Hamiltonian Locality Testing via Trotterized Postselection

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Abstract

The (tolerant) Hamiltonian locality testing problem, introduced in [Bluhm, Caro, Oufkir '24], is to determine whether a Hamiltonian H is ε_1-close to being k-local (i.e. can be written as the sum of weight-k Pauli operators) or ε_2-far from any k-local Hamiltonian, given access to its time evolution operator and using as little total evolution time as possible, with distance typically defined by the normalized Frobenius norm. We give the tightest known bounds for this problem, proving an $\mathrm{O}\left(\sqrt{\frac{\varepsilon_2}{(\varepsilon_2-\varepsilon_1)^5}}\right)$ evolution time upper bound and an $\Omega(1/(\varepsilon_2-\varepsilon_1))$ lower bound. Our algorithm does not require reverse time evolution or controlled application of the time evolution operator, although our lower bound applies to algorithms using either tool.

Furthermore, we show that if we *are* allowed reverse time evolution, this lower bound is tight, giving a matching $\mathrm{O}(1/(\varepsilon_2-\varepsilon_1))$ evolution time algorithm.

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1 Introduction

When dealing with large or expensive-to-measure objects, learning the entire object may be too costly. Property testing algorithms instead attempt to distinguish between the object having a given property, or being far from any object with the property. More generally, one can consider *tolerant testing*, where one attempts to distinguish between the object being within ε_1-close to having a property, or being at least ε_2-far from any object with the property. Such algorithms have been extensively studied in quantum and classical settings (see [18] for an overview of the quantum case), but [6] was the first to consider it for Hamiltonians accessed via their time evolution operator e^{-iHt}. In this setting the natural measure of cost is *total evolution time*, $\sum_j t_j$ where the j^{th} application of the time evolution operator is e^{-iHt_j}.[1]

The property they considered was k-locality, a problem initially raised (but not studied) in [18, Section 7] as well [19]. A Hamiltonian H is k-local if and only if it can be written as $\sum_j H_j$, where each H_j operates on only k qubits. Such locality constraints (perhaps even geometrically locality constraints) are considered to be physically relevant. Local Hamiltonians also appear to be theoretically relevant, as nearly all general learning algorithms for Hamiltonians assume that the Hamiltonian is local, whether they use the time evolution operator [15, 14, 5], or copies of the Gibbs state [2, 4]. Local Hamiltonians are also conducive to efficient simulation on quantum computers, using the technique of Trotterization to break up the Hamiltonian into local quantum gate operations [16]. Finally, local Hamiltonians play an important role in quantum complexity theory, such as QMA-completeness and the Quantum PCP conjecture [1].

The initial version of [6] gave an $\mathrm{O}\big(n^{k+1}/(\varepsilon_2)^3\big)$ evolution time algorithm when distance is measured by the *normalized* (divided by $2^{n/2}$ for a Hamiltonian acting on n qubits) Frobenius norm, improved in [12] to $\mathrm{O}\big((\varepsilon_2-\varepsilon_1)^{-7}\big)$ and then in a later version of [6] to $\mathrm{O}\big((\varepsilon_2-\varepsilon_1)^{-2.5}\varepsilon_2^{-0.5}\big)$.[2][3] This left open the question: how hard is locality testing? Is it possible to achieve linear (a.k.a. Heisenberg) scaling in $1/\varepsilon$ for evolution time, and is such a scaling optimal in all error regimes? In this work we make progress towards resolving the complexity of this problem, improving the best known upper and lower bounds. Our algorithm is based on a technique we refer to as *Trotterized post-selection*, in which we suppress the effect of local terms in the Hamiltonian evolution by repeatedly evolving for a short time period and post-selecting on the non-local part of the time evolution operator.

1.1 Our Results

Our main result is a improved upper bound for the Hamiltonian locality testing problem. As with past works, our algorithm is also time-efficient and non-adaptive, though it does requires n qubits of quantum memory, like [12, 3].

▶ **Theorem 1.** *Let $0 \le \varepsilon_1 < \varepsilon_2 \le 1$, $\delta \in (0,1)$, and $k \in \mathbb{N}$. There is an algorithm that distinguishes whether an n-qubit Hamiltonian H is (1) within ε_1 of some k-local Hamiltonian or (2) ε_2-far from all k-local Hamiltonians, with probability $1-\delta$. The algorithm uses $\mathrm{O}\left(\sqrt{\frac{\varepsilon_2}{(\varepsilon_2-\varepsilon_1)^7}}\log(1/\delta)\right)$ non-adaptive queries to the time evolution operator with $\mathrm{O}\left(\sqrt{\frac{\varepsilon_2}{(\varepsilon_2-\varepsilon_1)^5}}\log(1/\delta)\right)$ total evolution time.*

We pair it with the first lower bound in the tolerant testing setting. While our upper bound uses only forward time evolution and does not require controlled application of e^{-itH}, our lower bound also applies to algorithms using either of these tools.

[1] Another cost measure that can be considered is total query count, the number of individual applications of the time evolution operator. Our algorithm also uses the fewest number of queries of any known algorithm.

[2] The original [6] algorithm only worked in the intolerant setting of $\varepsilon_1 = 0$.

[3] [12] was later subsumed by [3], which gives an $\mathrm{O}\big((\varepsilon_2-\varepsilon_1)^{-3}\big)$ analysis.

▶ **Theorem 2.** *Let $0 \le \varepsilon_1 < \varepsilon_2 \le 1$ and $k \in \mathbb{N}$. Then any algorithm that can distinguish whether an n-qubit Hamiltonian H is (1) within ε_1 of some k-local Hamiltonian or (2) ε_2-far from all k-local Hamiltonians, must use $\Omega\left(\frac{1}{\varepsilon_2-\varepsilon_1}\right)$ evolution time in expectation to achieve constant success probability.*

▶ Remark 3. [6, Theorem 3.6] gives a hardness result for the *unnormalized* Frobenius norm (as well as other Schatten norms) in the *non-tolerant* setting that scales as $\Omega\left(\frac{2^{n/2}}{\varepsilon}\right)$. Once normalized, this also gives a $\Omega\left(\frac{1}{\varepsilon}\right)$ lower bound. However, this hardness result only holds for exponentially small ε, due to the fact that the "hard" Hamiltonian in [6, Lemma 3.2] no longer has $\|H\|_\infty \le 1$ when the *unnormalized* Frobenius distance to k-local is super-constant. Therefore Theorem 2 is, to the authors' knowledge, the first lower bound that works for arbitrary values of ε, in addition to being the first for the tolerant setting. Our proof is also considerably simpler, and still extends to all of the distance measures considered in [6] and more.

Finally, we show that, when reverse time evolution and controlled operations are allowed, it is possible to saturate this lower bound even in the tolerant case (proof in the appendix).

▶ **Theorem 4.** *Let $0 \le \varepsilon_1 < \varepsilon_2 \le 1$, $\delta \in (0,1)$, and $k \in \mathbb{N}$. There is an algorithm that tests whether an n-qubit Hamiltonian H is (1) ε_1-close to some k-local Hamiltonian or (2) ε_2-far from all k-local Hamiltonians, with probability $1-\delta$. The algorithm uses $\mathrm{O}\left(\frac{\log(1/\delta)}{(\varepsilon_2-\varepsilon_1)^2}\right)$ non-adaptive queries to the time evolution operator and its inverse, with $\mathrm{O}\left(\frac{\log(1/\delta)}{\varepsilon_2-\varepsilon_1}\right)$ total evolution time.*

2 Proof Overview

2.1 Upper Bound

For simplicity, we will consider the intolerant case ($\varepsilon_1 = 0$, $\varepsilon_2 = \varepsilon$) for this proof overview; the same techniques apply in the tolerant case but require somewhat more care. First we start with the intuition behind the algorithm of [12, 3].

We will need the fact that the space of $2n$ qubit states $\mathbb{C}^{2^{2n}}$ has the Bell basis $(|\sigma_P\rangle)_P$, where P spans the n-fold Paulis, $|\sigma_{I^{\otimes n}}\rangle$ is the maximally entangled state $\frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}|x\rangle \otimes |x\rangle$, and $|\sigma_P\rangle = (I^{\otimes n} \otimes P)|\sigma_{I^{\otimes n}}\rangle$. Therefore, for any unitary U, if we apply $I^{\otimes n} \otimes U$ to $|\sigma_{I^{\otimes n}}\rangle$ and then measure in the Bell basis, we are able to sample from the (squared) Pauli spectrum[4] of U (the squares of the Pauli decomposition coefficients always sum to 1 for a unitary [17]).

For any Hamiltonian H, the closest k-local Hamiltonian is given by dropping all of the non-local Paulis from its Pauli decomposition. Therefore, as by the first-order Taylor series expansion,

$$e^{-iHt} \approx I^{\otimes n} - iHt$$

for small enough t, we can set $U = e^{-iH\cdot t}$ in the aforementioned procedure, and if H is ε-far from local we will sample a non-local Pauli term with $\approx (t\cdot\varepsilon)^2$ probability. Conversely, if H is local we should sample no non-local terms, giving us a distinguishing algorithm if the process is repeated $\mathrm{O}\big((t\cdot\varepsilon)^{-2}\big)$ times, for a total time evolution of $\mathrm{O}\big(t^{-1}\cdot\varepsilon^{-2}\big)$.

[4] That is, α_P^2 when U is written as $\sum_P \alpha_p P$.

So ideally we would like t to be $\Theta(1/\varepsilon)$ and only repeat a constant number of times, leading to a total time evolution of $\mathrm{O}(\varepsilon^{-1})$, which would be optimal by Theorem 2.

Unfortunately, these higher-order terms in the Taylor series cannot be ignored at larger values of t. As we have $\|H\|_\infty \leq 1$, we can bound the k^{th} order term of the Taylor series expansion of H by $\mathrm{O}(t^k)$, and so we must set t to be at most $\Theta(\varepsilon)$, resulting in the total time evolution of $\mathrm{O}(\varepsilon^3)$ obtained in previous work [12, 3].

To evade this barrier, we will instead show that it is possible to (approximately) simulate evolving by $H_{>k}$, which is composed of only the *non-local* terms of the Pauli decomposition of H. Note that if H is k-local, this is 0, while if it is not, $H_>k$ is the difference between H and the closest k-local Hamiltonian. Suppose we could evolve by the time evolution operator of this Hamiltonian. Then performing the Bell sampling procedure from before would return $|\sigma_{I^{\otimes n}}\rangle$ with probability

$$\begin{aligned}
&\left|\langle\sigma_{I^{\otimes n}}|\left(I^{\otimes n}\otimes e^{-iH_{>k}t}\right)|\sigma_{I^{\otimes n}}\rangle\right|^2\\
&=\left|\langle\sigma_{I^{\otimes n}}|\left(I^{\otimes n}\otimes\left(\sum_{\ell=0}^{\infty}(H_{>k})^{\ell}\frac{(it)^{\ell}}{\ell!}\right)\right)|\sigma_{I^{\otimes n}}\rangle\right|^2\\
&=\left|1+\langle\sigma_{I^{\otimes n}}|\left(I^{\otimes n}\otimes\left(\sum_{\ell=2}^{\infty}(H_{>k})^{\ell}\frac{(it)^{\ell}}{\ell!}\right)\right)|\sigma_{I^{\otimes n}}\rangle\right|^2\\
&=1-\langle\sigma_{I^{\otimes n}}|\left(I^{\otimes n}\otimes(H_{>k})^2\right)|\sigma_{I^{\otimes n}}\rangle+\sum_{\ell=3}^{\infty}\mathrm{O}\left(t^{\ell}\cdot\left|\langle\sigma_{I^{\otimes n}}|\left(I^{\otimes n}\otimes(H_{>k})^{\ell}\right)|\sigma_{I^{\otimes n}}\rangle\right|\right)
\end{aligned}$$

as H contains no identity term.

To tame this infinite series, imagine that $\|H_{>k}\|_\infty \leq 1$ (we will eventually evolve by a related operator A that *does* satisfy $\|A\|_\infty \leq 1$). Then we have

$$\left|\langle\sigma_{I^{\otimes n}}|\left(I^{\otimes n}\otimes(H_{>k})^{\ell}\right)|\sigma_{I^{\otimes n}}\rangle\right|\leq\langle\sigma_{I^{\otimes n}}|\left(I^{\otimes n}\otimes(H_{>k})^{2}\right)|\sigma_{I^{\otimes n}}\rangle$$

for all integers $\ell \geq 2$, so as long as t is a sufficiently small constant, we have that $\left|\langle\sigma_{I^{\otimes n}}|\left(I^{\otimes n}\otimes e^{-iH_{>k}t}\right)|\sigma_{I^{\otimes n}}\rangle\right|^2$ is at least

$$1-0.99\cdot\langle\sigma_{I^{\otimes n}}|\left(I^{\otimes n}\otimes(H_{>k})^{2}\right)|\sigma_{I^{\otimes n}}\rangle=1-0.99\cdot\mathrm{Tr}\left((H_{>k})^2\right)/2^n,$$

where $\mathrm{Tr}\left((H_{>k})^2\right)/2^n=\varepsilon^2$ is exactly the squared normalized Frobenius distance of H from being k-local. So if we apply $e^{-iH_{>k}t}$ with $t=\Theta(1)$, we are left with a $\approx\varepsilon^2$ probability of sampling a non-local Pauli term if H is non-local, and are guaranteed to measure identity if H is local (as then $e^{-iH_{>k}\cdot t}$ is the identity). This means we can distinguish locality from non-locality with $\mathrm{O}(\varepsilon^{-2})$ repetitions, requiring $\mathrm{O}(\varepsilon^{-2})$ total evolution time.[5]

Now, we cannot actually apply $e^{-iH_{>k}t}$. However, when starting at $|\sigma_{I^{\otimes n}}\rangle$, we can approximate it up to $t=\Theta(1)$ by the use of a process reminiscent of the Elitzur-Vaidman bomb-tester [9] and Quantum Zeno effect [10], which we refer to as *Trotterized post-selection*.

Let D be the subspace of Bell states corresponding to non-local Paulis *or* identity and let Π_D be the projector onto that subspace. Starting with $|\sigma_{I^{\otimes n}}\rangle$ once again, we apply $I^{\otimes n}\otimes e^{-iHt'}$ for $t'=\mathrm{O}(\varepsilon)$, measure with $\{\Pi_D, I^{\otimes 2n}-\Pi_D\}$, and then post-select on the measurement result Π_D. We then repeat our application of $I^{\otimes n}\otimes e^{-iHt'}$ and post-selection, for $\mathrm{O}(1/t')$ iterations, provided our post-selection succeeds each time.

[5] Unfortunately, even with access to the time evolution operator of $H_{>k}$ we cannot set t to the optimal $\Theta(1/\varepsilon)$, as we lose control of the higher-order terms of the Taylor expansion.

As we start with $|\sigma_{I^{\otimes n}}\rangle$, then make small adjustments (i.e., $e^{-iHt} \approx I^{\otimes 2n}$ for small t), the chance of failing the post-selection is small: only $\mathrm{O}(\varepsilon^2)$ at each iteration, and so as long as we only use $\mathrm{O}(1/\varepsilon)$ iterations, we will succeed with probability $1 - \mathrm{O}(\varepsilon)$. Now, as we are taking small steps, we can approximate each iteration of $\Pi_D \left(I^{\otimes n} \otimes e^{-iH\cdot \mathrm{O}(\varepsilon)}\right) \Pi_D$ as

$$\Pi_D \left(I^{\otimes n} \otimes e^{-iH\cdot \mathrm{O}(\varepsilon)}\right) \Pi_D = \Pi_D \left(I^{\otimes n} \otimes \sum_{\ell=0}^{\infty} H^\ell \frac{(-i)^\ell\, \mathrm{O}(\varepsilon^\ell)}{\ell!}\right) \Pi_D = e^{-iA\cdot \mathrm{O}(\varepsilon)} + R$$

where we define $A \coloneqq \Pi_D(I^{\otimes n} \otimes H)\Pi_D$ and choose some $\|R\|_\infty \leq \mathrm{O}(\varepsilon^2)$.[6]

Now, in general, $A \neq I^{\otimes n} \otimes H_{>k}$, but as long as H has no identity term in its Pauli decomposition[7], by construction $A|\sigma_{I^{\otimes n}}\rangle = (I^{\otimes n} \otimes H_{>k})\,|\sigma_{I^{\otimes n}}\rangle$, and so $\langle\sigma_{I^{\otimes n}}|A^2|\sigma_{I^{\otimes n}}\rangle = \langle\sigma_{I^{\otimes n}}|I \otimes (H_{>k})^2\,|\sigma_{I^{\otimes n}}\rangle$. Combined with the fact that $\|A\|_\infty = \|\Pi_D\,(I^{\otimes n} \otimes H)\,\Pi_D\|_\infty \leq \|H\|_\infty \leq 1$, we can argue that, if we iterate t/t' times

$$\begin{aligned}
\langle\sigma_{I^{\otimes n}}|\prod_{i=1}^{t/t'} e^{-iA\cdot t'}|\sigma_{I^{\otimes n}}\rangle &= \langle\sigma_{I^{\otimes n}}|e^{-iA\cdot t}|\sigma_{I^{\otimes n}}\rangle \\
&= \langle\sigma_{I^{\otimes n}}|\left(\sum_{\ell=0}^{\infty} A^\ell \frac{(-it)^\ell}{\ell!}\right)|\sigma_{I^{\otimes n}}\rangle \\
&= 1 - t^2\langle\sigma_{I^{\otimes n}}|H_{>k}^2|\sigma_{I^{\otimes n}}\rangle + \mathrm{O}(t^3 \cdot \varepsilon^2)
\end{aligned}$$

where the final inequality follows from the fact that for all $k > 2$,

$$\left|\langle\sigma_{I^{\otimes n}}|A^k|\sigma_{I^{\otimes n}}\rangle\right| \leq \|A\|_\infty^{k-2}\langle\sigma_{I^{\otimes n}}|A^2|\sigma_{I^{\otimes n}}\rangle \leq \langle\sigma_{I^{\otimes n}}|\left(I^{\otimes n} \otimes (H_{>k})^2\right)|\sigma_{I^{\otimes n}}\rangle = \varepsilon^2.$$

So as our method based on access to the time evolution operator of $H_{>k}$ only required distinguishing between $\langle\sigma_{I^{\otimes n}}|H_{>k}|\sigma_{I^{\otimes n}}\rangle$ being $\Theta(\varepsilon^2)$ and 0 we can emulate it with access to e^{-iAt} without losing too much accuracy, as long as we take t to be a small enough constant. We can therefore test locality with a total time evolution of $\mathrm{O}(\varepsilon^{-2})$.

2.2 Lower Bound

To prove the lower bound, it suffices to show that for any k there exists Hamiltonians H_1 and H_2 such that a query to the time t evolution of H_1 and H_2 differ in diamond distance by at most $\mathrm{O}((\varepsilon_2 - \varepsilon_1)t)$, with H_1 ε_1-close to being k-local and H_2 ε_2-far from being k-local.

We achieve this by considering the weight-k Pauli $Z_{1:k}$ that is Z on the first k qubits, and identity on the last $n-k$ qubits. We then set $H_1 \coloneqq \varepsilon_1 Z_{1:k}$ and $H_2 \coloneqq \varepsilon_2 Z_{1:k}$. Because $Z_{1:k}$ is diagonal, so is $e^{-i\varepsilon Z_{1:k}\cdot t}$, making it straightforward to bound the diamond distance of the two time evolution operators by $\mathrm{O}(t(\varepsilon_2 - \varepsilon_1))$. By the sub-additivity of diamond distance, the total time evolution required to distinguish the two Hamiltonians with constant probability is therefore at least $\Omega\left((\varepsilon_2 - \varepsilon_1)^{-1}\right)$.

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[6] Note that the Π_D on the right does nothing besides make A obviously Hermitian, assuming our invariant of our post-selection succeeding.

[7] We can assume this without loss of generality, as our algorithm never uses controlled application of $e^{-iH\cdot t}$, and so any identity term would manifest as an undetectable global phase.

3 Preliminaries

3.1 Quantum Information

A Hamiltonian on n-qubits is a $2^n \times 2^n$ Hermitian matrix. The time evolution operator of a Hamiltonian H for time $t \geq 0$ is the unitary matrix

$$e^{-iHt} := \sum_{k=0}^{\infty} H^k (-i)^k \frac{t^k}{k!}.$$

We define the n-qubit Pauli matrices to be $\mathcal{P}^{\otimes n} := \{I, X, Y, Z\}^{\otimes n}$, where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For any Pauli P, we denote the locality $|P|$ to be the number of non-identity terms in the tensor product. Let the Frobenius inner product between matrices A and B be $\langle A, B\rangle := \mathrm{Tr}(A^\dagger B)$. The orthogonality of Pauli matrices under the Frobenius inner product is implied by the fact that any product of Paulis is another Pauli (up to sign) and the fact that among them only the identity has non-zero trace. Given a matrix $A = \sum_{P \in \mathcal{P}^{\otimes n}} \alpha_P P$, the locality of A is the largest $|P|$ such that $\alpha_P \neq 0$. If A is a Hamiltonian (i.e., Hermitian) then all α_P are real-valued. The *normalized* Frobenius norm is given by

$$\|A\|_2 = \sqrt{\frac{\langle A, A\rangle}{2^n}} = \sqrt{\frac{\mathrm{Tr}(A^\dagger A)}{2^n}} = \sqrt{\sum_{P \in \mathcal{P}^{\otimes n}} |\alpha_P|^2},$$

and will be used as our distance to k-locality, in keeping with the previous literature [6, 12, 3]. The other important norm will be the (unnormalized) spectral norm $\|A\|_\infty$, which is the largest singular value of A. For any matrix A, $\|A\|_2 \leq \|A\|_\infty$, recalling that $\|\cdot\|_2$ is the *normalized* Frobenius norm. As a form of normalization and to be consistent with the literature, we will assume that $\|H\|_\infty \leq 1$ for any Hamiltonian referenced. We will also WLOG assume that $\mathrm{Tr}(H) = 0$ for any Hamiltonian, since it does not affect the time evolution unitary beyond a global phase, and so as our algorithms do not use controlled application of the unitary, they cannot be affected by it.

We define $A_{>k} := \sum_{|P|>k} \alpha_P P$ and subsequently $A_{\leq k} := \sum_{|P|\leq k} \alpha_P P$. By the orthogonality of the Pauli matrices under the Frobenius inner product, $A_{\leq k}$ is the k-local Hamiltonian that is closest to A with distance $\|A - A_{\leq k}\|_2 = \|A_{>k}\|_2$.

Let $B = \{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}$ denote the set containing the four Bell states. We will view $B^{\otimes n}$ as a basis of $\mathbb{C}^{2^n} \otimes \mathbb{C}^{2^n}$, in which for each copy of B, one qubit is assigned to the left register and one to the right. Note that, up to phase, every state in $B^{\otimes n}$ is equal to $(I^{\otimes n} \otimes P)|\Phi^+\rangle^{\otimes n}$ for a unique $P \in \mathcal{P}^{\otimes n}$. We will write $|\sigma_P\rangle$ for this basis element. As an example,

$$|\Phi^+\rangle^{\otimes n} = |\sigma_{I^{\otimes n}}\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \otimes |x\rangle.$$

If $U = \sum_{P \in \mathcal{P}^{\otimes n}} \alpha_P P$ is a unitary matrix, then by Parseval's identity, $\sum_{P \in \mathcal{P}^{\otimes n}} |\alpha_P|^2 = 1$, i.e. $|\alpha_P|^2$ gives a probability distribution over the Paulis. Applying $I^{\otimes n} \otimes U$ to the state $|\sigma_{I^{\otimes n}}\rangle = |\Phi^+\rangle^{\otimes n}$ and measuring in the Bell basis $B^{\otimes n}$ allows one to sample from this distribution [17].

For a quantum channel that takes as input an n-qubit state, we will let the diamond norm refer to $\|\Lambda\|_\diamond := \max_\rho \|(I^{\otimes n} \otimes \Lambda)(\rho)\|_1$ where the maximization is over all $2n$-qubit states ρ. The diamond distance famously characterizes the maximum statistical distinguishability (i.e., induced trace distance) between quantum channels [21, Section 9.1.6], even with ancillas.

3.2 Probability

▶ **Fact 5** (Multiplicative Chernoff Bound). *Suppose $X_1, \ldots, X_m$ are independent Bernoulli random variables. Let X denote their sum and let $\mu := \mathbb{E}[X]$. Then for any $t > 0$*

$$\Pr\left[X \leq (1-t)\mu\right] \leq e^{-t^2\mu/2}.$$

We will not need a particularly tight form of this bound, so for ease of analysis we state the following (loose) corollary.

▶ **Corollary 6.** *Suppose $X_1, \ldots, X_m$ are i.i.d. Bernoulli random variables with probability p, and*

$$m = \frac{2}{p}\left(d + \log(1/\delta)\right).$$

Then

$$\Pr\left[\sum_{i=1}^{m} X_i < d\right] \leq \delta.$$

Proof. Let $\mu := \mathbb{E}[\sum_{i=1}^m X_i] = mp$ and let $\gamma := 1 - \frac{d}{\mu}$. By the Multiplicative Chernoff Bound,

$$\begin{aligned}\Pr\left[\sum_{i=1}^{m} X_i < d\right] &= \Pr\left[\sum_{i=1}^{m} X_i < (1-\gamma)\mu\right] \\ &\leq \exp\left(-\frac{\mu}{2}\gamma^2\right) = \exp\left(-\frac{\mu}{2} - \frac{d^2}{2\mu} + d\right) \leq \exp\left(-\frac{mp}{2} + d\right).\end{aligned}$$

Hence, as long as

$$m \geq \frac{2\log(1/\delta) + 2d}{p},$$

then $\sum_{i=1}^m X_i \leq d$ with probability at most δ. ◀

▶ **Fact 7** (Bernstein's inequality). *Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables. Let X denote their sum and let μ and σ^2 be the expectation and variance of X respectively. Then for $t \in (0, n)$*

$$\Pr\left[X - \mu \geq t\right] \leq e^{-\frac{\frac{t^2}{2}}{\sigma^2 + \frac{t}{3}}} \text{ and } \Pr\left[X - \mu \leq -t\right] \leq e^{-\frac{\frac{t^2}{2}}{\sigma^2 + \frac{t}{3}}}.$$

4 Upper Bound

We will frequently use the truncation of the Taylor series of the matrix exponential to analyze our algorithm. The following will allow us to then bound the error of the truncation.

▶ **Fact 8** ([8, Lemma F.2]). *If $\lambda \in \mathbb{C}$ then $\left|\sum_{k=\ell}^{\infty} \frac{\lambda^k}{k!}\right| \leq \frac{|\lambda|^\ell}{\ell!} e^{|\lambda|}$.*

▶ **Corollary 9.** *For n-qubit Hamiltonian H with $\|H\|_\infty \leq 1$, the first order Taylor series expansion of the matrix exponential gives*

$$e^{-iHt} = I^{\otimes n} - iHt + \frac{e^t \cdot t^2}{2} R$$

for $\|R\|_\infty \leq 1$.

Proof. By the triangle inequality and the fact that $\|H^k\|_\infty \le \|H\|_\infty \le 1$ for $k \ge 1$:

$$\|e^{-iHt} - (I^{\otimes n} - iHt)\|_\infty = \left\| \sum_{k=2}^{\infty} (-i)^k \frac{H^k t^k}{k!} \right\|_\infty \le \sum_{k=2}^{\infty} \frac{\|H^k\|_\infty t^k}{k!} \le \sum_{k=2}^{\infty} \frac{t^k}{k!} \le \frac{e^t \cdot t^2}{2},$$

using Fact 8 at the end. Setting $R := \frac{2}{e^t \cdot t^2} \left(e^{-iHt} - (I^{\otimes n} - iHt) \right)$ completes the proof. ◀

We also prove the related fact to bound the real and imaginary terms.

▶ **Fact 10.** *If $\lambda \in \mathbb{C}$ then*

$$\left| \sum_{k=\ell}^{\infty} \frac{\lambda^{2k}}{(2k)!} \right| \le \frac{|\lambda|^{2\ell}}{(2\ell)!} \cosh(|\lambda|)$$

and

$$\left| \sum_{k=\ell}^{\infty} \frac{\lambda^{2k+1}}{(2k+1)!} \right| \le \frac{|\lambda|^{2\ell+1}}{(2\ell+1)!} \cosh(|\lambda|).$$

Proof.

$$\left| \sum_{k=\ell}^{\infty} \frac{\lambda^{2k}}{(2k)!} \right| \le \sum_{k=\ell}^{\infty} \frac{|\lambda^{2k}|}{(2k)!} = |\lambda|^{2\ell} \sum_{k=0}^{\infty} \frac{|\lambda|^{2k}}{(2k+2\ell)!} \le \frac{|\lambda|^{2\ell}}{(2\ell)!} \sum_{k=0}^{\infty} \frac{|\lambda|^{2k}}{(2k)!} = \frac{|\lambda|^{2\ell}}{(2\ell)!} \cosh(|\lambda|)$$

and

$$\begin{aligned} \left| \sum_{k=\ell}^{\infty} \frac{\lambda^{2k+1}}{(2k+1)!} \right| &\le \sum_{k=\ell}^{\infty} \frac{|\lambda^{2k+1}|}{(2k+1)!} = |\lambda|^{2\ell+1} \sum_{k=0}^{\infty} \frac{|\lambda|^{2k}}{(2k+2\ell+1)!} \\ &\le \frac{|\lambda|^{2\ell+1}}{(2\ell+1)!} \sum_{k=0}^{\infty} \frac{|\lambda|^{2k}}{(2k)!} = \frac{|\lambda|^{2\ell+1}}{(2\ell+1)!} \cosh(|\lambda|). \end{aligned}$$

◀

4.1 Algorithm

▶ **Definition 11.** *We will use D to denote the subspace of $\mathbb{C}^{2^n} \otimes \mathbb{C}^{2^n}$ spanned by $|\sigma_P\rangle$ for Pauli strings P that are either the identity or are* not *k-local, and Π_D to denote the projector onto D. We define $A := \Pi_D \, (I^{\otimes n} \otimes H) \, \Pi_D$.*

We start by giving an algorithm that returns a Bernoulli random variable $X \in \{0, 1\}$, where $\mathbb{E}[X]$ approximates the distance of H from being k-local. It does so by iteratively applying $e^{-i\alpha H}$ sandwiched by $\{\Pi_D, I^{\otimes 2n} - \Pi_D\}$ measurements.

Algorithm 1 Hamiltonian Locality Estimator via Trotterized Postselection.

Start with $|\phi\rangle = |\sigma_{I^{\otimes n}}\rangle$.
for $\frac{50}{\sqrt{\varepsilon_2^2 - \varepsilon_1^2}}$ iterations **do**
    Apply $(I^{\otimes n} \otimes e^{-i\alpha H}$ to $|\phi\rangle$ for $\alpha = \frac{\varepsilon_2^2 - \varepsilon_1^2}{100\varepsilon_2}$.
    Measure $|\phi\rangle$ with the projectors $\Pi_D, I^{\otimes 2n} - \Pi_D$, terminating and returning $\perp$ if the result is $I^{\otimes 2n} - \Pi_D$.
end for
Measure $|\phi\rangle$ in the Bell basis, returning 0 if the result is $|\sigma_{I^{\otimes n}}\rangle$ and 1 otherwise.

Let $\alpha \coloneqq \frac{\varepsilon_2^2-\varepsilon_1^2}{100\varepsilon_2}$ be the step-size used in Algorithm 1, $t \coloneqq \frac{\sqrt{\varepsilon_2^2-\varepsilon_1^2}}{2\varepsilon_2}$ be the total time evolution used in Algorithm 1, and let $m \coloneqq t/\alpha = \frac{50}{\sqrt{\varepsilon_2^2-\varepsilon_1^2}}$ be the number of iterations used in Algorithm 1. In our analysis will frequently use the fact that $\alpha \le \frac{\varepsilon_2}{100} \le \frac{1}{100}$ and $t \le 0.5$ to simplify higher-order terms.

▶ Remark 12. While we attempted to keep the constants in the algorithm reasonable, no attempt was made to optimize them. We observe that t should remain $\Theta\left(\frac{\sqrt{\varepsilon_2^2-\varepsilon_1^2}}{\varepsilon_2}\right)$ for optimal scaling, but α can be made arbitrarily small to (marginally) improve the constants in the total time evolution used. This has a cost in the total number of queries used, scaling roughly proportional to α^{-1}.

First we show that the final state of the Trotterized postselection algorithm corresponds to evolving $|\sigma_{I^{\otimes n}}\rangle$ by e^{-iAt}, with a bounded error term. There are two main sources of error: (1) the error from higher-order terms in the respective Taylor series of $e^{-iA\alpha}$ and $\Pi_D\left(I^{\otimes n} \otimes e^{-iH\alpha}\right)\Pi_D$ not matching and (2) the error from post-selection causing normalization issues. The following technical lemma allows us to tackle the error from (1). This is done by showing that $e^{-itA} = \Pi_D\left(I^{\otimes n} \otimes e^{-itH}\right)\Pi_D \pm \mathrm{O}\left(\alpha^2\right)$ for sufficiently small α. By chaining these together, the triangle inequality will eventually show in Lemma 14 that the accumulated error is then at most $\mathrm{O}\left(\alpha^2 m\right) = \mathrm{O}(\alpha t)$.

▶ **Lemma 13.** *Let $H = \sum_{P \in \mathcal{P}^{\otimes n}} \alpha_P P$ be any Hamiltonian with $\|H\|_\infty \le 1$. Then,*

$$\Pi_D(I^{\otimes n} \otimes e^{-i\alpha H})\Pi_D = e^{-i\alpha A} + \eta$$

where $\|\eta\|_\infty \le e^\alpha \cdot \alpha^2$.

Proof. By Taylor expanding the complex exponential of $e^{-i\alpha H}$ and applying Corollary 9, we get

$$\begin{aligned}\Pi_D(I^{\otimes n} \otimes e^{-i\alpha H})\Pi_D &= \Pi_D\left(I^{\otimes n} \otimes \left(I^{\otimes n} - i\alpha H + \frac{e^\alpha \cdot \alpha^2}{2} R\right)\right)\Pi_D \\ &= I^{\otimes 2n} - i\alpha A + \frac{e^t \cdot \alpha^2}{2} R'\end{aligned}$$

where $\|R'\|_\infty \le \|I^{\otimes n} \otimes R\|_\infty = \|R\|_\infty \le 1$.

Next, we observe that $\|A\|_\infty \le \|I^{\otimes n} \otimes H\|_\infty = \|H\|_\infty \le 1$ and that A is Hermitian by symmetry. We can then Taylor expand $e^{-i\alpha A}$ to get

$$e^{-i\alpha A} = I^{\otimes 2n} - i\alpha A + \frac{e^\alpha \cdot \alpha^2}{2} Q$$

where $\|Q\|_\infty \le 1$. By the triangle inequality, the difference

$$\eta \coloneqq \Pi_D(I^{\otimes n} \otimes e^{-i\alpha H})\Pi_D - e^{-i\alpha A}$$

between these two linear transformations satisfies

$$\|\eta\|_\infty \le \|R'\|_\infty \cdot \frac{e^\alpha \cdot \alpha^2}{2} + \|Q\|_\infty \cdot \frac{e^\alpha \cdot \alpha^2}{2} \le e^\alpha \cdot \alpha^2.$$ ◀

Luckily, the error from (2) is mostly a non-issue, using a process similar to the Elitzur-Vaidman bomb [9]: by taking small steps between applications of Π_D, we ensure that we are barely changing our system, and so the post-selection nearly always succeeds. This also means that the normalization error can be suppressed to be arbitrarily small, at the cost of linearly increasing the number of times we have to query the time evolution operator. Using these facts together, we show that Algorithm 1 approximately applies the time evolution operator of A.

▶ **Lemma 14.** *Algorithm 1 terminates before the final measurement with probability at most* $\frac{99}{98}\alpha t$*. If it does not,* $|\phi\rangle = e^{-iAt}|\sigma_{I^{\otimes n}}\rangle + |\Delta\rangle$ *just before the final measurement, with* $\||\Delta\rangle\|_2 \leq \frac{7}{4}\alpha t$*.*

Proof. Note that the algorithm can only be terminated early if, in one of the loop iterations, the measurement in Algorithm 1 returns $I^{\otimes 2n} - \Pi_D$. At the start of the iteration $|\phi\rangle = |\sigma_{I^{\otimes n}}\rangle \in D$. Since $|\phi\rangle$ remains within D after each successful iteration, by Taylor expanding the exponential, and applying Corollary 9 to obtain a suitable R with $\|R\|_\infty \leq 1$, the probability of failure at each iteration is at most

$$\begin{aligned}
&\left\|(I^{\otimes 2n} - \Pi_D)\left(I^{\otimes n} \otimes e^{-iH\alpha}\right)\Pi_D|\phi\rangle\right\|_2^2 \\
&= \left\|(I^{\otimes 2n} - \Pi_D)\left(I^{\otimes n} \otimes \left(I^{\otimes n} - i\alpha H + \frac{\alpha^2}{2}e^{\alpha}R\right)\right)|\phi\rangle\right\|_2^2 \\
&= \left\|(I^{\otimes 2n} - \Pi_D)\left(-i\alpha(I^{\otimes n} \otimes H) + \frac{\alpha^2}{2}e^{\alpha}(I^{\otimes n} \otimes R)\right)|\phi\rangle\right\|_2^2 \\
&\leq \left(\alpha\|H\|_\infty + \frac{\alpha^2 e^{\alpha}}{2}\|R\|_\infty\right)^2 \\
&\leq \left(1 + \alpha e^{\alpha} + \frac{\alpha^2}{4}e^{2\alpha}\right)\alpha^2 \\
&< \frac{99}{98}\alpha^2
\end{aligned}$$

where the third line follows from $|\phi\rangle \in D$, the fourth from the triangle inequality combined with the definition of the spectral norm, and the final line from $\alpha \leq 0.01$. By a union bound over the m iterations, the first part of the lemma follows, noting that $t \coloneqq \alpha \cdot m$.

For the second part pertaining to accuracy, first we note that in each iteration, if the measurement in Algorithm 1 does *not* make the algorithm terminate, the iteration had the effect of taking $|\phi\rangle \in D$ to

$$\Pi_D\left(I^{\otimes n} \otimes e^{-i\alpha H}\right)|\phi\rangle = \Pi_D\left(I^{\otimes n} \otimes e^{-i\alpha H}\right)\Pi_D|\phi\rangle,$$

normalized to length 1. After the m iterations of the loop of Algorithm 1, $|\phi\rangle$ is then

$$\prod_{i=1}^{m}\Pi_D\left(I^{\otimes n} \otimes e^{-i\alpha H}\right)\Pi_D|\sigma_{I^{\otimes n}}\rangle$$

normalized to length 1. By Lemma 13, before normalization this is equivalent to

$$\prod_{i=1}^{m}\left(e^{-i\alpha A} + \eta\right)|\sigma_{I^{\otimes n}}\rangle = \left(\sum_{k=0}^{m}\binom{m}{k}e^{-i\alpha A(m-k)} \cdot \eta^k\right)|\sigma_{I^{\otimes n}}\rangle$$

for $\|\eta\|_\infty \leq \alpha^2 e^{\alpha}$. The distance of the un-normalized vector from $e^{-iAt}|\sigma_{I^{\otimes n}}\rangle$ is then

$$\left\|e^{-iAt}|\sigma_{I^{\otimes n}}\rangle - \prod_{i=1}^{m}\left(e^{-iAt} + \eta\right)|\sigma_{I^{\otimes n}}\rangle\right\|_2 = \left\|\left(\sum_{k=1}^{m}\binom{m}{k}e^{-i\alpha A(m-k)} \cdot \eta^k\right)|\sigma_{I^{\otimes n}}\rangle\right\|_2$$
$$\leq \sum_{k=1}^{m} m^k\|\eta\|_\infty^k \leq \sum_{k=1}^{m}\left(m\alpha^2 e^{\alpha}\right)^k \leq \sum_{k=1}^{\infty}\left(m\alpha^2 e^{\alpha}\right)^k = m\alpha^2 e^{\alpha}\frac{1}{1 - m\alpha^2 e^{\alpha}} = \alpha t e^{\alpha}\frac{1}{1 - \alpha t e^{\alpha}}.$$

Finally, to bound the error introduced by normalization, for each $r \in [m]$, write $|\phi_r\rangle := \prod_{i=1}^{r} \Pi_D(I^{\otimes n} \otimes e^{-i\alpha H})\Pi_D|\sigma_{I^{\otimes n}}\rangle$ for the projected state at iteration r. We note that, by the same argument proving that the probability of the measurement at any given step returning the $I^{\otimes 2n} - \Pi_D$ result is at most $\frac{99}{98}\alpha^2$, $|\phi_r\rangle$ is separated from $e^{-iAt}|\phi_{r-1}\rangle$ by an *orthogonal* vector of length at most $\sqrt{\frac{99}{98}}\alpha\|e^{-iAt}|\phi_{r-1}\rangle\|_2 = \sqrt{\frac{99}{98}}\alpha\||\phi_{r-1}\rangle\|_2$. Therefore,

$$\||\phi_r\rangle\|_2 \geq \||\phi_{r-1}\rangle\|_2\sqrt{1 - \frac{99}{98}\alpha^2} \geq \||\phi_{r-1}\rangle\|_2 - 0.6\frac{99}{98}\alpha^2$$

where the last inequality follows from the fact that $1 - \sqrt{1-x} \leq 0.6x$ for $x \in [0, \frac{5}{9}]$ and $\frac{99}{98}\alpha^2 < \frac{5}{9}$. The total additional error from the normalization is then at most $\frac{297}{490}\alpha^2 m = \frac{297}{490}\alpha t$. By the triangle inequality, the total distance from $e^{-iAt}|\sigma_{I^{\otimes n}}\rangle$ is at most

$$\frac{297}{490}\alpha t + t\alpha e^{\alpha}\frac{1}{1 - \alpha t e^{\alpha}} \leq \frac{7}{4}\alpha t. \qquad \blacktriangleleft$$

We now show that (approximately) applying e^{-iAt} instead of $I^{\otimes n} \otimes e^{-iHt}$ allows us to suppress the higher-order terms that were preventing us from increasing the evolution time t when testing for locality. We will need the following results that let us characterize the individual terms of the Taylor expansion.

▶ **Fact 15.** *For any matrix M, $\langle\sigma_P|(I \otimes M)|\sigma_Q\rangle = \frac{\mathrm{Tr}(PMQ)}{2^n}$.*

Proof.

$$\langle\sigma_P|(I \otimes M)|\sigma_Q\rangle = \frac{1}{2^n}\sum_{x,y\in\{0,1\}^n} (\langle x| \otimes \langle x|P)\,(|y\rangle \otimes MQ|y\rangle)$$
$$= \frac{1}{2^n}\sum_{x,y\in\{0,1\}^n} \langle x|y\rangle \cdot \langle x|PMQ|y\rangle = \frac{1}{2^n}\sum_{x\in\{0,1\}^n} \langle x|PMQ|x\rangle = \frac{\mathrm{Tr}(PMQ)}{2^n} \qquad \blacktriangleleft$$

▶ **Lemma 16.** $\langle\sigma_{I^{\otimes n}}|A|\sigma_{I^{\otimes n}}\rangle = 0$.

Proof.

$$\langle\sigma_{I^{\otimes n}}|A|\sigma_{I^{\otimes n}}\rangle = \langle\sigma_{I^{\otimes n}}|\Pi_D\left(I^{\otimes n} \otimes H\right)\Pi_D|\sigma_{I^{\otimes n}}\rangle = \langle\sigma_{I^{\otimes n}}|I^{\otimes n} \otimes H|\sigma_{I^{\otimes n}}\rangle$$
$$= \frac{1}{2^n}\mathrm{Tr}\,(H) = 0 \qquad \text{(Fact 15)}$$

recalling that we have assumed that $\mathrm{Tr}(H) = 0$. ◀

▶ **Lemma 17.** *For $k \geq 2$, $|\langle\sigma_{I^{\otimes n}}|A^k|\sigma_{I^{\otimes n}}\rangle| \leq \langle\sigma_{I^{\otimes n}}|A^2|\sigma_{I^{\otimes n}}\rangle = \|H_{>k}\|_2^2$.*

Proof. The first inequality follows because $\|A\|_\infty \leq \|H\|_\infty \leq 1$, and the fact that H is Hermitian and so A is too, meaning that every eigenvalue of A^k is non-increasing in magnitude as a function of k, and non-negative when k is even.

For the second equality, we observe that

$$A|\sigma_{I^{\otimes n}}\rangle = \Pi_D\left(I^{\otimes n} \otimes H\right)\Pi_D|\sigma_{I^{\otimes n}}\rangle = \Pi_D\left(I^{\otimes n} \otimes H\right)|\sigma_{I^{\otimes n}}\rangle = \left(I^{\otimes n} \otimes H_{>k}\right)|\sigma_{I^{\otimes n}}\rangle,$$

as H has no identity component. By Fact 15,

$$\langle\sigma_{I^{\otimes n}}|A^2|\sigma_{I^{\otimes n}}\rangle = \langle\sigma_{I^{\otimes n}}|I^{\otimes n} \otimes (H_{>k})^2|\sigma_{I^{\otimes n}}\rangle = \frac{1}{2^n}\mathrm{Tr}\left((H_{>k})^2\right) = \|H_{>k}\|_2^2. \qquad \blacktriangleleft$$

Combining Lemmas 14, 16, and 17, we are able to give bounds on the acceptance probability of Algorithm 1 (assuming it does not terminate early) based on how close or far H is from being k-local. This gives us an algorithm for testing locality, through repetition of Algorithm 1 and concentration of measure.

▶ **Lemma 18.** *Let $\varepsilon := \|H_{>k}\|_2$. The probability that Algorithm 1 outputs 1, conditioned on not terminating early, is at least $\varepsilon^2 t^2 \left(1 - \frac{3}{10}\varepsilon^2 t^2\right) - \frac{7}{2}\varepsilon\alpha t^2$ and no more than $\varepsilon^2 t^2 \left(1 + \frac{1}{10}t^2\right) + \frac{287}{80}\varepsilon\alpha t^2 + \frac{49}{1600}\varepsilon_2\alpha t^2$.*[8]

Proof. At the end of Algorithm 1 (assuming it did not terminate early), the final state lies in D. By Lemma 14 and the definition of the final measurement, the probability that the algorithm outputs 1 is the squared length of the component of $|\psi\rangle := e^{-iAt}|\sigma_{I^{\otimes n}}\rangle + |\Delta\rangle$ along the complement of $|\sigma_{I^{\otimes n}}\rangle$, for some Δ such that $\||\Delta\rangle\|_2 \leq 2\alpha t$. So by the triangle inequality, $\Pr[X = 1]$ is in the range[9]

$$\left(\left(\sqrt{1 - |\langle\sigma_{I^{\otimes n}}|e^{-iAt}|\sigma_{I^{\otimes n}}\rangle|^2} - \||\Delta\rangle\|_2\right)^2, \left(\sqrt{1 - |\langle\sigma_{I^{\otimes n}}|e^{-iAt}|\sigma_{I^{\otimes n}}\rangle|^2} + \||\Delta\rangle\|_2\right)^2\right).$$

To analyze $\left|\langle\sigma_{I^{\otimes n}}|e^{-iAt}|\sigma_{I^{\otimes n}}\rangle\right|$, we note that because A is Hermitian, $\langle\sigma_{I^{\otimes n}}|A^k|\sigma_{I^{\otimes n}}\rangle$ is real-valued for all $k \geq 0$. By splitting up the Taylor expansion of the matrix exponential into real and imaginary terms, we see that

$$\begin{aligned}
&\left|\langle\sigma_{I^{\otimes n}}|e^{-iAt}|\sigma_{I^{\otimes n}}\rangle\right|^2 = \left|\langle\sigma_{I^{\otimes n}}|\left(\sum_{m=0}^{\infty}(-i)^m\frac{A^m t^m}{m!}\right)|\sigma_{I^{\otimes n}}\rangle\right|^2 \\
&= \left|\langle\sigma_{I^{\otimes n}}|\left(\sum_{m=0}^{\infty}(-1)^m\frac{A^{2m}t^{2m}}{(2m)!}\right)|\sigma_{I^{\otimes n}}\rangle\right|^2 + \left|\langle\sigma_{I^{\otimes n}}|\left(\sum_{m=0}^{\infty}(-1)^{m+1}\frac{A^{2m+1}t^{2m+1}}{(2m+1)!}\right)|\sigma_{I^{\otimes n}}\rangle\right|^2.
\end{aligned}$$

Analyzing the first term, we see that

$$\begin{aligned}
&\left|\langle\sigma_{I^{\otimes n}}|\left(\sum_{m=0}^{\infty}(-1)^m\frac{A^{2m}t^{2m}}{(2m)!}\right)|\sigma_{I^{\otimes n}}\rangle\right| \\
&= \left|\langle\sigma_{I^{\otimes n}}|\left(I^{\otimes 2n} - \frac{t^2}{2}A^2 + \sum_{m=2}^{\infty}(-1)^m\frac{A^{2m}t^{2m}}{(2m)!}\right)|\sigma_{I^{\otimes n}}\rangle\right| \\
&= \left|\frac{\mathrm{Tr}(I^{\otimes n})}{2^n} - \frac{t^2}{2}\langle\sigma_{I^{\otimes n}}|A^2|\sigma_{I^{\otimes n}}\rangle + \langle\sigma_{I^{\otimes n}}|\left(\sum_{m=2}^{\infty}(-1)^m\frac{A^{2m}t^{2m}}{(2m)!}\right)|\sigma_{I^{\otimes n}}\rangle\right| && \text{(Fact 15)} \\
&= \left|1 - \frac{\varepsilon^2 t^2}{2} + \sum_{m=2}^{\infty}(-1)^m\langle\sigma_{I^{\otimes n}}|\frac{A^{2m}t^{2m}}{(2m)!}|\sigma_{I^{\otimes n}}\rangle\right| && \text{(Lemma 17)} \\
&= 1 - \frac{\varepsilon^2 t^2}{2} + \eta_{\mathrm{real}}
\end{aligned}$$

where $|\eta_{\mathrm{real}}| \leq \frac{\varepsilon^2 t^4}{24}\cosh(t) \leq \frac{\varepsilon^2 t^4}{20}$ by Fact 10, Lemma 17, the triangle inequality, and the fact that $t \leq \frac{1}{2}$.

8 The ε_2 in the $\frac{49}{1600}\varepsilon_2\alpha t^2$ term of the upper bound is intended and *not* a typo.

9 One might think to use $1 - \left|\langle\sigma_{I^{\otimes n}}|\left(e^{-iAt}|\sigma_{I^{\otimes n}}\rangle + |\Delta\rangle\right)\right|^2$ followed by the triangle inequality, but this actually leads to a lossy analysis of the number of queries used.

Then, for the second term, we have

$$\begin{aligned}
\eta_{\text{imaginary}} &\coloneqq \left| \langle \sigma_{I^{\otimes n}} | \left(\sum_{m=0}^{\infty} (-1)^m \frac{A^{2m+1} t^{2m+1}}{(2m+1)!} \right) | \sigma_{I^{\otimes n}} \rangle \right| \\
&= \left| \langle \sigma_{I^{\otimes n}} | \left(A + \sum_{m=1}^{\infty} (-1)^{m+1} \frac{A^{2m+1} t^{2m+1}}{(2m+1)!} \right) | \sigma_{I^{\otimes n}} \rangle \right| \\
&= \left| \langle \sigma_{I^{\otimes n}} | \left(\sum_{m=1}^{\infty} (-1)^m \frac{A^{2m+1} t^{2m+1}}{(2m+1)!} \right) | \sigma_{I^{\otimes n}} \rangle \right| && \text{(Lemma 16)} \\
&\leq \varepsilon^2 \sum_{m=1}^{\infty} \frac{t^{2m+1}}{(2m+1)!} && \text{(Lemma 17)} \\
&\leq \varepsilon^2 \frac{t^3}{6} \cosh(t) \leq \frac{1}{10} \varepsilon^2 t^2. && \text{(Fact 10)}
\end{aligned}$$

Since

$$\left| \langle \sigma_{I^{\otimes n}} | e^{-iAt} | \sigma_{I^{\otimes n}} \rangle \right|^2 = \left(1 - \frac{\varepsilon^2 t^2}{2} + \eta_{\text{real}} \right)^2 + \eta_{\text{imaginary}}^2,$$

we can upper bound it by $\left(1 - \frac{\varepsilon^2 t^2}{2} + |\eta_{\text{real}}|\right)^2 + \eta_{\text{imaginary}}^2$ and, as $\eta_{\text{imaginary}} \geq 0$, lower bound it by $\left(1 - \frac{\varepsilon^2 t^2}{2} - |\eta_{\text{real}}|\right)^2$.

We can therefore upper bound the probability of Algorithm 1 accepting by

$$\begin{aligned}
&\left(\sqrt{1 - |\langle \sigma_{I^{\otimes n}} | e^{-iAt} | \sigma_{I^{\otimes n}} \rangle|^2} + \| |\Delta\rangle \|_2 \right)^2 \\
&\leq \left(\sqrt{1 - \left(1 - \frac{\varepsilon^2 t^2}{2} - |\eta_{\text{real}}|\right)^2} + \frac{7}{4} \alpha t \right)^2 && \text{(Lemma 14)} \\
&\leq \left(\sqrt{\varepsilon^2 t^2 + 2|\eta_{\text{real}}|} + \frac{7}{4} \alpha t \right)^2 \\
&\leq \varepsilon^2 t^2 + 2|\eta_{\text{real}}| + \frac{7}{2} \alpha t \sqrt{\varepsilon^2 t^2 + \frac{1}{10} \varepsilon^2 t^4} + \frac{49}{16} \alpha^2 t^2 \\
&\leq \varepsilon^2 t^2 \left(1 + \frac{1}{10} t^2\right) + \frac{287}{80} \varepsilon \alpha t^2 + \frac{49}{1600} \varepsilon_2 \alpha t^2 && \left(t \leq 0.5,\ \alpha \leq \frac{\varepsilon_2}{100}\right)
\end{aligned}$$

and lower bound it by

$$\begin{aligned}
&\left(\sqrt{1 - |\langle \sigma_{I^{\otimes n}} | e^{-iAt} | \sigma_{I^{\otimes n}} \rangle|^2} - \| |\Delta\rangle \|_2 \right)^2 \\
&\geq \left(\sqrt{1 - \left(1 - \frac{\varepsilon^2 t^2}{2} + |\eta_{\text{real}}|\right)^2 - \eta_{\text{imaginary}}^2} - \| |\Delta\rangle \|_2 \right)^2 \\
&\geq \varepsilon^2 t^2 - \left(\frac{\varepsilon^2 t^2}{2} + |\eta_{\text{real}}| \right)^2 - |\eta_{\text{imaginary}}|^2 - \frac{7}{2} \varepsilon \alpha t^2 \\
&\geq \varepsilon^2 t^2 \left(1 - \frac{3}{10} \varepsilon^2 t^2\right) - \frac{7}{2} \varepsilon \alpha t^2.
\end{aligned}$$

◀

▶ **Theorem 1.** *Let $0 \le \varepsilon_1 < \varepsilon_2 \le 1$, $\delta \in (0,1)$, and $k \in \mathbb{N}$. There is an algorithm that distinguishes whether an n-qubit Hamiltonian H is (1) within ε_1 of some k-local Hamiltonian or (2) ε_2-far from all k-local Hamiltonians, with probability $1-\delta$. The algorithm uses $\mathrm{O}\left(\sqrt{\frac{\varepsilon_2}{(\varepsilon_2-\varepsilon_1)^7}}\log(1/\delta)\right)$ non-adaptive queries to the time evolution operator with $\mathrm{O}\left(\sqrt{\frac{\varepsilon_2}{(\varepsilon_2-\varepsilon_1)^5}}\log(1/\delta)\right)$ total evolution time.*

Proof. By Lemma 18 the output of Algorithm 1, conditioned on succeeding, is a Bernoulli random variable X_i with bounded expectation. That is, when $\varepsilon \ge \varepsilon_2$ then

$$\mathbb{E}[X_i] \ge \varepsilon_2^2 t^2\left(1-\frac{3}{10}\varepsilon_2^2 t^2\right) - \frac{7}{2}\varepsilon_2\alpha t^2$$

and when $\varepsilon \le \varepsilon_1$ then

$$\mathbb{E}[X_i] \le \varepsilon_1^2 t^2\left(1+\frac{1}{10}t^2\right) + \frac{287}{80}\varepsilon_1\alpha t^2 + \frac{49}{1600}\varepsilon_2\alpha t^2.$$

Let

$$\tau \coloneqq \frac{1}{2}\left[\varepsilon_2^2 t^2\left(1-\frac{3}{10}\varepsilon_2^2 t^2\right) - \frac{7}{2}\varepsilon_2\alpha t^2 + \varepsilon_1^2 t^2\left(1+\frac{1}{10}t^2\right) + \frac{287}{80}\varepsilon_1\alpha t^2 + \frac{49}{1600}\varepsilon_2\alpha t^2\right]$$

then be the halfway point these two values, and our decision threshold. And for convenience let

$$\xi \coloneqq \frac{1}{2}\left[\varepsilon_2^2 t^2\left(1-\frac{3}{10}\varepsilon_2^2 t^2\right) - \frac{7}{2}\varepsilon_2\alpha t^2 - \varepsilon_1^2 t^2\left(1+\frac{1}{10}t^2\right) - \frac{287}{80}\varepsilon_1\alpha t^2 - \frac{49}{1600}\varepsilon_2\alpha t^2\right]$$

be a lower bound on the distance from τ to our bounds on $\mathbb{E}[X_i]$. Observe that $\varepsilon_1 < \varepsilon_2 \le 1$, $\varepsilon_2\alpha = \frac{\varepsilon_2^2-\varepsilon_1^2}{100}$ and $t = \frac{\sqrt{\varepsilon_2^2-\varepsilon_1^2}}{2\varepsilon_2}$ so:

$$\frac{9}{80}\frac{(\varepsilon_2^2-\varepsilon_1)^2}{\varepsilon_2^2} \le \frac{1}{2}(\varepsilon_2^2-\varepsilon_1^2)t^2 - \frac{1}{5}\varepsilon_2^2 t^4 \le \xi \le \frac{\varepsilon_2^2-\varepsilon_1^2}{2}t^2 \le \frac{\varepsilon_2^2 t^2}{2}.$$

Now say that we have i.i.d samples $\{X_1,\ldots,X_s\}$ from *successful* runs of Algorithm 1 for s to be determined and let $X \coloneqq \sum_{i=1}^{s} X_i$. If $\varepsilon \ge \varepsilon_2$, then by Bernstein's inequality the probability that $X \le s\tau$ is at most:

$$\begin{aligned}
\Pr\left[\sum_{i=1}^{s} X_i \le s\tau\right] &= \Pr\Big[X - \mathbb{E}[X] \le s\tau - \mathbb{E}[X]\Big] \\
&\le \exp\left[-\frac{\frac{(s\tau-\mathbb{E}[X])^2}{2}}{s\,\mathbb{E}[X]\,(1-\mathbb{E}[X]) + \frac{\mathbb{E}[X]-s\tau}{3}}\right] \\
&\le \exp\left[-\frac{\left(s\tau-\mathbb{E}[X]\right)^2}{2\left(s\,\mathbb{E}[X] + \frac{\mathbb{E}[X]-s\tau}{3}\right)}\right] \\
&\le \exp\left[-\frac{s\xi^2}{2\left(\varepsilon_2^2 t^2 + \frac{\xi}{3}\right)}\right] \\
&\le \exp\left[-\frac{3s\xi^2}{7\varepsilon_2^2 t^2}\right] \\
&\le \exp\left[-\frac{s}{46.5}\frac{(\varepsilon_2^2-\varepsilon_1^2)^3}{\varepsilon_2^4}\right]
\end{aligned}$$

where the fourth line follows due to the expression in the exponential being monotonically increasing with respect to $\mathbb{E}[X] \in (\tau, 1]$. Likewise, if $\varepsilon \le \varepsilon_1$ then the probability that $X \ge s\tau$ is at most:

$$
\begin{aligned}
\Pr\left[\sum_{i=1}^{s} X_i \ge s\tau\right] &= \Pr\left[X - \mathbb{E}[X] \ge s\tau - \mathbb{E}[X]\right] \\
&\le \exp\left[-\frac{\frac{(s\tau-\mathbb{E}[X])^2}{2}}{s\,\mathbb{E}[X]\,(1-\mathbb{E}[X]) + \frac{s\tau-\mathbb{E}[X]}{3}}\right] \\
&\le \exp\left[-\frac{(s\tau - \mathbb{E}[X])^2}{2\left(s\,\mathbb{E}[X] + \frac{s\tau-\mathbb{E}[X]}{3}\right)}\right] \\
&\le \exp\left[-\frac{s\xi^2}{2\left(\varepsilon_1^2 t^2\left(1+\frac{1}{10}t^2\right) + \frac{287}{80}\varepsilon_1\alpha t^2 + \frac{49}{1600}\varepsilon_2\alpha t^2 + \frac{\xi}{3}\right)}\right] \\
&\le \exp\left[-\frac{s\xi^2}{2\left(\varepsilon_2^2 t^2\left(1+\frac{1}{40}+\frac{287}{800}+\frac{49}{16000}+\frac{1}{6}\right)\right)}\right] \\
&\qquad\qquad\qquad\qquad (\varepsilon_1 < \varepsilon_2,\ t \le 0.5,\ \alpha \le \frac{\varepsilon_2}{100}) \\
&\le \exp\left[-\frac{s}{55.9}\frac{(\varepsilon_2^2-\varepsilon_1^2)^3}{\varepsilon_2^4}\right]
\end{aligned}
$$

where the fourth line also follows due to the expression in the exponential being monotonically decreasing with respect to $\mathbb{E}[X] \in [0, \tau)$. Therefore, setting

$$
s = 55.9\frac{\varepsilon_2^4}{(\varepsilon_2^2 - \varepsilon_1^2)^3}\ln(2/\delta)
$$

suffices for us to succeed at distinguishing the two cases with probability at most $1 - \delta/2$.

Algorithm 1 has an $\frac{99}{98}\alpha t < \frac{99}{19600}\frac{(\varepsilon_2^2-\varepsilon_1^2)^{3/2}}{\varepsilon_2^2} \le \frac{99}{19600}$ chance of failure. By applying Corollary 6,

$$
s' = \frac{2}{1 - \frac{99}{19600}}\left(s + \ln(2/\delta)\right) \le 115\frac{\varepsilon_2^4}{(\varepsilon_2^2 - \varepsilon_1^2)^3}\ln(2/\delta)
$$

suffices to achieve s successful runs with probability $1 - \delta/2$. By a union bound, we will correctly differentiate the two cases with probability at least $1 - \delta$.

The total time complexity used is then

$$
\begin{aligned}
s't &\le 115\frac{\varepsilon_2^4}{(\varepsilon_2^2-\varepsilon_1^2)^3}\ln(2/\delta)\cdot\frac{\sqrt{\varepsilon_2^2-\varepsilon_1^2}}{2\varepsilon_2} \le 58\frac{\varepsilon_2^3}{((\varepsilon_2-\varepsilon_1)(\varepsilon_2+\varepsilon_1))^{5/2}}\log(2/\delta) \\
&\le 58\sqrt{\frac{\varepsilon_2}{(\varepsilon_2-\varepsilon_1)^5}}\log(2/\delta) = \mathrm{O}\left(\sqrt{\frac{\varepsilon_2}{(\varepsilon_2-\varepsilon_1)^5}}\log(1/\delta)\right),
\end{aligned}
$$

with a total number of queries of

$$
\begin{aligned}
s'm &= \frac{s't}{\alpha} \le 58\frac{\varepsilon_2^3}{(\varepsilon_2^2-\varepsilon_1^2)^{5/2}}\log(2/\delta)\cdot\frac{100\varepsilon_2}{\varepsilon_2^2-\varepsilon_1^2} \le 5800\frac{\varepsilon_2^4}{(\varepsilon_2^2-\varepsilon_1^2)^{7/2}}\log(2/\delta) \\
&\le 5800\sqrt{\frac{\varepsilon_2}{(\varepsilon_2-\varepsilon_1)^7}} = \mathrm{O}\left(\sqrt{\frac{\varepsilon_2}{(\varepsilon_2-\varepsilon_1)^7}}\right).
\end{aligned}
$$

◀

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A Lower Bound

We will utilize the following fact about diamond distance of unitaries that will make calculations easier, at a loss of some constant factors.

▶ **Fact 19** ([13, Proposition 1.6]). *For all unitaries U and V of equal dimension,*

$$\frac{1}{2}\|U - V\|_\diamond \le \min_{\theta\in[0,2\pi)} \|e^{i\theta}U - V\|_\infty \le \|U - V\|_\diamond.$$

We now show our lower bound for k-locality testing, simply by showing that the statistical distance of the resulting unitaries (i.e., diamond distance) only grows linearly with time.

▶ **Definition 20.** *For $0 \le k \le n$, we define*

$$Z_{1:k} := \bigotimes_{i=1}^{k} Z \otimes \bigotimes_{j=k+1}^{n} I$$

to be the tensor product of Z on the first k qubits and identity on the last $n - k$ qubits.

▶ **Lemma 21.** *For $0 \le \varepsilon_1 \le \varepsilon_2$*

$$\|e^{-iZ_{1:k}\varepsilon_1 t} - e^{-iZ_{1:k}\varepsilon_2 t}\|_\diamond \le 2(\varepsilon_1 - \varepsilon_2)t.$$

Proof. Since $Z_{1:k}$ is diagonal with ± 1 entries, $e^{-iZ_{1:k}\varepsilon t}$ is diagonal with entries $e^{\mp i\varepsilon t}$. Therefore, the eigenvalues of $e^{i\theta} \cdot e^{-iZ_{1:k}\varepsilon_1 t} - e^{-iZ_{1:k}\varepsilon_2 t}$ can be directly calculated, giving us

$$\begin{aligned}
&\min_{\theta\in[0,2\pi)} \|e^{i\theta} \cdot e^{-iZ_{1:k}\varepsilon_1 t} - e^{-iH\varepsilon_2 t}\|_\infty \\
&= \min_{\theta\in[0,2\pi)} \max\left(|e^{i(\theta-\varepsilon_1 t)} - e^{-i\varepsilon_2 t}|, |e^{i(\theta+\varepsilon_1 t)} - e^{i\varepsilon_2 t}|\right) \\
&= \min\left(|e^{-i\varepsilon_1 t} - e^{-i\varepsilon_2 t}|, |e^{-i\varepsilon_1 t} + e^{-i\varepsilon_2 t}|\right) \\
&= 2\min\left(\left|\sin\left(\frac{(\varepsilon_2-\varepsilon_1)t}{2}\right)\right|, \left|\cos\left(\frac{(\varepsilon_2-\varepsilon_1)t}{2}\right)\right|\right) \\
&\le (\varepsilon_2 - \varepsilon_1)t,
\end{aligned}$$

where one of $\theta \in \{0, \pi\}$ minimizes the value via symmetry. By Fact 19, $\|e^{-iZ_{1:k}\varepsilon_1 t} - e^{-iZ_{1:k}\varepsilon_2 t}\|_\diamond \le 2(\varepsilon_1 - \varepsilon_2)t$.[10] ◀

[10] A direct calculation of the diamond distance will give an upper bound of $(\varepsilon_2 - \varepsilon_1)t$, without the factor of 2 from Fact 19. See [14, Proof of Proposition 1.6].

▶ Remark 22. Lemma 21 easily extends to the scenario where one is allowed to make calls to the inverse oracle, controlled versions of the oracle, the complex conjugate of the oracle, and any combination of these augmentations, as the diamond distance between the corresponding unitaries can be bounded as a function of time evolution.

We are now ready to prove our tolerant locality testing lower bound by reducing to Lemma 21.

▶ **Theorem 2.** *Let $0 \leq \varepsilon_1 < \varepsilon_2 \leq 1$ and $k \in \mathbb{N}$. Then any algorithm that can distinguish whether an n-qubit Hamiltonian H is (1) within ε_1 of some k-local Hamiltonian or (2) ε_2-far from all k-local Hamiltonians, must use $\Omega\left(\frac{1}{\varepsilon_2-\varepsilon_1}\right)$ evolution time in expectation to achieve constant success probability.*

Proof. Observe that for any $k' > k$, $H_1 \coloneqq \varepsilon_1 Z_{1:k'}$ is within ε_1 of being k-local and $H_2 \coloneqq \varepsilon_2 Z_{1:k'}$ is likewise ε_2-far from being k-local. $\|H_1\|_\infty \leq \|H_2\|_\infty \leq 1$ is also satisfied. Let t_i be the time evolution for each query in our algorithm. By Lemma 21, the diamond distance between the time evolution of these two cases is at most $2(\varepsilon_2 - \varepsilon_1)t_i$ for each query. By the sub-additivity of diamond distance, a total time evolution of $\sum_i t_i = \Omega\big((\varepsilon_2 - \varepsilon_1)^{-1}\big)$ is required to distinguish H_1 and H_2 with constant probability. ◀

▶ Remark 23. Theorem 2 also holds when the distance to k-locality is determined by operator norm $\|\cdot\|_\infty$, any *normalized* schatten p-norm $\|X\|_p \coloneqq \frac{1}{2^{n/p}} \mathrm{Tr}\left(|X|^p\right)^{\frac{1}{p}}$, or any Pauli decomposition p-norm $\|X\|_{\mathrm{Pauli},p} \coloneqq \left(\sum_{P\in\mathcal{P}^{\otimes n}} |\alpha_P|^p\right)^{\frac{1}{p}}$ for $X = \sum_{P\in\mathcal{P}^{\otimes n}} \alpha_P P$, improving upon that of [6, Theorem 3.6]. This is simply because the distance of $\varepsilon Z_{1:k'}$ (for $k' > k$) from being k-local is exactly ε for all of these distance measures.

B Optimal Tolerant Testing with Inverse Queries

In this section we augment the tolerant testing algorithm in [12, 3], with amplitude estimation to get an optimal tolerant tester when given access to controlled versions of the forward and reverse time evolution.[11]

We begin with the following crucial result of Gutiérrez.

▶ **Lemma 24** ([3, Lemma 3.1]). *Let $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq 1$. Let $\alpha \coloneqq \frac{\varepsilon_2-\varepsilon_1}{3c}$ and H be an n-qubit Hamiltonian with $\|H\|_\infty = 1$. Define $U \coloneqq e^{-iH\alpha}$, and let $U_{>k}$ be $U|\sigma_{I^{\otimes n}}\rangle$ projected onto onto the space spanned by $\{(I\otimes P)|\sigma_{I^{\otimes n}}\rangle : P \in \{I,X,Y,Z\}^{\otimes n}, |P| > k\}$. We have that if H is ε_1-close to being k-local, then*

$$\|U_{>k}\|_2^2 \leq \left((\varepsilon_2 - \varepsilon_1)\frac{2\varepsilon_1 + \varepsilon_2}{9c}\right)^2,$$

and if H is ε_2-far from being k-local, then

$$\|U_{>k}\|_2^2 \geq \left((\varepsilon_2 - \varepsilon_1)\frac{\varepsilon_1 + 2\varepsilon_2}{9c}\right)^2.$$

We also cite the following result of [11], which itself follows as a corollary of the celebrated Quantum Amplitude Estimation [7, Theorem 12] result.

[11] Using the multiplicative error form from [20] should allow for one to remove the need for controlled access while remaining non-adaptive, though it causes the constants to blow-up.

▶ **Lemma 25** (Quantum Amplitude Estimation [11, Corollary 29]). *Let Π be a projector and $|\psi\rangle$ be an n-qubit pure state such that $\langle\psi|\Pi|\psi\rangle = \eta$. Given access to the unitary transformations $R_\Pi = 2\Pi - I$ and $R_\psi = 2|\psi\rangle\langle\psi| - I$, there exists a quantum algorithm that outputs $\widehat{\eta}$ such that*

$$|\widehat{\eta} - \eta| \leq \xi$$

with probability at least $\frac{8}{\pi^2}$. The algorithm makes no more than $\pi\frac{\sqrt{\eta(1-\eta)+\xi}}{\xi}$ calls to the controlled versions of R_Π and R_ψ.

In particular, this implies that if we have (controlled) query access to U, U^* for some unitary U, and a known state $|\phi\rangle$, we can estimate $\eta = \|\Pi U|\phi\rangle\|_2^2$ to ζ accuracy by defining $|\psi\rangle := U|\phi\rangle$ and implementing R_ψ with controlled applications of U.

We are now ready to state the algorithm, which can be seen as the algorithm of [12, 3] augmented with Lemma 25.

▶ **Theorem 4.** *Let $0 \leq \varepsilon_1 < \varepsilon_2 \leq 1$, $\delta \in (0,1)$, and $k \in \mathbb{N}$. There is an algorithm that tests whether an n-qubit Hamiltonian H is (1) ε_1-close to some k-local Hamiltonian or (2) ε_2-far from all k-local Hamiltonians, with probability $1-\delta$. The algorithm uses $\mathrm{O}\left(\frac{\log(1/\delta)}{(\varepsilon_2-\varepsilon_1)^2}\right)$ non-adaptive queries to the time evolution operator and its inverse, with $\mathrm{O}\left(\frac{\log(1/\delta)}{\varepsilon_2-\varepsilon_1}\right)$ total evolution time.*

Proof. Let $U := e^{-iH\alpha}$ as in Lemma 24. We apply Lemma 24 with Π the projector onto the space spanned by $\{(I \otimes P)|\sigma_{I^{\otimes n}}\rangle : P \in \{I, X, Y, Z\}^{\otimes n}, |P| > k\}$ to estimate $\|U_{>k}\|_2^2$. Observe that the absolute difference between the two terms in Lemma 24 is

$$\left((\varepsilon_2 - \varepsilon_1)\frac{\varepsilon_1 + 2\varepsilon_2}{9c}\right)^2 - \left((\varepsilon_2 - \varepsilon_1)\frac{2\varepsilon_1 + \varepsilon_2}{9c}\right)^2 = \frac{(\varepsilon_2 - \varepsilon_1)^3(\varepsilon_2 + \varepsilon_1)}{27c^2}.$$

Therefore, we can distinguish the two cases to constant success probability by estimating $\eta = \|U_{>k}\|_2^2$ to error $\zeta = \frac{(\varepsilon_2-\varepsilon_1)^3(\varepsilon_2+\varepsilon_1)}{54c^2}$. By Lemma 25, the number of queries is then no more than

$$\begin{aligned}
&\pi\frac{\sqrt{(\varepsilon_2 - \varepsilon_1)^2(\varepsilon_1 + 2\varepsilon_2)^2/(81c^2) + (\varepsilon_2 - \varepsilon_1)^3(\varepsilon_1 + \varepsilon_2)/(54c^2)}}{(\varepsilon_2 - \varepsilon_1)^3(\varepsilon_1 + \varepsilon_2)/(54c^2)}\\
=&\frac{54\pi c}{(\varepsilon_2 - \varepsilon_1)^2}\frac{\sqrt{(\varepsilon_1 + 2\varepsilon_2)^2/81 + (2\varepsilon_2 - 2\varepsilon_1)(2\varepsilon_1 + 2\varepsilon_2)/216}}{\varepsilon_1 + \varepsilon_2}\\
\leq&\frac{54\pi c}{(\varepsilon_2 - \varepsilon_1)^2}\frac{\sqrt{(2\varepsilon_1 + 2\varepsilon_2)^2/81 + (2\varepsilon_1 + 2\varepsilon_2)^2/216}}{\varepsilon_1 + \varepsilon_2}\\
\leq&\frac{54\pi c}{(\varepsilon_2 - \varepsilon_1)^2}\frac{\sqrt{11(2\varepsilon_1 + 2\varepsilon_2)^2/648}}{\varepsilon_1 + \varepsilon_2}\\
\leq&\frac{3\sqrt{22}\pi c}{(\varepsilon_2 - \varepsilon_1)^2}.
\end{aligned}$$

Since the Hamiltonian is applied for $\alpha := \frac{\varepsilon_2-\varepsilon_1}{3c}$ for each query, the total evolution of the Hamiltonian is at most

$$\frac{3\sqrt{22}\pi c}{(\varepsilon_2 - \varepsilon_1)^2}\frac{\varepsilon_2 - \varepsilon_1}{3c} = \frac{\sqrt{22}\pi}{\varepsilon_2 - \varepsilon_1}.$$

By standard error reduction, we can reduce the constant failure probability to at most δ using $\log(1/\delta)$ repetitions.

Finally, observe that constructing R_Π (and its controlled version), as in Lemma 25 is free, as Π is a known projector onto the low locality Paulis. On the other hand, R_ψ requires us to take (a version of) the Grover Diffusion operator $D := 2|0\rangle\langle 0| - I$ and conjugate it by U. This is the step that requires access to $U^\dagger := e^{iH\alpha}$. ◀

Since this matches the lower bound of Theorem 2, Theorem 4 is optimal.