Let S_1 and S_2 be two squares in $\mathcal{R}_1$. We say that S_1 and S_2 conflict (up to renaming) if one of the following conditions is true:*
- *$\mathcal{FC}(S_1) = \mathcal{FC}(S_2)$, and S_1 and S_2 are not twin squares.*
- *$\mathcal{FC}(S_1)$ contains $\mathcal{FC}(S_2)$, and S_1 intersects $\mathcal{FC}(S_2)$.*

▶ **Lemma 18.** *The conflict graph G_3 is 9-colorable.*

The proof can be found in Appendix A.

▶ **Lemma 19.** *Any independent set of G_3 is guillotine separable.*

Proof. Similarly to Lemma 9, we show a recursive cutting strategy. Starting with an independent set $\mathcal{R}_2$ of G_3, we first separate two squares with the "biggest" fitting cell. Let $S_1, S_2 \in \mathcal{R}_2$ be the squares such that $\mathsf{size}(\mathcal{FC}(S_1)) \geq \mathsf{size}(\mathcal{FC}(S_2))$, and for any other square $S \in \mathcal{R}_2$, $\mathsf{size}(\mathcal{FC}(S_2)) \geq \mathsf{size}(\mathcal{FC}(S))$. If $\mathcal{FC}(S_1) \neq \mathcal{FC}(S_2)$, then S_1 and S_2 can be separated using the proof of Lemma 9 (see Figure 4). Otherwise, S_1, S_2 must be twin squares with a common fitting cell (let $C = \mathcal{FC}(S_1)$), since they do not conflict. Then, we can cut along the grid lines of C (see Figure 8). Since for any square $S \in \mathcal{R}_2$, $\mathsf{size}(C) \geq \mathsf{size}(\mathcal{FC}(S))$, we can safely perform these cuts without intersecting any square in $\mathcal{R}_2$.

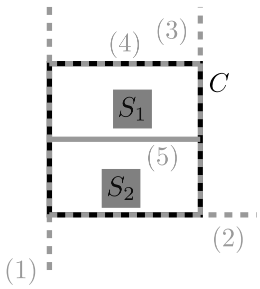


Figure 8 Cutting to separate twin squares; we first cut the dashed grey lines 1 to 4 to isolate C. We cut along the 5^{th} light grey line to separate S_1 and S_2. No square can intersect the last cut.

Let i be the level of C. Since we have isolated C with the first 4 cuts, further cuts of C can be done independently from the rest of the graph. We use the horizontal (or vertical) level-$(i+1)$ grid line, which certifies that S_1 and S_2 are twins (see Definition 16), as a cut to separate S_1 from S_2 (see Figure 8). Let $S \in \mathcal{R}_2$ be any square inside C intersecting the cut. Since S is contained in C then $\mathsf{size}(\mathcal{FC}(S)) \leq \mathsf{size}(C)$. S also cannot have a level-$(i+1)$ (or smaller) cell as its fitting cell because of intersecting the level-$(i+1)$ grid line. Therefore $\mathcal{FC}(S) = C$. Since twin squares S_1 and S_2 intersect the vertical level-$(i+1)$ grid line and their fitting cell is also C, then S conflicts with them. This implies that $S \notin \mathcal{R}_2$ and the cut is safe to perform. Similarly to the proof of Lemma 9, we recursively apply this process to separate $\mathcal{R}_2$. ◀

4.2 Half-Cells for a $1/32$-separation

In this section, we redefine the notion of cells to include rectangular cells.

▶ **Definition 20** (Rectangular Cells). *Any two consecutive horizontal and vertical grid lines of level i and j form a unique rectangle, called a level-(i, j) cell.*

Let *square-cells* be all level-(i, i) cells for some i (our old notion of cells), and *half-cells* be all level-$(i, i + 1)$ cells for some i. Note that half-cells of level $(i, i + 1)$ can be obtained by splitting square-cells of level (i, i) with the level-$(i + 1)$ vertical line within them.

We will use $\mathsf{height}(C)$ and $\mathsf{width}(C)$ to refer to the height and width of a cell C, respectively. For any pair of cells C_1, C_2, we will say $\mathsf{size}(C_1) \le \mathsf{size}(C_2)$, if $\mathsf{height}(C_1) \le \mathsf{height}(C_2)$ and $\mathsf{width}(C_1) \le \mathsf{width}(C_2)$. Note that among square-cells and half-cells, any pair of cells C_1, C_2 is comparable (either $\mathsf{size}(C_1) \le \mathsf{size}(C_2)$ or $\mathsf{size}(C_2) \le \mathsf{size}(C_1)$), since we are only considering the tall halves. Adding half-cells to the laminar family $\mathcal{L}_1$, the resulting family $\mathcal{L}_2$ remains laminar. In this section, let fitting cells be defined over the laminar family $\mathcal{L}_2$ (Definition 10), while the scaling factor r remains 2. Examples of the new fitting cells are depicted on Figure 9.

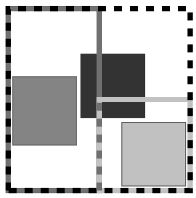


Figure 9 The black dashed square-cell is the original cell of all the squares. The fitting cell of the black square is its original cell. The dark and light gray squares have a half-cell and a higher-level square-cell as their fitting cells, respectively.

▶ **Fact 21.** *For a level-i square S, the fitting cell $\mathcal{FC}(S)$ is*
- *A square-cell of level j, for $i \le j \le i + 1$, or*
- *A half-cell of level $(i, i + 1)$*

Proof. The first part follows directly from Fact 11. Let us consider the second part. When we define fitting cells over $\mathcal{L}_2$, the surviving set remains the same; therefore, when the fitting cell is a half-cell, it is always two times narrower than if we defined fitting cells over $\mathcal{L}$. Similarly to the proof of Fact 11, a half-cell C of level $(i, i + 1)$ has $\mathsf{width}(S) > \frac{1}{2}\mathsf{width}(C)$, therefore a half-cell C_1 of level $(i + 1, i + 2)$ would not be able to contain S. ◀

Twin squares. Similarly to the original definition Definition 16, we define twin squares $S_1, S_2 \in \mathcal{R}_1$ to be a pair of squares in the same fitting cell C_1 (which is a level-i square-cell) that are separated by a level-$(i + 1)$ grid line.

If their fitting cell is a half-cell, we do not consider $S_1, S_2 \in \mathcal{R}_1$ twin squares. Additionally, twin squares $S_1, S_2 \in \mathcal{R}_1$ cannot be separated by a vertical grid line, otherwise their fitting cells would be separate half-cells.

Conflict graph G_4. We define the new conflict graph G_4 based on the same Definition 17 of conflicts with respect to the new fitting cells of $\mathcal{L}_2$. See Figure 10 for illustration.

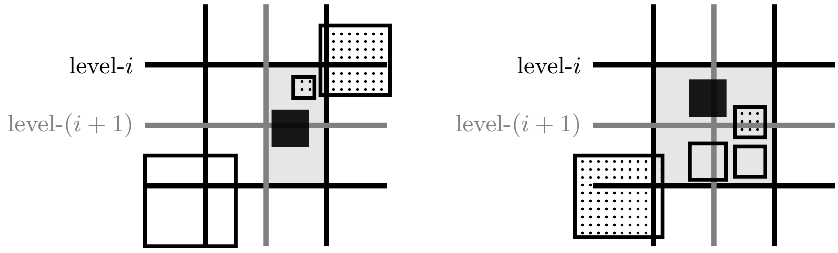


Figure 10 The dot-filled squares conflict with the black square, and the empty squares do not. The fitting cell of the black square is the shaded half-cell on the left and the shaded square-cell on the right.

▶ **Lemma 22.** *The conflict graph G_4 is 8-colorable.*

Proof. This proof is easier to show by using a stronger statement. We will show that there exists an 8-coloring of G_4, such that for each cell $C \in \mathcal{L}_2$, the squares $\mathsf{Squares}_{\mathcal{FC}}(C)$ are colored with at most 2 colors.

We will see this proof by induction over the number of squares ($|V(G_4)|$). The base case holds trivially. We again pick the smallest non-empty cell $C_1 \in \mathcal{L}_2$ for induction. Then $\mathsf{Squares}_{\mathcal{FC}}(C_1) \neq \emptyset$ and for any cell C_2 smaller than C_1 ($\mathsf{size}(C_2) \leq \mathsf{size}(C_1)$), we have $\mathsf{Squares}_{\mathcal{FC}}(C_2) = \emptyset$.

We assume that the induction hypothesis holds for $G_4 \setminus \mathsf{Squares}_{\mathcal{FC}}(C_1)$ and show that we can use 2 colors to color $\mathsf{Squares}_{\mathcal{FC}}(C_1)$, with at most eight colors in total. We will consider the cases where C_1 is a square-cell or a half-cell separately.

Firstly, consider that C_1 is a square-cell of level i. Equivalently to Lemma 8, it can be seen that there are at most 6 squares that can conflict with any square in $\mathsf{Squares}_{\mathcal{FC}}(C_1)$. Since these squares use at most 6 colors, at least 2 colors remain to color $\mathsf{Squares}_{\mathcal{FC}}(C_1)$. We will count the number of squares C_1 contains, which, similarly to previous proofs, is the same as $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ due to how C_1 was chosen. All squares in C_1 must intersect the vertical grid line of level $i+1$, splitting C_1; otherwise, those squares would be contained in a subcell of C_1. According to Fact 4, these squares can be at most 3. These squares can be colored with 2 colors since the squares above and below the horizontal grid line of the level $i+1$ are twin squares and can share a color (Figure 11).

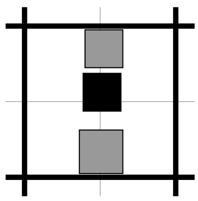


Figure 11 Two colors (black and gray) suffice to color all squares contained in C_1 since if squares are above and below the horizontal grid line crossing C_1, these are twin squares.

Secondly, we consider that C_1 is a level-$(i, i+1)$ half-cell. Again, the squares that C_1 contains among $\mathcal{R}_1$ are exactly $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ due to how C_1 was chosen. These squares must intersect the horizontal grid line of level $i+1$ within C_1; otherwise, they would fit into a subcell of C_1. The squares in $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ must be of level i to have survived, and the number of level-i squares intersecting the level-$(i+1)$ grid line in C_1 can be at most 1 (Fact 21, Fact 4). Since $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ cannot be empty, it must contain exactly one square. Let S be that square.

It remains to bound the number of squares conflicting with S. Note that all the conflicting squares must intersect the boundary of C_1 (Definition 17). Let C_0 be the level-i square-cell such that C_1 is a subcell of C_0.

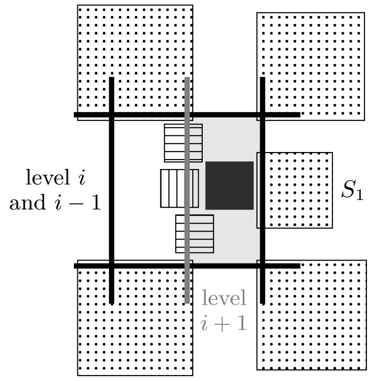


Figure 12 Cell C_1 is shaded, and cell C_0 is bordered by the black grid lines of level i. We depict that the left vertical grid line of level i is also of level $i-1$.

At most, 4 squares can intersect with the corners of cell C_1. Next, consider the squares intersecting the level-i horizontal grid lines bordering C_1. For these squares to survive, their level must be at most $i-1$. This implies they are too large to intersect only the level-i horizontal grid line without intersecting a corner (Figure 12). Next, let us consider the level-i vertical grid lines bordering C_0. Similarly to Lemma 8, Figure 3b, one of those grid lines must also be of level $i-1$ and therefore cannot have a surviving square intersecting it without also intersecting a corner. This leaves us with one square S_1 as depicted on Figure 12. Finally, let us consider the squares intersecting the level-$(i+1)$ vertical grid line bordering C_1 (Figure 12). Since we already counted the squares intersecting the borders of C_0 in addition to C_1, these squares must be in $\mathsf{Squares}_{\mathcal{FC}}(C_0)$. By induction hypothesis, they can be colored with 2 colors. If the colors of the corner squares and S_1 are different from these two colors, we have at most 7 colors used for the set of squares that conflict with S. By coloring S with a different color, we have a valid 8-coloring with at most 2 colors used for $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ as stated. ◀

▶ **Lemma 23.** *Any independent set of G_4 is guillotine separable.*

Proof. Similarly to Lemma 19, we show a recursive cutting strategy. Starting with an independent set $\mathcal{R}_2$ of G_4, we first separate two squares with the "biggest" fitting cell. Let $S_1, S_2 \in \mathcal{R}_2$ be the squares such that $\mathsf{size}(\mathcal{FC}(S_1)) \geq \mathsf{size}(\mathcal{FC}(S_2))$, and for any other square $S \in \mathcal{R}_2$, $\mathsf{size}(\mathcal{FC}(S_2)) \geq \mathsf{size}(\mathcal{FC}(S))$. If $\mathcal{FC}(S_1) \neq \mathcal{FC}(S_2)$, then S_1 and S_2 can be separated using the proof of Lemma 9 (see Figure 4). Next, we consider the case that $\mathcal{FC}(S_1) = \mathcal{FC}(S_2)$, e.g S_1 and S_2 share the same fitting cell. Let this fitting cell be C and its level be i.

First, consider the case that C is a half-cell. This implies that S_1 and S_2 intersect the same horizontal grid line of level $i+1$. Otherwise, they would fit in smaller subcells. Since there are no twin squares in half-cells, S_1 and S_2 conflict, which contradicts them being in the same independent set $\mathcal{R}_2$.

Secondly, consider the case that C is a square-cell. Similarly to the proof of Lemma 19, the square-cell can be isolated from the rest of the graph by the first 4 cuts as depicted on Figure 8. Again, these cuts will not intersect any squares outside of C, because the fitting cells for other squares must be smaller than or equal to the size of C (due to our choice of S_1 and S_2). Similarly to Lemma 19, the fifth cut is a level-$(i+1)$ horizontal grid line if i is the level of C. Let us now consider the squares inside C since the fifth cut will be performed only on C. For $\mathsf{Squares}_{\mathcal{FC}}(C)$, we can see similarly to Lemma 19 that they are not cut. For a square S whose fitting cell is a half-cell subcell of C, there is a conflict between S and S_i, because $\mathcal{FC}(S_1) \geq \mathcal{FC}(S)$ and S_1 intersects $\mathcal{FC}(S)$. This means that S cannot exist in the independent set $\mathcal{R}_2$, and the cutting strategy doesn't cut any squares.

Similarly to the proof of Lemma 9, we recursively apply this process to separate $\mathcal{R}_2$. ◀

5 Using Coarser Grid ($r = 4$) for Squares to get $\frac{9}{256}$-separation

We consider a set $\mathcal{R}$ containing n non-overlapping axis-parallel squares. In Theorem 27, we show that we can separate at least $\frac{9}{256}$ fraction of the squares with guillotine cuts.

For the coarser grid, we set $r = 4$, and get $\frac{9}{16}$ fraction of surviving squares $\mathcal{R}_1 \in \mathcal{R}$ by applying Theorem 7. Then, we introduce a conflict graph G_5, showing that it is 16-colorable. Finally, we show that any independent set of this graph is Guillotine separable, leading to a 9/256-separation.

Similar to Section 4.2, we consider the laminar family $\mathcal{L}_3$ containing all the level-i cells together with all the level-$(i, i+1)$ half-cells for any $i \geq 0$. For each square S, we define the fitting cell $\mathcal{FC}(S)$ over the laminar family $\mathcal{L}_3$ (Definition 10).

Since, in this case, $r = 4$, the number of squares associated with a fitting cell increases. In this section, let the surviving set $\mathcal{R}_1$ be redefined to be the surviving set of the new grid structure.

▶ **Fact 24.** *For any square $S \in \mathcal{R}_1$, one of the following must be true:*

- *The fitting cell $\mathcal{FC}(S)$ is a level-i square-cell for some i. Then S intersects the level-$(i+1)$ vertical grid line, cutting $\mathcal{FC}(S)$ in half, and $\mathcal{FC}(S)$ can be a fitting cell for at most seven squares (see Figure 13a).*
- *The fitting cell $\mathcal{FC}(S)$ is a level-$(i, i+1)$ half-cell for some i. Then S intersects the level-$(i+1)$ horizontal grid line, cutting $\mathcal{FC}(S)$ in half, and $\mathcal{FC}(S)$ can be a fitting cell for at most three squares (see Figure 13b).*

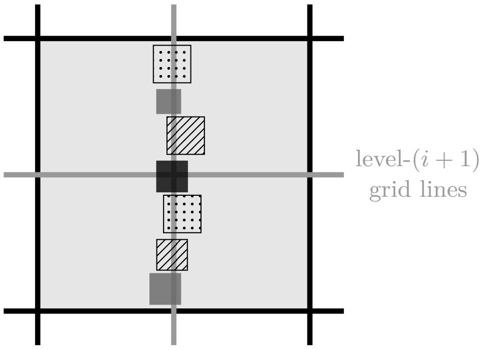
Proof. If the intersection property did not hold for either of the cases above, the square S would fit into some smaller cell in $\mathcal{L}_3$, leading to a contradiction.

Since S intersects a level-$(i+1)$ grid line in both cases, S must be level-j for $j \leq i$. Also, $\mathsf{size}(S)$ must be strictly more than $\frac{1}{8}$ fraction of the height of any level-i cell, because of the Fact 4.

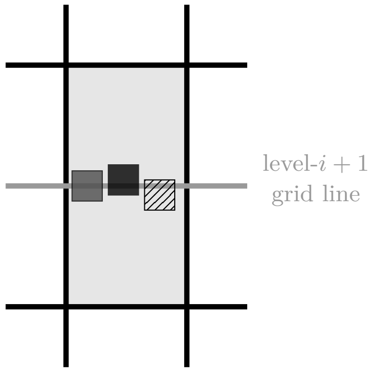
In case $\mathcal{FC}(S)$ is of level i, any such square must intersect the level-$(i+1)$ vertical grid line cutting $\mathcal{FC}(S)$ in half. Hence, at most seven squares can fit into $\mathcal{FC}(S)$.

Otherwise, if $\mathcal{FC}(S)$ is a level-$(i, i+1)$ cell, the width of $\mathcal{FC}(S)$ is half the width of any level-i cell. Since S must intersect the level-$(i+1)$ horizontal grid line cutting $\mathcal{FC}(S)$ into half, then at most three squares can fit into $\mathcal{FC}(S)$. ◀

Conflict Graph G_5. We use the same Definition 17 of conflicting squares to define our conflict graph G_5 on $\mathcal{R}_1$. Similar to Section 4.2, there are no twin squares separated by the vertical line cutting through the cell; otherwise, the two squares would have different fitting



(a) Square-cell.



(b) Half-cell.

Figure 13 A coloring for the squares in $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ when C_1 is a level-i square-cell or level-$(i, i+1)$ half-cell on the left and right respectively. C_1 is shaded.

cells. The following lemma shows the coloring of G_5, similar to the proof of Lemma 22, involving more counting arguments. Since we have $r = 4$ instead of $r = 2$, this grid is coarser, leading to more squares inside and intersecting the boundary of any cell C in $\mathcal{L}_3$.

▶ **Lemma 25.** *The graph G_5 is 16-colorable.*

The proof can be found in Appendix A.

▶ **Lemma 26.** *Any independent set of G_5 is guillotine separable.*

The proof of this lemma is similar to the proof of Lemma 23 in Section 4.

▶ **Theorem 27.** *For an n-size set $\mathcal{R}$ of non-overlapping axis-parallel squares, there exists a $\frac{9}{256}$-separation with guillotine cuts.*

Proof. Using Theorem 7, we can place a grid on $\mathcal{R}$ and define the original cells for a subset $\mathcal{R}_1 \subset \mathcal{R}$ containing at least $\frac{9n}{16}$ squares, with $r = 4$. Then, Lemma 25 ensures that the conflict graph G_5 for the squares is 16-colorable. Thus, we can extract a subset $\mathcal{R}_2 \subset \mathcal{R}_1$ containing at least $\frac{9n}{256}$ squares, which do not conflict with any other square in $\mathcal{R}_2$.

Finally, Lemma 26 provides a cutting strategy for $\mathcal{R}_2$, showing that $\mathcal{R}_2$ is guillotine separable. ◀

References

1 Fidaa Abed, Parinya Chalermsook, José Correa, Andreas Karrenbauer, Pablo Pérez-Lantero, José A. Soto, and Andreas Wiese. On Guillotine Cutting Sequences. In Naveen Garg, Klaus Jansen, Anup Rao, and José D. P. Rolim, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2015)*, volume 40 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 1–19, Dagstuhl, Germany, 2015. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. `doi:10.4230/LIPICS.APPROX-RANDOM.2015.1`.

2 Anna Adamaszek, Sariel Har-Peled, and Andreas Wiese. Approximation schemes for independent set and sparse subsets of polygons. *Journal of the ACM (JACM)*, 66(4):29, 2019.

3 Anna Adamaszek and Andreas Wiese. Approximation schemes for maximum weight independent set of rectangles. In *Proceedings of the 2013 IEEE 54th Annual Symposium on Foundations of Computer Science*, FOCS '13, pages 400–409. IEEE Computer Society, 2013. `doi:10.1109/FOCS.2013.50`.

4 Anna Adamaszek and Andreas Wiese. *A QPTAS for Maximum Weight Independent Set of Polygons with Polylogarithmically Many Vertices*, pages 645–656. Society for Industrial and Applied Mathematics, 2014. doi:10.1137/1.9781611973402.49.

5 Mir-Bahador Aryanezhad, Nima Fakhim Hashemi, Ahmad Makui, and Hasan Javanshir. A simple approach to the two-dimensional guillotine cutting stock problem. *Journal of Industrial Engineering International*, 8(1):21, September 2012.

6 Parinya Chalermsook and Julia Chuzhoy. Maximum independent set of rectangles. In *Proceedings of the twentieth annual ACM-SIAM symposium on Discrete algorithms*, pages 892–901. Society for Industrial and Applied Mathematics, 2009. doi:10.1137/1.9781611973068.97.

7 Parinya Chalermsook and Bartosz Walczak. Coloring and maximum weight independent set of rectangles. In *Proceedings of the 2021 ACM-SIAM symposium on discrete algorithms (SODA)*, pages 860–868. SIAM, 2021. doi:10.1137/1.9781611976465.54.

8 Timothy M Chan and Sariel Har-Peled. Approximation algorithms for maximum independent set of pseudo-disks. *Discrete & Computational Geometry*, 48(2):373–392, 2012. doi:10.1007/S00454-012-9417-5.

9 CH Cheng, BR Feiring, and TCE Cheng. The cutting stock problem – A survey. *International Journal of Production Economics*, 36(3):291–305, 1994.

10 Thomas Erlebach, Klaus Jansen, and Eike Seidel. Polynomial-time approximation schemes for geometric intersection graphs. *SIAM Journal on Computing*, 34(6):1302–1323, 2005. doi:10.1137/S0097539702402676.

11 Waldo Gálvez, Fabrizio Grandoni, Arindam Khan, Diego Ramírez-Romero, and Andreas Wiese. Improved approximation algorithms for 2-dimensional knapsack: Packing into multiple l-shapes, spirals, and more. *arXiv preprint arXiv:2103.10406*, 2021. arXiv:2103.10406.

12 Arindam Khan and Madhusudhan Reddy Pittu. On Guillotine Separability of Squares and Rectangles. In Jaroslaw Byrka and Raghu Meka, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2020)*, volume 176 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 47:1–47:22, Dagstuhl, Germany, 2020. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPICS.APPROX/RANDOM.2020.47.

13 Andrea Lodi, Silvano Martello, and Michele Monaci. Two-dimensional packing problems: A survey. *European journal of operational research*, 141(2):241–252, 2002. doi:10.1016/S0377-2217(02)00123-6.

14 Joseph Mitchell. Approximating maximum independent set for rectangles in the plane. *SIAM Journal on Computing*, 0(0):FOCS21–299–FOCS21–322, 0. doi:10.1137/22M1475521.

15 Nthabiseng Ntene and Jan H van Vuuren. A survey and comparison of guillotine heuristics for the 2d oriented offline strip packing problem. *Discrete Optimization*, 6(2):174–188, 2009. doi:10.1016/J.DISOPT.2008.11.002.

16 János Pach and Gábor Tardos. Cutting glass. In *Proceedings of the Sixteenth Annual Symposium on Computational Geometry*, SoCG '00, pages 360–369. ACM, 2000. doi:10.1145/336154.336223.

17 Jorge Urrutia. Problem presented at ACCOTA'96: Combinatorial and computational aspects of optimization. *Topology, and Algebra, Taxco, Mexico*, 1996.

A Deferred proofs

▶ **Theorem 7** (Claim 2.1, [1]). *The horizontal and vertical shifts $a, b \in [0, M)$ can be chosen such that the surviving set $\mathcal{R}_1$ is of size at least $(1 - \frac{1}{r})^2 |\mathcal{R}|$.*

Proof. We will show that this result can be reached in expectation by sampling $a, b \in [0, M)$ independently and uniformly at random.

Let $S \in \mathcal{R}$ be any square of level i. Let P_h and P_v be the probabilities that a level-i horizontal or vertical grid line is drawn on S, respectively. By using Fact 4, we get that $P_h, P_v \leq \frac{1}{r}$. Thus, the probability that S is in the surviving set (survives), is $(1-P_h)(1-P_v) \geq (1-\frac{1}{r})^2$. In expectation, $(1-\frac{1}{r})^2|\mathcal{R}|$ rectangles survive. This implies that there must exist a and b such that at least $(1-\frac{1}{r})^2|\mathcal{R}|$ squares survive. ◀

▶ **Lemma 14.** *The conflict graph G_2 is 11-colorable.*

Proof. Similar to Lemma 8, we will prove this by induction on the number of squares ($|V(G_2)|$). The base case is trivial. For the inductive step, consider the smallest cell C_1 that contains a square ($\mathsf{Squares}_{\mathcal{FC}}(C_1) \neq \emptyset$). This implies that for any cell C_2 smaller than C_1, $\mathsf{Squares}_{\mathcal{FC}}(C_2) = \emptyset$. By induction hypothesis, $G_2 \setminus \mathsf{Squares}_{\mathcal{FC}}(C_1)$ is 11-colorable, and we will show that we can color the squares of $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ while maintaining a valid 11-coloring of G_2.

First, we bound the number of squares contained within C_1. Let i be the level of C_1. Since C_1 is the smallest cell containing any square, any square S within C_1 must cross the level-$(i+1)$ grid lines within C_1, or otherwise S would be contained in a smaller level-$(i+1)$ cell. According to Fact 11 and Fact 4, at most 3 consecutive squares can fit into C_1 width- or height-wise. Since all contained squares must intersect the level-$(i+1)$ grid lines, it is easy to verify that the number of these squares can be at most 5 in total (Figure 14). Note that this set of squares equals $\mathsf{Squares}_{\mathcal{FC}}(C_1)$.

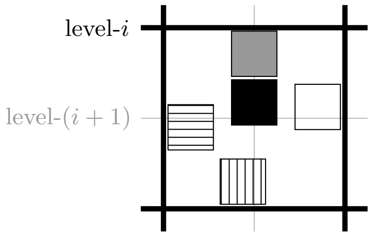


Figure 14 We color five squares of $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ using different colors as shown in the figure.

We count all conflicts involving $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ for coloring. The number of colors that cannot be used for coloring $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ based on the second condition of Definition 12 can be counted equivalently to Lemma 8, resulting in 6 colors that $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ must avoid. Since the counted set of squares contained within C_1 is equal to $\mathsf{Squares}_{\mathcal{FC}}(C_1)$, five additional colors suffice for coloring $\mathsf{Squares}_{\mathcal{FC}}(C_1)$. This gives us a valid 11-coloring of G_2. ◀

▶ **Lemma 18.** *The conflict graph G_3 is 9-colorable.*

Proof. This proof follows a similar inductive argument on the size of G_3 with an improvement on the proof of Lemma 14, as the definition of conflict in G_3 is relaxed.

The base case is trivially true, and for the inductive step, we again pick the smallest non-empty cell C_1, such that $\mathsf{Squares}_{\mathcal{FC}}(C_1) \neq \emptyset$ and for any cell C_2 contained in C_1, $\mathsf{Squares}_{\mathcal{FC}}(C_2) = \emptyset$. Let i be the level of C_1. By the induction hypothesis, $G_2 \setminus \mathsf{Squares}_{\mathcal{FC}}(C_1)$ is 11-colorable, and we show that we can color the squares of $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ while maintaining a valid 11-coloring of G_2. By induction hypothesis, $G_3 \setminus \mathsf{Squares}_{\mathcal{FC}}(C_1)$ is 9-colorable, and we show that we can color the squares of $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ while maintaining this property. The proof of Lemma 14, implies that $|\mathsf{Squares}_{\mathcal{FC}}(C_1)| \leq 5$. However, using Definition 17, there is at most one square on each side of the vertical and horizontal level-$(i+1)$ line cutting through the center of C_1. Hence, it is always possible to pair up the twin squares by giving them the same color, such that we can color the graph $G_3[\mathsf{Squares}_{\mathcal{FC}}(C_1)]$ using at most three colors as shown in Figure 15.

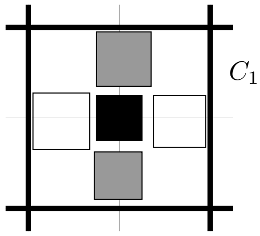


Figure 15 A 3-coloring for $G_3[\mathsf{Squares}_{\mathcal{FC}}(C_1)]$ using the same color for twin squares.

Using the bound of six (from proof of Lemma 8) on the number of squares in $G_3 \setminus \mathsf{Squares}_{\mathcal{FC}}(C_1)$ conflicting with any square of $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ leaves three out of nine used to color $G_3 \setminus \mathsf{Squares}_{\mathcal{FC}}(C_1)$ to color $\mathsf{Squares}_{\mathcal{FC}}(C_1)$. Eventually, we have a 9-coloring for G_3. ◀

▶ **Lemma 25.** *The graph G_5 is 16-colorable.*

Proof. We show a 16-coloring of G_5 by induction on the size of the graph with the property that for any cell $C \in \mathcal{L}_3$, all the squares in $\mathsf{Squares}_{\mathcal{FC}}(C)$ are colored with at most four colors.

The base case is trivial. Let us consider the smallest cell C_1 so that $\mathsf{Squares}_{\mathcal{FC}}(C_1) \neq \emptyset$, and for any cell C_2 contained in C_1 we have $\mathsf{Squares}_{\mathcal{FC}}(C_2) = \emptyset$. By induction hypothesis, $G \setminus \mathsf{Squares}_{\mathcal{FC}}(C_1)$ is 16-colorable. We show that we can color the squares of $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ while maintaining the mentioned property. There are two possible cases:

Case 1. Consider C_1 to be a level-i square-cell. Like Lemma 22, we start by counting the number of colors needed for the squares inside C_1, which are precisely $\mathsf{Squares}_{\mathcal{FC}}(C_1)$. According to Fact 24, at most seven such squares can intersect the level-$(i+1)$ vertical grid line cutting through C_1. Since there can be at most three squares above and below the level-$(i+1)$ horizontal grid line, we can pair up the twin squares by giving them the same color, such that we can color the graph $G[\mathsf{Squares}_{\mathcal{FC}}(C_1)]$ with at most four colors (see Figure 13a for illustration). Now, we consider the squares not in $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ that conflict with any square in $\mathsf{Squares}_{\mathcal{FC}}(C_1)$. Any such square must intersect the boundary of the cell C_1 (see Figure 16).

- There can be at most four squares that contain a corner of C_1.
- We now look at the squares that do not contain any corner of C_1. For the two level-$(i-1)$ grid lines, there can be only one square intersecting each line without intersecting the corner of the cell: Squares crossing a grid line of level $(i-1)$ must be of level-j for some $j \leq i-2$, and according to Fact 4, their size is strictly bigger than half of the size of a level-i cell (specifically, C_1) with $r = 4$. Thus, at most, two additional conflicting squares can intersect the level-$(i-1)$ grid lines.
- Let us consider the two other sides of C_1 formed by the level-i grid lines. There are at most three squares intersecting those sides but not the corners, since the size of those squares is strictly more than $\frac{1}{4}$ of the size of C_1. In total, there are at most six additional conflicting squares intersecting the level-i grid lines.

We have shown that the squares in $G_5 \setminus \mathsf{Squares}_{\mathcal{FC}}(C_1)$ conflicting with $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ can be colored by at most twelve colors. Considering the additional four colors needed to color $\mathsf{Squares}_{\mathcal{FC}}(C_1)$, the graph is 16-colorable.

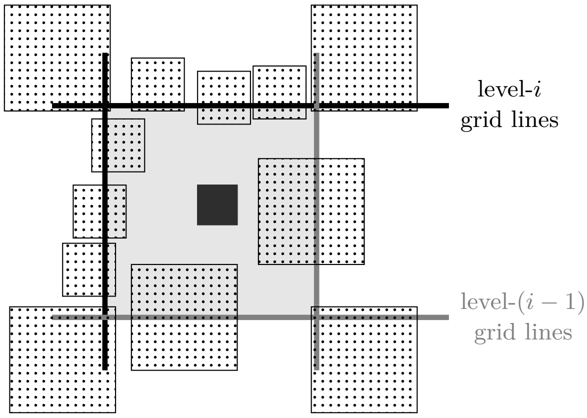


Figure 16 The black square is a square of $\mathsf{Squares}_{\mathcal{FC}}(C_1)$, where C_1 (shaded) is a level-i square-cell. The dot-filled squares are the squares of $G \backslash \mathsf{Squares}_{\mathcal{FC}}(C_1)$ that can conflict with any square in $\mathsf{Squares}_{\mathcal{FC}}(C_1)$.

Case 2. Consider C_1 to be a level-$(i, i+1)$ half-cell. The squares contained by C_1 are precisely the squares in $\mathsf{Squares}_{\mathcal{FC}}(C_1)$. According to Fact 24, at most three squares can intersect the level-$(i+1)$ horizontal grid line cutting through C_1, and they can be colored using three colors as shown in Figure 13b. Now, we bound the number of colors needed to color the squares not in $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ conflicting with any square in $\mathsf{Squares}_{\mathcal{FC}}(C_1)$. Note that those squares must intersect the boundary of C_1 (see Figure 16).

- There are at most four squares covering a corner of C_1.
- Now, consider the squares not containing any of the corners of C_1. Let C_0 be the smallest level-i square-cell containing C_1. Up to three squares can intersect the vertical level-i grid line that C_1 shares with C_0, without intersecting a corner of C_1 (Figure 17). Since the width of C_1 is half that of its height, there can be at most one square intersecting the level-i grid lines at the top and the bottom of C_1 without intersecting a corner. Hence, at most, two squares intersect the top or bottom grid lines of C_1 without intersecting its corners.

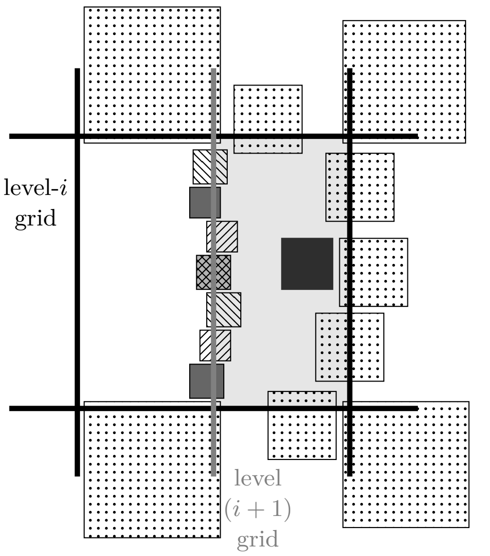
  Next, we examine the fourth side of C_1, which is formed by a level-$(i+1)$ grid line (the gray lines on Figure 17). We consider two possible scenarios depending on whether all conflicting squares intersecting the level-$(i+1)$ grid line have C_0 as their fitting cell.

  (a) If all these squares belong to $\mathsf{Squares}_{\mathcal{FC}}(C_0)$, then by the induction hypothesis, they are colored with at most four colors (see Figure 17a).

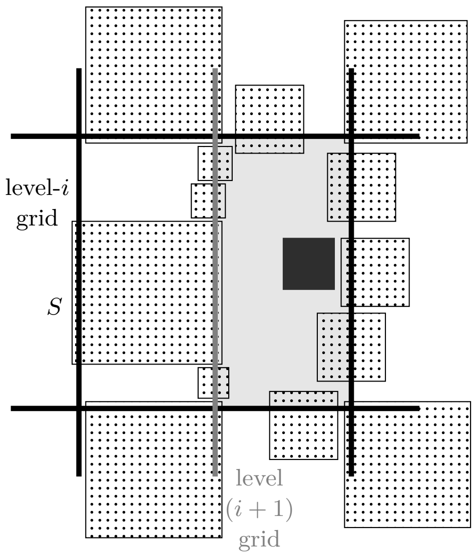
  (b) Otherwise, there exists a square S that also intersects a vertical side of C_0 (and not a corner of C_1) such that C_0 is not its fitting cell (see Figure 17b). Then, the size of S is bigger than half of $\mathsf{height}(C_1)$. Moreover, the size of the other squares crossing this side of C_1 (a level-$(i+1)$ grid line) is strictly more than $\frac{1}{8}$ of $\mathsf{height}(C_1)$. Thus, at most three more squares can fit into this space.

  In both scenarios, the squares intersecting the fourth side of C_1 can be colored with at most four colors.

We showed that all the squares in $G_5 \setminus \mathsf{Squares}_{\mathcal{FC}}(C_1)$ conflicting with any square in $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ can be colored using at most thirteen colors. Since we need at most three more colors for the squares in $\mathsf{Squares}_{\mathcal{FC}}(C_1)$ (Figure 13b), then G_5 is 16-colorable in this case as well. ◀



(a) All the squares intersecting the gray level-$(i+1)$ grid line and not a corner are in $\mathsf{Squares}_{\mathcal{FC}}(C_0)$.



(b) The square S, intersecting the level-$(i+1)$ grid line and not a corner, is not in $\mathsf{Squares}_{\mathcal{FC}}(C_0)$.

Figure 17 The black square is a square in $\mathsf{Squares}_{\mathcal{FC}}(C_1)$, and C_1 is a level-$(i, i+1)$ cell (shaded). The cell formed by the bold black lines is C_0. The remaining squares are the squares in $G_5 \setminus \mathsf{Squares}_{\mathcal{FC}}(C_1)$ that conflict with any element of $\mathsf{Squares}_{\mathcal{FC}}(C_1)$. They can all have different colors in G_5.