On Estimating the Quantum ℓ_α Distance

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Abstract

We study the computational complexity of estimating the quantum ℓ_α distance $\mathrm{T}_\alpha(\rho_0,\rho_1)$, defined via the Schatten α-norm $\|A\|_\alpha \coloneqq \mathrm{tr}(|A|^\alpha)^{1/\alpha}$, given $\mathrm{poly}(n)$-size state-preparation circuits of n-qubit quantum states ρ_0 and ρ_1. This quantity serves as a lower bound on the trace distance for $\alpha > 1$. For any constant $\alpha > 1$, we develop an efficient *rank-independent* quantum estimator for $\mathrm{T}_\alpha(\rho_0,\rho_1)$ with time complexity $\mathrm{poly}(n)$, achieving an *exponential* speedup over the prior best results of $\exp(n)$ due to Wang, Guan, Liu, Zhang, and Ying (*IEEE Trans. Inf. Theory* 2024). Our improvement leverages efficiently computable *uniform* polynomial approximations of *signed* positive power functions within quantum singular value transformation, thereby eliminating the dependence on the rank of the states.

Our quantum algorithm reveals a dichotomy in the computational complexity of the Quantum State Distinguishability Problem with Schatten α-norm (QSD_α), which involves deciding whether $\mathrm{T}_\alpha(\rho_0,\rho_1)$ is at least $2/5$ or at most $1/5$. This dichotomy arises between the cases of constant $\alpha > 1$ and $\alpha = 1$:

- For any $1+\Omega(1) \leq \alpha \leq O(1)$, QSD_α is BQP-complete.
- For any $1 \leq \alpha \leq 1+\frac{1}{n}$, QSD_α is QSZK-complete, implying that no efficient quantum estimator for $\mathrm{T}_\alpha(\rho_0,\rho_1)$ exists unless $\mathsf{BQP} = \mathsf{QSZK}$.

The hardness results follow from reductions based on new rank-dependent inequalities for the quantum ℓ_α distance with $1 \leq \alpha \leq \infty$, which are of independent interest.

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1 Introduction

Closeness testing of quantum states is a central topic in quantum property testing [58], which aims to develop (efficient) quantum testers for properties of quantum objects. This problem is also closely related to verifying the functionality of quantum devices, such as Q_0 and Q_1, which are commonly designed to prepare the respective n-qubit (mixed) quantum states ρ_0 and ρ_1. The goal of (tolerant) quantum state testing is to design efficient quantum

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algorithms that test whether ρ_0 is 2/5-far from or 1/5-close to ρ_1 with respect to a given closeness measure. Notably, this problem generalizes classical (tolerant) distribution testing (see [20] and [35, Chapter 11]) from a non-commutative perspective.

When the "source codes" of distribution- or state-preparation circuits are given, a surprising correspondence was established between such closeness testing problems – measured by the ℓ_1 norm distance [66, 37] or entropy difference [36] – and interactive proof systems that admit statistical zero-knowledge (SZK). This correspondence links closeness testing problems to both complexity theory and cryptography. A similar correspondence was later identified in the quantum world: closeness testing of quantum states with respect to the trace distance (given by Schatten 1-norm) [77, 78], denoted by QSD, or the von Neumann entropy difference [10] was shown to be QSZK-complete.[1]

In contrast, when the closeness measure follows an ℓ_2-norm-like definition, such as the Hilbert-Schmidt distance or the quantum linear entropy, the corresponding closeness testing problems are in BQP using the SWAP test [17, 27]. Taken together, these results reveal a dichotomy in the complexity of closeness testing: when the measure is ℓ_1-norm-like, the problems are QSZK-hard and their query or sample complexities have *polynomial* dependence on the dimension or rank of the states; whereas for ℓ_2-norm-like measures, the problems are contained in BQP and their query or sample complexities are *rank-independent*.

What about the closeness testing problems with respect to generalizations that approximates the trace distance or the von Neumann entropy? The quantum ℓ_α distance, defined as $\mathrm{T}_\alpha(\rho_0, \rho_1) := \frac{1}{2}\operatorname{tr}(|\rho_0 - \rho_1|^\alpha)^{1/\alpha}$ via the Schatten α-norm, generalizes both the trace distance ($\alpha = 1$) and the Hilbert-Schmidt distance ($\alpha = 2$). Similarly, the quantum q-Tsallis entropy $\mathrm{S}_q(\rho)$ extends both von Neumann entropy ($q = 1$) and quantum linear entropy ($q = 2$).

Interestingly, prior results show a divergence in behavior for closeness measures looser than the ℓ_2 norm: The closeness testing problem with respect to $\mathrm{T}_\alpha(\rho_0, \rho_1)$, denoted by QSD_α (see Definition 17), is in BQP only for *even* integer $\alpha \geq 2$ via the Shift test [27]; while for odd integers $\alpha \geq 3$, the query and sample complexities generally depend on the rank [71]. However, the techniques in [27] yield BQP algorithms for estimating $\mathrm{S}_q(\rho)$ for *all* integer $q \geq 2$. A recent work [53] further explored the closeness testing problem with respect to $\mathrm{S}_q(\rho_0) - \mathrm{S}_q(\rho_1)$, and extended the observed dichotomy from integers – where the transition occurs between $q = 1$ and $q \geq 2$ – to a continuous setting, showing a sharp distinction between $q = 1$ and any constant $q > 1$. These results naturally lead to an intriguing question:

▶ **Problem 1.** What is the computational complexity of the closeness testing problem with respect to $\mathrm{T}_\alpha(\rho_0, \rho_1)$? Does a similar dichotomy hold between $\alpha = 1$ and constants $\alpha > 1$, or does the complexity vary largely depending on whether α is even or odd?

Why study ℓ_α problems for possibly non-integer $\alpha > 1$? The trace distance ($\alpha = 1$) is a fundamental closeness measure of quantum states, capturing the maximum success probability of quantum state discrimination [41, 40] and playing a key role in applications such as the security of quantum key distribution [11, 63]. For $\alpha > 1$, such as $\alpha = 1.001$, the quantum ℓ_α distance provides a natural lower bound on the trace distance, and addressing Problem 1 could make this bound efficiently computable. Moreover, insights from ℓ_α problems have previously contributed to progress on well-studied ℓ_1 problems, as seen in [49].

Beyond their connections to ℓ_1 problems, ℓ_α problems for $\alpha > 1$ are of independent interest. In classical scenarios, they have applications in machine learning (e.g., [46]), as well as in streaming and sketching algorithms (e.g. [43]). In quantum scenarios, the Hilbert-

[1] The QSZK containment of the closeness testing problem with respect to the trace distance, denoted by $\mathrm{QSD}[a(n), b(n)]$, holds only in the polarizing regime $a(n)^2 - b(n) \geq 1/O(\log n)$, as shown in [77, 78]. A recent work [51] slightly improved the parameter regime for this containment.

Schmidt distance ($\alpha = 2$) is widely used in quantum information theory (e.g., [42, 61]), and more recently, has been leveraged in designing near-term (variational) quantum algorithms (e.g., [5, 28]). Consequently, positive answers to Problem 1 may offer new opportunities to refine, extend, or develop techniques relevant to these areas.

A classical counterpart to Problem 1 was investigated in [69] nearly a decade ago. The main takeaway aligns with [53]: For constant $\alpha > 1$, the sample complexity for distinguishing whether $\mathrm{TV}_\alpha(D_0, D_1)$ is at least $2/5$ or at most $1/5$ is *independent* of the dimension of the probability distributions D_0 and D_1,[2] fewer samples are needed as α increases. Classically, these upper bounds can be achieved by drawing a polynomial number of samples and computing the ℓ_∞ norm distance between the resulting empirical distributions. However, this approach does not directly extend to the quantum world: (1) quantum states ρ_0 and ρ_1 are generally not simultaneously diagonalizable; and (2) even when they are, estimating their eigenvalues associated with the unknown common eigenbasis remains challenging. Addressing these challenges is central to resolving Problem 1, which is the focus of our work.

1.1 Main results

We begin by stating our first main theorem, which provides a positive answer to Problem 1 when α lies in the range $1 + \Omega(1) \leq \alpha \leq O(1)$:

▶ **Theorem 2** (Quantum estimator for T_α, informal). *Given quantum query access to the state-preparation circuits of n-qubit quantum states ρ_0 and ρ_1, for any constant $\alpha > 1$, there is a quantum algorithm for estimating $\mathrm{T}_\alpha(\rho_0, \rho_1)$ within additive error $1/5$ and query complexity $O(1)$. If the state-preparation circuits have $\mathrm{poly}(n)$-size descriptions, then the algorithm' time complexity is $\mathrm{poly}(n)$. Thus, for any constant $\alpha > 1$, QSD_α is in* BQP.

More precisely, for a given additive error ϵ, the explicit query complexity of Theorem 2 is $O(1/\epsilon^{\alpha+1+\frac{1}{\alpha-1}})$ (see Theorem 14). In combination with the samplizer [74, 75], estimating $\mathrm{T}_\alpha(\rho_0, \rho_1)$ can be done using $\widetilde{O}(1/\epsilon^{3\alpha+2+\frac{2}{\alpha-1}})$ samples of ρ_0 and ρ_1 (see Theorem 16). Both upper bounds can be expressed as $\mathrm{poly}(1/\epsilon)$ for the regime $1+\Omega(1) \leq \alpha \leq O(1)$. In addition, if the state-preparation circuits of ρ_0 and ρ_1 have size $L(n) = \mathrm{poly}(n)$, then Theorem 2 implies a quantum algorithm with time complexity $\widetilde{O}(L/\epsilon^{\alpha+1+\frac{1}{\alpha-1}})$, or equivalently, $\mathrm{poly}(n, 1/\epsilon)$.

Previous quantum algorithms for estimating the quantum ℓ_α distance for constant $\alpha > 1$ have all relied on its powered variant, specifically the powered quantum ℓ_α distance:

$$\Lambda_\alpha(\rho_0, \rho_1) \coloneqq \frac{1}{2}\,\mathrm{tr}(|\rho_0 - \rho_1|^\alpha) = 2^{\alpha-1} \cdot \mathrm{T}_\alpha(\rho_0, \rho_1)^\alpha.$$

Thus, for $1 < \alpha \leq O(1)$, the estimates of $\mathrm{T}_\alpha(\rho_0, \rho_1)$ and $\Lambda_\alpha(\rho_0, \rho_1)$ are polynomially related.

When $\alpha > 1$ is an even integer, estimating $\mathrm{T}_\alpha(\rho_0, \rho_1)$ follows from a folklore result via the Shift test [27], using $O(1/\epsilon)$ queries or $O(1/\epsilon^2)$ samples.[3] However, for odd integer $\alpha > 1$, no efficient quantum algorithm is known in general. Closeness testing of quantum states with respect to $\mathrm{T}_\alpha(\rho_0, \rho_1)$ for $\alpha = 3$, with query complexity $O(1/\epsilon^{3/2})$, has been noted only in [32]. For general non-integer constants $\alpha > 1$, the quantum query complexity of estimating $\mathrm{T}_\alpha(\rho_0, \rho_1)$ was studied in [71], with polynomial dependence on the maximum rank r of ρ_0 and ρ_1. A technical comparison of our approach with this result is provided in Section 1.2.

[2] The closeness measure $\mathrm{TV}_\alpha(D_0, D_1)$ represents the classical ℓ_α distance based on the ℓ_α norm and generalizes the total variation distance, which is recovered at $\alpha = 1$.
[3] The sample complexity was noted in [62, Equations (83) and (84)].

By combining our efficient quantum estimator for $\mathrm{T}_\alpha(\rho_0, \rho_1)$ in the regime $1 + \Omega(1) \leq \alpha \leq O(1)$ (Theorem 2) with our hardness results for QSD_α (Theorem 3), we identify a sharp phase transition between the case of $\alpha = 1$ and constant $\alpha > 1$, addressing Problem 1. For clarity, we summarize our main theorems and the quantitative bounds on quantum query and sample complexities, derived from both our results and prior work, in Table 1.

Table 1 Computational, query, and sample complexities of QSD_α for $1 \leq \alpha \leq O(1)$.

		$\alpha = 1$	$1 < \alpha \leq 1 + \frac{1}{n^{1+\delta}}$	$1 + \frac{1}{n^{1+\delta}} < \alpha \leq 1 + \frac{1}{n}$	$1 + \Omega(1) \leq \alpha \leq O(1)$
QSD_α		QSZK-complete$^{(*)}$ [77, 78]	QSZK-complete$^{(*)}$ Theorem 3(2)		BQP-complete Theorems 2 and 3(1)
Query	Upper Bound	$\widetilde{O}\left(r/\epsilon^2\right)$ [72]	$\widetilde{O}\left(r^{3+\frac{2}{\alpha}}/\epsilon^{4\alpha+2}\right)$ [71]		$O\left(1/\epsilon^{\alpha+1+\frac{1}{\alpha-1}}\right)$ Theorem 14
Compl.	Lower Bound	$\widetilde{\Omega}\left(r^{1/2}\right)$ [18]	$\widetilde{\Omega}\left(r^{1/2}\right)$ Theorem 27(2)	$\Omega\left(r^{1/3}\right)$ Theorem 27(1)	$\Omega(1/\epsilon)$ Theorem 22(1)
Sample	Upper Bound	$\widetilde{O}\left(r^2/\epsilon^5\right)$ [72]	$\mathrm{poly}(r, 1/\epsilon)$ Noted in [71, Footnote 2]		$\widetilde{O}\left(1/\epsilon^{3\alpha+2+\frac{2}{\alpha-1}}\right)$ Theorem 16
Compl.	Lower Bound	$\Omega\left(r/\epsilon^2\right)$ [60]	$\Omega\left(r/\epsilon^2\right)$ Theorem 28		$\Omega\left(1/\epsilon^2\right)$ Theorem 22(2)

(*) For any $\alpha(n) \in \left[1, 1 + \frac{1}{n}\right]$, the promise problem $\mathrm{QSD}_\alpha[a, b]$ is contained in QSZK only under the polarizing regime $a(n)^2 - b(n) \geq 1/O(\log n)$, which can be slightly improved when $\alpha = 1$ (see Footnote 1). However, establishing containment in a complexity class typically requires the natural regime $a(n) - b(n) \geq 1/\mathrm{poly}(n)$, as in Theorem 2.

Finally, we present our second main theorem, which addresses the computational hardness of QSD_α, as outlined in Theorem 3. In this context, $\textsc{Pure}\mathrm{QSD}_\alpha$ refers to a restricted variant of QSD_α (see also Definition 17), where the states of interest are pure.

▶ **Theorem 3** (Computational hardness of QSD_α). *The promise problem QSD_α captures the computational power of the respective complexity classes in the corresponding regimes of α:*

(1) **Easy regimes:** *For any $1 \leq \alpha \leq \infty$, $\textsc{Pure}\mathrm{QSD}_\alpha$ is* BQP*-hard. As a corollary, QSD_α is* BQP*-complete for $1 + \Omega(1) \leq \alpha \leq O(1)$.*

(2) **Hard regimes:** *For any $1 \leq \alpha \leq 1 + \frac{1}{n}$, QSD_α is* QSZK*-complete, where the* QSZK *containment of $\mathrm{QSD}_\alpha[a,b]$ only holds for the polarizing regime $a(n)^2 - b(n) \geq 1/O(\log n)$.*

1.2 Proof techniques: BQP containment for α constantly above 1

At a high level, Quantum Singular Value Transformation (QSVT) [34] implies that the main challenge in designing a quantum algorithm based on a smooth function – e.g., Grover search [38] and the OR function, or the HHL algorithm [39] and the multiplicative inverse function (see [57] for more examples) – reduces to finding an efficiently computable polynomial approximation. Once such an approximation is obtained, the algorithm follows directly using techniques from [34], with its efficiency determined entirely by the polynomial's properties.

Now we focus on quantum algorithms for estimating the powered quantum ℓ_α distance. We begin by reviewing [71] and then provide a high-level overview of our approach.

The quantum query complexity of estimating the quantum ℓ_α distance for non-integer α was first considered in [71, Theorem IV.1]. Their approach begins with the identity

$$2\Lambda_\alpha(\rho_0, \rho_1) = \|\rho_0 - \rho_1\|_\alpha^\alpha = \mathrm{tr}\left(|\nu_-|^{\alpha/2} \Pi_{\nu_+} |\nu_-|^{\alpha/2}\right),$$

where $\nu_\pm = \rho_0 \pm \rho_1$ and Π_{ν_+} denotes the projector onto the support subspace of ν_+. According to this identity, they aim to prepare a quantum state that is a block-encoding of (normalized) $|\nu_-|^{\alpha/2}\Pi_{\nu_+}|\nu_-|^{\alpha/2}$.[4] To this end, they first prepare a quantum state that is a block-encoding of Π_{ν_+}, and then perform a unitary operator that is a block-encoding of $|\nu_-|^{\alpha/2}$ on it.[5] Finally, the (unnormalized) powered quantum ℓ_α distance, $\Lambda_\alpha(\rho_0, \rho_1)$, can be obtained by estimating the trace of $|\nu_-|^{\alpha/2}\Pi_{\nu_+}|\nu_-|^{\alpha/2}$ using quantum amplitude estimation [16]. After the error analysis, their approach was shown to have query complexity $\widetilde{O}(r^{3+1/\{\alpha/2\}}/\epsilon^{4+1/\{\alpha/2\}}) = \mathrm{poly}(r, 1/\epsilon)$.[6] The dependence on the rank is *inherent* in the approach of [71], as they have to prepare a rank-dependent quantum state that is a block-encoding of Π_{ν_+}, making the rank parameters unavoidable in the error analysis.

To overcome this technical issue, we utilize an identity different from theirs:

$$2\Lambda_\alpha(\rho_0, \rho_1) = \|\rho_0 - \rho_1\|_\alpha^\alpha = \mathrm{tr}\Big(\rho_0 \cdot \mathrm{sgn}(\nu_-) \cdot |\nu_-|^{\alpha-1}\Big) - \mathrm{tr}\Big(\rho_1 \cdot \mathrm{sgn}(\nu_-) \cdot |\nu_-|^{\alpha-1}\Big).$$

The idea is to estimate the terms $\mathrm{tr}(\rho_j \cdot \mathrm{sgn}(\nu_-) \cdot |\nu_-|^{\alpha-1})$ for $j \in \{0, 1\}$ individually, and then combine them to obtain an estimate of $\Lambda_\alpha(\rho_0, \rho_1)$. Our algorithm is sketched as follows:

1. Find a good approximation polynomial for $\mathrm{sgn}(x) \cdot |x|^{\alpha-1}$.
2. Implement a unitary block-encoding U of $\mathrm{sgn}(\nu_-) \cdot |\nu_-|^{\alpha-1}$ using Quantum Singular Value Transformation (QSVT) [34] and Linear Combinations of Unitaries (LCU) [23, 15], given the state-preparation circuits of ρ_0 and ρ_1.
3. Perform the Hadamard test [3] on U and ρ_j with outcome $b_j \in \{0,1\}$ for each $j \in \{0,1\}$.
4. Estimate $\Lambda_\alpha(\rho_0, \rho_1)$ by computing the expected value of $b_0 - b_1$.

Our algorithm is actually inspired by the trace distance estimation in [72], which corresponds to the case of $\alpha = 1$. However, the approach in [72] still has a rank-dependent query complexity of $\widetilde{O}(r/\epsilon^2)$, compared to the $\widetilde{O}(r^5/\epsilon^6)$ in [71].[7] Nevertheless, we discover an approach for estimating the quantum ℓ_α distance with a *rank-independent* complexity as long as α is constantly greater than 1. Specifically, we use the best uniform approximation polynomial $P_d(x)$ (of degree d) for the function $\mathrm{sgn}(x) \cdot |x|^q$, as given in [31, Theorem 8.1.1]:

$$\max_{x\in[-1,1]} \big|P_d(x) - \mathrm{sgn}(x) \cdot |x|^q\big| \to \frac{1}{d^q}, \text{ as } d \to \infty.$$

Our use of the best uniform approximation by polynomials is inspired by the recent work [53] on estimating the q-Tsallis entropy of quantum states for non-integer q, where they used the best uniform approximation polynomial for x^q in the non-negative range $[0, 1]$ (given in [67]). The difference is that in our case, we have to further consider the sign of x, thereby requiring the polynomial approximation to behave well in the negative part. It turns out that the polynomial approximation given in [31] is suitable for our purpose. Having noticed this, we then use the now standard techniques (used in [48, 53]) such as Chebyshev truncations and the de La Vallée Poussin partial sum (cf. [65]) to construct efficiently computable asymptotically best approximation polynomials such that

$$\max_{x\in[-1,1]} \left|P(x) - \frac{1}{2}\,\mathrm{sgn}(x) \cdot |x|^q\right| \le \epsilon, \ \max_{x\in[-1,1]} |P(x)| \le 1, \text{ and } \deg(P) = O\left(\frac{1}{\epsilon^{1/q}}\right).$$

[4] See Definition 10 for the formal definition of block-encoding.
[5] This is because of the evolution of subnormalized density operators [71, Lemma II.2].
[6] Here, $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part of x.
[7] Some readers may wonder if our approach applies to trace distance estimation ($\alpha = 1$) to remove the rank dependence in quantum query and sample complexities in [72]. However, the answer is generally no, as the rank dependence is intrinsic to trace distance estimation due to the polynomial dependence of the rank in the quantum query and sample complexities lower bounds (see [52, Section 2.2.2] in the full version).

Using this efficiently computable polynomial (with $q = \alpha - 1$) and with further analysis, we can then estimate the quantum ℓ_α distance to within additive error ϵ with the desired query upper bound in Theorem 2. Moreover, using the (multi-)samplizer [74, 75], a quantum query-to-sample simulation, we can also achieve the desired sample upper bound.

1.3 Proof techniques: QSZK completeness for $\alpha > 1$ near 1

To establish the BQP- and QSZK-hardness results in Theorem 3, we reduce the promise problems QSD and PureQSD ($\alpha = 1$) to the corresponding promise problems QSD_α and $\mathrm{PureQSD}_\alpha$ for appropriate ranges of α. The key technique underlying these reductions is the following rank-dependent inequalities that generalize the case of $\alpha = 2$ from [24, 25]:

▶ **Theorem 4** (T_α vs. T, informal). *For any states ρ_0 and ρ_1 and $\alpha \in [1, \infty]$, it holds that:*

$$2^{1-\frac{1}{\alpha}} \cdot \mathrm{T}_\alpha(\rho_0, \rho_1) \leq \mathrm{T}(\rho_0, \rho_1) \leq 2\big(\mathrm{rank}(\rho_0)^{1-\alpha} + \mathrm{rank}(\rho_1)^{1-\alpha}\big)^{-\frac{1}{\alpha}} \cdot \mathrm{T}_\alpha(\rho_0, \rho_1). \tag{1}$$

For $\alpha = \infty$, the inequalities hold in the limit as $\alpha \to \infty$.

The proof of Theorem 4 follows from considering orthogonal positive semi-definite matrices ς_0 and ς_1 satisfying $\rho_0 - \rho_1 = \varsigma_0 - \varsigma_1$, and analyzing their properties carefully.

We then illuminate the hardness results in Theorem 3:

- For the easy regime, Equation (1) becomes an equality when both ρ_0 and ρ_1 are pure states. This equality implies the BQP-hardness of $\mathrm{PureQSD}_\alpha$, along with the query and sample complexity lower bounds for all $1 \leq \alpha \leq \infty$, establishing Theorem 3(1).
- For the hard regime, Equation (1) is *sensitive* to α. Particularly, for $\alpha = 1+\frac{1}{n}$, if quantum states ρ_0 and ρ_1 are τ-far, meaning $\mathrm{T}(\rho_0, \rho_1) \geq \tau$, it follows only that $\mathrm{T}_{\alpha=1+\frac{1}{n}}(\rho_0, \rho_1) \geq \tau/2$. However, when $\alpha \leq 1 + \frac{1}{n^{1+\delta}}$ for any arbitrarily small constant δ, the same trace distance condition ensures that $\mathrm{T}_\alpha(\rho_0, \rho_1) \geq \tau$ as $n \to \infty$, leading to the QSZK hardness result in Theorem 3(2) and distinct query complexity lower bounds in Table 1.

Lastly, we explain the QSZK containment in the hard regime. Simply combining Theorem 4 and the QSZK containment of QSD from [77, 78] does not work, as the resulting QSZK containment of $\mathrm{QSD}_\alpha[a, b]$ holds only for $a(n)^2/2 - b(n) \geq 1/O(\log n)$, which is even weaker than the polarizing regime defined in Footnote 1. To address this, we establish a *partial* polarization lemma for T_α (Lemma 25), which ensures that for quantum states ρ_0 and ρ_1 where $\mathrm{T}(\rho_0, \rho_1)$ is either at least a or at most b, we can construct new quantum states $\widetilde{\rho}_0$ and $\widetilde{\rho}_1$ such that $\mathrm{T}_\alpha(\widetilde{\rho}_0, \widetilde{\rho}_1)$ is either at least $\frac{1}{2} - \frac{1}{2}e^{-k}$ or at most $1/16$, as long as the parameters a and b are in the polarizing regime. Theorem 3(2) follows by combining this partial polarization lemma for T_α with the polarization lemma for T in [77].

1.4 Discussion and open problems

While the quantum ℓ_α distance $\mathrm{T}_\alpha(\cdot, \cdot)$ and its powered version $\Lambda_\alpha(\cdot, \cdot)$ are almost computationally interchangeable for $1 \leq \alpha \leq O(1)$, their behavior differs greatly when $\alpha = \infty$:

- The quantity $\mathrm{T}_\infty(\rho_0, \rho_1)$ corresponds to the largest eigenvalue $\lambda_{\max}$ of $(\rho_0 - \rho_1)/2$. The associated promise problem QSD_∞ is BQP-hard and contains in QMA.[8] However, establishing a BQP containment appears challenging, as $(\rho_0 - \rho_1)/2$ does not directly admit an efficiently computable basis – unlike its classical counterpart in [69], which does.

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[8] The verification circuit in the QMA containment simply follows from phase estimation [45], where a (normalized) eigenvector corresponding to $\lambda_{\max}$ serves as a witness state.

- The quantity $\Lambda_\infty(\rho_0, \rho_1)$ takes values in $\{0, 1/2, 1\}$ for any states ρ_0 and ρ_1, and is nonzero if and only if the states are orthogonal, with at least one being pure. Thus, even the pure-state-restricted variant of the associated promise problem, PUREPOWEREDQSD$_\infty[1, 0]$, is $\mathsf{C_=P}$-hard (see [52, Appendix A] in the full version). Here, $\mathsf{C_=P} = \mathsf{coNQP}$ [2, 80], a subclass of PP that provides a precise variant of BQP, ensuring acceptance for all *yes* instances.

This fundamental difference between these quantities raises an intriguing question on QSD_∞:

(a) What is the computational complexity of the promise problem QSD_∞, defined by $\mathrm{T}_\infty(\cdot, \cdot)$? Can we show that QSD_∞ is also in BQP, or is it inherently more difficult?

Another open problem concerns quantitative bounds for QSD_α:

(b) Can the query and sample bounds in Table 1 be improved, particularly for the regime $1 + \Omega(1) \leq \alpha \leq O(1)$? Moreover, can tight bounds be established when the states have *small support*, analogous to the classical case in [69, Table 1]?

1.5 Related works

Schatten p-norm estimation $\mathrm{tr}(|A|^p)$ of $O(\log n)$-local Hermitian A on n qubits to within additive error $2^{n-p}\epsilon\|A\|^p$ for $\epsilon(n) \leq 1/\mathrm{poly}(n)$ and real $p(n) \leq \mathrm{poly}(n)$ was shown to be $\mathsf{DQC1}$-complete in [19]. Given a unitary block-encoding of a matrix A, in [56], they presented a quantum algorithm that estimates the Schatten p-norm $(\mathrm{tr}(|A|^p))^{1/p}$ to relative error ϵ for integer p, where a condition number κ satisfying $A \geq I/\kappa$ is required for the case of odd p.

The query complexity of N-dimensional quantum state certification (i.e., determine whether two quantum states are identical or ϵ-far) with respect to trace distance was shown to be $O(N/\epsilon)$ in [32]. The query complexity of trace distance estimation was shown to be $\widetilde{O}(r^5/\epsilon^6)$ in [71] and later improved to $\widetilde{O}(r/\epsilon^2)$ in [72], where r is the rank of the quantum states, confirming a conjecture in [25] that low-rank trace distance estimation is in BQP. Both Low-rank trace distance and fidelity estimations are known to be BQP-complete [1, 72]. Based on the approach of [72], space-bounded quantum state discrimination with respect to trace distance was shown to be BQL-complete in [48]. In addition to trace distance, fidelity is another important measure of the closeness between quantum states. The query complexity of fidelity estimation was shown to be $\widetilde{O}(r^{12.5}/\epsilon^{13.5})$ in [76] and later improved to $\widetilde{O}(r^{5.5}/\epsilon^{6.5})$ in [71] and to $\widetilde{O}(r^{2.5}/\epsilon^5)$ in [33]. Recently, the query complexity of pure-state trace distance and fidelity estimations was shown to be $\Theta(1/\epsilon)$ in [70] and was recently extended in [29] to estimating fidelity of a mixed state to a pure state.

In addition to the query complexity, the sample complexity has also been studied in the literature. In [7], the sample complexity of N-dimensional quantum state certification was shown to be $\Theta(N/\epsilon^2)$ with respect to trace distance and $\Theta(N/\epsilon)$ with respect to fidelity. The sample complexity of trace distance estimation is known to be $\widetilde{O}(r^2/\epsilon^5)$ in [72] and that of fidelity estimation is known to be $\widetilde{O}(r^{5.5}/\epsilon^{12})$, where r is the rank of quantum states. The sample complexity of pure-state squared fidelity estimation is known to be $\Theta(1/\epsilon^2)$ via the SWAP test [17], where the matching lower bound was given in [4]. Recently, the sample complexity of pure-state trace distance and fidelity estimations was shown to be $\Theta(1/\epsilon^2)$ in [73], which was achieved by using the samplizer in [75].

2 Preliminaries

We assume a fundamental knowledge of quantum computation and quantum information theory. For an introduction, we refer the reader to the textbook [59].

We adopt the following notations throughout the paper: (1) $[n] \coloneqq \{1, 2, \dots, n\}$; (2) $\widetilde{O}(f)$ denotes $O(f \operatorname{polylog}(f))$, while $\widetilde{\Omega}(f)$ denotes $\Omega(f/\operatorname{polylog}(f))$; and (3) $|\bar{0}\rangle$ represents $|0\rangle^{\otimes a}$ for integer $a > 1$. In addition, the Schatten α-norm of a matrix A is defined as $\|A\|_\alpha \coloneqq \big(\operatorname{tr}(|A|^\alpha)\big)^{1/\alpha}$, where $|A| \coloneqq \big(A^\dagger A\big)^{1/2}$. For simplicity, we use the notation $\|A\|$ to denote the operator norm (equivalently, the Schatten ∞-norm) of a matrix A.

2.1 Closeness measures for quantum states

We start by defining the trace distance and providing useful properties of this distance:

▶ **Definition 5** (Trace distance). *Let ρ_0 and ρ_1 be two quantum states that are mixed in general. The trace distance between ρ_0 and ρ_1 is defined by*

$$\mathrm{T}(\rho_0, \rho_1) \coloneqq \frac{1}{2} \operatorname{tr}(|\rho_0 - \rho_1|) = \frac{1}{2} \operatorname{tr}\Big(\big((\rho_0 - \rho_1)^\dagger (\rho_0 - \rho_1)\big)^{1/2}\Big).$$

Notably, the trace distance is a distance metric (e.g., [79, Lemma 9.1.8]).

Next, we define the quantum ℓ_α distance and its *powered* version, which generalize the trace distance ($\alpha = 1$) using the Schatten norm. Notably, the quantum ℓ_α distance coincides with the Hilbert-Schmidt distance when $\alpha = 2$:

▶ **Definition 6** (Quantum ℓ_α distance and its powered version). *Let ρ_0 and ρ_1 be two quantum states that are mixed in general. The quantum ℓ_α distance $\mathrm{T}_\alpha(\cdot,\cdot)$ and its powered version $\Lambda_\alpha(\cdot,\cdot)$ between ρ_0 and ρ_1 are defined as follows*:

$$\mathrm{T}_\alpha(\rho_0, \rho_1) \coloneqq \frac{1}{2}\|\rho_0 - \rho_1\|_\alpha \quad \textit{and} \quad \Lambda_\alpha(\rho_0, \rho_1) \coloneqq \frac{1}{2}\|\rho_0 - \rho_1\|_\alpha^\alpha.$$

Here, the Schatten α-norm of $\rho_0 - \rho_1$ is given by $\|\rho_0 - \rho_1\|_\alpha \coloneqq \left(|\rho_0 - \rho_1|^\alpha\right)^{1/\alpha}$.

By the monotonicity of the Schatten norm, e.g., [6, Equation (1.31)], it holds that:

$$\forall \alpha \geq 1, \quad 0 \leq \Lambda_\alpha(\rho_0, \rho_1) \leq \mathrm{T}_\alpha(\rho_0, \rho_1) \leq \mathrm{T}(\rho_0, \rho_1) \leq 1.$$

As a corollary of the Davis convexity theorem [26], the quantum ℓ_α distance $\mathrm{T}_\alpha(\cdot,\cdot)$ also serves as a distance metric, whereas its powered version $\Lambda_\alpha(\cdot,\cdot)$ does not. Further inequalities for the trace distance and the quantum ℓ_α distance are given in the full version, particularly in [52, Section 2.1].

Lastly, we require the following relationship for additive error estimation between the quantum ℓ_α distance and its powered version. The proof is provided in the full version, specifically in [52, Proposition 2.5].

▶ **Proposition 7** (T_α vs. powered T_α). *The quantum ℓ_α distance $\mathrm{T}_\alpha(\cdot,\cdot)$ and its powered version $\Lambda_\alpha(\cdot,\cdot)$ are related through the equality $\mathrm{T}_\alpha(\rho_0, \rho_1) = 2^{\frac{1}{\alpha}-1} \cdot \Lambda_\alpha(\rho_0, \rho_1)^{\frac{1}{\alpha}}$. Accordingly, if x is an estimate of $\Lambda_\alpha(\rho_0, \rho_1)$ to within additive error ϵ, then $2^{\frac{1}{\alpha}-1} \cdot x^{\frac{1}{\alpha}}$ serves as an estimate of $\mathrm{T}_\alpha(\rho_0, \rho_1)$ to within additive error $2^{\frac{1}{\alpha}-1} \cdot \epsilon^{\frac{1}{\alpha}}$.*

2.2 Closeness testing of quantum states via state-preparation circuits

We begin by defining the closeness testing of quantum states with respect to the trace distance, denoted as $\mathrm{QSD}[a, b]$,[9] and two variants of this problem:

[9] While Definition 8 aligns with the classical counterpart of QSD defined in [66, Section 2.2], it is slightly less general than the definition in [77, Section 3.3]. Specifically, Definition 8 assumes that the input length m and the output length n are *polynomially equivalent*, whereas [77, Section 3.3] allows for cases where the output length (e.g., a single qubit) is *much smaller* than the input length.

▶ **Definition 8** (Quantum State Distinguishability Problem, QSD, adapted from [77, Section 3.3]). *Let Q_0 and Q_1 be quantum circuits acting on m qubits ("input length") and having n specified output qubits ("output length"), where $m(n)$ is a polynomial function of n. Let ρ_i denote the quantum state obtained by running Q_i on state $|0\rangle^{\otimes m}$ and tracing out the non-output qubits. Let $a(n)$ and $b(n)$ be efficiently computable functions. Decide whether:*

- Yes: *A pair of quantum circuits (Q_0, Q_1) such that $\mathrm{T}(\rho_0, \rho_1) \geq a(n)$;*
- No: *A pair of quantum circuits (Q_0, Q_1) such that $\mathrm{T}(\rho_0, \rho_1) \leq b(n)$.*

Furthermore, we denoted the restricted version, where ρ_0 and ρ_1 are pure states, as PureQSD.

In this work, we consider the *purified quantum access input model*, as defined in [77], in both white-box and black-box scenarios:

- **White-box input model**: The input of the problem QSD consists of descriptions of polynomial-size quantum circuits Q_0 and Q_1. Specifically, for $b \in \{0,1\}$, the description of Q_b includes a sequence of polynomially many 1- and 2-qubit gates.
- **Black-box input model**: In this model, instead of providing the descriptions of the quantum circuits Q_0 and Q_1, only query access to Q_b is allowed, denoted as O_b for $b \in \{0,1\}$. For convenience, we also allow query access to $Q_b^\dagger$ and controlled-Q_b, denoted by $O_b^\dagger$ and controlled-O_b, respectively.

In addition to query complexity, defined within the black-box input model, *sample complexity* refers to the number of copies of quantum states ρ_0 and ρ_1 needed to accomplish a specific closeness testing task. Useful lemmas on computational hardness and quantitative lower bounds for query and sample complexities are available in the full version, specifically in [52, Section 2.2].

2.3 Polynomial approximations

We now present useful tools on best uniform polynomial approximations. In addition, definitions and lemmas on Chebyshev expansion and truncations can be found in the full version, particularly in [52, Section 2.3.2].

Let $f(x)$ be a continuous function defined on the interval $[-1,1]$ that we aim to approximate using a polynomial of degree at most d. We define P_d^* as a *best uniform approximation* on $[-1,1]$ to f of degree d if, for any degree-d polynomial approximation P_d of f, it holds that $\max_{x\in[-1,1]}|f(x) - P_d^*(x)| \leq \max_{x\in[-1,1]}|f(x) - P_d(x)|$.

The best uniform (polynomial) approximation of positive (constant) powers $|x|^\alpha$ was first established by Serge Bernstein [14, 13]. However, the focus here is on the best uniform approximation of *signed* positive powers $\mathrm{sgn}(x)|x|^\alpha$, as stated in Lemma 9. This result is often attributed to Bernstein's work (see, e.g., [68, Equation (10.2)]), and a proof of a more general version can be found in [31, Theorem 8.1.1].

▶ **Lemma 9** (Best uniform approximation of signed positive powers, adapted from [31, Theorem 8.1.1]). *For any positive real (constantly large) order α, let $P_d^* \in \mathbb{R}[x]$ be the best uniform polynomial approximation for $f(x) = \mathrm{sgn}(x)|x|^\alpha$ of degree $d = \left\lceil (\beta_\alpha/\epsilon)^{1/\alpha} \right\rceil$, where β_α is a constant depending on α. Then, for sufficiently small ϵ, $\max_{x\in[-1,1]} |P_d^*(x) - f(x)| \leq \epsilon$.*

2.4 Quantum algorithmic toolkit

In this subsection, we recap the quantum singular value transformation. Additional quantum algorithmic tools, including useful quantum algorithmic subroutines and the quantum samplizer can be found in the full version, specifically in [52, Sections 2.4.2 and 2.4.3].

2.4.1 Quantum singular value transformation

We start by defining block-encoding:

▶ **Definition 10** (Block-encoding). *A linear operator A on an $(n+a)$-qubit Hilbert space is said to be an (α, a, ϵ)-block-encoding of an n-qubit linear operator B, if $\|\alpha(\langle 0|^{\otimes a} \otimes I_n)A(|0\rangle^{\otimes a} \otimes I_n) - B\| \leq \epsilon$, where I_n is the n-qubit identity operator and $\|\cdot\|$ is the operator norm.*

Then, we state the quantum singular value transformation:

▶ **Lemma 11** (Quantum singular value transformation, [34, Theorem 31]). *Suppose that unitary operator U is a (α, a, ϵ)-block-encoding of Hermitian operator A, and $P \in \mathbb{R}[x]$ is a polynomial of degree d with $|P(x)| \leq \frac{1}{2}$ for $x \in [-1, 1]$. Then, we can implement a quantum circuit $\tilde{U}$ that is a $(1, a+2, 4d\sqrt{\epsilon/\alpha} + \delta)$-block-encoding of $P(A/\alpha)$, by using $O(d)$ queries to U and $O((a+1)d)$ one- and two-qubit quantum gates. Moreover, the classical description of $\tilde{U}$ can be computed in deterministic time* $\mathrm{poly}(d, \log(1/\delta))$.

3 Efficient quantum algorithms for estimating quantum ℓ_α distance

In this section, we present efficient quantum algorithms for estimating the quantum ℓ_α distance $\mathrm{T}_\alpha(\rho_0, \rho_1)$ when $\alpha \geq 1 + \Omega(1)$. These algorithms utilize either queries to state-preparation circuits or samples of the states ρ_0 and ρ_1. The core of our approach is an *efficient* uniform approximation to *signed* positive constant power functions (Lemma 12), which provides a uniform error bound over the entire interval $[-1, 1]$.

This uniform polynomial approximation enables a query-efficient quantum algorithm for estimating $\mathrm{T}_\alpha(\rho_0, \rho_1)$ through its powered version $\Lambda_\alpha(\rho_0, \rho_1)$, as shown in Theorem 14. As a result, we establish a BQP containment of the promise problem QSD_α defined in Section 4. Additionally, by leveraging the multi-samplizer in [73], we devise a sample-efficient quantum algorithm for estimating $\mathrm{T}_\alpha(\rho_0, \rho_1)$, detailed in Theorem 16.

3.1 Efficient uniform approximations of signed positive powers

Using the averaged Chebyshev truncation specified in [52, Section 2.3.2] in the full version, we provide an *efficiently computable* uniform polynomial approximation of *signed* positive constant powers (see also [52, Lemma 3.1]):

▶ **Lemma 12** (Efficient uniform polynomial approximation of signed positive powers). *Let α be a positive real (constantly large) number. For any $\epsilon \in (0, 1/2)$, there is a degree-d polynomial $P_d \in \mathbb{R}[x]$, where $d = \left\lceil (\beta'_\alpha/\epsilon)^{1/\alpha} \right\rceil$ and β'_α is a constant depending on α, that can be deterministically computed in $\widetilde{O}(d)$ time. For sufficiently small ϵ, it holds that:*

$$\max_{x \in [-1,1]} \left| \frac{1}{2} \operatorname{sgn}(x) |x|^\alpha - P_d(x) \right| \leq \epsilon \quad \text{and} \quad \max_{x \in [-1,1]} |P_d(x)| \leq 1.$$

More specifically, the degree-d uniform approximation of signed positive powers (see Lemma 9) ensures that the degree-$(2d-1)$ averaged Chebyshev truncation (given in [52, Equation 2.3]) achieves almost the same approximation error bound, with efficiently computable Chebyshev coefficients. The complete proof is available in the full version, specifically in [52, Section 3.1].

3.2 Quantum ℓ_α distance estimation for constantly large $\alpha > 1$

3.2.1 Query-efficient quantum algorithm for estimating powered T_α

We now present efficient quantum query algorithms for estimating $\Lambda_\alpha(\rho_0, \rho_1)$ and $\mathrm{T}_\alpha(\rho_0, \rho_1)$, as presented in [52, Lemma 3.2] in the full version, and the complete proof is provided in [52, Section 3.2.1]:

▶ **Lemma 13** (Powered quantum ℓ_α distance estimation via queries). *Suppose that Q_0 and Q_1 are unitary operators that prepare purifications of mixed quantum states ρ_0 and ρ_1, respectively. For every constantly large $\alpha \geq 1 + \Omega(1)$, there is a quantum query algorithm that estimates $\Lambda_\alpha(\rho_0, \rho_1)$ to within additive error ϵ by using $O(1/\epsilon^{1+\frac{1}{\alpha-1}})$ queries to Q_0 and Q_1.*

By combining Proposition 7 with Lemma 13 for additive error ϵ^α, we obtain a quantum query algorithm for estimating $\mathrm{T}_\alpha(\rho_0, \rho_1)$ when $\alpha \geq 1 + \Omega(1)$ is constantly large, as also stated in [52, Theorem 3.3] in the full version:

▶ **Theorem 14** (Quantum ℓ_α distance estimation via queries). *Suppose that Q_0 and Q_1 are unitary operators that prepare purifications of mixed quantum states ρ_0 and ρ_1, respectively. For every constantly large $\alpha \geq 1 + \Omega(1)$, there is a quantum query algorithm that estimates $\mathrm{T}_\alpha(\rho_0, \rho_1)$ to within additive error ϵ by using $O(1/\epsilon^{\alpha+1+\frac{1}{\alpha-1}})$ queries to Q_0 and Q_1.*

3.2.2 Sample-efficient quantum algorithm for estimating powered T_α

We proceed by describing efficient quantum sample algorithms for $\Lambda_\alpha(\rho_0, \rho_1)$ and $\mathrm{T}_\alpha(\rho_0, \rho_1)$, as stated in Lemma 15 and in [52, Lemma 3.4] in the full version. Our sample algorithms are obtained by combining the quantum query algorithm from Lemma 13 with the samplizer [74, 75]. The corresponding explanatory framework is presented in in [52, Algorithm 1], and the detailed proof is given in [52, Section 3.2.2].

▶ **Lemma 15** (Powered quantum ℓ_α distance estimation via samples). *For every constantly large $\alpha \geq 1 + \Omega(1)$, $\Lambda_\alpha(\rho_0, \rho_1)$ can be estimated to within additive error ϵ on a quantum computer by using $\widetilde{O}(1/\epsilon^{3+\frac{2}{\alpha-1}})$ samples of ρ_0 and ρ_1.*

By combining Proposition 7 with Lemma 15 for additive error ϵ^α, we obtain a quantum sample algorithm for estimating $\mathrm{T}_\alpha(\rho_0, \rho_1)$ when $\alpha \geq 1 + \Omega(1)$ is constantly large, as also stated in [52, Theorem 3.5] in the full version:

▶ **Theorem 16** (Quantum ℓ_α distance estimation via samples). *For every constantly large $\alpha \geq 1 + \Omega(1)$, there is a quantum sample algorithm that estimates the quantum ℓ_α distance $\mathrm{T}_\alpha(\rho_0, \rho_1)$ to within additive error ϵ by using $\widetilde{O}(1/\epsilon^{3\alpha+2+\frac{2}{\alpha-1}})$ samples of ρ_0 and ρ_1.*

4 Hardness and lower bounds for α constantly above 1

We begin by introducing a generalization of QSD from [77], where the trace distance is replaced with the quantum ℓ_α distance as the closeness measure:

▶ **Definition 17** (Quantum State Distinguishability Problem with Schatten α-norm, QSD_α). *Let Q_0 and Q_1 be quantum circuits acting on m qubits ("input length") and having n specified output qubits ("output length"), where $m(n)$ is a polynomial function of n. Let ρ_i denote the quantum state obtained by running Q_i on state $|0\rangle^{\otimes m}$ and tracing out the non-output qubits. Let $a(n)$ and $b(n)$ be efficiently computable functions. Decide whether:*

- Yes: *A pair of quantum circuits (Q_0, Q_1) such that $\mathrm{T}_\alpha(\rho_0, \rho_1) \geq a(n)$;*
- No: *A pair of quantum circuits (Q_0, Q_1) such that $\mathrm{T}_\alpha(\rho_0, \rho_1) \leq b(n)$.*

Moreover, we denoted the restricted version, where ρ_0 and ρ_1 are pure states, as $\textsc{PureQSD}_\alpha$.

In the remainder of this section, we establish rank-dependent inequalities between the quantum ℓ_α distance and the trace distance (Theorem 18) in Section 4.1. These inequalities facilitate reductions that demonstrate the BQP hardness (Theorem 21) and derive quantitative lower bounds on queries and samples (Theorem 22) for $\textsc{PureQSD}_\alpha$ in Section 4.2.

4.1 Rank-dependent inequalities between T_α and the trace distance

We generalize the rank-dependent inequalities between the (squared) Hilbert-Schmidt distance and the trace distance, as demonstrated in [24, Appendix G] and [25, Theorem 1] for the case of $\alpha = 2$, to all $1 \leq \alpha \leq \infty$:

▶ **Theorem 18** (T_α vs. T). *Let ρ_0 and ρ_1 be quantum states. The following holds*:
(1) *For any α in the range $1 \leq \alpha < \infty$,*

$$2^{1-\frac{1}{\alpha}} \cdot \mathrm{T}_\alpha(\rho_0,\rho_1) \leq \mathrm{T}(\rho_0,\rho_1) \leq 2\big(\mathrm{rank}(\rho_0)^{1-\alpha} + \mathrm{rank}(\rho_1)^{1-\alpha}\big)^{-\frac{1}{\alpha}} \cdot \mathrm{T}_\alpha(\rho_0,\rho_1).$$

(2) *For $\alpha = \infty$, $2 \cdot \mathrm{T}_\infty(\rho_0,\rho_1) \leq \mathrm{T}(\rho_0,\rho_1) \leq 2\min\{\mathrm{rank}(\rho_0),\mathrm{rank}(\rho_1)\} \cdot \mathrm{T}_\infty(\rho_0,\rho_1)$.*

It is worth noting that Item 1 and Item 2 in Theorem 18 are consistent, specifically

$$\lim_{\alpha\to\infty} \big(\mathrm{rank}(\rho_0)^{1-\alpha} + \mathrm{rank}(\rho_1)^{1-\alpha}\big)^{-\frac{1}{\alpha}} = \min\{\mathrm{rank}(\rho_0),\mathrm{rank}(\rho_1)\}.$$

Additionally, the inequalities in Theorem 18 sharpen the inequalities between the trace norm and the Schatten norm (see, e.g., [6, Equation (1.31)]):

$$\forall 1 \leq p \leq \infty, \quad \|A\|_p \leq \|A\|_1 \leq r_A^{1-1/p} \cdot \|A\|_p. \tag{2}$$

By considering the maximum rank of ρ_0 and ρ_1, we can derive a simplified form of Theorem 18 for convenience:

▶ **Corollary 19** (T_α vs. T, simplified). *For any quantum states ρ_0 and ρ_1, the following holds*:

$$\forall 1 \leq \alpha < \infty, \quad 2^{1-\frac{1}{\alpha}} \cdot \mathrm{T}_\alpha(\rho_0,\rho_1) \leq \mathrm{T}(\rho_0,\rho_1) \leq (2\max\{\mathrm{rank}(\rho_0),\mathrm{rank}(\rho_1)\})^{1-\frac{1}{\alpha}} \cdot \mathrm{T}_\alpha(\rho_0,\rho_1).$$

Moreover, for pure quantum states, Theorem 18 yields the following equality:

▶ **Corollary 20** ($\mathrm{T}_\alpha = \mathrm{T}$ for pure states). *For any pure states $|\psi_0\rangle\langle\psi_0|$ and $|\psi_1\rangle\langle\psi_1|$, we have*:

$$\forall 1 \leq \alpha \leq \infty, \quad 2^{1-\frac{1}{\alpha}} \cdot \mathrm{T}_\alpha(|\psi_0\rangle\langle\psi_0|, |\psi_1\rangle\langle\psi_1|) = \mathrm{T}(|\psi_0\rangle\langle\psi_0|, |\psi_1\rangle\langle\psi_1|).$$

The proof of Theorem 18 (see also [52, Theorem 4.2] in the full version) relies on the positive semi-definite matrices ς_0 and ς_1, defined for each $b \in \{0,1\}$ as $\varsigma_b := \frac{1}{2}\big((-1)^b(\rho_0-\rho_1) + |\rho_0-\rho_1|\big)$. These matrices are orthogonal and satisfy the relation $\rho_0 - \rho_1 = \varsigma_0 - \varsigma_1$. The complete proof is provided in [52, Section 4.1].

4.2 Computational hardness and lower bounds

We first state the computational hardness result of $\textsc{PureQSD}_\alpha$ with $1 \leq \alpha \leq \infty$, obtained by a reduction from $\textsc{PureQSD}$, which is BQP hard [64], as stated in Theorem 21 and in [52, Theorem 4.5]. The proof can be found in the full version, specifically in [52, Section 4.2].

▶ **Theorem 21** ($\textsc{PureQSD}_\alpha$ is BQP-hard). *For any $1 \leq \alpha \leq \infty$ and $n \geq 2$, it holds that*:

$$\textsc{PureQSD}_\alpha\Big[2^{\frac{1}{\alpha}-1} \cdot \big(1-2^{-n}\big),\ 2^{\frac{1}{\alpha}-1-n}\Big] \text{ is BQP-hard.}$$

Next, we provide lower bounds on the query and sample complexities for $\textsc{PureQSD}_\alpha$ by reducing to those for $\textsc{PureQSD}$ in [70], as presented in Theorem 22 and in [52, Theorem 4.6]. The proof is provided in the full version, particularly in [52, Section 4.2].

▶ **Theorem 22** (Quantitative lower bounds for $\textsc{PureQSD}_\alpha$). *For any $1 \leq \alpha \leq \infty$ and $0 < \epsilon < 2^{\frac{1}{\alpha}-2}$, there exist n-qubit pure states $|\psi_0\rangle$ and $|\psi_1\rangle$ such that deciding whether $\mathrm{T}_\alpha(|\psi_0\rangle\langle\psi_0|, |\psi_1\rangle\langle\psi_1|)$ is at least ϵ or exactly 0 requires:*

(1) **Queries**: *In the purified quantum access model, the quantum query complexity is $\Omega(1/\epsilon)$.*

(2) **Samples**: *The quantum sample complexity is $\Omega(1/\epsilon^2)$.*

5 Quantum ℓ_α distance estimation for $\alpha > 1$ near 1

In this section, we establish that QSD_α is QSZK-complete for $1 \leq \alpha \leq 1 + \frac{1}{n}$ (see also [52, Theorem 5.1] in the full version), extending the prior result that QSD ($\alpha = 1$) is QSZK-complete, as shown in [77]:

▶ **Theorem 23** (QSD_α is QSZK-complete for $\alpha > 1$ near 1). *Let $a(n)$ and $b(n)$ be efficiently computable functions such that $0 \leq b < a \leq 1$. Then, for any $1 \leq \alpha \leq 1 + \frac{1}{n}$, it holds that:*

For any $a(n)^2 - b(n) \geq 1/O(\log n)$, $\mathrm{QSD}_\alpha[a, b]$ is in QSZK.

Moreover, $\mathrm{QSD}_\alpha[a, b]$ is QSZK-hard if $a(n) \leq 1/2 - 2^{-n^\tau - 1}$ and $b(n) \geq 2^{-n^\tau - \frac{1}{n+1}}$ for every constant $\tau \in (0, 1/2)$ and sufficiently large integer n.

The main challenge in proving Theorem 23 is to establish a QSZK containment of QSD_α under the polarizing regime $a(n)^2 - b(n) \geq 1/O(\log n)$.[10] A direct approach, combining the inequalities between T and T_α (Corollary 19) with the QSZK containment of QSD from [77, 78], only yields a QSZK containment of $\mathrm{QSD}_\alpha[a, b]$ under a *weaker* regime, $a(n)^2/2 - b(n) \geq 1/O(\log n)$. To circumvent this, we provide a (partial) polarization lemma for T_α (Lemma 25), which enables us to achieve the desired QSZK containment in Theorem 23.

The remainder of this section establishes the QSZK containment of QSD_α in Section 5.1. We then show the QSZK hardness of QSD_α (Theorem 26) and derive quantitative lower bounds on query complexity (Theorem 27) and sample complexity (Theorem 28) in Section 5.2.

5.1 QSZK containment via a partial polarization lemma for T_α

▶ **Theorem 24** (QSD_α is in QSZK). *Let $a(n)$ and $b(n)$ be efficiently computable functions satisfying $0 \leq b < a \leq 1$. Then, the following holds:*

$$\text{For any } \alpha \in \left[1, 1 + \frac{1}{n}\right] \text{ and any } a(n)^2 - b(n) \geq \frac{1}{O(\log n)}, \mathrm{QSD}_\alpha[a, b] \text{ is in QSZK.}$$

The proof of Theorem 24 (see also [52, Theorem 5.2] in the full version) can be found in [52, Section 5.1]. The main technical tool is a *partial* polarization lemma for T_α (see also [52, Lemma 5.3]), which ensures that any pair (a, b) within the polarizing regime can be made *constantly* separated:

[10] Notably, similar to the classical cases [12], by reducing to the Quantum Jensen-Shannon Divergence Problem (QJSP) or the Measured Quantum Triangular Discrimination Problem (measQTDP) introduced in [51], the QSZK containment of QSD_α holds slightly beyond the polarizing regime.

▶ **Lemma 25** (A partial polarization lemma for T_α)**.** *Let Q_0 and Q_1 be quantum circuits that prepare quantum states ρ_0 and ρ_1, respectively. There is a deterministic procedure that, given an input (Q_0, Q_1, a, b, k) where $a(n)^2 - b(n) \geq 1/O(\log n)$, outputs new quantum circuits $\widetilde{Q}_0$ and $\widetilde{Q}_1$ that prepare the states $\widetilde{\rho}_0$ and $\widetilde{\rho}_1$, respectively. The resulting states satisfy that:*

$$\text{For any } \alpha \in [1, 1+1/n], \quad \mathrm{T}_\alpha(\rho_0,\rho_1) \geq a \implies \mathrm{T}_\alpha(\widetilde{\rho}_0,\widetilde{\rho}_1) \geq (1-e^{-k})/2,$$
$$\mathrm{T}_\alpha(\rho_0,\rho_1) \leq b \implies \mathrm{T}_\alpha(\widetilde{\rho}_0,\widetilde{\rho}_1) \leq 1/16.$$

Here, the states $\widetilde{\rho}_0$ and $\widetilde{\rho}_1$ are defined over $\widetilde{O}\Big(nk^{O\left(\frac{b\ln(2/a^2)}{a^2-b}\right)}\Big)$ qubits. Moreover, when $k \leq O(1)$ or $a^2 - b \geq \Omega(1)$, the time complexity of the procedure is $\mathrm{poly}\Big(|Q_0|, |Q_1|, k, \exp\big(\frac{b\ln(1/a^2)}{a^2-b}\big)\Big)$.

In analogy with polarization lemmas for various classical [12, 21, 66] and quantum [51, 77] closeness measures, the proof of Lemma 25 proceeds by separately reducing the errors on both sides of the problem QSD_α. This is achieved using the XOR lemma ([52, Lemma 5.4]) and the direct product lemma ([52, Lemma 5.5]) for T_α. The full statements of these two lemmas, along with the full proofs of all three, are given in the full version, particularly in [52, Section 5.1].

5.2 Computational hardness and lower bounds for $\alpha > 1$ near 1

▶ **Theorem 26** (QSD_α is QSZK-hard)**.** *For any positive constant $\delta > 0$ that can be made arbitrarily small, the following holds for sufficiently large n:*

(1) *For any $1 \leq \alpha \leq 1+\frac{1}{n^{1+\delta}}$, $\forall \tau \in (0,1/2)$, $\mathrm{QSD}_\alpha\big[1-\gamma_{\delta,\tau}(n), \gamma'_{\delta,\tau}(n)\big]$ is QSZK-hard, where $\gamma_{\delta,\tau}(n) \coloneqq 1 - 2^{-\frac{n+1}{n^{1+\delta}+1}} + 2^{-n^\tau - \frac{n+1}{n^{1+\delta}+1}}$ and $\gamma'_{\delta,\tau}(n) \coloneqq 2^{-n^\tau - \frac{1}{n^{1+\delta}+1}}$.*

(2) *For any $1+\frac{1}{n^{1+\delta}} < \alpha \leq 1+\frac{1}{n}$, $\forall \tau \in (0,1/2)$, $\mathrm{QSD}_\alpha\big[\frac{1}{2}-2^{-n^\tau-1}, 2^{-n^\tau-\frac{1}{n+1}}\big]$ is QSZK-hard.*

The proof of Theorem 26 (see also [52, Theorem 5.6]), based on reductions to QSD that is QSZK-hard [77, 78], is provided in the full version, particularly in [52, Section 5.2]. For any n-qubit state ρ of rank r, let ρ_{U} be the corresponding n-qubit state whose eigenvalues are uniformly distributed over the support of ρ. Next, we can establish the following lower bounds (see also [52, Theorem 5.7]), and the proof also can be found in [52, Section 5.2]:

▶ **Theorem 27** (Query complexity lower bounds for QSD_α)**.** *The following query complexity lower bounds hold in the purified quantum query access model, depending on the range of α, where $\delta > 0$ is a constant that can be made arbitrarily small:*

(1) *For any $1+\frac{1}{n^{1+\delta}} < \alpha \leq 1+\frac{1}{n}$ and $0 < \epsilon \leq 2^{\frac{1}{\alpha}-2}$, there exist an n-qubit state ρ of rank r and the corresponding state ρ_{U} such that deciding whether $\mathrm{T}_\alpha(\rho,\rho_{\mathrm{U}})$ is at least ϵ or exactly 0 requires $\Omega(r^{1/3})$ queries.*

(2) *For any $1 \leq \alpha \leq 1+\frac{1}{n^{1+\delta}}$, there exist a constant $\epsilon > 0$ such that, for some n-qubit state ρ of rank r and the corresponding state ρ_{U}, estimating $\mathrm{T}_\alpha(\rho,\rho_{\mathrm{U}})$ to within additive error ϵ requires $\widetilde{\Omega}(r^{1/2})$ queries.*

By leveraging the same reduction used to prove Theorem 27(1), the rank-dependent sample complexity lower bound (given in [52, Lemma 2.10(2)] in the full version) for estimating the trace distance $\mathrm{T}(\cdot,\cdot)$ can be extended to the quantum ℓ_α distance $\mathrm{T}_\alpha(\cdot,\cdot)$ with $1 \leq \alpha \leq 1+\frac{1}{n}$, as also stated in [52, Theorem 5.8]:

▶ **Theorem 28** (Sample complexity lower bound for QSD_α)**.** *For any $1 \leq \alpha \leq 1+\frac{1}{n}$ and $0 \leq \epsilon \leq 2^{\frac{1}{\alpha}-2}$, there exists an n-qubit state ρ of rank r and the corresponding state ρ_{U} such that deciding whether $\mathrm{T}_\alpha(\rho,\rho_{\mathrm{U}})$ is at least ϵ or exactly 0 requires $\Omega\big(r/\epsilon^2\big)$ samples of ρ.*

References

1 Jayadev Acharya, Ibrahim Issa, Nirmal V. Shende, and Aaron B. Wagner. Estimating quantum entropy. *IEEE Journal on Selected Areas in Information Theory*, 1(2):454–468, 2020. Preliminary version in *ISIT 2019*. doi:10.1109/JSAIT.2020.3015235.

2 Leonard M. Adleman, Jonathan Demarrais, and Ming-Deh A. Huang. Quantum computability. *SIAM Journal on Computing*, 26(5):1524–1540, 1997. doi:10.1137/S0097539795293639.

3 Dorit Aharonov, Vaughan Jones, and Zeph Landau. A polynomial quantum algorithm for approximating the Jones polynomial. *Algorithmica*, 55(3):395–421, 2009. Preliminary version in *STOC 2006*. doi:10.1007/s00453-008-9168-0.

4 Anurag Anshu, Zeph Landau, and Yunchao Liu. Distributed quantum inner product estimation. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*, pages 44–51, 2022. doi:10.1145/3519935.3519974.

5 Andrew Arrasmith, Lukasz Cincio, Andrew T. Sornborger, Wojciech H. Zurek, and Patrick J. Coles. Variational consistent histories as a hybrid algorithm for quantum foundations. *Nature communications*, 10(1):3438, 2019. doi:10.1038/s41467-019-11417-0.

6 Guillaume Aubrun and Stanisław J. Szarek. *Alice and Bob Meet Banach: The Interface of Asymptotic Geometric Analysis and Quantum Information Theory*, volume 223 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2017. doi:10.1090/surv/223.

7 Costin Bădescu, Ryan O'Donnell, and John Wright. Quantum state certification. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, pages 503–514, 2019. doi:10.1145/3313276.3316344.

8 Bernhard Baumgartner. An inequality for the trace of matrix products, using absolute values. ArXiv e-prints, 2011. arXiv:1106.6189.

9 Aleksandrs Belovs. Quantum algorithms for classical probability distributions. In *Proceedings of the 27th Annual European Symposium on Algorithms, ESA 2019*, volume 144 of *LIPIcs*, pages 16:1–16:11. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2019. doi:10.4230/LIPICS.ESA.2019.16.

10 Avraham Ben-Aroya, Oded Schwartz, and Amnon Ta-Shma. Quantum expanders: motivation and construction. *Theory of Computing*, 6(3):47–79, 2010. Preliminary version in *CCC 2008*. doi:10.4086/toc.2010.v006a003.

11 Michael Ben-Or, Michał Horodecki, Debbie W. Leung, Dominic Mayers, and Jonathan Oppenheim. The universal composable security of quantum key distribution. In *Theory of Cryptography, Second Theory of Cryptography Conference, TCC 2005, Cambridge, MA, USA, February 10-12, 2005, Proceedings*, pages 386–406. Springer, 2005. doi:10.1007/978-3-540-30576-7_21.

12 Itay Berman, Akshay Degwekar, Ron D. Rothblum, and Prashant Nalini Vasudevan. Statistical difference beyond the polarizing regime. In *Proceedings of the 17th International Conference on Theory of Cryptography Conference*, pages 311–332. Springer, 2019. ECCC:TR19-038. doi:10.1007/978-3-030-36033-7_12.

13 Serge Bernstein. Sur la meilleure approximation de $|x-c|^p$. *Doklady Akademii Nauk SSSR*, 18:379–384, 1938.

14 Serge Bernstein. Sur la meilleure approximation de $|x|^p$ par des polynômes de degrés très élevés. *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, 2(2):169–190, 1938. URL: https://www.mathnet.ru/eng/im3513.

15 Dominic W. Berry, Andrew M. Childs, Richard Cleve, Robin Kothari, and Rolando D. Somma. Simulating Hamiltonian dynamics with a truncated Taylor series. *Physical Review Letters*, 114(9):090502, 2015. doi:10.1103/PhysRevLett.114.090502.

16 Gilles Brassard, Peter Høyer, Michele Mosca, and Alain Tapp. Quantum amplitude amplification and estimation. In Samuel J. Lomonaco, Jr. and Howard E. Brandt, editors, *Quantum Computation and Information*, volume 305 of *Contemporary Mathematics*, pages 53–74. AMS, 2002. doi:10.1090/conm/305/05215.

17 Harry Buhrman, Richard Cleve, John Watrous, and Ronald de Wolf. Quantum fingerprinting. *Physical Review Letters*, 87(16):167902, 2001. doi:10.1103/PhysRevLett.87.167902.

18 Mark Bun, Robin Kothari, and Justin Thaler. The polynomial method strikes back: tight quantum query bounds via dual polynomials. *Theory of Computing*, 16(10):1–71, 2020. Preliminary version in *STOC 2018*. doi:10.4086/toc.2020.v016a010.

19 Chris Cade and Ashley Montanaro. The quantum complexity of computing Schatten p-norms. In *Proceedings of the 13th Conference on the Theory of Quantum Computation, Communication and Cryptography*, volume 111 of *LIPIcs*, pages 4:1–4:20. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2018. doi:10.4230/LIPICS.TQC.2018.4.

20 Clément L. Canonne. A survey on distribution testing: your data is big. but is it blue? In *Theory of Computing Library*, number 9 in Graduate Surveys, pages 1–100. University of Chicago, 2020. ECCC:TR15-063. doi:10.4086/toc.gs.2020.009.

21 André Chailloux, Dragos Florin Ciocan, Iordanis Kerenidis, and Salil Vadhan. Interactive and noninteractive zero knowledge are equivalent in the help model. In *Proceedings of the Fifth Theory of Cryptography Conference*, pages 501–534. Springer, 2008. IACR ePrint:2007/467. doi:10.1007/978-3-540-78524-8_28.

22 Sourav Chakraborty, Eldar Fischer, Arie Matsliah, and Ronald de Wolf. New results on quantum property testing. In *30th Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2010*, volume 8 of *LIPIcs*, pages 145–156. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2010. doi:10.4230/LIPICS.FSTTCS.2010.145.

23 Andrew M. Childs and Nathan Wiebe. Hamiltonian simulation using linear combinations of unitary operations. *Quantum Information and Computation*, 12(11–12):901–924, 2012. doi:10.26421/QIC12.11-12-1.

24 Patrick J. Coles. Unification of different views of decoherence and discord. *Physical Review A*, 85(4), 2012. doi:10.1103/physreva.85.042103.

25 Patrick J. Coles, M. Cerezo, and Lukasz Cincio. Strong bound between trace distance and Hilbert-Schmidt distance for low-rank states. *Physical Review A*, 100(2):022103, 2019. doi:10.1103/physreva.100.022103.

26 Chandler Davis. All convex invariant functions of Hermitian matrices. *Archiv der Mathematik*, 8(4):276–278, 1957. doi:10.1007/bf01898787.

27 Artur K. Ekert, Carolina Moura Alves, Daniel K. L. Oi, Michał Horodecki, Paweł Horodecki, and Leong Chuan Kwek. Direct estimations of linear and nonlinear functionals of a quantum state. *Physical Review Letters*, 88(21):217901, 2002. doi:10.1103/physrevlett.88.217901.

28 Nic Ezzell, Elliott M. Ball, Aliza U. Siddiqui, Mark M. Wilde, Andrew T. Sornborger, Patrick J. Coles, and Zoë Holmes. Quantum mixed state compiling. *Quantum Science and Technology*, 8(3):035001, 2023. doi:10.1088/2058-9565/acc4e3.

29 Wang Fang and Qisheng Wang. Optimal quantum algorithm for estimating fidelity to a pure state. In *Proceedings of the 33rd Annual European Symposium on Algorithms (ESA)*, 2025. arXiv:2506.23650.

30 Christopher A. Fuchs and Jeroen van de Graaf. Cryptographic distinguishability measures for quantum-mechanical states. *IEEE Transactions on Information Theory*, 45(4):1216–1227, 1999. doi:10.1109/18.761271.

31 Michael I. Ganzburg. *Limit theorems of polynomial approximation with exponential weights*, volume 192 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2008. doi:10.1090/memo/0897.

32 András Gilyén and Tongyang Li. Distributional property testing in a quantum world. In *Proceedings of the 11th Innovations in Theoretical Computer Science Conference*, pages 25:1–25:19, 2020. doi:10.4230/LIPIcs.ITCS.2020.25.

33 András Gilyén and Alexander Poremba. Improved quantum algorithms for fidelity estimation. ArXiv e-prints, 2022. arXiv:2203.15993.

34 András Gilyén, Yuan Su, Guang Hao Low, and Nathan Wiebe. Quantum singular value transformation and beyond: exponential improvements for quantum matrix arithmetics. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, pages 193–204, 2019. doi:10.1145/3313276.3316366.

35 Oded Goldreich. *Introduction to Property Testing*. Cambridge University Press, 2017. doi:10.1017/9781108135252.

36 Oded Goldreich, Amit Sahai, and Salil Vadhan. Honest-verifier statistical zero-knowledge equals general statistical zero-knowledge. In *Proceedings of the 30th Annual ACM Symposium on Theory of Computing*, pages 399–408, 1998. doi:10.1145/276698.276852.

37 Oded Goldreich and Salil Vadhan. Comparing entropies in statistical zero knowledge with applications to the structure of SZK. In *Proceedings of the Fourteenth Annual IEEE Conference on Computational Complexity*, pages 54–73. IEEE, 1999. ECCC:TR98-063. doi:10.1109/CCC.1999.766262.

38 Lov K. Grover. A fast quantum mechanical algorithm for database search. In *Proceedings of the 28th Annual ACM Symposium on Theory of Computing*, pages 212–219, 1996. doi:10.1145/237814.237866.

39 Aram W. Harrow, Avinatan Hassidim, and Seth Lloyd. Quantum algorithm for linear systems of equations. *Physical Review Letters*, 103(15):150502, 2009. doi:10.1103/PhysRevLett.103.150502.

40 Carl W. Helstrom. Detection theory and quantum mechanics. *Information and Control*, 10(3):254–291, 1967. doi:10.1016/S0019-9958(67)90302-6.

41 Alexander S. Holevo. Statistical decision theory for quantum systems. *Journal of Multivariate Analysis*, 3(4):337–394, 1973. doi:10.1016/0047-259X(73)90028-6.

42 Zdeněk Hradil, Jaroslav Řeháček, Jaromír Fiurášek, and Miroslav Ježek. Maximum-likelihood methods in quantum mechanics. *Quantum State Estimation*, pages 59–112, 2004. doi:10.1007/978-3-540-44481-7_3.

43 Piotr Indyk. Stable distributions, pseudorandom generators, embeddings, and data stream computation. *Journal of the ACM (JACM)*, 53(3):307–323, 2006. Preliminary version in *FOCS 2000*. doi:10.1145/1147954.1147955.

44 Shelby Kimmel, Cedric Yen-Yu Lin, Guang Hao Low, Maris Ozols, and Theodore J. Yoder. Hamiltonian simulation with optimal sample complexity. *npj Quantum Information*, 3(1):1–7, 2017. doi:10.1038/s41534-017-0013-7.

45 A. Yu. Kitaev. Quantum measurements and the Abelian stabilizer problem. ArXiv e-prints, 1995. arXiv:quant-ph/9511026.

46 Marius Kloft, Ulf Brefeld, Sören Sonnenburg, and Alexander Zien. ℓ_p-norm multiple kernel learning. *The Journal of Machine Learning Research*, 12:953–997, 2011. doi:10.5555/1953048.2021033.

47 Hirotada Kobayashi. Non-interactive quantum perfect and statistical zero-knowledge. In *Proceedings of the 14th International Symposium on Algorithms and Computation*, pages 178–188. Springer, 2003. doi:10.1007/978-3-540-24587-2_20.

48 François Le Gall, Yupan Liu, and Qisheng Wang. Space-bounded quantum state testing via space-efficient quantum singular value transformation. ArXiv e-prints, 2023. doi:10.48550/arXiv.2308.05079.

49 James R Lee and Assaf Naor. Embedding the diamond graph in ℓ_p and dimension reduction in ℓ_1. *Geometric & Functional Analysis GAFA*, 14(4):745–747, 2004. doi:10.1007/s00039-004-0473-8.

50 Nana Liu, Qisheng Wang, Mark M. Wilde, and Zhicheng Zhang. Quantum algorithms for matrix geometric means. *npj Quantum Information*, 11:101, 2025. doi:10.1038/s41534-025-00973-7.

51 Yupan Liu. Quantum state testing beyond the polarizing regime and quantum triangular discrimination, 2025. *To appear in* computational complexity. arXiv:2303.01952.

52 Yupan Liu and Qisheng Wang. On estimating the quantum ℓ_α distance. ArXiv e-prints, 2025. The full version of this paper also includes references [8, 9, 21, 22, 30, 44, 47, 50, 54, 55]. arXiv:2505.00457v2.

53 Yupan Liu and Qisheng Wang. On estimating the trace of quantum state powers. In *Proceedings of the 2025 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 947–993. SIAM, 2025. doi:10.1137/1.9781611978322.28.

54 Seth Lloyd, Masoud Mohseni, and Patrick Rebentrost. Quantum principal component analysis. *Nature Physics*, 10(9):631–633, 2014. doi:10.1038/nphys3029.

55 Guang Hao Low and Isaac L. Chuang. Hamiltonian simulation by qubitization. *Quantum*, 3:163, 2019. doi:10.22331/q-2019-07-12-163.

56 Alessandro Luongo and Changpeng Shao. Quantum algorithms for spectral sums. ArXiv e-prints, 2020. arXiv:2011.06475.

57 John M. Martyn, Zane M. Rossi, Andrew K. Tan, and Isaac L. Chuang. Grand unification of quantum algorithms. *PRX quantum*, 2(4):040203, 2021. doi:10.1103/PRXQuantum.2.040203.

58 Ashley Montanaro and Ronald de Wolf. A survey of quantum property testing. In *Theory of Computing Library*, number 7 in Graduate Surveys, pages 1–81. University of Chicago, 2016. doi:10.4086/toc.gs.2016.007.

59 Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2010. doi:10.1017/CBO9780511976667.

60 Ryan O'Donnell and John Wright. Quantum spectrum testing. *Communications in Mathematical Physics*, 387(1):1–75, 2021. Preliminary version in *STOC 2015*. doi:10.1007/s00220-021-04180-1.

61 Palash Pandya, Omer Sakarya, and Marcin Wieśniak. Hilbert-Schmidt distance and entanglement witnessing. *Physical Review A*, 102(1):012409, 2020. doi:10.1103/PhysRevA.102.012409.

62 Yihui Quek, Eneet Kaur, and Mark M. Wilde. Multivariate trace estimation in constant quantum depth. *Quantum*, 8:1220, 2024. doi:10.22331/Q-2024-01-10-1220.

63 Renato Renner and Robert König. Universally composable privacy amplification against quantum adversaries. In *Theory of Cryptography, Second Theory of Cryptography Conference, TCC 2005, Cambridge, MA, USA, February 10-12, 2005, Proceedings*, pages 407–425. Springer, 2005. doi:10.1007/978-3-540-30576-7_22.

64 Soorya Rethinasamy, Rochisha Agarwal, Kunal Sharma, and Mark M. Wilde. Estimating distinguishability measures on quantum computers. *Physical Review A*, 108(1):012409, 2023. doi:10.1103/PhysRevA.108.012409.

65 Theodore J. Rivlin. *Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory*. Courier Dover Publications, 1990.

66 Amit Sahai and Salil Vadhan. A complete problem for statistical zero knowledge. *Journal of the ACM*, 50(2):196–249, 2003. Preliminary version in *FOCS 1997*. ECCC:TR00-084. doi:10.1145/636865.636868.

67 Aleksandr F. Timan. *Theory of Approximation of Functions of a Real Variable*, volume 34 of *International Series of Monographs on Pure and Applied Mathematics*. Pergamon Press, 1963. doi:10.1016/c2013-0-05307-8.

68 Vilmos Totik. *Metric properties of harmonic measures*, volume 184 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2006. doi:10.1090/memo/0867.

69 Bo Waggoner. ℓ_p testing and learning of discrete distributions. In *Proceedings of the 2015 Conference on Innovations in Theoretical Computer Science*, pages 347–356, 2015. doi:10.1145/2688073.2688095.

70 Qisheng Wang. Optimal trace distance and fidelity estimations for pure quantum states. *IEEE Transactions on Information Theory*, 70(12):8791–8805, 2024. doi:10.1109/TIT.2024.3447915.

71 Qisheng Wang, Ji Guan, Junyi Liu, Zhicheng Zhang, and Mingsheng Ying. New quantum algorithms for computing quantum entropies and distances. *IEEE Transactions on Information Theory*, 70(8):5653–5680, 2024. doi:10.1109/TIT.2024.3399014.

72 Qisheng Wang and Zhicheng Zhang. Fast quantum algorithms for trace distance estimation. *IEEE Transactions on Information Theory*, 70(4):2720–2733, 2024. doi:10.1109/TIT.2023.3321121.

73 Qisheng Wang and Zhicheng Zhang. Sample-optimal quantum estimators for pure-state trace distance and fidelity via samplizer. ArXiv e-prints, 2024. doi:10.48550/arXiv.2410.21201.

74 Qisheng Wang and Zhicheng Zhang. Quantum lower bounds by sample-to-query lifting, 2025. *To appear in* SIAM Journal on Computing. arXiv:2308.01794.

75 Qisheng Wang and Zhicheng Zhang. Time-efficient quantum entropy estimator via samplizer. *IEEE Transactions on Information Theory*, 2025. Preliminary version in *ESA 2024*. doi:10.1109/TIT.2025.3576137.

76 Qisheng Wang, Zhicheng Zhang, Kean Chen, Ji Guan, Wang Fang, Junyi Liu, and Mingsheng Ying. Quantum algorithm for fidelity estimation. *IEEE Transactions on Information Theory*, 69(1):273–282, 2023. doi:10.1109/TIT.2022.3203985.

77 John Watrous. Limits on the power of quantum statistical zero-knowledge. In *Proceedings of the 43rd Annual IEEE Symposium on Foundations of Computer Science*, pages 459–468. IEEE, 2002. doi:10.1109/SFCS.2002.1181970.

78 John Watrous. Zero-knowledge against quantum attacks. *SIAM Journal on Computing*, 39(1):25–58, 2009. Preliminary version in *STOC 2006*. doi:10.1137/060670997.

79 Mark M. Wilde. *Quantum Information Theory*. Cambridge University Press, 2013. doi:10.1017/9781316809976.

80 Tomoyuki Yamakami and Andrew C. Yao. $\mathsf{NQP}_{\mathbb{C}} = \mathsf{coC}_{=}\mathsf{P}$. *Information Processing Letters*, 71(2):63–69, 1999. doi:10.1016/s0020-0190(99)00084-8.