

On Algorithmic Applications of \mathcal{F} -Branchwidth

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Abstract

\mathcal{F} -branchwidth is a framework for width measures of graphs, recently introduced by Eiben et al. [ITCS 2022], that captures tree-width, co-tree-width, clique-width, and mim-width, and several of their generalizations and interpolations. In this work, we search for algorithmic applications of \mathcal{F} -branchwidth measures that do not have an equivalent counterpart in the literature so far. Our first contribution is a minimal set of eleven \mathcal{F} -branchwidth measures such that each of the infinitely many \mathcal{F} -branchwidth measures is equivalent to one of the eleven. We observe that for the FO MODEL CHECKING problem, each \mathcal{F} -branchwidth is either equivalent to clique-width (and therefore has an FPT-algorithm by formula length plus the width) or the problem remains as hard as on general graphs even on graphs of constant width. Next, we study the number of equivalence classes of the neighborhood equivalence in a decomposition, which upper bounds the run time of the model checking algorithm for A&C DN logic recently introduced by Bergounoux et al. [SODA 2023]. We give structural lower bounds that show that for each \mathcal{F} -branchwidth, an efficient model checking algorithm was already known or cannot be obtained via this method. Lastly, we classify the complexity of INDEPENDENT SET parameterized by any \mathcal{F} -branchwidth except for one open case. Also here, our contributions are lower bounds. In this context, we also prove that INDEPENDENT SET on graphs of mim-width w cannot be solved in time $n^{o(w)}$ unless the Exponential Time Hypothesis fails, answering an open question in the literature.

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1 Introduction

Eiben et al. [12] recently introduced \mathcal{F} -branchwidth, a framework of graph width measures that allows for an infinite number of instantiations. Among them are width measures that are asymptotically equivalent to treewidth, clique-width, and mim-width. These three have wide algorithmic applications which immediately begs the following question:

Are there any *unknown* \mathcal{F} -branchwidth measures that have interesting algorithmic applications?



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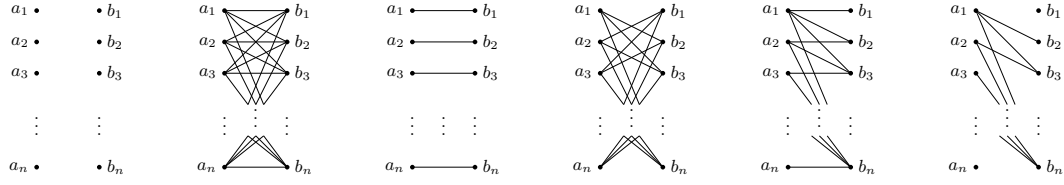
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The goal of this paper is to address this question – to which we provide negative answers in several contexts.

For a family of bipartite graphs \mathcal{F} , the \mathcal{F} -branchwidth of a graph G bounds the size of any member of \mathcal{F} that appears as a semi-induced subgraph in a recursive partitioning of the vertex set of G . Therefore, any family of bipartite graphs \mathcal{F} gives rise to a new \mathcal{F} -branchwidth measure. To avoid chaotic definitions, some technical restrictions are imposed on \mathcal{F} , but there is still an infinite number of families \mathcal{F} satisfying them. However, as already observed in [12], there is a finite number of classes of \mathcal{F} -branchwidth measures that are bounded on different families of graphs. This means that we can restrict the search for new algorithmically useful width measures to the representatives of these classes. These are obtained as any combination of the six families shown in Figure 1.



■ **Figure 1** The six canonical families usable as building blocks for \mathcal{F} -branchwidth. From left to right: edgeless (\emptyset), complete (\mathbb{K}), matchings (\equiv), antimatchings (\otimes), chains (\cong), and skewed chains (\bowtie).

This brings down the number of cases to consider from infinite to constant, if one wants to determine the complexity of a problem parameterized by *any* of the possible \mathcal{F} -branchwidth measures. However, a priori there are still 2^6 cases to consider. Our first contribution is to provide a family of eleven \mathcal{F} -branchwidth, called *The Eleven*, that captures the modeling power of every possible \mathcal{F} -branchwidth in the following sense. For an illustration see Figure 2.

► **Theorem 1.** *For each \mathcal{F} -branchwidth, one of the following holds.*

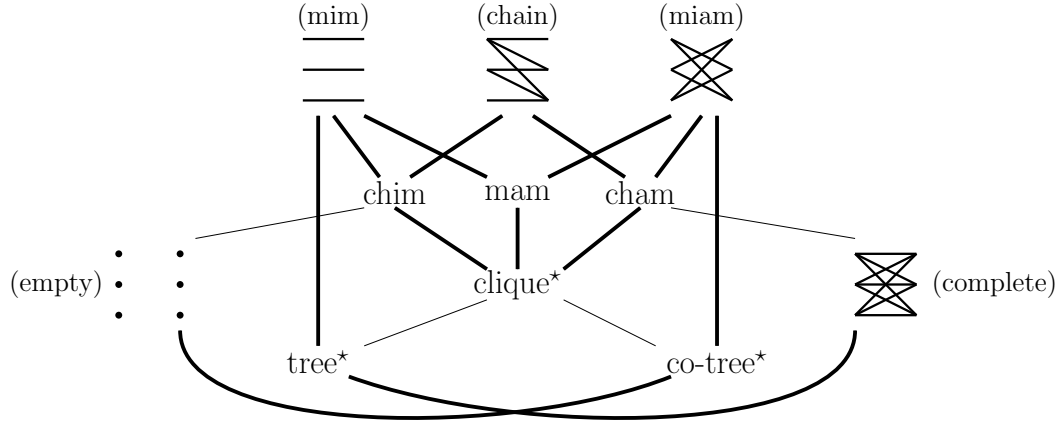
- (i) *There is a computable function f such that for each graph G with $\mathcal{F}\text{-bw}(G) \leq k$, $|V(G)| \leq f(k)$.*
- (ii) *\mathcal{F} -branchwidth is asymptotically equivalent to \mathcal{F}^* -branchwidth, where \mathcal{F}^* is one of the Eleven branchwidths (presented in Figure 2 and formally defined in Table 1).*

We provide families of obstructions to bounded width that show that the classification in Theorem 1 is in fact minimal, in the sense that no pair of width measures can be asymptotically equivalent to each other.

► **Theorem 2.** *For each pair of distinct $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{F}$ there is some $i \in [2]$ such that there is a graph class of bounded \mathcal{F}_i -width but unbounded \mathcal{F}_{3-i} -width.*

Throughout the following, we refer to the \mathcal{F} -branchwidth measures based on the families in \mathfrak{F} as the *canonical* ones. Notice that the second item of Theorem 1 says that \mathcal{F} -branchwidth and \mathcal{F}^* -branchwidth are bounded on the same graph classes, but the exact bounds may differ. Theorem 1, along with the hierarchy among the eleven relevant width measures, allows us to determine the complexity of a problem parameterized by any \mathcal{F} -branchwidth by considering these cases alone.

A common way to classify the algorithmic power of a width measure is to determine for which logic L , the L -MODEL CHECKING problem can be solved efficiently whenever the input graph is given together with a decomposition of constant width. For instance, an



■ **Figure 2** The Eleven canonical \mathcal{F} -branchwidth measures, ordered according to expressive power from top (high) to bottom (low). The five base cases show the singleton classes, and bold edges show combinations. For instance, chim-width is $(\equiv \cup \equiv)$ -branchwidth, clique*-width is $(\equiv \cup \equiv \cup \equiv)$ -branchwidth, and tree*-width is $(\equiv \cup \equiv)$ -branchwidth. The remaining edges show some relationships given by the structures of the families. As one may expect, tree*-width is asymptotically equivalent to tree-width, clique*-width is asymptotically equivalent to clique-width, and co-tree*-width is asymptotically equivalent to co-tree-width.

L-formula ϕ may express that a graph has an independent set of size k . Then, the evaluation of ϕ at a graph G is true, in symbols $G \models \phi$ (read: “ G models ϕ ”), if and only if G has an independent set of size k . Typically, one aims for efficient parameterized algorithms in the parameterization formula length plus width.

L-MODEL CHECKING parameterized by $|\phi| + w$

Input: Graph G given together with a decomposition of width w , L-formula ϕ .
Question: Does $G \models \phi$?

Classical examples of such meta-theorems are Courcelle’s Theorem [8] which gives an FPT-algorithm when L is MSO_2 logic and the width is tree-width and its extension to the more general clique-width while restricting the logic to MSO_1 [9]. More recent examples include an FPT-algorithm for FO logic and twin-width [6], and an XP-algorithm for A&C DN logic and mim-width [3]. The following hierarchy holds for the first three logics: $\text{FO} \subsetneq \text{MSO}_1 \subsetneq \text{MSO}_2$.

We consider the MODEL CHECKING problem for any of these logics parameterized by any of the \mathcal{F} -branchwidth measures. Our first result is that for MSO/FO , we cannot expect any new tractability islands via \mathcal{F} -branchwidth, so in fact, our current understanding of the complexity of MSO/FO MODEL CHECKING is tight for any \mathcal{F} -branchwidth. We illustrate the following theorem for the canonical \mathcal{F} -branchwidth measures in Figure 3.

► **Theorem 3.** *For each \mathcal{F} -branchwidth, one of the following holds.*

1. *There is a computable function f such that each graph of \mathcal{F} -branchwidth k has clique-width at most $f(k)$, implying that FO MODEL CHECKING is FPT parameterized by formula length plus k .*
2. *There is a constant c such that FO MODEL CHECKING parameterized by formula length is $\text{AW}[*]$ -hard even when the input graph is given with one of its decompositions of \mathcal{F} -branchwidth c .*

Next, we move on to A&C DN logic which was recently introduced by Bergougnoux et al. [3] as a logic to capture the problems that are solvable in XP time parameterized by the mim-width of a given decomposition of the input graph. This logic is a good target to study under the lens of \mathcal{F} -branchwidth. It is the right fit for mim-width which is \equiv -branchwidth where \equiv is the family of matchings – one of the base families of \mathcal{F} -branchwidth. A&C DN logic is essentially¹ based on existential MSO₁ and a *neighborhood operator* which, for a given set variable X , returns the set $\bigcup_{v \in X} N(v)$, and predicates checking acyclicity and connectivity. The runtime of the algorithm for A&C DN MODEL CHECKING is bounded by the index of the *neighborhood equivalence relation* (nec), first studied by Bui-Xuan et al. [7]. To get efficient algorithms, it therefore suffices to show, for graphs of width k , that the index of the neighborhood equivalence grows relatively slowly in terms of n and k . For instance, a bound of $f(k) \cdot n^{\mathcal{O}(1)}$ implies an $f(k) \cdot n^{\mathcal{O}(1)}$ time algorithm for A&C DN MODEL CHECKING, while a bound of $n^{g(k)}$ implies an $n^{\mathcal{O}(g(k))}$ time algorithm, using the meta-theorem of [3].

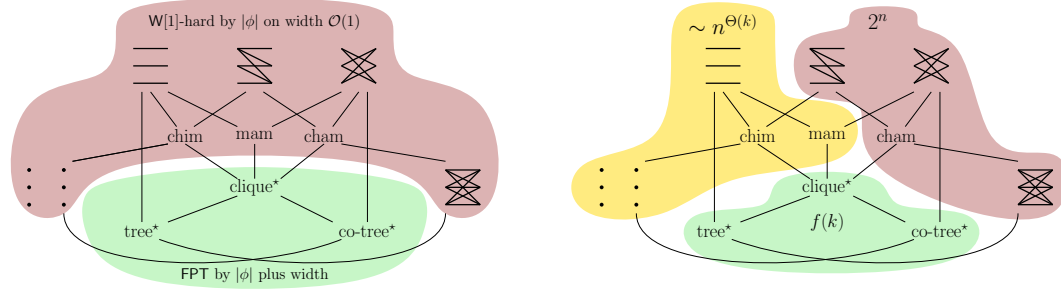
Several such bounds on the index of this equivalence relation are already known in the literature. First, mim-width k implies that there are at most n^k equivalence classes [2]. Furthermore, there are bounds of the form $2^{\mathcal{O}(k)}$ when k is either the tree-width, clique-width, or co-tree-width (see, for instance, [3]). Next, we cannot hope to obtain any meaningful bound for any \mathcal{F} -branchwidth that generalizes complete-width. When a cut consists of matching with n vertices on each side, the complete-width is just 1, while the number of neighborhoods across the cut is 2^n . Indeed, for each subset of vertices contained in one side of the cut, there is a different subset across the cut whose neighborhood is exactly that subset.

The cases that remain are the ones asymptotically equivalent to empty-width, chim-width, and mam-width. As mim-width generalizes all of them, we immediately have XP-algorithms parameterized by each of them. Also here, we provide a number of negative results. For chim-width and mam-width, not much improvement can be made over the bound given by mim-width. We show that there are graphs of chim-width (or mam-width) k such that each of their decompositions induces a cut with at least $(n/(k^2 \log n))^{\Omega(k)}$ neighborhoods. By a probabilistic argument, we show that there are graphs whose decompositions have cuts of empty-width k that have $n^{\Omega(k)}$ neighborhoods. We conclude that the technology from [3] cannot be applied directly to any \mathcal{F} -branchwidth measure to extend the range of tractability of A&C DN MODEL CHECKING, see Figure 3 for an illustration.

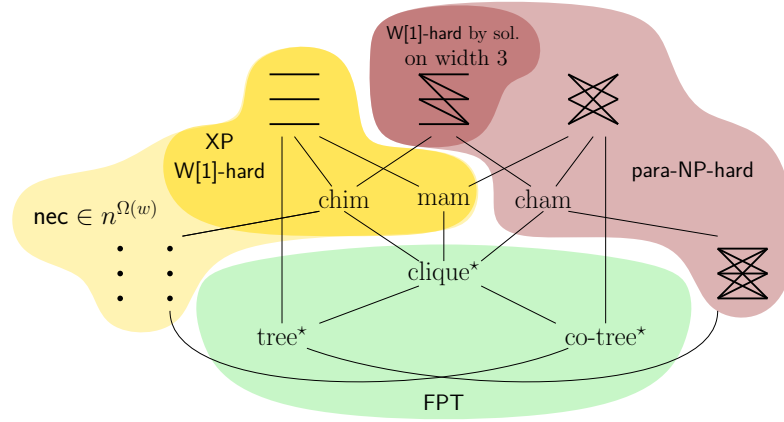
► **Theorem 4.** *For each \mathcal{F} -branchwidth, one of the following holds.*

1. *There is a computable function f such that each graph of \mathcal{F} -branchwidth k has clique-width at most $f(k)$, implying that A&C DN MODEL CHECKING is FPT parameterized by formula length plus the width of a given decomposition of the input graph.*
2. *Each graph of \mathcal{F} -branchwidth k has mim-width at most k , implying that A&C DN MODEL CHECKING is XP parameterized by formula length plus the width of a given decomposition of the input graph. On the other hand, there are cuts of \mathcal{F} -branchwidth k with $n^{\Omega(k)}$ neighborhood equivalence classes.*
3. *There is a computable function f such that each graph of complete-width at most k has \mathcal{F} -branchwidth at most $f(k)$, implying that there are cuts on n vertices of constant \mathcal{F} -branchwidth with $2^{\Omega(n)}$ neighborhood equivalence classes.*

¹ The neighborhood operator in [3] is somewhat more general, but for the sake of this introduction it suffices to think of simple neighborhoods.



■ **Figure 3** Results about the complexity of model checking problems on graphs of bounded \mathcal{F} -branchwidth. Left: Complexity of FO MODEL CHECKING. Right: Number of neighborhoods in cuts of bounded width.



■ **Figure 4** The complexity of INDEPENDENT SET parameterized by any \mathcal{F} -branchwidth measure. Note that the complexity of INDEPENDENT SET parameterized by empty-width is still open, but the XP lower bound on the number of equivalence classes in the neighborhood equivalence implies that known techniques are not applicable to show fixed-parameter tractability in this case. Moreover, it is worth looking into the complexity of IS parameterized by solution size on graphs of constant miam-width, cham-width, or complete-width.

For the canonical \mathcal{F} -branchwidth measures, we also take a more fine-grained look at the index of the neighborhood equivalence. The upper bounds on the number of equivalence classes in terms of clique*-width and tree*-width follow from known bounds [3] together with Ramsey-theory based arguments from [12]. This results in upper bounds that are double-exponential in the width k , i.e., bounds of the form $2^{2^{\text{poly}(k)}}$. For tree*-width and co-tree*-width k , we give improved, single-exponential, bounds of $2^{\mathcal{O}(k^3 \log k)}$ each, and a lower bound of $2^{\Omega(k \log k)}$ for tree*-width k .

For the lower bound statements of Theorem 4, observe two things. First, for chim-width and mam-width, the bounds in fact hold for *any* decomposition of graphs within a family we construct. Second, lower bounds on the number of equivalence classes of the neighborhood equivalence only rule out direct applications of the meta-theorem from [3], and are not hardness results per se.

Due to that, we explicitly consider the INDEPENDENT SET and DOMINATING SET problems as two of the most fundamental problems expressible in A&C DN logic. Here, we provide several new hardness results. First, we show W[1]-hardness parameterized by chim-width and mam-width, and we show that under the Exponential Time Hypothesis, no $n^{o(k)}$ time

algorithm can exist. The latter gives a negative answer to the question explicitly asked for instance in [1, 4] whether the linear dependence on the mim-width in the exponent of the algorithms from [7] can be lowered significantly.

► **Theorem 5.** *Unless the Exponential Time Hypothesis fails, neither INDEPENDENT SET nor DOMINATING SET can be solved in $f(k)n^{o(k)}$ time for any computable function f , where k is either the chim-width, the mam-width, or the mim-width of a given decomposition of the n -vertex input graph. Moreover, both problems parameterized by either the chim-width or the mam-width are $W[1]$ -hard.*

Since INDEPENDENT SET is NP-complete on cubic graphs, it is para-NP-hard when parameterized by any width measure in whose obstructions the degree increases proportionally to the size. This is the case for complete bipartite graphs, anti-matchings, and chains. However, one might hope for efficient algorithms for INDEPENDENT SET parameterized by solution size on graphs of constant width. For chain-width, we rule out this possibility by proving that the graphs from a reduction due to Fomin, Golovach, and Raymond [13] have bounded chain-width.

► **Theorem 6.** *INDEPENDENT SET parameterized by solution size is $W[1]$ -hard even when the input graph is provided with a linear decomposition of chain-width 3.*

We summarize the results for the INDEPENDENT SET problem in Figure 4.

This paper is organized as follows. In the next section, we introduce our notations and the \mathcal{F} -branchwidths framework, as well as some basic facts on these parameters. Theorem 2 is proved in Section 3. Theorem 3 is proved in Section 4. Theorem 4 is proved in Section 5. Finally, Theorem 5 is proved in Section 5. Missing proofs are in the full version.

2 Preliminaries

For an integer i , we let $[i] = \{1, 2, \dots, i\}$.

We use standard graph terminology and notation [11], some of which we briefly recall in the following. In every notation with a graph subscript, we may omit it if the graph is clear from the context.

A *cut* of a graph G is a partition of $V(G)$ into two sets A, B . We denote by $G[A, B]$ the subgraph of G with edge set $\{uv \in E(G) \mid u \in A, v \in B\}$. A graph G is *bipartite* if it can be written as $G[A, B]$ for some cut A, B . In this case we also use (A, B, E) to denote G . If $|A| = |B|$, then we call $|V(G)/2| (= |A| = |B|)$ the *half-order* of G .

For a graph G , its *(edge) complement* \overline{G} is given by $(V(G), \{uv \mid u, v \in V(G) \wedge uv \notin E(G)\})$. Given a set $A \subseteq V(G)$, we use \overline{A} as shorthand for $V(G) \setminus A$. For a bipartite graph $G = (A, B, E)$, its *bipartite (edge) complement* is given by $(A, B, \{uv \mid u \in A \wedge v \in B \wedge uv \notin E\})$.

Branch decomposition. A *branch decomposition* or *tree layout* (or simply *layout*) of a graph G is a pair (T, λ) where T is a ternary tree (i.e., every internal node of T has degree 3) and λ is a bijection from $V(G)$ to the leaves of T . An edge e of T *induces* or *defines* a cut (X_e, Y_e) of G , where X_e and Y_e are the preimages by λ of the leaves in the two components of $T - e$. For a symmetric function $f_G: 2^{V(G)} \rightarrow \mathbb{Z}$ which we refer to as a *cut function* of G , the *f-width* of e is $f_G(X_e)$, the *f-width* of the branch decomposition T is the maximum width over all edges of T , and the *f-branchwidth* of G , denoted by $\text{f-width}(G)$, is the minimum width of a branch decomposition of G .

The following lemma encapsulates a standard argument on branch decompositions that we need to obtain some of our lower bound results.

► **Lemma 7.** *Let G be a graph and $X \subseteq V(G)$ such that $|X| \geq 2$. Any branch decomposition of G has an edge that displays a bipartition (L, R) of $V(G)$ so that $|L \cap X| \geq |X|/3$ and $|R \cap X| \geq |X|/3$.*

Corresponding *linear branchwidth* constrains T to be a path with a pendant for each internal node. The linear f -branchwidth of G is denoted by $\text{lin-}f\text{-width}(G)$. It is equivalent to view the layout given by such a tree as a permutation of $V(G)$ where the considered cuts for linear branchwidth are between vertices before and vertices after each point in the permutation.

Besides \mathcal{F} -branchwidth, we will also use the cut function nec_G (and its associated branchwidth) defined as follows. Given $A \subseteq V(G)$, two subsets X and Y of A are neighborhood-equivalent over A if $N(X) \setminus A = N(Y) \setminus A$, we define $\text{nec}_G(A)$ as the number of equivalence relation over A .

We focus on the branchwidths proposed in [12] that arise as follows. Let \mathcal{F} be an infinite family of bipartite graphs where each graph in \mathcal{F} has bipartition $(A = (a_1, \dots, a_n), B = (b_1, \dots, b_n))$ such that for each $2n$ -vertex graph in \mathcal{F} and each subset L of $[n]$, the graph induced on $\{a_i, b_i \mid i \in L\}$ is isomorphic to a graph in \mathcal{F} . Given a graph G , we define the \mathcal{F} -cut function f_G of G such that $f_G(X)$ is the largest k such that a $2k$ -vertex graph in \mathcal{F} is isomorphic to an induced subgraph of $G[X, V(G) \setminus X]$. It is known that for any such choice of \mathcal{F} , \mathcal{F} -branchwidth is asymptotically equivalent to \mathcal{F}' -branchwidth where \mathcal{F}' is a union of any of the following six graph classes which are called *si ph* (short for *size identifiable partner hereditary*): \equiv the class of all bipartite induced matchings, \bowtie the class of all bipartite complements of induced matchings, also called *antimatchings* (or co-matchings), \boxtimes the class of all complete bipartite graphs with an equal number of vertices on each side of the bipartition, \vdots the class of all bipartite edgeless graphs with an equal number of vertices on each side of the bipartition, which we simply refer to as *bipartite edgeless graphs*, \boxminus the class of all bipartite graphs with an equal number of vertices on each side of the bipartition such that the vertices on each side permit an indexing under which each vertex u on one side has an edge to a vertex v other the index of u is at most the index of v , which we refer to as *chains*, and \preceq the class of all bipartite graphs with an equal number of vertices on each side of the bipartition such that the vertices on each side permit an indexing under which each vertex u on one side has an edge to a vertex v other the index of u is less than the index of v , which we refer to as *skewed chains*.

It will be useful to introduce the following notation for some bipartite graphs:

- $\vdots_k = ((a_1, \dots, a_k), (b_1, \dots, b_k), E)$, where $E = \emptyset$.
- $\bowtie_k = ((a_1, \dots, a_k), (b_1, \dots, b_k), E)$, where $E = \{a_i b_j \mid i, j \in [k]\}$.
- $\equiv_k = ((a_1, \dots, a_k), (b_1, \dots, b_k), E)$, where $E = \{a_i b_i \mid i \in [k]\}$.
- $\boxtimes_k = ((a_1, \dots, a_k), (b_1, \dots, b_k), E)$, where $E = \{a_i b_j \mid i, j \in [k], i \neq j\}$.
- $\boxminus_k = ((a_1, \dots, a_k), (b_1, \dots, b_k), E)$, where $E = \{a_i b_j \mid 1 \leq i \leq j \leq k\}$.
- $\preceq_k = ((a_1, \dots, a_k), (b_1, \dots, b_k), E)$, where $E = \{a_i b_j \mid 1 \leq i < j \leq k\}$.

2.1 Co-/degeneracy, empty-width/complete-width, and subdivisions

In this subsection, we explore the relations between degeneracy [16], subdivisions, complete-width and empty-width. All the corollary of this subsection follows from the fact that the (linear) complete-width of a graph is exactly the (linear) empty-width of its complement. This will be useful to establish relationships between \mathcal{F} -branchwidths in the next section.

► **Proposition 8.** *Let G be a graph with degeneracy at most k . Then, G has linear complete-width at most k .*

► **Corollary 9.** *Let G be a graph with co-degeneracy at most k . Then, G has linear empty-width at most k .*

On the other hand, hypercubes have unbounded degeneracy but complete-width at most 2.

► **Proposition 10.** *For each $d \in \mathbb{N}$, the hypercube of dimension d has minimum degree d and linear complete-width at most two.*

► **Corollary 11.** *For each $d \in \mathbb{N}$, the hypercube of dimension d has minimum co-degree d and linear empty-width at most two.*

For a graph G and a positive integer q , the q -subdivision of G is the graph obtained from G by subdividing each edge q times.

► **Observation 12.** *Let G be a simple graph, and let G' be obtained from G by subdividing each edge at least once. Then, any linear ordering of G' has complete-width at most one.*

► **Corollary 13.** *Let G be a simple graph and let $\overline{G'}$ be the complement of a graph obtained from G by subdividing each edge at least once. Then, any linear ordering of G' has empty-width at most one.*

3 The Eleven

In this section we show that we can restrict ourselves further to eleven different \mathcal{F} -branchwidth measures, called *the Eleven* (shown in Figure 2) and still obtain complete classifications. We begin by restating a lemma from [12] to justify that we do not need to consider the family of skewed chains.

► **Lemma 14** (Lemma 4.9 in [12]). *Let \mathcal{F} be a union of *si ph* classes. If $\mathbb{N} \subseteq \mathcal{F}$, then for every graph G it holds that $|\mathcal{F}\text{-bw}(G) - \mathcal{F}'\text{-bw}(G)| \leq 1$, where $\mathcal{F}' = (\mathcal{F} \setminus \mathbb{N}) \cup \mathbb{N}$.*

The remaining five singleton-classes that we restrict our attention to from now on are each interesting in their own right. The following observation will come in useful.

► **Observation 15.** *Let $H = ((a_1, \dots, a_k), (b_1, \dots, b_k), E)$ be an ordered bipartite graph.*

1. *If $H = \mathbb{N}_k$, then H contains an edgeless bipartite graph of half-order $\lfloor k/2 \rfloor$.*
2. *If $H = \mathbb{N}_k$ or $H = \mathbb{N}_k$, then H contains an edgeless bipartite graph of half-order $\lfloor k/2 \rfloor$.*
3. *If $H = \mathbb{N}_k$, or $H = \mathbb{N}_k$, or $H = \mathbb{N}_k$, then H contains a complete bipartite graph of half-order $\lfloor k/2 \rfloor$.*

We move on to pairs, first observing that we can restrict ourselves to the case when at most one of \mathbb{N} or \mathbb{N} is contained in \mathcal{F} .

► **Lemma 16.** *Let \mathcal{F} be a union of *si ph* classes such that $\mathbb{N} \cup \mathbb{N} \subseteq \mathcal{F}$. There is a computable function f such that for each graph G with $\mathcal{F}\text{-bw}(G) \leq k$, $|V(G)| < f(k)$.*

When $\mathbb{N} \subseteq \mathcal{F}$, the presence or absence of \mathbb{N} or \mathbb{N} in \mathcal{F} does not matter asymptotically either.

► **Lemma 17.** *Let \mathcal{F} be a union of *si ph* classes such that $\mathbb{N} \subseteq \mathcal{F}$. Let \mathcal{F}^* be any (possibly empty) combination of $\{\mathbb{N}, \mathbb{N}\}$, and let $\mathcal{F}' = \mathcal{F} \cup \mathcal{F}^*$. Then, for any graph G , $\mathcal{F}\text{-bw}(G) \leq \mathcal{F}'\text{-bw}(G) \leq 2 \cdot \mathcal{F}\text{-bw}(G) + 1$.*

Similarly, we have the following.

► **Lemma 18.** *Let \mathcal{F} be a union of *si ph* classes such that $\boxtimes \subseteq \mathcal{F}$. Let \mathcal{F}^* be any (possibly empty) combination of $\{\boxtimes, \boxminus\}$, and let $\mathcal{F}' = \mathcal{F} \cup \mathcal{F}^*$. Then, for any graph G , $\mathcal{F}\text{-bw}(G) \leq \mathcal{F}'\text{-bw}(G) \leq 2 \cdot \mathcal{F}\text{-bw}(G) + 1$.*

We are now ready to define the Eleven— see Table 1 — and prove Theorem 1.

■ **Table 1** Definitions of the Eleven \mathcal{F} -branchwidth and introduction of the notations for their associated \mathcal{F} -cut functions.

\mathcal{F}	Name	\mathcal{F} -cut function	\mathcal{F}	Name	\mathcal{F} -cut function
\boxtimes	complete-width	complete_G	$\boxtimes \cup \equiv$	tree*-width	t^*_G
$\vdots \vdots$	empty-width	empty_G	$\vdots \vdots \cup \boxtimes$	co-tree*-width	co-t^*_G
\equiv	mim-width	mim_G	$\equiv \cup \boxtimes$	mam-width	mam_G
\boxtimes	miam-width	miam_G	$\boxtimes \cup \boxminus$	chim-width	chim_G
\boxminus	chain-width	chain_G	$\equiv \cup \boxminus$	cham-width	cham_G
			$\equiv \cup \boxtimes \cup \boxminus$	clique*-width	c^*_G

Observe that tree*-width is asymptotically equivalent to tree-width and clique*-width is asymptotically equivalent [12, Section 4].

► **Theorem 1.** *For each \mathcal{F} -branchwidth, one of the following holds.*

- (i) *There is a computable function f such that for each graph G with $\mathcal{F}\text{-bw}(G) \leq k$, $|V(G)| \leq f(k)$.*
- (ii) *\mathcal{F} -branchwidth is asymptotically equivalent to \mathcal{F}^* -branchwidth, where \mathcal{F}^* is one of the Eleven branchwidths (presented in Figure 2 and formally defined in Table 1).*

Next, we argue that the relationships between the Eleven as shown in Figure 2 indeed faithfully represent their hierarchy. That is, whenever two width measures are connected, then the one on top *strictly* generalizes the one the bottom, and whenever there is no connection (in terms of paths) between two, then they are incomparable. The fact that the depicted generalizations hold up follows directly from the definition and in some cases Observation 15.

For the lower bounds on width measures, we need the following canonical classes of graphs of unbounded width. For each $n \in \mathbb{N}$, we consider the following graphs on vertex set $[n] \times [n]$.

- The $n \times n$ *grid*, denoted by \boxplus_n , has edges $\{(i, j), (i', j')\}$ where $|i - i'| + |j - j'| = 1$.
- The *complement of the $n \times n$ grid* is denoted by \boxminus_n .
- The $n \times n$ *comparability grid*, denoted by \boxtimes_n , has edges $\{(i, j), (i', j')\}$ where $i \leq i'$ and $j \leq j'$. For an illustration see Figure 5. Note that comparability grids appear in the study of rank-width naturally [15].

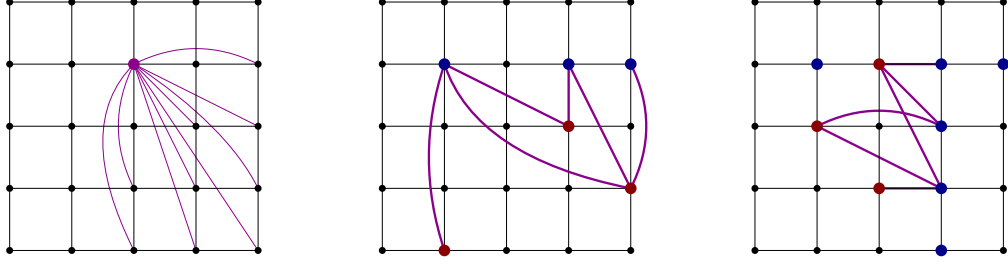
Grids have unbounded mim-width [17], which in the language of \mathcal{F} -branchwidth can be expressed in a more general form as follows.

► **Proposition 19.** *Let \mathcal{F} be a union of *si ph* families. If $(\equiv \cup \vdots \vdots) \cap \mathcal{F} \neq \emptyset$, then for each $n \in \mathbb{N}$, $\mathcal{F}\text{-bw}(\boxplus_n) \in \Omega(n)$.*

By complementation, we have the following consequence.

► **Corollary 20.** *Let \mathcal{F} be a union of *si ph* families. If $(\boxtimes \cup \boxminus) \cap \mathcal{F} \neq \emptyset$, then for each $n \in \mathbb{N}$, $\mathcal{F}\text{-bw}(\boxminus_n) \in \Omega(n)$.*

Lastly, we show that comparability grids are a family of unbounded chain-width, while their mam-width is bounded by a constant.



■ **Figure 5** Illustration of comparability grids by the example of \mathbb{B}_5 . On the left, the grid structure plus the additional edges incident with one highlighted vertex. In the middle and on the right, some chains that appear in cuts of a branch decomposition.

■ **Table 2** Each cell shows families of graphs whose width is bounded by a constant by the width measure whose cut-function indexes the row and unbounded by the width measure whose cut-function indexes the column. Here, \boxplus stands for the family of grids, \boxminus for the complements of grids, and \boxtimes for the comparability grids. For instance, the entry \boxplus in row “chain” and column “mim” indicates that the family of grids has constant chain-width but unbounded mim-width. Furthermore, K_n denotes the family of complete graphs, and \emptyset_n the family of edgeless graphs.

	chain	mim	miam	chim	mam	cham	empty	complete	$c^*/t^*/co-t^*$
chain		\boxplus	\boxminus	\boxplus	$\boxplus \boxplus$	\boxplus	\boxtimes	\boxplus	$\boxtimes \boxplus$
mim	\boxtimes		\boxminus	\boxtimes	\boxminus	$\boxtimes \boxplus$	\boxtimes	\boxminus	$\boxplus \boxtimes$
miam	\boxplus	\boxtimes		$\boxplus \boxtimes$	\boxtimes	\boxplus	$\boxtimes \boxplus$	\boxplus	$\boxplus \boxplus$
chim			\boxplus		\boxtimes	\boxtimes	\emptyset_n	\boxtimes	\boxtimes
mam	\boxtimes			\boxtimes		\boxplus	\boxplus	\boxplus	\boxplus
cham		\boxplus		\boxplus	\boxplus		\boxplus	K_n	\boxplus
empty			\boxplus		\boxplus	\boxplus		\boxplus	\boxplus
complete		\boxplus		\boxplus	\boxplus		\boxtimes		\boxplus
c^*							\emptyset_n	K_n	$/ K_n / \emptyset_n$
t^*							\emptyset_n		$/ / \emptyset_n$
$co-t^*$								K_n	$/ K_n /$

► **Proposition 21.** For each $n \in \mathbb{N}$, $\text{lin-mam-width}(\mathbb{B}_n) = 1$ and $\text{chain-width}(\mathbb{B}_n) \geq n/4$.

Next, we observe upper bounds on \mathcal{F} -branchwidth of grids, their complements, and comparability grids.

► **Proposition 22.** Let \mathcal{F} be a union of *si ph* families. If $\mathcal{F} \subseteq \boxtimes \cup \boxtimes \cup \boxtimes$, then, for each $n \in \mathbb{N}$, $\mathcal{F}\text{-bw}(\boxplus_n) \in \mathcal{O}(1)$.

By complementation and Corollary 9, we get the following.

► **Corollary 23.** Let \mathcal{F} be a union of *si ph* families. If $\mathcal{F} \subseteq \boxplus \cup \boxminus \cup \boxtimes$, then, for each $n \in \mathbb{N}$, $\mathcal{F}\text{-bw}(\boxplus_n) \in \mathcal{O}(1)$.

Using Propositions 19, 21, and 22 and Corollaries 20 and 23, we can argue almost all of the desired relationships, see Table 2. The remainder is discussed in the full version which concludes the proof of Theorem 2.

4 FO logic

It is known that FO MODEL CHECKING is AW[*]-hard on interval graphs [14]. Furthermore, interval graphs are known to have linear mim-width 1, witnessed by a linear order that can be computed in polynomial time [2]. Next, we can observe that for any graph G and $A \subseteq V(G)$,

$$\text{mim}_G(A) \leq 1 \Leftrightarrow \text{miam}_G(A) \leq 1.$$

This is because a matching of half-order 2 and an anti-matching of half-order 2 are isomorphic. Therefore, we have the following consequence.

► **Corollary 24** (of [2, 14]). *FO-MODEL CHECKING parameterized by the length of the formula is AW[*]-hard on graphs of linear mam-width one, even if a width-1 linear order of the input graph is given.*

Due to Corollary 24, the only remaining cases where we could hope for FPT-algorithms for FO-MODEL CHECKING are complete-width and empty-width. However, we show by two trivial reductions that the problem remains W[1]-hard parameterized by formula length on graphs of width 1 as well. Note that the following reduction is folklore, but we state it here for completeness.

► **Proposition 25.** *FO-MODEL CHECKING parameterized by the length of the formula is AW[*]-hard on graphs of linear complete-width one, even if a width-1 linear order of the input graph is given.*

Let (G, ϕ) be an instance of FO-MODEL CHECKING such that G has linear complete-width one. Observe that \overline{G} has linear empty-width one. We replace each occurrence of $x_i \sim x_j$ in ϕ with $\neg x_i \sim x_j$ to obtain a formula ϕ' . Then, $\overline{G} \models \phi'$ if and only if $G \models \phi$ and we obtain the following.

► **Corollary 26.** *FO-MODEL CHECKING parameterized by the length of the formula is AW[*]-hard on graphs of linear empty-width one, even if a width-1 linear order of the input graph is given.*

5 Bounds on nec in terms of \mathcal{F} -branchwidth

The run time of the algorithm for A&C DN MODEL CHECKING from [3] is polynomial in the number of equivalence classes of the neighborhood equivalence relation. Therefore, any bound on this quantity immediately translates to an efficient algorithm for A&C DN MODEL CHECKING.

5.1 Upper Bounds

We first give some upper bounds. For tree-width, co-tree-width, and clique-width, single-exponential upper bounds on nec are known (e.g., [3]). Pipelined through the arguments from [12], this gives bounds for the equivalent \mathcal{F} -branchwidth parameters, but they might end up double-exponential due to the Ramsey arguments in [12]. Here, we give single-exponential upper bounds in terms of these width parameters by giving more direct arguments.

► **Lemma 27.** *Let G be a graph and $A \subseteq V(G)$ such that $\mathfrak{t}_G^*(A) \leq k$. Let M be a matching in $G[A, \overline{A}]$. Then, $|M| < (k + 1)^{k+2}$.*

► **Lemma 28.** *Let G be a graph, $A \subseteq V(G)$ with $\mathfrak{t}_G^*(A) \leq k$. There is a set $\Omega \subseteq A$ of size at most $2^{\mathcal{O}(k^2 \log k)}$ such that $\{N(v) \cap \bar{A} \mid v \in A\} = \{N(v) \cap \bar{A} \mid v \in \Omega\}$.*

► **Proposition 29.** *Let G be a graph and $A \subseteq V(G)$ with $\mathfrak{t}_G^*(A) \leq k$. Then, $\text{nec}_G(A) \in 2^{\mathcal{O}(k^3 \log k)}$.*

► **Proposition 30.** *Let G be a graph and $A \subseteq V(G)$ with $\text{co-}\mathfrak{t}_G^*(A) \leq k$. Then, $\text{nec}_G(A) \leq 2^{\mathcal{O}(k^3 \log k)}$.*

5.2 Lower bounds

We first exhibit some cuts with high nec in terms of the respective width parameters. Later, we then prove that there are families of graphs where each of their branch decompositions has to have a cut with large nec .

► **Observation 31.** *Let G be obtained by taking the disjoint union of an antimatching with $2n$ vertices and a matching with $2k$ vertices and completely connecting the sides A and \bar{A} of the resulting bipartite graph. It holds that $\text{empty-width}(G) \leq k$, and $\text{nec}_G(A) \geq 2^k \cdot n$.*

For the following observation, take the disjoint union of k chains of half-order k .

► **Observation 32.** *There is a bipartite graph $G = (A, B, E)$ with $\mathfrak{t}_G^*(A) = k$ and $\text{nec}_G(A) \in 2^{\Omega(k \log k)}$.*

► **Observation 33.** *Let $H = H_{\equiv}^n$ with bipartition (A, B) . Then, $\text{nec}_H(A) = 2^n$.*

► **Observation 34.** *For each $n \in \mathbb{N}$ there is a graph G on $2n$ vertices and $A \subseteq V(G)$ such that $\text{complete}_G(A) = 1$ and $\text{nec}_G(A) = 2^n$.*

► **Proposition 35.** *There is an infinite family of graphs G with $\text{complete-width}(G) = 1$ and such that each branch decomposition of G induces a cut (A, \bar{A}) with $\text{nec}_G(A) \in 2^{\Omega(\sqrt{|V(G)|})}$.*

► **Proposition 36.** *For each $n, k \in \mathbb{N}$ with $n \geq k$ there is a bipartite graph $G = (A, B, E)$ on $2n$ vertices such that $\text{mam}_G(A) = k$ and $\text{nec}_G(A) = \lfloor n/k \rfloor^k$.*

By taking a disjoint union of k antimatchings of half-order n/k in the previous proof, we get the following corollary.

► **Corollary 37.** *For each $n, k \in \mathbb{N}$ with $n \geq k$ there is a bipartite graph $G = (A, B, E)$ on $2n$ vertices such that $\text{chim}_G(A) = k$ and $\text{nec}_G(A) = \lfloor n/k \rfloor^k$.*

In the following, we provide several lower bounds on the nec -width of graphs of low chim -width, mam -width, tree^* -width, or co-tree^* -width. The proofs strategy are based on the proof of [5, Theorem 1.3] stating that there exists arbitrary large graphs of rank-width k with Boolean-width $\Omega(k^2)$.

We start by proving a lower bound on nec -width in terms of chim -width. The following construction is based on the reduction used to prove Theorem 5 for chim -width.

► **Lemma 38.** *For each $k \in \mathbb{N}$ with $k > 0$, there exist arbitrary large graphs G of linear chim -width at most $2k$ and with $\text{nec-width}(G) \in (n/(k^2 \log n))^{\Omega(k)}$.*

It is easy to obtain the same result as Lemma 38 for mam -width by replacing anti-matchings by chains in the construction of the graph G .

► **Lemma 39.** *For each $k \in \mathbb{N}$ with $k > 0$, there exist arbitrary large graphs G of linear mam -width at most $k + 1$ and with $\text{nec-width}(G) \in (n/(k^2 \log n))^{\Omega(k)}$.*

► **Lemma 40.** *For each $k \in \mathbb{N}$ with $k > 0$, there exist arbitrary large graphs G of linear tree*-width at most $2k$ and with $\text{nec-width}(G) \in k^{\Omega(k)}$.*

► **Lemma 41.** *For each $n, k \in \mathbb{N}$ with n sufficiently large, there is a graph $G = (A, B, E)$ on $2n$ vertices such that $\text{empty-width}(G) \leq k$ and with $\text{nec-width}(G) \in |V(G)|^{\Omega(k)}$.*

6 Independent Set parameterized by \mathcal{F} -Branchwidth

In this section, we discuss the complexity of INDEPENDENT SET parameterized by any \mathcal{F} -branchwidth measure. On top of that, we strengthen known hardness results for INDEPENDENT SET and DOMINATING SET (full version) on graphs of bounded mim-width under the ETH. Concretely, we show the following.

► **Theorem 5.** *Unless the Exponential Time Hypothesis fails, neither INDEPENDENT SET nor DOMINATING SET can be solved in $f(k)n^{o(k)}$ time for any computable function f , where k is either the chim-width, the mam-width, or the mim-width of a given decomposition of the n -vertex input graph. Moreover, both problems parameterized by either the chim-width or the mam-width are $W[1]$ -hard.*

These lower bounds are tight because the algorithm – based on the d -neighbor equivalence – from [7] solves INDEPENDENT SET and DOMINATING SET in time $n^{O(w)}$ where w is the mim-width of a given decomposition which is always smaller than its chim-width and mam-width.

It is well-known that INDEPENDENT SET is FPT parameterized by clique-width, and therefore clique*-width, so for all \mathcal{F} -branchwidth measures that are at most as general as clique*-width, INDEPENDENT SET is FPT.

6.1 Independent Set on graphs of bounded complete-width and chain-width

Since INDEPENDENT SET is NP-complete on cubic graphs, it is paraNP-hard when parameterized by any width measure in whose obstructions the degree increases proportionally to the size. This is the case for complete bipartite graphs, anti-matchings, and chains.

► **Observation 42.** *INDEPENDENT SET is NP-hard on graphs with complete-width at most 3.*

Nevertheless, the previous result does not rule out efficient parameterized algorithms for INDEPENDENT SET when the solution size is an additional part of the parameter. Note that here, the bounded-degree argument does not apply, since INDEPENDENT SET parameterized by solution size plus maximum degree can easily be shown to be FPT [10]. However, we show that INDEPENDENT SET remains $W[1]$ -hard parameterized by solution size on graphs of (the more general) chain-width at most 3. (The same reduction in fact shows $W[1]$ -hardness parameterized by solution size plus chim-width.)

This can be shown by observing that the graphs obtained in the reduction that proved $W[1]$ -hardness for INDEPENDENT SET parameterized by mim-width plus solution size [13] in fact have a linear order whose mim-width is bounded by a function of the parameter and whose chim-width is at most three.

► **Corollary 43.** *INDEPENDENT SET is $W[1]$ -hard parameterized by*

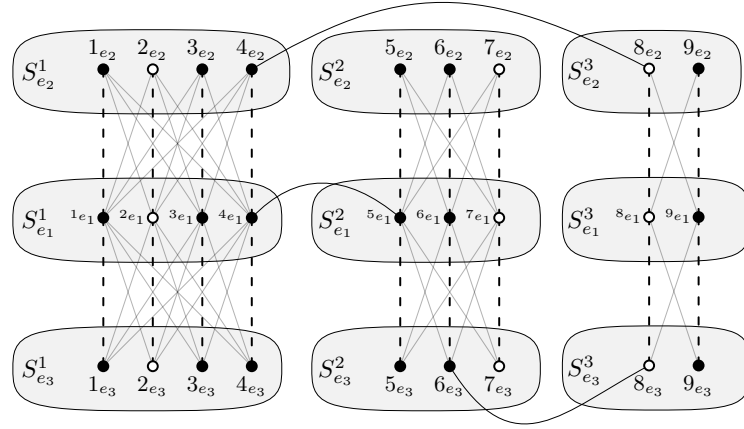
- *the solution size (plus the linear mim-width) on graphs of linear chain-width 3, or*
- *the solution size plus the linear chim-width*

of the input graph, even if a linear order of bounded width is provided with the input.

6.2 Independent Set on graphs of bounded chim-width

To prove Theorem 5 for chim-width and INDEPENDENT SET, we provide a reduction from MULTICOLORED INDEPENDENT SET in general graphs to Independent Set in graphs of low chim-width. Briefly, an instance of MULTICOLORED INDEPENDENT SET consists of a graph whose vertex set is the disjoint union of k sets V_1, \dots, V_k and the problem asks for a *multicolored* independent set which is an independent set with exactly one vertex in each V_i . Let H a graph with $V(H) = V_1 \uplus \dots \uplus V_k$ be an instance of MULTICOLORED INDEPENDENT SET. We denote by N the number of vertices of H and by M the number of edges of H . In the following, we show how to construct in polynomial time a graph G with at most $n = NM$ vertices and a linear order of chim-width at most $2k$ such that H admits a multicolored independent set iff G admits an independent set of size at least kM . This is sufficient to prove Theorem 5 for chim-width. Indeed, with a $f(w)n^{o(w)}$ time algorithm for INDEPENDENT SET for some computable function f , we would be able to solve MULTICOLORED INDEPENDENT SET in time $f(k)N^{o(k)}$ which is not possible under the ETH [10]. Furthermore, the W[1]-hardness of INDEPENDENT SET parameterized by chim-width follows from the W[1]-hardness of MULTICOLORED INDEPENDENT SET parameterized by k [10].

Intuitively, we start by creating a *selection gadget* S_{uv} for each edge uv of H such that the maximum independent sets of $G[S_{uv}]$ are bijectively mapped to the *potential solution* of H – set of vertices with exactly one vertex in each V_i – which do not contains both u and v . Then we glue the selection gadgets together in such a way that the maximum independent sets in two glued gadgets are non-adjacent if and only if they are associated with the same potential solution of H . See Figure 6 for an illustration of the following construction.



■ **Figure 6** Illustration of the reduction for chim-width with $k = 3$, $V_1 = \{1, 2, 3, 4\}$, $V_2 = \{5, 6, 7\}$ and $V_3 = \{8, 9\}$. Here $M = 3$ and the edges of H consists of $e_1 = 45$, $e_2 = 48$ and $e_3 = 68$. Dashed lines represent the non-edges of the anti-matchings between the selection gadgets. To improve the legibility, we omit the edges of the cliques $S_{e_i}^j$: $i, j \in [3]$. The independent set I consisting of the white filled vertices have size kM and $I_H = \{2, 7, 8\}$ is a multicolored independent set of H .

Selection gadgets. For each edge $e \in E(H)$, we create the selection gadget S_e which consists of the disjoint union of k cliques S_e^1, \dots, S_e^k such that for each $j \in [k]$, $S_e^j := \{x_e \mid x \in V_j\}$. We finish the construction of S_e by adding the edge between u_e and v_e where u and v are the endpoints of e . Given a set $X \subseteq S_e$, we denote by X_H the set $\{x \in V(H) \mid x_e \in X\}$.

► **Observation 44.** *For every edge e of H , X is a maximum independent set I of $G[S_e]$ if and only if X_H has exactly one vertex in each V_i and does not contain both endpoints of e .*

Propagation. The final step of this construction is to add edges between the selection gadgets such that if H admits a multicolored independent sets then every maximum independent set I of G intersects the selections gadgets in the same way, i.e. for every pair of edges (e_1, e_2) of H , we have $(I \cap S_{e_1})_H = (I \cap S_{e_2})_H$. We achieve this without blowing up the chim-width of G as follows. We fix an edge e_1 of H and for every edge $e \neq e_1$ of H and $i \in [k]$, we first add all the edges between $S_{e_1}^i$ and S_e^i and then for each vertex $x \in V_i$, we remove the edge $x_{e_1}x_e$. Consequently, the bipartite graph $G[S_{e_1}^i, S_e^i]$ is an anti-matching of half-order $|V_i|$. As $S_{e_1}^i$ and S_e^i are cliques in G , the maximum independent sets of $G[S_{e_1}^i \cup S_e^i]$ are exactly the pairs $\{x_{e_1}, x_e\}$ with $x \in V_i$. This leads to the following observation.

► **Observation 45.** *For every edge $e \neq e_1$ of H and subsets X, Y of size k of respectively S_{e_1} and S_e , there is no edge between X and Y in G if and only if we have $X_H = Y_H$.*

We deduce from Observations 44 and 45 the correctness of our reduction.

► **Lemma 46.** *The graph G admits an independent set of size at least kM if and only if H admits a multicolored independent set.*

Finally, we prove that we can construct in polynomial time a linear order of G of chim-width at most $2k + 2$. The following proof relies on the fact that induced matchings and chains in anti-matchings have size at most 2, which implies that the chim-width of the cut $(S_{e_1}, V(G) \setminus S_{e_1})$ is at most $2k$.

► **Lemma 47.** *We can compute in polynomial time a linear order of G of chim-width at most $2k + 2$.*

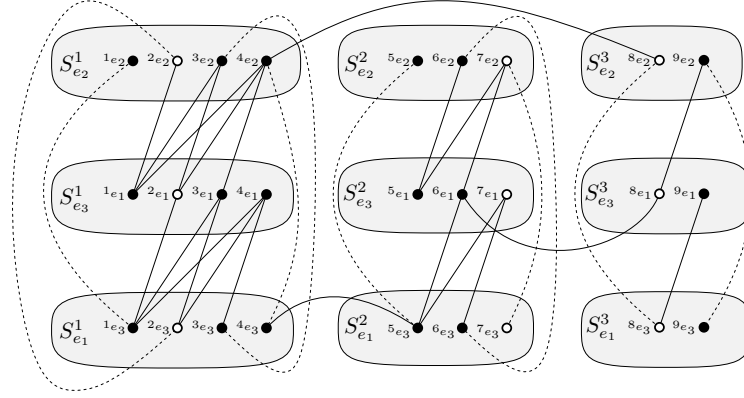
6.3 Independent Set on graphs of bounded mam-width

The proof of Theorem 5 for mam-width is similar to the one for chim-width, we start from the same instance of MULTICOLORED INDEPENDENT SET and construct a graph G . Compared to chim-width, the following construction is a bit more involved and uses skewed chains instead of anti-matchings to glue the selection gadgets together. See Figure 7 for an illustration of the following construction.

Selection gadgets. We reuse the selection gadget $S_{e_1}, S_{e_2}, \dots, S_{e_M}$ from the reduction for chim-width where e_1, \dots, e_M are the edges of H .

Propagation. First, we add anti-matchings between S_{e_1} and S_{e_2} just as we did in the reduction of chim-width. That is for every $i \in [k]$, we do the following. We add all the edges between $S_{e_1}^i$ and $S_{e_2}^i$. Finally, for every vertex $v \in V_i$, we remove the edge $v_{e_1}v_{e_2}$. Note that $G[S_{e_1}^i, S_{e_2}^i]$ is big anti-matching, but this does not blow-up the mam-width. Indeed, we can insert in the permutation of Lemma 47 each vertex of $S_{e_2}^i$ next to its non-neighbor in $S_{e_1}^i$, this way every cut of the resulting permutation contains at most one non-edge of $G[S_{e_1}^i, S_{e_2}^i]$ and the mam-width stays low. Note that Observation 45 holds for $e = e_2$.

Finally, we add skewed chains between $S_{e_1} \cup S_{e_2}$ and the other selection gadgets, by doing the following. We take a arbitrary strict linear order $<_H$ on the vertices of H such that for each $i, j \in [k]$ with $i < j$, we have $u <_H v$ for each $u \in V_i$ and $v \in V_j$. For every $i \in [M]$ with $i > 2$, $j \in [k]$, and $x, y \in V_j$ such that $x <_H y$, we make x_{e_1} adjacent to y_{e_i} and y_{e_2} adjacent to x_{e_i} .



■ **Figure 7** Illustration of the reduction for mam-width with $k = 3$, $V_1 = \{1, 2, 3, 4\}$, $V_2 = \{5, 6, 7\}$ and $V_3 = \{8, 9\}$. Here $M = 3$ and the edges of H consists of $e_1 = 45$, $e_2 = 48$ and $e_3 = 68$. To improve the legibility, we omit the edges of the cliques $S^j_{e_i} : i, j \in [3]$. Moreover, we represent only the non-edges (with dashed lines) of the anti-matchings between S_{e_1} and S_{e_2} . The independent set I consisting of the white filled vertices have size kM and $I_H = \{2, 7, 8\}$ is a multicolored independent set of H .

Observe that, for every $i \in [M]$, $j \in [t]$ and $X \in \{R, S\}$, the graphs $G[S^j_{e_1}, S^j_{e_i}]$ and $G[S^j_{e_2}, S^j_{e_i}]$ are skewed chain graphs of half-order $|V_i|$. Moreover, the maximum independent sets of $G[S^j_{e_1} \cup S^j_{e_2} \cup S^j_{e_i}]$ are exactly the sets $\{v_{e_1}, v_{e_2}, v_{e_i}\}$ with $v \in V_i$.

► **Observation 48.** For every $i \in [M]$ and subsets X, Y and Z of respectively S_{e_1}, S_{e_2} and S_{e_i} , each of size k , there is no edge between any two of these subsets in G if and only if $X_H = Y_H = Z_H$.

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