On Planar Straight-Line Dominance Drawings

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Abstract

We study the following question, which has been considered since the 90's: Does every *st*-planar graph admit a planar straight-line dominance drawing? We show concrete evidence for the difficulty of this question, by proving that, unlike upward planar straight-line drawings, planar straight-line dominance drawings with prescribed y-coordinates do not always exist and planar straight-line dominance drawings cannot always be constructed via a contract-draw-expand inductive approach. We also show several classes of *st*-planar graphs that always admit a planar straight-line dominance drawing. These include *st*-planar 3-trees in which every stacking operation introduces two edges incoming into the new vertex, *st*-planar graphs in which every vertex is adjacent to the sink, and *st*-planar graphs in which no face has the left boundary that is a single edge.

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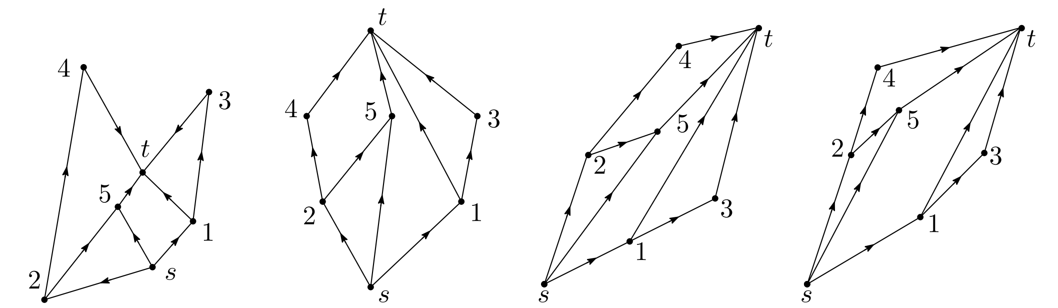
1 Introduction

Drawings of directed graphs are an evergreen research topic in the graph drawing literature. Early papers on the subject go back to the 80's [13, 14, 34] and the number of papers on the topic published since 2023 is in double digits [1, 2, 3, 4, 6, 7, 9, 10, 11, 12, 16, 20, 22, 23, 24, 25, 28, 30]. From an applicative perspective, many domains require techniques for visualizing directed graphs, such as visualization tools for biological networks and SIEM systems for cyber threat intelligence. Many standards for drawing directed graphs have been defined, and in most of them the drawing is *upward*, i.e., each edge is represented by a Jordan arc whose y-coordinates monotonically increase from the tail to the head of the edge. Di Battista and Tamassia [14] proved that every *upward planar graph* (that is, a directed graph that admits an upward planar drawing) admits an upward planar *straight-line* drawing, a result analogous to Fary's celebrated result about the geometric realizability of planar graphs [19]. In order to prove the geometric realizability of upward planar graphs, it suffices to look at upward planar graphs whose faces are delimited by 3-cycles. Indeed, every upward planar graph is a subgraph of an *st-planar graph* [14] (that is, an upward planar graph with a single source s and a single sink t), which in turn is a subgraph of a *maximal st-planar graph* [14] (that is, an st-planar graph to which no edge can be added without losing simplicity or upward planarity).

One of the easiest algorithms, if not *the* easiest algorithm, for constructing upward planar straight-line drawings is due to Di Battista, Tamassia and Tollis [15]. This algorithm assigns x- and y-coordinates to the vertices simply by performing two pre-order traversals of the input st-planar graph. Moreover, the algorithm constructs upward planar straight-line drawings that are actually *dominance drawings*. These are *xy-monotone drawings* (that is, each edge is represented by a Jordan arc whose x- and y-coordinates monotonically increase from the tail to the head of the edge) such that, for any pair of vertices u, v, there exists a directed path from u to v in the graph if and only if $x(u) \leq x(v)$ and $y(u) \leq y(v)$ hold in the drawing. Dominance drawings constitute an interesting graph drawing style because they express the reachability between vertices by their dominance relationship, i.e., by the coordinates assigned to them; this allows one to answer reachability queries in constant time, see, e.g, [27, 35]. For more about dominance drawings, see, e.g., [5, 8, 18, 26, 31, 32, 34]. Figure 1 shows planar straight-line drawings of an st-planar graph that are non-upward, upward (and not xy-monotone), xy-monotone (and not dominance), and dominance.



(a) Non-upward. **(b)** Upward. **(c)** xy-monotone. **(d)** Dominance.

Figure 1 Four planar straight-line drawings of an st-planar graph G. (a) A non-upward drawing. (b) An upward drawing. (c) An xy-monotone drawing. (d) A dominance drawing.

Di Battista, Tamassia and Tollis's algorithm [15] does not actually construct an upward planar straight-line drawing of every st-planar graph. Indeed, it may construct a non-planar drawing if the input st-planar graph contains *transitive* edges, where an edge is transitive if the graph contains a directed path from the tail to the head of the edge. By subdividing each transitive edge with a new vertex, their algorithm constructs a planar dominance drawing of any st-planar graph in which each edge is either a straight-line segment (if it is non-transitive) or a 1-bend polyline (if it is transitive). Whether this bend per edge can be eliminated by designing an algorithm different from the one in [15] is the question we study in this paper.

Formally, we ask: Does every *st*-planar graph admit a planar straight-line dominance drawing? Apart from the *st*-planar graphs without transitive edges, the question is known to have a positive answer for *series-parallel digraphs* [8]. We prove the following results.

- In Section 3, we prove a remarkable difference between dominance and upward drawings. We revisit the two main approaches for the construction of upward planar straight-line drawings of *st*-planar graphs and prove that they cannot be successfully applied to construct planar straight-line dominance drawings. The first approach [14] contracts an internal edge of the graph, constructs a drawing inductively, and then expands the previously contracted edge to be a "short" segment. We show that there exist *st*-planar graphs in which no edge can be used in the contract-draw-expand approach so to get a planar straight-line dominance drawing. The second method [17, 21, 33] prescribes the y-coordinates of the vertices, so that the tail of any edge is assigned a smaller y-coordinate than its head. This additional constraint on the drawing allows for easier recursive schemes for its construction. We prove that planar straight-line dominance drawings with prescribed y-coordinates do not always exist. We believe that these results provide solid evidence for the difficulty of constructing planar straight-line dominance drawings.
- In Section 4, we study *st*-planar graphs whose underlying graph is a planar 3-tree. Planar 3-trees, also known as stacked triangulations and Apollonian networks, constitute a common benchmark for planar graph drawing problems, as they allow for easy inductive constructions; for example, every planar 3-tree with at least four vertices can be constructed by "stacking" a vertex inside a face of a smaller planar 3-tree. For our question, the study of *st*-planar 3-trees turns out to be complicated, as inductive drawing constructions do not cope well with the dominance relationship that needs to be ensured between vertices that are "far away" in the graph. We show how to construct planar straight-line dominance drawings for two classes of *st*-planar 3-trees, the first one with a constraint on the orientation (every stacking operation introduces two edges incoming into the new vertex) and the second one with a constraint on the graph structure (every stacking operation happens in a face incident to the sink). The latter graph class coincides with the maximal *st*-planar graphs in which the sink is adjacent to every vertex.
- In Section 5, we improve the mentioned result by Di Battista, Tamassia and Tollis [15], by proving that a planar straight-line dominance drawing always exists for any *st*-planar graph in which every transitive edge is to the right of every directed path from the tail to the head of the edge. This result is obtained via an ear decomposition of the graph. This shows that the problem of constructing planar straight-line dominance drawings is made difficult by the interaction between "left transitive edges" and "right transitive edges".

All our algorithms construct drawings whose resolution is exponentially small (or worse). This drawback is sometimes necessary for upward planar straight-line drawings [15], and hence also for planar straight-line dominance drawings. However, for the graph classes we considered, we do not know whether an exponentially-small resolution is actually required in order to construct planar straight-line dominance drawings.

2 Preliminaries

A *drawing* of a graph maps each vertex to a distinct point in the plane and each edge to a Jordan arc between its endpoints. A drawing is *straight-line* if each edge is represented by a straight-line segment and *planar* if no two edges intersect, except at common endpoints. Two planar drawings of a connected graph are *plane-equivalent* if they define the same clockwise order of the edges incident to each vertex and the same clockwise order of the vertices and edges along the boundary of the outer face. A *plane embedding* is an equivalence class of planar drawings and a *plane graph* is a graph with a plane embedding. Whenever we talk about planar drawings of a plane graph, we always assume that they are in the equivalence class associated with the plane graph. An *st-plane graph* is an *st*-planar graph with a plane embedding (for its underlying graph) in which s and t are incident to the outer face. An *st*-plane graph is *maximal* if no edge can be added to it while maintaining it an *st*-plane graph. Since every *st*-planar graph can be augmented (by adding vertices and edges) to maximal without altering the reachability between vertices [14], the existence of a planar straight-line dominance drawing for all *st*-planar graphs can be decided by only looking at maximal *st*-planar graphs. Note that, in any planar straight-line dominance drawing of an *st*-planar graph, the vertex placements can be perturbed so that no two vertices share the same x- or the same y-coordinate and so that the drawing remains planar, straight-line and dominance. Hence, throughout the paper, every considered dominance drawing has distinct x- and distinct y-coordinates for its vertices. Two vertices in a directed graph are *incomparable* if no directed path goes from any of the vertices to the other one.

As a warm-up result, we prove that every Hamiltonian *st*-planar graph has a planar straight-line dominance drawing. A directed graph is *Hamiltonian* if it contains a directed path $(v_1 = s, v_2, \dots, v_n = t)$, where $\{v_1, v_2, \dots, v_n\}$ is the vertex set of the graph.

▶ **Theorem 1.** *Hamiltonian st-planar graphs admit planar straight-line dominance drawings.*

Proof. Consider a Hamiltonian *st*-planar graph G. Construct an upward planar straight-line drawing Γ of G; this always exists [14]. Stretch Γ vertically, so that the slope of every edge is in the range $(45°, 135°)$. Now rotate Γ in clockwise direction by $45°$. Since the slope of every edge is now in the range $(0°, 90°)$, we have that Γ is xy-monotone. Since vertical stretch and rotation are affine transformations, Γ is planar, as well. Finally, since G contains a Hamiltonian path $(v_1, \dots, v_n)$, vertex v_j is reachable from vertex v_i, for every $1 \leq i < j \leq n$. Since the slope of the edge (v_k, v_{k+1}) is in the range $(0°, 90°)$, for $k = i, \dots, j-1$, vertex v_j is in the first quadrant of vertex v_i, hence Γ is a dominance drawing. ◀

3 Planar Straight-line Dominance Drawings are Difficult to Get

In this section, we revisit the two main approaches for the construction of upward planar straight-line drawings of *st*-planar graphs and prove that they cannot be enhanced (or at least not in a direct way) to construct planar straight-line dominance drawings.

3.1 Constructing Drawings via Contractions and Expansions

Di Battista and Tamassia [14] first proved that every *st*-plane graph admits an upward planar straight-line drawing. Their proof extends to directed graphs a well-known proof by Fáry [19], showing that every (undirected) plane graph admits a planar straight-line drawing. We briefly describe the algorithm by Di Battista and Tamassia [14].

An internal edge (u, v) of a maximal *st*-plane graph G is *contractible* if it satisfies the following conditions: (1) The vertices u and v have exactly two common neighbors, denoted by z_1 and z_2; note that the cycles (u, v, z_1) and (u, v, z_2) delimit internal faces of G. (2) For $i = 1, 2$, the edges connecting u and v with z_i are both incoming or both outgoing at z_i.

The *contraction* of a contractible edge (u, v) constructs a graph G' by identifying u and v into a vertex w with the following adjacencies (see Fig 2a). For every neighbor $z \notin \{u, v, z_1, z_2\}$ of u (of v), we have that G' contains an edge between w and z, which is outgoing at z if and only if the edge between u and z (resp. between v and z) is outgoing at z. Also, for $i = 1, 2$, we have that G' contains an edge between w and z_i, which is outgoing at z_i if and only if the edges connecting u and v with z_i are both outgoing at z_i. It is easy to see that G' is a maximal st-plane graph.

The core of Di Battista and Tamassia's algorithm lies in the following two statements[1]: (i) every maximal st-plane graph G has a contractible edge (u, v), whose contraction results in a maximal st-plane graph G'; and (ii) an upward planar straight-line drawing Γ of G can be obtained from an upward planar straight-line drawing Γ' of G' by *expanding* w, that is, by replacing w with a sufficiently small segment (with a suitable slope) representing (u, v).

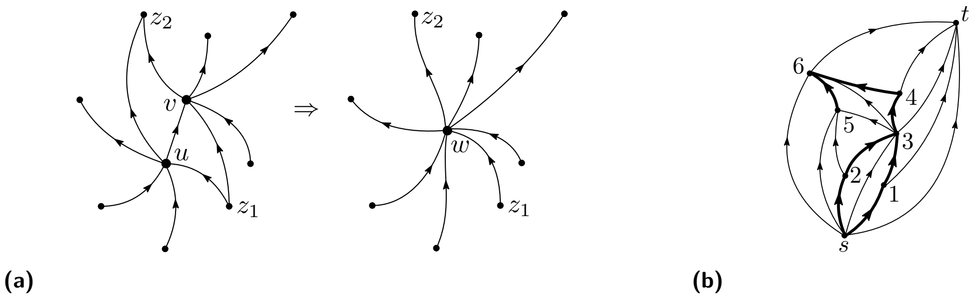


Figure 2 (a) The contraction of an edge (u, v) in a maximal st-plane graph. (b) A maximal st-plane graph with no dominance-expandable edge. Thin edges are not contractible, while fat edges are contractible but not dominance-expandable; for example, $(1, 3)$ is not dominance-expandable, because vertex 2 is a predecessor of vertex 3 but not a predecessor of vertex 1.

Since, depending on the geometric placement of the neighbors of w in Γ', the edge (u, v) might need to be an arbitrarily small segment in Γ, in order for Γ to be a dominance drawing we need u and v to have the same successors and predecessors. That is, let $\mathcal{S}(z)$ be the set of successors of a vertex z, that is, the set of all vertices z' such that there exists a directed path from z to z'. Analogously, let $\mathcal{P}(z)$ be the sets of predecessors of a vertex z. A contractible edge (u, v) is *dominance-expandable* if $\mathcal{S}(u) = \mathcal{S}(v) \cup \{v\}$ and $\mathcal{P}(v) = \mathcal{P}(u) \cup \{u\}$. Di Battista and Tamassia's approach could be enhanced to construct planar straight-line dominance drawings if every maximal st-plane graph contained a dominance-expandable edge. However, we can prove that there exist maximal st-plane graphs with no dominance-expandable edge, as the one in Fig 2b, which constitutes a barrier for this approach we cannot overcome.

We remark that, for every graph class for which we can prove the existence of planar straight-line dominance drawings in the upcoming sections, there exist graphs in the class that do not have a dominance-expandable edge or such that the contraction of any dominance-expandable edge would result in a graph not in the same class.

3.2 Constructing Drawings by Prescribing the y-Coordinates

Eades, Feg, Lin, and Nagamochi [17] and, independently, Pach and Tóth [33] proved that every upward planar drawing can be straightened while preserving the y-coordinates of the vertices. This implies that every st-plane graph admits an upward planar straight-line drawing with prescribed y-coordinates (as long as these respect the reachability between

[1] Di Battista and Tamassia's proof actually distinguishes the case in which G contains a *separating triangle* (a 3-cycle with vertices in its interior) from the case in which it does not, performing different constructions in the two cases. However, the first case is unnecessary, as a contractible edge in an st-planar graph can always be found, similarly to what was noted by Wood [36] for undirected graphs.

vertices). This result was strengthened by Hong and Nagamochi [21], who proved that every internally-triconnected st-plane graph admits an upward planar straight-line *convex* drawing with prescribed y-coordinates and prescribed outer face. It is interesting that, while more constrained, drawings with prescribed y-coordinates (and a prescribed outer face) allow for an easier recursive construction.

We now show that, unlike upward planar straight-line drawings, planar straight-line dominance drawings with given y-coordinates do not always exist.

▶ **Theorem 2.** *For every $n \geq 7$, there exists an st-planar graph G_n with vertex set $\{v_1, v_2, \ldots, v_n\}$ such that:*

- *there exists a planar dominance drawing of G_n such that $y(v_i) = i$, for $i = 1, \ldots, n$; and*
- *there exists no planar straight-line dominance drawing of G_n such that $y(v_i) = i$, for $i = 1, \ldots, n$.*

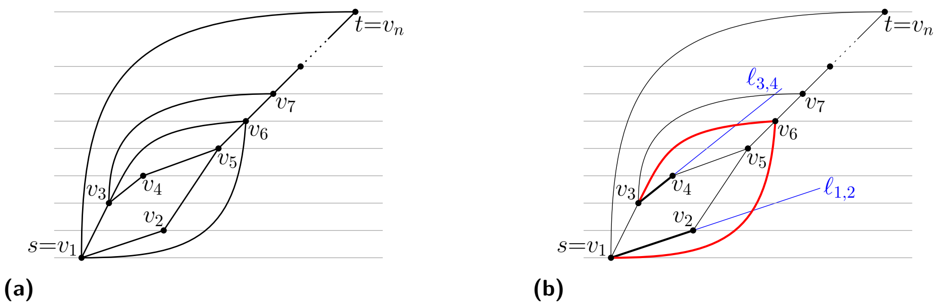


Figure 3 (a) The graph for the proof of Theorem 2. (b) The rays $\ell_{1,2}$ and $\ell_{3,4}$ diverge.

Proof. The st-planar graph G_n consists of the directed paths $(s = v_1, v_2, v_5)$, (v_1, v_3, v_4, v_5), $(v_5, v_6, \ldots, v_n = t)$, and of the edges (v_1, v_6), (v_3, v_6), (v_3, v_7), and (v_1, v_n). Fig 3a shows a planar dominance drawing of G_n with $y(v_i) = i$, for $i = 1, \ldots, n$. Suppose, for a contradiction, that a planar straight-line dominance drawing Γ of G_n exists with $y(v_i) = i$, for $i = 1, \ldots, n$. We prove that the plane embedding in Γ of the underlying graph of G_n is the one in Fig 3a, except, possibly, for the position of the edge (s, t). Obviously, the path $(v_1, v_2, v_5, v_6, \ldots, v_n)$ has a unique plane embedding. Since v_2 and v_4 are incomparable and $y(v_2) < y(v_4)$, we have $x(v_4) \leq x(v_2)$, hence the clockwise order of the vertices along the cycle $\mathcal{C} := (v_1, v_3, v_4, v_5, v_2)$ is v_1, v_3, v_4, v_5, v_2. From that, we get that the edges (v_3, v_6) and (v_3, v_7) lie above the path $(v_3, v_4, v_5, v_6, v_7)$, and finally that the edge (v_1, v_6) lies below the path (v_1, v_2, v_5, v_6). For any distinct $i, j \in \{1, \ldots, n\}$, let $\ell_{i,j}$ be the ray starting at v_i and passing through v_j. Since $x(v_1) < x(v_3) < x(v_4) \leq x(v_2)$, we have $x(v_2) - x(v_1) > x(v_4) - x(v_3)$. Also, we have $y(v_2) - y(v_1) = y(v_4) - y(v_3) = 1$. Hence, the ray $\ell_{1,2}$ has smaller slope than the ray $\ell_{3,4}$; that is, such rays diverge, see Fig 3b. Since the ray $\ell_{1,6}$ has smaller slope than $\ell_{1,2}$, and since the ray $\ell_{3,6}$ has larger slope than $\ell_{3,4}$, it follows that $\ell_{1,6}$ and $\ell_{3,6}$ also diverge, while they meet at v_6, a contradiction which proves the theorem. Note that vertices $v_8, \ldots, v_n$ only serve the purpose of creating an infinite graph family. ◀

We can similarly show that one cannot, in general, prescribe the x-coordinates of a planar straight-line dominance drawing.

Also, we can strengthen Theorem 2 by proving that, for every $n \geq 10$ and for every sequence $y_1 < \cdots < y_n$ of y-coordinates, there exists an st-planar graph G'_n with vertex set $\{v_1, \ldots, v_n\}$ such that there exists a planar dominance drawing of G'_n with $y(v_i) = y_i$, for $i = 1, \dots, n$, and there exists no planar straight-line dominance drawing of G'_n with $y(v_i) = y_i$, for $i = 1, \dots, n$. That is, the y-coordinates prescribed by Theorem 2 do not need to be uniformly distributed.

The key point for this is the observation that the proof of Theorem 2 works as long as $y(v_2) - y(v_1) \leq y(v_4) - y(v_3)$. Hence, we can consider the four lines $y = y_i$, with $i = 4, 5, 6, 7$, and then distinguish two cases. If $y_5 - y_4 \leq y_7 - y_6$, we let our st-planar graph G'_n contain the graph G_7 from the proof of Theorem 2 and we set $y(v_i) = y_{i+3}$, for $i = 1, \dots, 7$, where $v_1, \dots, v_7$ is the vertex set of G_7. Otherwise, that is, if $y_7 - y_6 < y_5 - y_4$, we let our st-planar graph G'_n contain the graph obtained by reversing the edge directions of the graph G_7 and we set $y(v_i) = y_{8-i}$, for $i = 1, \dots, 7$, where $v_1, \dots, v_7$ is the vertex set of G_7.

4 st-plane 3-trees

A *plane 3-tree* is a plane graph recursively defined as follows. A 3-cycle embedded in the plane is a plane 3-tree. Any plane 3-tree with $n \geq 4$ vertices can be obtained from a plane 3-tree with $n-1$ vertices by *stacking* a new vertex into an internal face, that is, by connecting the new vertex to the three vertices incident to the face. An *st-plane 3-tree* is an st-plane graph whose underlying graph is a plane 3-tree. In our opinion, st-plane 3-trees constitute a very challenging class of st-plane graphs for our problem. Indeed, the "natural" strategies for drawing the graphs in this class are to either recursively construct and then combine the drawings of three smaller st-plane 3-trees, or to iteratively add a single vertex to a previously constructed drawing of a smaller st-plane 3-tree; both these strategies do not cope well with the geometric relationship that has to be ensured for incomparable vertices. Nevertheless, in this section we show how to obtain planar straight-line dominance drawings of two classes of st-plane 3-trees.

4.1 Upper st-plane 3-trees

Consider the construction of an st-plane 3-tree G via repeated stacking operations. If a vertex u is stacked into a face delimited by a cycle (a, b, c), where a and c are the source and the sink of the cycle, respectively, then the edge (a, u) is directed from a to u, the edge (u, c) is directed from u to c, while the edge (b, u) might be directed either way. We say that G is an *upper st-plane 3-tree* if, at every stacking operation, the edge that can be directed either way is always directed towards the newly inserted vertex. We have the following.

▶ **Theorem 3.** *Upper st-plane 3-trees admit planar straight-line dominance drawings.*

Proof. Let G be an n-vertex upper st-plane 3-tree whose outer face is delimited by the cycle (s, m, t). Let Δ be any triangle with vertices p_s, p_m, p_t, where $x(p_s) < x(p_m) < x(p_t)$ and $y(p_s) < y(p_m) < y(p_t)$. Also, let D be a closed disk in the interior of Δ such that, for any point p in D, we have $x(p_m) < x(p) < x(p_t)$ and $y(p_m) < y(p) < y(p_t)$; see Figs 4a and 4c. We prove by induction that G admits a planar straight-line dominance drawing such that:

- s lies at p_s, m lies at p_m, and t lies at p_t; and
- every internal vertex of G lies in the interior of D.

The statement clearly implies the theorem. In the base case, in which $n = 3$, the triangle Δ is the required drawing of G and the statement is trivially true.

Suppose now that $n > 3$. Let r be the first stacked vertex in the construction of G; that is, r is the unique vertex of G adjacent to s, m, and t. Note that the edge (m, r) is directed away from m, given that G is an upper st-plane 3-tree. Let G_1, G_2, and G_3 be the subgraphs

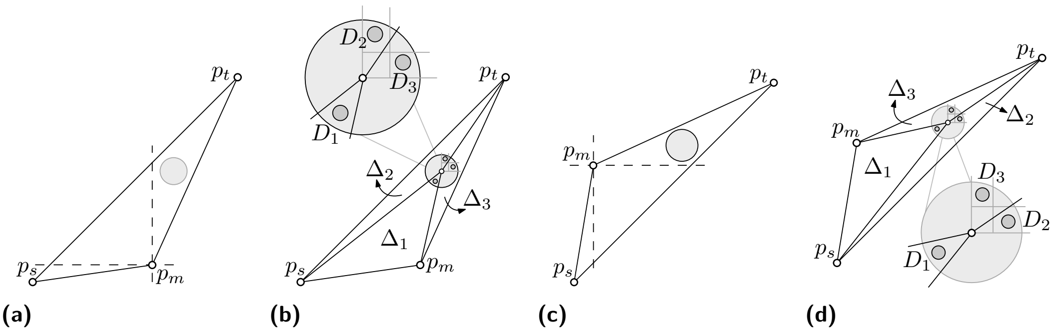


Figure 4 (a) and (c) Triangle Δ and disk D for the input to the induction. (b) and (d) Placing point p_r (white) and disks D_1, D_2, and D_3 (gray) inside D; an enlarged view of the placement of r and of disks D_1, D_2, and D_3 inside D is also shown.

of G inside the cycles (s, m, r), (s, r, t), and (m, r, t), respectively. Note that G_1 is an upper sr-plane 3-tree, G_2 is an upper st-plane 3-tree, and G_3 is an upper mt-plane 3-tree. Also, each of G_1, G_2, and G_3 has less than n vertices. Let p_r be any point inside D and let Δ_1, Δ_2, and Δ_3 be the triangles (p_s, p_m, p_r), (p_s, p_r, p_t), and (p_m, p_r, p_t), respectively. Place r at p_r; by the properties of D, we have $x(p_m) < x(p_r)$ and $y(p_m) < y(p_r)$, which complies with the orientation of (m, r). Let D_1, D_2, and D_3 be closed disks such that (see Figs 4b and 4d):

- disk D_1 lies in the interior of $\Delta_1 \cap D$, disk D_2 lies in the interior of $\Delta_2 \cap D$, and disk D_3 lies in the interior of $\Delta_3 \cap D$;
- for any point $p \in D_2 \cup D_3$, we have $x(p_r) < x(p)$ and $y(p_r) < y(p)$; and
- for any point $p_2 \in D_2$ and any point $p_3 \in D_3$, if the clockwise order of the vertices of Δ is p_s, p_t, p_m, then we have $x(p_2) < x(p_3)$ and $y(p_3) < y(p_2)$, otherwise we have $y(p_2) < y(p_3)$ and $x(p_3) < x(p_2)$.

Clearly, disks D_1, D_2, and D_3 with the above properties always exist. By induction, G_1, G_2, and G_3 have planar straight-line dominance drawings Γ_1, Γ_2, and Γ_3 with s, m, r, and t drawn at p_s, p_m, p_r, and p_t, respectively, so that the internal vertices of G_1, G_2, and G_3 lie in the interior of D_1, D_2, and D_3, respectively. This results in a straight-line drawing Γ of G.

Since p_r, D_1, D_2, and D_3 lie in the interior of D, all the internal vertices of G lie in the interior of D, as required. The upward planarity of Γ follows from the ones of Γ_1, Γ_2, and Γ_3. In order to prove that Γ is a dominance drawing, consider any pair of vertices u and v.

- If u and v belong to the same graph G_i, for some $i \in \{1, 2, 3\}$, then their placement complies with their dominance relationship, by induction.
- If one of u and v is s, say $u = s$, then u is a predecessor of v, and indeed we have $x(u) < x(v)$ and $y(u) < y(v)$. The case $u = t$ can be discussed similarly.
- If neither of u and v is s or t, and one of u and v is m, say $u = m$, then u is a predecessor of v, since G is an upper st-plane 3-tree. Since $x(p_m) < x(p)$ and $y(p_m) < y(p)$, for any point $p \in D$, we have $x(u) < x(v)$ and $y(u) < y(v)$.
- If u is an internal vertex of G_1 and v is an internal vertex of G_2 or G_3, then u is a predecessor of v, since G is an upper st-plane 3-tree. Since, for any point $p \in D_1$ and any point $q \in D_2 \cup D_3$, we have $x(p) < x(p_r) < x(q)$ and $y(p) < x(y_r) < y(q)$, the placement of u and v complies with their dominance relationship.
- Finally, if u is an internal vertex of G_2 and v is an internal vertex of G_3, then u and v are incomparable, since G is an upper st-plane 3-tree. Since, for any point $p_2 \in D_2$ and any point $p_3 \in D_3$, we have $x(p_2) < x(p_3)$ and $y(p_3) < y(p_2)$, or $y(p_2) < y(p_3)$ and $x(p_3) < x(p_2)$, the placement of u and v complies with their dominance relationship.

This completes the induction and the proof of the theorem. ◀

An analogous result holds true for st-plane 3-trees such that, at every stacking operation, the edge that can be directed either way is always directed out of the newly inserted vertex.

Trying to use a similar strategy in order to construct a planar straight-line dominance drawing of every st-plane 3-tree might be tempting. However, the “types” of internal vertices in a general st-plane 3-tree are more than three. Namely, referring to the notation introduced in the proof of the theorem, the internal vertices of G_2 and G_3 are not all successors of r, but rather some are predecessors, some are successors, and some are incomparable to r. Hence, the “three-disks schema” fails, and more complex geometric invariants seem to be needed.

4.2 Sink-dominant st-plane 3-trees

We next look at the st-plane 3-trees in which every stacking operation happens in a face incident to the sink t of the graph. This results in an st-plane 3-tree in which the sink is adjacent to every vertex. We call this a *sink-dominant st-plane 3-tree*. It is easy to observe that every n-vertex maximal st-plane graph in which the sink has degree $n-1$ is a sink-dominant st-plane 3-tree (and vice versa). We have the following.

▶ **Theorem 4.** *Sink-dominant st-plane 3-trees admit planar straight-line dominance drawings.*

Proof. Let G be an n-vertex sink-dominant st-plane 3-tree whose outer face is delimited by the cycle (s, m, t).

Assumption. If $n > 3$, then let r be the internal vertex of G adjacent to s, m and t. If r is a predecessor of m, as in Fig 5a, we add a new source s' adjacent to s, m and t in the outer face of G, so that the outer face of the resulting graph G' is delimited by the 3-cycle (s', s, t). Now m is the internal vertex of G' adjacent to s', s and t; furthermore, m is a successor of s. Hence, by possibly adding a vertex and three edges to G and changing some labels, we can assume w.l.o.g. that the internal vertex r that is adjacent to the three vertices s, m and t incident to the outer face of our input st-plane 3-tree G is a successor of m.

Inductive hypothesis. The proof is similar in spirit to, however more involved than, the proof of Theorem 3. Let Δ be any triangle with vertices p_s, p_m and p_t, where $x(p_s) < x(p_m) < x(p_t)$ and $y(p_s) < y(p_m) < y(p_t)$. If p_m lies above the line through p_s and p_t, we say that Δ is of *type A* (see Fig 5b), otherwise it is of *type B* (see Fig 5c). Let D and E be closed disks contained in the interior of Δ such that:

- if Δ is of type A, then D and E are horizontally aligned, that is, they have the same two vertical tangents, while if Δ is of type B, then D and E are vertically aligned;
- if Δ is of type A, then D is strictly below and to the right of p_m, while if Δ is of type B, then D is strictly above and to the left of p_m; and
- E is strictly above and to the right of p_m.

We prove, by induction on n, that G admits a planar straight-line dominance drawing such that:

- s lies at p_s, m lies at p_m, and t lies at p_t;
- every internal vertex of G that is a successor of m lies in the interior of E; and
- every internal vertex of G that is incomparable to m lies in the interior of D.

Note that, because of the assumption that r is a successor of m, no internal vertex of G is a predecessor of m.

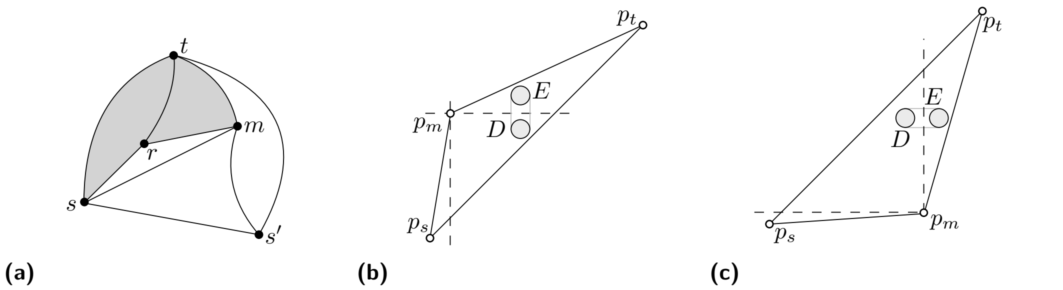


Figure 5 (a) Augmenting G so that the vertex adjacent to the three vertices on the outer face is a successor of two of them. (b) and (c) Triangle Δ and disks D and E for the input to the induction. In (b) Δ is of type A, while in (c) it is of type B.

The statement clearly implies the theorem. In the base case, in which $n = 3$, the triangle Δ is the required drawing of G and the statement is true. Suppose now that $n > 3$. We only show the construction for the case in which Δ is of Type B, as the other case is analogous.

Graph structure. Recall that r is the unique vertex of G adjacent to s, m and t, and that the edge (m, r) is directed towards r; refer to Fig 6.

Let $P_m := (v_0 = m, v_1, \dots, v_\ell = r)$ be the longest directed path from m to r. Since every vertex is adjacent to t and since r is a successor of m, we have that P_m exists and is unique. For $j = 1, \dots, \ell$, let M_j be the subgraph of G induced by the vertices inside or on the boundary of the 3-cycle (v_{j-1}, v_j, t) and note that M_j is a sink-dominant st-plane 3-tree. By the fact that P_m is the longest directed path from m to r, we have that, if M_j contains internal vertices, then the internal vertex of M_j that is adjacent to v_{j-1}, v_j and t is a successor of v_j. This implies that M_j can be drawn recursively.

Also, let $P_s := (u_0 = s, u_1, \dots, u_k = r)$ be the longest directed path from s to r in G that does not pass through m. For $i = 1, \dots, k$, the subgraph S_i of G induced by the vertices inside or on the boundary of the 3-cycle (u_{i-1}, u_i, t) is a sink-dominant st-plane 3-tree. Further, if S_i contains internal vertices, then the internal vertex of S_i that is adjacent to u_{i-1}, u_i and t is a successor of u_i, hence S_i can be drawn recursively.

Since every vertex of G is adjacent to t, the interior of the cycle $\mathcal{C}_{sm} := P_s \cup P_m \cup (s, m)$ does not contain any vertices, while it might contain some edges (and in fact it does, unless $\mathcal{C}_{sm} = (s, m, r)$). We are going to draw $\mathcal{C}_{sm}$ as a convex curve (hence the edges in its interior will not cause crossings).

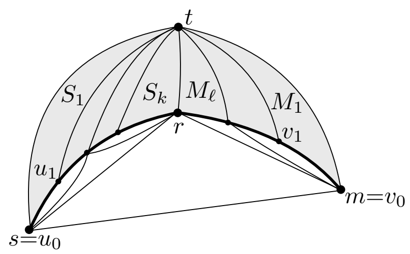


Figure 6 Paths P_m and P_s (as thick lines) and graphs $M_1, \dots, M_\ell, S_1, \dots, S_k$ (with gray interior).

Construction. We now draw P_s and P_m. We also draw disks inside the triangles representing the cycles (u_{i-1}, u_i, t), for $i = 1, \dots, k$, and (v_{j-1}, v_j, t), for $j = 1, \dots, \ell$, so that induction can be applied in order to draw the subgraphs S_i and M_j recursively. Refer to Fig 7 for an illustration of the relative placement of the vertices of P_s and P_m and of the desired disks.

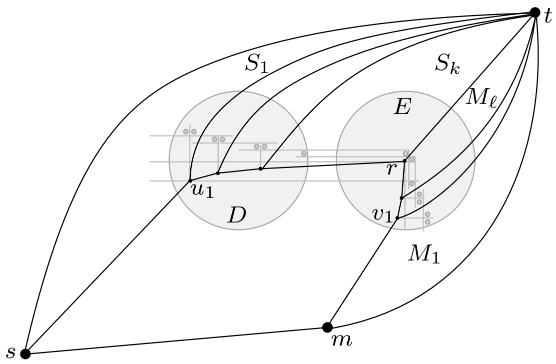


Figure 7 Drawing paths P_s and P_m, disks $D_1^u, E_1^u, \dots, D_k^u, E_k^u$, and disks $D_1^v, E_1^v, \dots, D_\ell^v, E_\ell^v$. For the sake of readability, some edges are drawn as curves, as the illustration is mainly meant to represent the relative placement of the vertices of the paths P_s and P_m and of the listed disks.

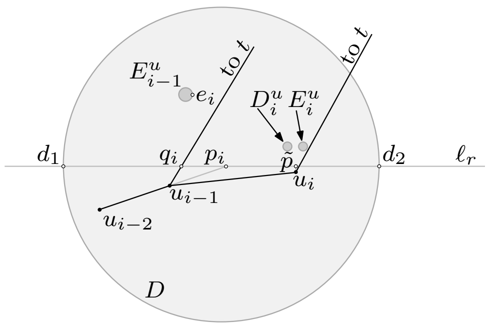


Figure 8 Drawing vertex u_i.

We start by placing r at the center of the disk E. Next, we draw the vertices $u_1, \dots, u_{k-1}$ in this order inside D. Let σ_r be the intersection of the horizontal line ℓ_r through r with D, and let d_1 and d_2 be the leftmost and rightmost endpoints of σ_r, respectively. For $i = 1, \dots, k-1$, by drawing u_i, we complete the drawing of the triangle Δ_i^u representing cycle (u_{i-1}, u_i, t). Then we also place suitable disks D_i^u and E_i^u inside Δ_i^u so that S_i can be drawn recursively.

When we have to draw u_i, for some $i \in \{1, \dots, k-1\}$, we assume that (see Fig 8):

- (C1) the polygonal line $(u_0, \dots, u_{i-1}, r)$ is convex and lies below ℓ_r;
- (C2) if $i > 1$, the line through u_{i-2} and u_{i-1} cuts σ_r in its interior, at a point p_i;
- (C3) if $i > 1$, the segment between u_{i-1} and t cuts σ_r in its interior, at a point q_i; and
- (C4) if $i > 1$, the disks D_{i-1}^u and E_{i-1}^u lie inside D and above ℓ_r.

We denote by e_i be the rightmost point of E_{i-1}^u. Note that conditions (C1)–(C4) are vacuous if $i = 1$ (i.e., before drawing u_1). In that case, for the sake of simplicity of the description, we let p_i, q_i, and e_i coincide with d_1.

We now explain how to draw u_i. Let $\overline{x} = \max\{x(p_i), x(q_i), x(e_i)\}$, let $\tilde{x} = (x(d_2) + \overline{x})/2$, where $\overline{x} < \tilde{x} < x(d_2)$, and let $\tilde{p}$ be the point of σ_r with $x(\tilde{p}) = \tilde{x}$. We place u_i at $(\tilde{x}, y(r) - \epsilon)$, where $\epsilon > 0$ is sufficiently small so that conditions (C1)–(C3) are satisfied when we have to draw u_{i+1}. Indeed, if $\epsilon = 0$, then u_i would be placed at $\tilde{p}$ and conditions (C1)–(C3) would be trivially satisfied when we have to draw u_{i+1}, hence they are also satisfied for some sufficiently small $\epsilon > 0$, by continuity.

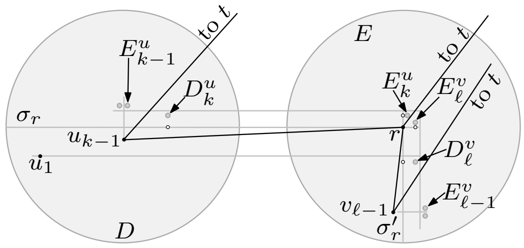


Figure 9 Illustration for the placement of the disks D^u_k, E^u_k, D^v_ℓ, and E^v_ℓ. White circles represent initial or intermediate placements for such disks.

We now place the disks D^u_i and E^u_i so that they have radius δ and centers at $(\tilde{x} \pm \epsilon', y(r) + \epsilon')$, where $\epsilon' > \delta > 0$ are sufficiently small so that:

- D^u_i and E^u_i lie inside the triangle $\Delta^u_i = (u_{i-1}, u_i, t)$;
- D^u_i and E^u_i are lower than D^u_{i-1} and E^u_{i-1}; and
- condition (C4) is satisfied when we have to draw u_{i+1}.

Indeed, if $\epsilon' = \delta = 0$ such disks would degenerate and coincide with $\tilde{p}$, which is inside D and also inside Δ^u_i, as the segment $\overline{u_{i-1}t}$ cuts σ_r at a point q_i to the left of $\tilde{p}$ and the segment $\overline{u_i t}$ cuts σ_r at a point to the right of $\tilde{p}$. Hence, such disks remain inside D and Δ^u_i if $\epsilon' > \delta > 0$ is sufficiently small, by continuity. Since $\epsilon' > \delta$, disks D^u_i and E^u_i lie above ℓ_r, hence ensuring condition (C4). Finally, choosing $\delta + \epsilon'$ smaller than the distance between E^u_{i-1} and ℓ_r ensures that D^u_i and E^u_i are lower than D^u_{i-1} and E^u_{i-1}.

We now draw the vertices $v_1, \dots, v_{\ell-1}$ in this order. For $j = 1, \dots, \ell - 1$, when we draw v_j, we have drawn the triangle Δ^v_j representing cycle (v_{j-1}, v_j, t). Then we also show how to place suitable disks D^v_j and E^v_j inside Δ^v_j so that M_j can be drawn recursively. This is done very similarly to the way vertices $u_1, \dots, u_{k-1}$ and disks $D^u_1, E^u_1, \dots, D^u_{k-1}, E^u_{k-1}$ were drawn (see again Fig 7), so we only highlight the differences here.

- First, all such vertices and disks lie inside E, rather than inside D.
- Second, the role previously played by σ_r is now played by a segment σ'_r along the vertical line ℓ'_r through r. The endpoints of σ'_r are the lowest intersection point e_1 of ℓ'_r with the boundary of E and the intersection point of ℓ'_r with the horizontal line through u_1. This is because the vertices $v_1, \dots, v_{\ell-1}$ and the disks $D^v_1, E^v_1, \dots, D^v_{\ell-1}, E^v_{\ell-1}, D^v_\ell$ have to be placed below (and to the right of) $u_1, \dots, u_{k-1}$, so to satisfy the constraints of a dominance drawing. When a vertex v_j is drawn, the segment $\overline{v_j t}$ crosses the interior of σ'_r.
- Third, the triangles Δ^v_j are of Type A, unlike the triangles Δ^u_i which are of Type B. Hence, the disks D^v_j and E^v_j are horizontally aligned, to the right of σ'_r. The disks D^v_j and E^v_j are above and to the left of the disks D^v_{j-1} and E^v_{j-1}.

After the vertices $v_1, \dots, v_{\ell-1}$ and the disks $D^v_1, E^v_1, \dots, D^v_{\ell-1}, E^v_{\ell-1}$ have been drawn, it only remains to draw the disks D^u_k and E^u_k inside $\Delta^u_k = (u_{k-1}, u_k, t)$ and the disks D^v_ℓ and E^v_ℓ inside $\Delta^v_\ell = (v_{\ell-1}, v_\ell, t)$. See Fig 9. We have to place D^u_k and E^u_k above σ_r and below E^u_{k-1}, with D^u_k in D and E^u_k in E; also, E^u_k has to be to the right of r. Analogously, we have to place D^v_ℓ and E^v_ℓ in E, to the right of σ'_r and to the left of $E^v_{\ell-1}$, with E^v_ℓ above r. Finally, E^u_k has to be above and to the left of E^v_ℓ.

We can again use continuity arguments to prove that such disk placements exist. Indeed, $\overline{u_{k-1}t}$ cuts the interior of σ_r, hence D^u_k can be initially set to be a point in the interior of σ_r, to the right of $\overline{u_{k-1}t}$. Analogously, D^v_ℓ can be initially set to be a point in the interior of σ'_r above $\overline{v_{\ell-1}t}$. Disks E^u_k and E^v_ℓ are initially set to coincide with r. Now D^u_k and E^u_k can be moved upward of a sufficiently small distance so that D^u_k does not collide with $\overline{u_{k-1}t}$ and remains below E^u_{k-1}; note that now D^u_k and E^u_k are in the interior of Δ^u_k. Analogously, disks

D^v_ℓ and E^v_ℓ can be moved rightward, of a sufficiently small distance so that they still are to the left of $E^v_{\ell-1}$ and they now both lie in the interior of Δ^v_ℓ. Next, we move E^u_k rightward and E^v_ℓ upward so that they are to the right and above r, respectively. This movement is sufficiently small so that E^u_k remains in Δ^u_k and E^v_ℓ in Δ^v_ℓ, and so that E^u_k remains above and to the left of E^v_ℓ. Finally, we enlarge the disks so that they have a positive radius. Such a radius can be set to be sufficiently small so that all the above listed properties, which were satisfied before such an enlargement, are still maintained.

The drawing Γ of G is completed by drawing the subgraphs $M_1, \dots, M_\ell$, $S_1, \dots, S_k$ recursively, with triangles $\Delta^v_1, \dots, \Delta^v_\ell, \Delta^u_1, \dots, \Delta^u_k$ representing their outer faces, and with disks $D^v_1, E^v_1, \dots, D^v_\ell, E^v_\ell, \dots, D^u_1, E^u_1, \dots, D^u_k, E^u_k$ inside such triangles.

Correctness. The drawing Γ is straight-line by construction.

The drawings of the subgraphs $M_1, \dots, M_\ell, S_1, \dots, S_k$ are planar by induction. Moreover, the construction guarantees that the cycle $\mathcal{C}_{sm}$ is represented by a convex curve which keeps in its exterior every edge from a vertex of $\mathcal{C}_{sm}$ to t. It follows that distinct subgraphs among $M_1, \dots, M_\ell, S_1, \dots, S_k$ do not cross each other, that the edges inside or on the boundary of $\mathcal{C}_{sm}$ do not cross the subgraphs $M_1, \dots, M_\ell, S_1, \dots, S_k$, and that the edges inside or on the boundary of $\mathcal{C}_{sm}$ do not cross each other. Hence, Γ is planar.

Finally, we prove that Γ is a dominance drawing.

- Vertices that are internal to the same subgraph among $M_1, \dots, M_\ell, S_1, \dots, S_k$ are in the correct dominance relationship, by induction.
- Vertices that are internal to distinct subgraphs among $M_1, \dots, M_\ell, S_1, \dots, S_k$ are incomparable. This is because, for any internal vertex v of a subgraph M_j or S_i, we have that t is the only vertex incident to the outer face of M_j or S_i, respectively, that is a successor of v, as a consequence of the fact that P_s and P_m are the longest paths between their end-vertices. By induction, vertices that are internal to distinct subgraphs among $M_1, \dots, M_\ell, S_1, \dots, S_k$ are placed into disks among $D^v_1, E^v_1, \dots, D^v_\ell, E^v_\ell, D^u_1, E^u_1, \dots, D^u_k$, E^u_k. Also, any two disks associated to distinct subgraphs among $M_1, \dots, M_\ell, S_1, \dots, S_k$ are one to the left and above the other one, hence such vertices are in the correct dominance relationship.
- By construction, $v_1, \dots, v_{\ell-1}$ are to the right and below $u_1, \dots, u_{k-1}$, which is the correct dominance relationship as any vertex among $v_1, \dots, v_{\ell-1}$ is incomparable with any vertex among $u_1, \dots, u_{k-1}$.
- Also by construction, we have that v_j is above and to the right of v_{j-1}, for $j = 1, \dots, \ell$, and that u_i is above and to the right of u_{i-1}, for $i = 1, \dots, \ell$, which is the correct dominance relationship because of the existence of the directed paths P_s and P_m.
- Each vertex u_i with $i = 1, \dots, k$ is below and to the right of every disk among $D^u_1, E^u_1, \dots, D^u_i$ and is below and to the left of every disk among $E^u_i, D^u_{i+1}, \dots, D^u_k, E^u_k$; this is indeed the correct dominance relationship, as all the vertices in the former sequence of disks are incomparable to u_i, while all the vertices in the latter sequence of disks are successors of u_i. That the vertices among $v_1, \dots, v_\ell$ are in the correct dominance relationship with respect to vertices inside disks $D^v_1, E^v_1, \dots, D^v_\ell, E^v_\ell$ can be argued similarly.
- Each vertex u_i with $i = 1, \dots, k-1$ is above and to the left of every disk among $D^v_1, E^v_1, \dots, D^v_\ell$; this is indeed the correct dominance relationship, as u_i is incomparable to every vertex internal to a subgraph M_j with $j = 1, \dots, \ell$, with the exception of the successors of r in M_ℓ, which are also successors of u_i; these vertices are in E^v_ℓ, which is indeed above and to the right of u_i. Similarly, each vertex v_j with $j = 1, \dots, \ell - 1$ is in the correct dominance relationship with respect to every vertex internal to a subgraph S_i with $i = 1, \dots, k$. ◀

Clearly, an analogous result holds true for *st*-plane 3-trees in which the source is adjacent to every vertex.

5 Left-non-transitive *st*-plane graphs

We now consider *left-non-transitive st-plane graphs*. These are the *st*-plane graphs such that the left boundary of every face is not a single edge. We show the following.

▶ **Theorem 5.** *Left-non-transitive st-plane graphs admit planar straight-line dominance drawings.*

Proof. Consider a left-non-transitive *st*-plane graph G. We are going to use a *right-to-left path decomposition of* G. This consists of a sequence of directed paths $P_1, P_2, \ldots, P_k$ such that the following properties are satisfied.

- P_1 is the right boundary of the outer face of G;
- for $i = 1, \ldots, k$, the graph $G_i := P_1 \cup P_2 \cup \cdots \cup P_i$ is an *st*-plane graph;
- for $i = 2, \ldots, k$, the path P_i is the left boundary of a face of G whose right boundary belongs to the left boundary of the outer face of G_{i-1}; the internal vertices of P_i do not belong to G_{i-1}; and
- $G_k = G$.

This decomposition can be found by ordering the faces of G as in a DFS of the dual of G; for a formal proof see, e.g., [29].

We are going to construct a planar straight-line dominance drawing Γ_i of G_i, for $i = 2, \ldots, k$. Then Γ_k is the desired drawing of G.

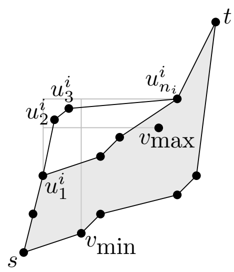


Figure 10 Constructing Γ_i from Γ_{i-1}. The interior of Γ_{i-1} is not shown and shaded gray.

For $i = 1, \ldots, k$, let $P_i = (u^i_1, u^i_2, \ldots, u^i_{n_i})$. Since G is left-non-transitive, $n_i \geq 3$ holds. The drawing Γ_1 of $G_1 = P_1$ is constructed as any straight-line drawing such that $x(u^1_1) < x(u^1_2) < \cdots < x(u^1_{n_1})$ and $y(u^1_1) < y(u^1_2) < \cdots < y(u^1_{n_1})$. Clearly, Γ_1 is a planar dominance drawing. Now suppose that a planar straight-line dominance drawing Γ_{i-1} of G_{i-1} has been constructed, for some $i \in \{2, \ldots, k\}$, so that no two vertices have the same x- or y-coordinate. We construct a planar straight-line dominance drawing Γ_i of G_i from Γ_{i-1} as follows; refer to Fig 10. Recall that u^i_1 and $u^i_{n_i}$ are vertices on the left boundary of G_{i-1}, while the internal vertices of P_i need to be inserted into Γ_{i-1} in order to define Γ_i. Among all the vertices of G_{i-1} that lie to the right of u^i_1 in Γ_{i-1}, let $v_{\min}$ be the one with smallest x-coordinate. Also, among all the vertices of G_{i-1} that lie below $u^i_{n_i}$ in Γ_{i-1}, let $v_{\max}$ be the one with largest y-coordinate. Note that $x(v_{\min}) \leq x(u^i_{n_i})$ and $y(u^i_1) \leq y(v_{\max})$. We assign coordinates to the internal vertices of P_i so that $x(u^i_1) < x(u^i_2) < \cdots < x(u^i_{n_i-1}) < x(v_{\min})$ and $y(v_{\max}) < y(u^i_2) < \cdots < y(u^i_{n_i-1}) < y(u^i_{n_i})$. This completes the construction of Γ_i.

We prove the planarity of Γ_i. Since the drawing of G_{i-1} in Γ_i coincides with Γ_{i-1} and since Γ_{i-1} is planar, it suffices to prove that the edges of P_i do not cross each other and do not cross Γ_{i-1}. The former follows directly from the fact that $x(u^i_1) < x(u^i_2) < \cdots < x(u^i_{n_i-1}) < x(u^i_{n_i})$ and $y(u^i_1) < y(u^i_2) < \cdots < y(u^i_{n_i-1}) < y(u^i_{n_i})$, by construction. We now deal with the latter.

- First, we prove that the edge (u^i_1, u^i_2) does not cross Γ_{i-1}. Let (u^i_1, w) be the edge of G_{i-1} outgoing from u^i_1 and incident to the left boundary of G_{i-1}. Such an edge has the outer face of Γ_{i-1} to its left, when traversed from u^i_1 to w. By construction, we have $x(u^i_2) < x(v_{\min}) \leq x(w)$ and $y(w) \leq y(v_{\max}) < y(u^i_2)$, hence the interval of x-coordinates spanned by the edge (u^i_1, u^i_2) is a subset of the one spanned by the edge (u^i_1, w) and the slope of the edge (u^i_1, u^i_2) is larger than the one of the edge (u^i_1, w). It follows that (u^i_1, u^i_2) lies in the outer face of Γ_{i-1}, and hence does not cross Γ_{i-1}.
- The proof that the edge $(u^i_{n_i-1}, u^i_{n_i})$ does not cross Γ_{i-1} is analogous.
- Finally, consider any edge (u^i_j, u^i_{j+1}) with $2 \leq j \leq n_i - 2$. By construction, the interval of x-coordinates spanned by (u^i_j, u^i_{j+1}) is a subset of the interval $(x(u^i_1), x(w))$, where w is defined as above. Also by construction, we have that $y(u^i_1) < y(w) \leq y(v_{\max}) < y(u^i_j) < y(u^i_{j+1})$. Hence, the edge lies above the edge (u^i_1, w), thus in the outer face of Γ_{i-1}, which is not crossed by it.

We now prove that Γ_i is a dominance drawing. Since the drawing of G_{i-1} in Γ_i coincides with Γ_{i-1} and since Γ_{i-1} is a dominance drawing, it suffices to prove that the placement of the internal vertices of P_i complies with the dominance relationships they are involved in. Consider any internal vertex u^i_j of P_i. For $h = 1, \dots, j-1$, vertex u^i_h is a predecessor of u^i_j and indeed we have $x(u^i_h) < x(u^i_j)$ and $y(u^i_h) < y(u^i_j)$, by construction. Analogously, for $h = j+1, \dots, n_i$, vertex u^i_h is a successor of u^i_j and indeed we have $x(u^i_j) < x(u^i_h)$ and $y(u^i_j) < y(u^i_h)$, by construction. Consider any vertex w of G_{i-1} different from u^i_1 and $u^i_{n_i}$.

- First, if w is a predecessor of u^i_1, then it is also a predecessor of u^i_j and indeed we have $x(w) < x(u^i_j)$ and $y(w) < y(u^i_j)$, given that $x(w) < x(u^i_1)$ and $y(w) < y(u^i_1)$ (since Γ_{i-1} is a dominance drawing) and that $x(u^i_1) < x(u^i_j)$ and $y(u^i_1) < y(u^i_j)$ (as proved above).
- Second, if w is a successor of $u^i_{n_i}$, then it is also a successor of u^i_j and indeed we have $x(u^i_j) < x(w)$ and $y(u^i_j) < y(w)$ given that $x(u^i_{n_i}) < x(w)$ and $y(u^i_{n_i}) < y(w)$ (since Γ_{i-1} is a dominance drawing) and that $x(u^i_j) < x(u^i_{n_i})$ and $y(u^i_j) < y(u^i_{n_i})$ (as proved above).
- Finally, if w is neither a predecessor of u^i_1 nor a successor of $u^i_{n_i}$, then it is incomparable with u^i_j. Note that $x(w) > x(u^i_1)$, as $x(w) < x(u^i_1)$ would imply $y(w) < y(u^i_1)$ (given that u^i_1 is on the left boundary of G_{i-1}), which is not possible since w is incomparable with u^i_1 and Γ_{i-1} is a dominance drawing. Analogously, we have $y(w) < y(u^i_{n_i})$. By construction, we have $x(u^i_j) < x(v_{\min}) \leq x(w)$ and $y(u^i_j) > y(v_{\max}) \geq y(w)$, hence the placement of w and u^i_j complies with their dominance relationship.

This concludes the proof that Γ_i is a planar straight-line dominance drawing, hence the induction and the proof of the theorem. ◀

Clearly, an analogous result holds true for *right-non-transitive st-plane graphs*, which are *st*-plane graphs such that the right boundary of every face is not a single edge.

6 Conclusions and Open Problems

In this paper, we tackled the following problem: Does every *st*-plane graph admit a planar straight-line dominance drawing? While we were not able to solve this question in its generality, our research advanced the state of the art in many directions.

First, we have provided concrete evidence for the difficulty in constructing planar straight-line dominance drawings. Most notably, we proved that planar straight-line dominance drawings with prescribed y-coordinates do not always exist. Our research in this direction indicates that, if an algorithm that constructs a planar straight-line dominance drawing of every st-plane graph exists, then it should use substantially different ideas than known algorithms for the construction of upward planar straight-line drawings.

Second, we have described several classes of st-plane graphs that admit a planar straight-line dominance drawing. A difficult benchmark here is, in our opinion, provided by the st-plane 3-trees. Hence, we believe it would be a major milestone to understand whether these graphs always admit planar straight-line dominance drawings.

We conclude with one more open problem. Does every (undirected) maximal planar graph admit a planar straight-line dominance drawing? That is, does it admit an st-orientation such that the resulting st-plane graph has a planar straight-line dominance drawing?

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