Hardness of Computation of Quantum Invariants on 3-Manifolds with Restricted Topology

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Abstract

Quantum invariants in low-dimensional topology offer a wide variety of valuable invariants about knots and 3-manifolds, presented by explicit formulas that are readily computable. Their computational complexity has been actively studied and is tightly connected to topological quantum computing. In this article, we prove that for any 3-manifold quantum invariant in the Reshetikhin-Turaev model, there is a deterministic polynomial time algorithm that, given as input an arbitrary closed 3-manifold M, outputs a closed 3-manifold M' with the same quantum invariant, such that M' is hyperbolic, contains no low genus embedded incompressible surface, and is presented by a strongly irreducible Heegaard diagram. Our construction relies on properties of Heegaard splittings and the Hempel distance. At the level of computational complexity, this proves that the hardness of computing a given quantum invariant of 3-manifolds is preserved even when severely restricting the topology and the combinatorics of the input. This positively answers a question raised by Samperton [44].

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1 Introduction

Quantum invariants are topological invariants defined using tools from physics, explicitly, from topological quantum field theories (TQFTs). These invariants have become of interest for modeling phenomena in condensed matter physics [5, 19], topological quantum computing [29], and experimental mathematics [14, 38], where many deep conjectures remain open. Thanks to their diversity and discriminating power to distinguish between non-equivalent topologies, they have also played an important role in the constitution of censuses of knots and 3-manifolds [12]. The invariants are constructed from the data of a fixed algebraic object, called a *modular category*, and a topological support, and take the form of a partition function, whose value depends solely on the topological type of the support and not on its combinatorial

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presentation. Generally, quantum invariants are defined for presentations of either knots or 3-manifolds, although there are types of invariants, such as the Reshetikhin-Turaev, whose definitions naturally encompass both objects.

The complexity of the exact and approximate computation of these invariants has attracted much interest, particularly in connection with quantum complexity classes. Most non-trivial quantum invariants turn out to be #P-hard to compute and, sometimes, even #P-hard to approximate [3, 20, 30] within reasonable precision. This is the case of the Jones polynomial for knots [20, 30] and the Turaev-Viro invariant of 3-manifolds associated with the Fibonacci category [3, 20]. On the positive side, polynomial time *quantum* algorithms exist for computing weak forms of approximations [2, 4, 20] and efficient *parameterized* algorithms have been designed [11, 13, 34, 36, 39] leading to polynomial time algorithms on certain families of instances. For the latter, the topology of the input knot or 3-manifold plays a central role in measuring the computational complexity of the problem, either directly with running times depending strongly on some topological parameter [15, 39], or indirectly where *simple topologies* guarantee the existence of simple combinatorial representations [23, 25, 37] that can be, in turn, algorithmically exploited.

There are other instances, however, where the hardness of computing the invariants is preserved even if the topology and combinatorics of the input are restricted. In [30], Kuperberg shows that, for certain quantum invariants that are *hard to approximate* on *links*, the hardness is preserved when restricted to *knots*. In a follow-up work, Samperton [44] proves that if computing a quantum invariant is hard for all input diagrams of any knot, then the computation remains hard when restricting the input to *hyperbolic* knots given by diagrams with a minimal number of crossings. In this article, we follow a similar path to both [30] and [44], this time for quantum invariants of 3-manifolds, by proving the hardness of computing invariants of irreducible presentations and hyperbolic manifolds.

We follow the strategy of Samperton [44], which consists of using Vafa's theorem [54] to efficiently complicate the topological structure of the input without changing the invariant. Nonetheless, while Samperton's process involves adding extra crossings to the knot diagrams, we increase the *Hempel distance* of a *Heegaard diagram* of some 3-manifold. Although there exists an extensive catalog of algorithms to increase Hempel distances [17, 22, 26, 28, 33, 42, 56], to the best of our knowledge, our work is the first to 1. explicitly compute the involved complexities, ensuring polynomial time; and, 2. keep some 3-manifold invariant constant throughout the process.

The main result is expressed in Theorem 1, whose precise statement can be found in Section 4. Here and throughout the paper, we denote by Σ_g the *closed surface of genus* g (unique up to homeomorphism) with some fixed orientation and assume $g \geq 2$. When referring to a general compact surface, potentially with boundary, we use Σ.

▶ **Theorem 1.** *Let $\mathcal{C}$ be a modular category and M a closed 3-manifold represented by a Heegaard diagram $(\Sigma_g, \alpha, \beta)$ of complexity m. There is a deterministic algorithm that constructs, in time $O(\text{poly}(m, g))$ and uniformly on the choice of $\mathcal{C}$, a strongly irreducible Heegaard diagram $(\Sigma_{g+1}, \alpha', \beta')$ representing a hyperbolic 3-manifold M' that shares with M the Reshetikhin-Turaev invariant over $\mathcal{C}$. Moreover, for a fixed choice of $k \in \mathbb{N}$, M' has no embedded orientable and incompressible surface of genus at most $2k$.*

▶ Remark 2. Hyperbolicity has historically been used as both a simplifying structure and an intermediate step for algorithms on 3-manifold, see for example [31, 50]. Similarly to [44] for knots, our result proves that hyperbolicity is of no help for the computational complexity of the quantum invariant. On the other hand, when producing hard instances of 3-manifolds in computational topology – e.g., in complexity reduction [1, 6] or the construction of combinatorially involved manifolds [24] – it is common to produce *Haken* 3-manifolds with low genus incompressible surfaces (generally, tori). It is an important open question to understand the hardness of computation for non-Haken 3-manifolds. Our result shows that the computational complexity of quantum invariants is preserved even when getting rid of low-genus incompressible surfaces.

The paper is divided into four parts: a review of background material (Section 2), the demonstration of some auxiliary algorithms (Section 3), the proof our main result (Section 4), and some illustration of its computational consequences for the hardness of computing quantum invariants (Section 5). Due to the conference's constraints, some of the demonstrations were omitted from this current article, but can be found in the extended version of the paper[1].

2 Background material

In the following review, we assume acquaintance with the basic ideas from geometric topology, such as boundaries, compactness, homeomorphisms, (free) homotopies, isotopies, and manifolds. For these topics, we refer the reader to [49] and [51]. Moreover, we shall use, without properly defining, some well-known concepts belonging to the theories of curves and surfaces, which can be found in [18].

2.1 Curves in surfaces

An *arc* in a surface Σ is the image of a proper embedding of the interval in Σ (i.e., its endpoints lie both on $\partial\Sigma$). Similarly, a *simple closed curve* is the image of a proper embedding $S^1 \hookrightarrow \Sigma$. We will often refer to a simple closed curve by only *closed curve* or just *curve*. A *multicurve* is a finite collection of disjoint properly embedded simple curves in Σ. We denote by $\#\gamma$ its number of connected components. Whenever possible, we distinguish simple curves from multicurves by using Greek letters to represent the latter.

A curve in the surface will be called *essential* if not homotopic to a point (or, equivalently, if it does not bound a disk on the surface), a puncture, or a boundary component. A multicurve is essential if all its components are essential. Unless otherwise stated, all curves and multicurves will be assumed essential.

We will often be interested not in a curve s, but in the equivalence classes of s up to *isotopies* in Σ_g, $[s]$. Being essential is preserved under isotopies, so we naturally extend the definition of essential curves to their isotopy classes. For two curves s, t in some surface Σ, their *geometric intersection number* is defined as the minimal number of their intersection points up to isotopy, that is $i(s,t) = \min\{|s' \cap t'| : s' \in [s], t' \in [t]\}$. Two curves in a surface Σ_g are isotopic if and only if they are (free) homotopic [18].

The *curve graph* of a closed surface Σ_g, $C(\Sigma_g)$, is the graph whose vertices are isotopy classes of essential curves and any two vertices $[s]$ and $[t]$ are connected by an edge if and only if $i(s,t) = 0$. The usual graph distance d defines a metric on the vertices of $C(\Sigma_g)$ that is interpreted as $d([s],[t]) = n$ implying the existence a sequence of essential curves $r_0, r_1, \ldots, r_n$ with $r_0 \in [s]$, $r_n \in [t]$, and $r_i \cap r_{i+1} = \emptyset$ for $0 \leq i < n$. The definitions related to the curve graph can be naturally extended to curves by identifying the elements of an isotopy class to the same vertex; in particular, $d(r,s) = 0$ is equivalent to r and s being isotopic.

[1] Available at `https://arxiv.org/abs/2503.02814`.

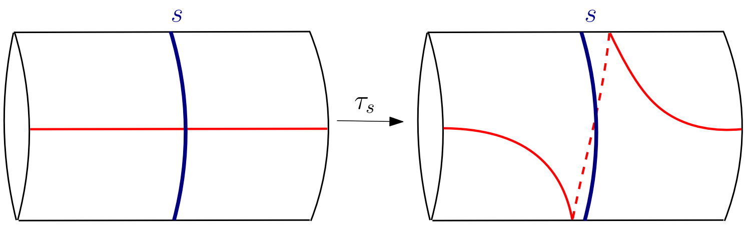

Figure 1 Illustration of the action of a Dehn twist about a curve s (blue) on some curve transversal to it (red). The homeomorphism is only different from the identity on a regular neighborhood of s.

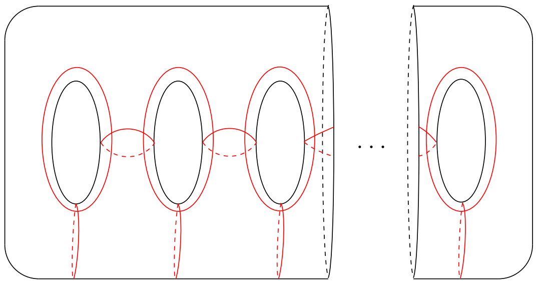

Figure 2 Diagram representing the Lickorish curves in a closed surface $\Sigma_g, g \geq 2$.

We recall that for each surface Σ_g, its *mapping class group*, $\text{Mod}(\Sigma_g)$, is the group of orientation-preserving homeomorphisms $\Sigma_g \to \Sigma_g$ up to isotopies. Its canonical action on the surface conserves the geometric intersection number between curves [18], therefore acting isometrically on $(C(\Sigma_g), d)$ by the induced map $\phi \cdot [s] = \phi_*([s])$. The mapping class groups always contain the (isotopy classes of) *Dehn twists*, homeomorphisms $\tau_s : \Sigma_g \to \Sigma_g$ defined by cutting off a local neighborhood of the (multi)curve s of Σ_g and gluing it back with a 2π counterclockwise twist, determined by the orientation of the surface (Figure 1). In particular, for each $g \geq 2$, $\text{Mod}(\Sigma_g)$ is generated as a group by (isotopy classes of) Dehn twists about the $3g-1$ recursively defined *Lickorish curves* in Σ_g (Figure 2) [32]. We can, consequently, always assume that an element of $\text{Mod}(\Sigma_g)$ is of form $\phi = \tau_{s_r}^{n_r} \circ \cdots \circ \tau_{s_1}^{n_1}$, where $n_i \in \mathbb{Z}$ and s_i are Lickorish curves for $1 \leq i \leq r$.

2.2 Curves on handlebodies and Heegaard splittings

An essential multicurve $\gamma \subset C(\Sigma_g)$ is a *(full) system* if no two components are isotopic to each other and $\Sigma_g - \gamma$ is a union of $\#\gamma - g + 1$ punctured spheres (here the minus sign indicates cutting the surface along γ). The minimum number of connected components that a full system may have is g, in which case $\Sigma_g - \gamma$ is a $2g$-punctured sphere and γ is called a *minimum* system. On the other extreme, a system γ is *maximum* or a *pants decomposition* when $\#\gamma = 3g - 3$ and $\Sigma_g - \gamma$ is the union of $2g - 2$ thrice punctured spheres, also known as *pairs of pants* (see Figure 4). When every connected component of a system γ is isotopic to a connected component of another system γ', we say that γ is *contained* in γ' and denote that by $\gamma \subseteq \gamma'$. A minimum system γ can always be extended to a pants decomposition $\rho \supseteq \gamma$, see Theorem 16.

The genus g *handlebody* defined for a full system γ in the surface Σ_g is the 3-manifold

$$V_\gamma = \Sigma_g \times [0,1] \cup_{\gamma \times \{0\}} \text{2-handles} \cup \text{3-handles}$$

built by attaching the 2-handles along the curves γ in $\Sigma \times \{0\}$ and then filling any resulting S^2 boundary component with 3-handles. By construction $\partial V_\gamma = \Sigma_g$. Because the curves in the system γ are assumed to be essential, each of its components will bound (non-trivial) compression disks in V_γ: they will be *meridians* of the handlebody. There are, however, many other meridians in V_γ, as it will be implied by the following definition.

▶ **Definition 3** (Disk graph and equivalent systems)**.** *Let V_γ be a handlebody constructed over Σ_g. Then the disk graph of V_γ, K_γ, is the subgraph of $C(\Sigma_g)$ whose vertices represent (isotopy classes of) meridians of V_γ. We say that two full systems, γ and γ', (potentially with $\#\gamma \neq \#\gamma'$) are equivalent if $\gamma' \subset K_\gamma$ and $\gamma \subset K_{\gamma'}$.*

Note that γ is equivalent to γ' if and only if they define the same handlebody. In particular, if $\gamma \subset \rho$, γ and ρ are equivalent.

In a seminal work, Hempel [22] studied the metric d of the disk graph canonically inherited from $C(\Sigma_g)$. This inspires the next definition. Here and throughout, whenever $A, B \subset C(\Sigma)$ and r is a curve in Σ, we let $d(r, A) = \min\{d(r, s) : s \in A\}$ and $d(A, B) = \min\{d(s, t) : s \in A, t \in B\}$. We assume a curve s on a handlebody V_γ to be fully contained in ∂V_γ.

▶ **Definition 4** (Diskbusting curves)**.** *An essential curve s on a handlebody V_γ is said to be diskbusting if $d(s, K_\gamma) \geq 2$, that is, if s intersects all meridians of V_γ.*

[52] provides a combinatorial condition to verify if a curve on a handlebody is diskbusting, which we quote using the language of [56]. Before, however, we will need a definition.

▶ **Definition 5** (Seams and seamed curves)**.** *An arc in a pair of pants P is called a* seam *if it has endpoints on two distinct components of ∂P. A curve s in a surface Σ_g with a pants decomposition ρ is said to be* seamed *for ρ if, for every component P of $\Sigma_g - \rho$, $s \cap P$ has at least one copy of each of the three types (up to isotopy) of seams in P.*

▶ **Theorem 6** (Theorems 1 of [52], Theorem 4.11 of [56])**.** *Let s be a curve on the handlebody V_γ. Then s is diskbusting in V_γ if and only if there is a pants decomposition ρ equivalent to γ such that s is seamed for ρ.*

A closed 3-manifold M is said to have a *Heegaard splitting* if it is the union of two handlebodies intersecting only at their common boundary. It is well-known [45] that there exists, for every closed 3-manifold M, a tuple $(\Sigma_g, \alpha, \beta)$ called a *Heegaard diagram (of genus g)*, where α and β are two full systems with the same cardinality in Σ_g, and a Heegaard splitting $M = V_\alpha \cup_{\Sigma_g} V_\beta$. Note, on the other hand, that neither the Heegaard diagram nor the splitting is unique: for example, isotopies to α or β yield the same splitting, whereas given a diagram $(\Sigma_g, \alpha, \beta)$, one can define another splitting for the same manifold, this time of diagram $(\Sigma_{g+1}, \alpha \cup \{c\}, \beta \cup \{c'\})$, where c and c' are curves with $i(c, c') = 1$ fully contained in the extra handle. This last process, known as *stabilization*, is topologically equivalent to directly summing a copy of S^3 to the original manifold M.

Some properties of 3-manifolds can be read straight-up from their Heegaard splittings. For example, a splitting $(\Sigma_g, \alpha, \beta)$ is called *irreducible* if V_α and V_β do not share a meridian. Haken's lemma [49, Theorem 6.3.5] implies that reducible closed 3-manifolds cannot have irreducible Heegaard splittings. Similarly, a splitting $(\Sigma_g, \alpha, \beta)$ is *strongly irreducible* if there are no two essential disjoint curves a and b in Σ_g such that a is a meridian of V_α and b is a meridian of V_β. Every strongly irreducible splitting is irreducible, but the converse is not true (Haken manifolds provide a list of counterexamples [49]).

Given two handlebodies V_α and V_β, we define their *Hempel distance* by $d(K_\alpha, K_\beta)$. In [22], Hempel argued that this distance could be seen as a measure of the complexity of a Heegaard splitting $V_\alpha \sqcup_{\Sigma_g} V_\beta$, which is translated into the following theorem, whose proof (a simple application of results by others) can be found in the full version of the paper.

▶ **Theorem 7.** *Let $(\Sigma_g, \alpha, \beta)$ be a Heegaard diagram of distance $d(K_\alpha, K_\beta) = k$. Then*
- $k \geq 1$ *if and only if* $(\Sigma_g, \alpha, \beta)$ *is irreducible;*
- $k \geq 2$ *if and only if* $(\Sigma_g, \alpha, \beta)$ *is strongly irreducible;*
- *if* $k \geq 3$*, then* M *is hyperbolic;*
- M *has no orientable and incompressible embedded surface of genus smaller than* $2k$*.*

▶ Remark 8. Scharlemann and Tomova [48] proved that, if $V_{\alpha'} \cup_{\Sigma_{g'}} V_{\beta'}$ is Heegaard splitting of genus smaller than $k/2$, then $V_\alpha \cup_{\Sigma_g} V_\beta$ is *isotopic* to $V_{\alpha'} \cup_{\Sigma_{g'}} V_{\beta'}$, potentially after finitely many stabilizations. In particular, if $k > 2g + 2$, the splitting is of minimum genus. Unfortunately, as we will see in the proof of Theorem 1, our algorithm is not polynomial time as function of k, which means that it does not imply an efficient reduction to a minimal genus splitting for every input.

Our proof of Theorem 1 will mainly consist of increasing the Hempel distance so that the hypotheses of Theorem 7 are satisfied. For such, we will extensively use the next two theorems due to Yoshizawa.

▶ **Theorem 9** (Theorem 5.8 of [56]). *Consider the full systems of curves α and β in Σ_g and $n = \max\{1, d(K_\alpha, K_\beta)\}$. Let $d_i = d(K_i, s)$ for $i = \alpha, \beta$, assume $d_i \geq 2$ and $d_\alpha + d_\beta - 2 > n$. Then, for any $k \in \mathbb{Z}^+$,*

$$\min(k, d_\alpha + d_\beta - 2) \leq d(K_\alpha, K_{\tau_s^{k+n+2}(\beta)}) \leq d_\alpha + d_\beta. \tag{1}$$

▶ **Theorem 10** (Theorem 6.2 of [56]). *Let $\gamma = \{c_1, \ldots, c_g\}$ be full in Σ_g and ρ a pants decomposition containing γ. Suppose s is seamed for ρ and define the multicurve $\tau_s^2(\gamma)$ of components $d_1, \ldots, d_g$. Then $d(K_\gamma, \tau_{d_g}^2 \circ \cdots \circ \tau_{d_1}^2(c_1)) \geq 3$.*

2.3 Quantum invariants for 3-manifolds

We will not review the technical construction of TQFTs here, referring the interested reader to [53]. For our purposes, it is enough to know that, for a fixed choice of modular category $\mathcal{C}$ (again, refer to [53] for the definition), the TQFT associates to every closed 3-manifold M a complex scalar known as its *Reshetikhin-Turaev* (*RT*) invariant. The RT invariant can be given as a function of a Heegaard diagram $(\Sigma_g, \alpha, \beta)$ of M and is denoted by $\langle M \rangle_{\mathcal{C}}^{RT}$ or $\langle (\Sigma_g, \alpha, \beta) \rangle_{\mathcal{C}}^{RT}$, depending on whether we want to emphasize the manifold or the diagram.

The algebraic structure imposed by the modular category sets some constraints on the quantum invariants. The next theorem, for example, is already somewhat folklore in the literature (see [41]) since it implies that non-homeomorphic 3-manifolds may share RT invariants. It can be deduced from Theorem 5.1 of [16] and [43, 53].

▶ **Theorem 11** (Vafa's theorem for 3-manifold TQFTs). *Let $\mathcal{C}$ be a modular category. Then there is an $N \in \mathbb{Z}^+$, depending only on $\mathcal{C}$, such that, for all $k \in \mathbb{Z}$ and every curve s in Σ_g, $\langle (\Sigma_g, \alpha, \beta) \rangle_{\mathcal{C}}^{RT} = \langle (\Sigma_g, \alpha, \tau_s^{kN}(\beta)) \rangle_{\mathcal{C}}^{RT}$.*

For a fixed modular category, we call the integer N the category's *Vafa's constant.*

2.4 Relevant data structure

An embedded graph $G = (V, E)$ in a surface Σ defines a *cellular complex* for Σ if $\Sigma - G$ is a union of open disks, which we call *faces.* Note that this implies that any boundary component of Σ fully lies within some set of edges in E. The *dual graph* of G is another graph embedded in Σ defined by assigning a vertex to each face of $\Sigma - G$ and an edge between the vertices

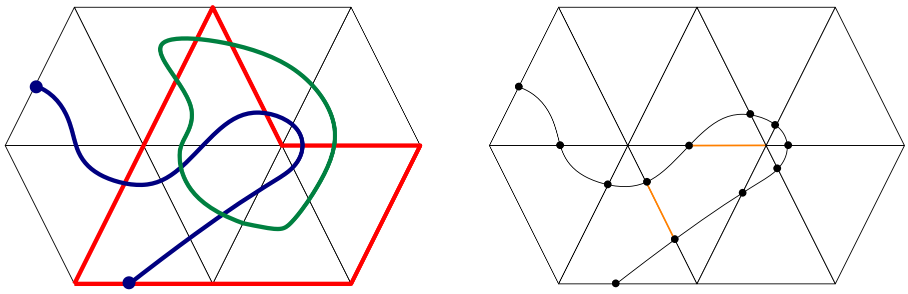

Figure 3 A triangulation of the disk with a curve represented by edge list (red), a standard curve (green), and a normal arc (blue).

if and only if the corresponding faces are separated by an edge in E. As a data structure, we represent a cellular embedding by the lists of faces, their incident edges, and vertices, allowing us to reconstruct Σ by gluing the appropriate pieces. Using this structure, one can compute the dual graph of the cellular embedding in linear time on the number of faces.

A *(generalized) triangulation* $T = (V, E)$ *of a surface* Σ is a cellular complex where all faces are bounded by exactly 3 edges. We denote the number of triangles in a triangulation by $|T|$; note that $|T| = O(|E|)$. We will most often consider *oriented triangulations* by giving an orientation to each triangle consistent with the orientation of the surface; we orient a triangle by imposing an order to its vertices as in the right-hand rule. We say that a triangulation T' of a surface Σ is a *subtriangulation* of another triangulation T of Σ, denoting $T \leq T'$, if the graph of T is embedded on T' (i.e., each vertex of T is a vertex of T' and each edge of T is a union of edges in T'). 'If T and T' are oriented, we also require the induced order of the vertices to be the same. Every face of T' is naturally contained in a face of T.

Using a triangulation $T = (V, E)$ of a surface Σ, we can describe an arc, curve, or multicurve s in Σ lying fully within the edges E by the list of the edges in $E \cap s$ (the red curve in Figure 3). We will call this list an *edge list representation* of the curve s, denote it by $E_T(s)$, and say that the number of edges in $E \cap s$ is the *edge complexity*, $\|E_T(s)\|$. We can always assume that $\|E_T(s)\| \leq |T|$.

For any fixed triangulation T of the surface Σ, there are always, however, isotopy classes of curves in Σ not representable as a subset of edges of T. To deal with this hindrance, we say, for a fixed oriented triangulation $T = (V, E)$ of the surface Σ, that a curve s is *standard* (to T) if it intersects T only transversely and at edges (green curve of Figure 3). If s is standard, we may represent it as an *intersection word* $I_T(s)$, taking E as an alphabet and traversing s along some arbitrary direction and, whenever we meet an edge $e \in E$, we append e to $I_T(s)$ if e is crossed according to the orientation of Σ and e^{-1} otherwise. While the edge representation is well-defined at the curve level, intersection words are defined only up to isotopies inside the triangulation's faces. Furthermore, isotopy classes of $I_T(s)$ are closed under cyclic permutations and taking inverses. The *complexity of an intersection word* $I_T(s)$, $\|I_T(s)\|$, is its length (i.e., the number of edges of T intersected by s, counted with multiplicity). If γ is a multicurve, we let $I_T(\gamma)$ be the set of $\#\gamma$ intersection words representing each component. Similarly, the complexity of a Heegaard diagram $(\Sigma_g, \alpha, \beta)$, where α and β are standard multicurves, equals $\|I_T(\alpha)\| + \|I_T(\beta)\|$. Standard arcs are treated accordingly.

A standard curve, multicurve or arc s is *normal* if no intersection word $I_T(s)$ contains a substring of form ee^{-1} or $e^{-1}e$ where $e \in E$ (the blue curve in Figure 3). When normal, the intersections of s and the faces of the triangulation are arcs connecting distinct edges of each triangle, which we call *fundamental arcs*. Normal curves are more convenient for tracing than their more general standard counterparts (refer to Proposition 14); moreover, any standard curve s can be made normal in time $O(\|I_T(s)\|)$, see the proof of Theorem 15.

▶ Remark 12. When a curve is normal for a triangulation, its isotopy class is fixed by the number of times it intersects each labeled edge of E [46], meaning that it can be described through a vector in $\mathbb{N}^{|E|}$ called the curve's *normal coordinates* [8, 9, 46, 47], but whose introduction is unnecessary to the results of Section 4.

3 Algorithms for curves in surfaces and handlebodies

3.1 Converting between representations of curves

In Section 2.4, we saw two representations of curves in a surface: edge lists and intersection words. While some topological operations such as cutting along a curve are easier to implement using edge list representations (one needs only to delete the edges crossing $E_T(s)$ from the dual graph of T to cut along s), others, such as doing Dehn twists (Theorem 15) are more suited to curves intersecting the triangulation transversely. Therefore, it will be convenient to have procedures to convert from one representation to the other.

First, we describe an algorithm to transform curves represented by edge lists into intersection words. For an edge list $E_T(s)$ not in a boundary of an oriented surface Σ, there are two choices of normal curves isotopic to s created by slightly displacing it either to the left or to the right of the edges (with respect to the orientation of Σ and some arbitrary orientation of s); we call them *twins born from s*. Given a choice of twin for s, say left, one can compute its intersection word by traversing s and appending the letters representing adjacent edges coming from the left side of the triangulation graph when embedded in Σ. If, however, s lies either partially or fully on the boundary of Σ, we can only consistently displace it to one side, which can be determined in time $O(\|E_T(s)\|)$. This algorithm does not change the triangulation. For convenience, we state this argument as the following proposition.

▶ **Proposition 13.** *Suppose $E_T(s)$ is an edge list representation of an arc, curve, or multicurve s in Σ. Then there is an algorithm to compute an intersection word $I_T(s)$ of complexity $O(|T|)$ in time $O(|T|)$.*

The next result, whose proof is in the paper's full version, describes an algorithm for the other direction, that is, for going from an intersection word to an edge list representation.

▶ **Proposition 14.** *Suppose that s is a normal arc, curve, or multicurve to some triangulation T, with $\|I_T(s)\| = m$. Then, there exists an algorithm that constructs, in time $O(m + |T|)$, a new triangulation $T' \geq T$ of size $O(m + |T|)$ with s in the edges of T'.*

3.2 Basic operations on curves, handlebodies, and Heegaard splittings

Before proceeding with the main results, we will establish algorithms for some basic operations. The first of these is only a minor modification of [47], whereas the second, in the context of curves in surfaces, is mostly due to [55, Theorem 7.1]. The proofs of both theorems are in the paper's full version.

▶ **Theorem 15.** *Suppose t and s are normal (multi)curves for some fixed triangulation T of Σ_g, given through intersection words $I_T(s)$ and $I_T(t)$ with $m = \max\{\|I_T(s)\|, \|I_T(t)\|\}$. Then an intersection word for $\tau_s^k(t)$ for all $k \in \mathbb{Z}$, with $\|I_T(\tau_s^k(t))\| \leq |k|m^3$, can be computed in time $O(|k|m^3)$.*

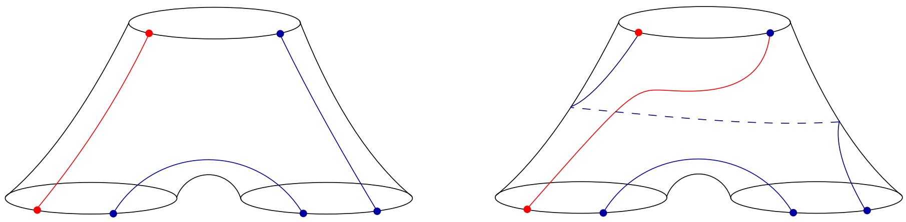

Figure 4 Left: the three types of seams, up to isotopy, on a pair of pants. Two connected components (a red and a blue one) of the multicurve σ are highlighted. Right: a surgery to connect the two distinct components.

▶ **Theorem 16.** *Suppose γ is a minimal system of edge curves in Σ_g. Then one can compute, in time $O(g|T|)$, a triangulation $T' \geq T$ of size $O(m|T|)$ and a pants decomposition ρ in the edges of T', with $\|E_{T'}(\rho)\| = O(g|T|)$, such that ρ contains γ.*

▶ **Theorem 17.** *Suppose γ is a normal minimal system in Σ_g with respect to a triangulation T, with $\|I_T(\gamma)\| = m$. Then there is an algorithm that outputs, in time $O((m + |T|)g^2)$, a disksbuting curve s for the disk graph K_γ, which is normal to T and is represented by intersection word of complexity $O(g|T|)$.*

Proof. Use Proposition 14 to get an edge list representation of γ in a new triangulation T' of complexity $O(m + |T|)$ and then Theorem 16 for an equivalent pants decomposition $\rho \supseteq \gamma$ of complexity $O(g|T'|)$ as an edge list in another triangulation T'' of Σ_g. We will construct a multicurve σ, seamed for the pants decomposition ρ, intersecting each component twice.

For each component r of ρ, select two edges contained in $E_T(r)$ to be points of intersection with σ. For every connected component P of $\Sigma_g - \rho$ (i.e., P is a pair of pants), we connect each intersection point of a boundary component of P to other intersection points in the two other components of ∂P. To avoid self-intersections of σ, we separately draw in each pair of pants a seam at a time, using breadth-first search in the dual graph, recording the intersection words, and changing the dual graph so that no seam can cross a previously traced seam (see the proof of Theorem 16). Even though we change the dual graph, making it more complicated at each drawing of a seam, because the seams are local within pairs of pants, each application of breadth-first search considers only $O(|T''|)$ nodes and edges. At the end of the process, we have a multicurve σ that, although seamed for the pants decomposition ρ, may have up to $g + 1$ connected components. It therefore remains to modify σ so it has just a single component.

Whenever there are still disconnected components in σ, there exists a pair of pants intersected by at least two distinct components of σ, refer to Figure 4. We can then do the surgery on the right side of Figure 4 to connect the two components. Using breadth-first, this takes a total time of $O(g|T''|)$ and yields a connected curve s of intersection word $I_{T''}(s)$. Because, by construction, s is seamed for each pair of pants from $\Sigma_g - \rho$, by Theorem 6, s is diskbusting for γ. Finally, we compute an intersection word of s with respect to T in time $O(|T''|)$ by deleting the edges in $T'' \setminus T$. Note that s is already normal for T as, by construction, s is standard to $T'' \geq T' \geq T$ and, since all paths are shortest, no cyclic reductions are possible. ◀

The proof of the following (slightly technical) result is also carried in the paper's extended version.

▶ **Lemma 18.** *Suppose $(\Sigma_g, \alpha, \beta)$ is a Heegaard diagram, with α and β given as intersection words in a triangulation T of Σ_g, where we $m = \max\{\|I_T(\alpha)\|, \|I_T(\beta)\|\}$. One can find, in time $O(m)$, a triangulation T' and new multicurves α', β' which represent a stabilization $(\Sigma_{g+1}, \alpha', \beta')$ of the original diagram, with $O(|T'|) = O(|T|)$ and $\|I_{T'}(\alpha')\|, \|I_{T'}(\beta')\| = O(m)$.*

4 Restricting the topology of 3-manifolds

Given a Heegaard diagram $(\Sigma_g, \alpha, \beta)$ and a fixed $k \in \mathbb{Z}^+$, the algorithm of Theorem 1 uses Theorems 9 and 11 to construct a new diagram $(\Sigma_{g+1}, \alpha, \tau_s^n(\beta))$ with Hempel distance at least k, without altering the associated RT-invariant in the process, where n is some integer multiple of the associated Vafa's constant and s is a curve distant enough from both K_α and K_β. The main challenge of the algorithm comes, however, from building such a curve s. We address the problem by first computing, through Proposition 20, two curves, say s_α and s_β, of distance at least k from K_α and K_β, respectively. We then use these two curves in Proposition 22 to construct a single curve s, whose distances to K_α and K_β satisfy the hypotheses of Theorem 9.

We establish Proposition 20 incrementally, first using Theorem 17 to find a curve s diskbusting to K_γ, increasing $d(K_\gamma, s)$ to at least 3 using Lemma 19, and finally making the distance bigger than k.

▶ **Lemma 19.** *Let γ be a minimal system in Σ_g, normal to a triangulation T of Σ_g, and given by an intersection word $I_T(\gamma)$ with $\|I_T(\gamma)\| = m$. One can compute, in time $O((gm|T|)^9)$, an intersection word of a normal curve s', of complexity $O((gm|T|)^9)$, with $d(K_\gamma, s) \geq 3$.*

This result is simply an application of Theorem 17 followed by Theorem 10; the details are carried out in the extended version.

▶ **Proposition 20.** *Let γ be a minimal system in Σ_g, normal with respect to some triangulation T. Then, for a fixed $3 \leq k \in \mathbb{Z}^+$, one can compute, in time $c_2^{O(k^{c_1})} \log k$, where $c_1 = \log 3$ and $c_2 = O(gm|T|k)$, the intersection word of a curve s, normal to T and of complexity $c_1^{O(k^{c_2})}$, for which $d(K_\gamma, s) > k$.*

Proof. Start using Lemma 19 to find an intersection word of a curve s_0 with $d(K_\gamma, s_0) \geq 3$. If $k = 3$ we are done, so assume otherwise. Define $M = \|I_T(s_0)\|$; recall that $M = O((gm|T|)^9)$. Let c be any connected component of γ and recursively define $s_{i+1} = \tau_{s_i}^{k_i+3}(c)$, where $k_i = 2^{i+1} + 2$. We claim that, for any $\ell \in \mathbb{Z}^+$, $d(K_\gamma, s_\ell) \geq k_{\ell-1}$. We prove this inducting on ℓ: for $\ell = 1$, $k_0 = 4$ and, by Theorem 9 with $\alpha = \beta = \gamma$ and $n = \min\{d(K_\gamma, K_\gamma), 1\} = 1$, we have that

$$d(K_\gamma, s_1) = d(K_\gamma, \tau_{s_0}^{k_0+n+2}(c)) \geq d(K_\gamma, K_{\tau_{s_0}^{k_0+n+2}(\gamma)}) \geq 4.$$

Now assume $d(K_\gamma, s_\ell) \geq 2^\ell + 2$. Again, by Theorem 9,

$$d(K_\gamma, s_{\ell+1}) = d(K_\gamma, \tau_{s_\ell}^{k_\ell+3}(c)) \geq d(K_\gamma, K_{\tau_{s_\ell}^{k_\ell+3}(\gamma)}) \geq 2d(K_\gamma, s_\ell) - 2 \geq 2(2^\ell + 2) - 2 = 2^{\ell+1} + 2,$$

so, by induction, $d(K_\gamma, s_\ell) \geq k_{\ell-1}$. Setting the total number of iterations at $\ell = \lceil \log(k-2) \rceil$ and $s = s_\ell$, we have that $d(K_\gamma, s) \geq k$.

We now estimate the total computational time and the output's complexity of the algorithm. First, we note that, by Theorem 15, $\|I_T(s_{i+1})\| \leq (k_i + 3)\|I_T(s_i)\|^3$. Recursively, this gives

$$
\begin{aligned}
\|I_T(s)\| &\le M^{3^\ell}(k_0+3)^{3^{\ell-1}}(k_1+3)^{3^{\ell-2}}\dots(k_{\ell-1}+3)\\
&\le M^{3^\ell}(2^1+5)^{3^{\ell-1}}(2^2+5)^{3^{\ell-2}}\dots(2^\ell+5)\\
&\le M^{3^\ell}(2^\ell+5)^{3^{\ell-1}}(2^\ell+5)^{3^{\ell-2}}\dots(2^\ell+5)\\
&\le M^{3^\ell}(2^\ell+5)^{3^{\ell-1}+3^{\ell-2}+\dots+3^0}\\
&\le M^{3^\ell}(2^\ell+5)^{\frac{1}{2}\times(3^\ell-1)}\\
&\le M^{3^{\log(k-2)+2}}(2^{\log(k-2)+2}+5)^{\frac{1}{2}\times(3^{\log(k-2)+2}-1)}\\
&\le M^{9\times 3^{\log(k-2)}}(4k-3)^{9/2\times 3^{\log(k-2)}}\\
&\le M^{9(k-2)^{\log 3}}(4k)^{5(k-2)^{\log 3}}\\
&\le M^{9k^{c_1}}(4k)^{5k^{c_1}}
\end{aligned}
$$

where we used the geometric series in the fifth line, $\lceil\log(k-2)\rceil \le \log(k-2)+2$ in the sixth line, the relation $a^{\log b} = 2^{\log a \log b} = b^{\log a}$ for any real $a, b > 1$ in the eighth line, and $c_1 = \log 3$ in the last line. The time complexity for computing $I_T(s)$ can be (very coarsely) estimated at $O(\ell \times (k_{\ell-1}+3)\|I_T(s_{\ell-1})\|^3) = O(\log k \times \|I_T(s)\|) = O((gm|T|)^{81k^{c_1}}(4k)^{5k^{c_1}}\log k)$ by noting that $\|I_T(s_i)\| \le \|I_T(s_{\ell-1})\|$ and $k_i \le k_{\ell-1} = k$ for all $1 \le i \le n$. ◀

We now show Proposition 22, starting with the following technical lemma, whose proof (a simple application of the triangle inequality) is carried out in the paper's full version.

▶ **Lemma 21.** *Consider some full systems γ and γ' in the surface Σ_g with $d(K_\gamma, K_{\gamma'}) \ge 4$ and a curve s with $d(K_\gamma, s) < 2$. Then $d(K_{\gamma'}, s) \ge 2$.*

▶ **Proposition 22.** *Fix an integer $k \ge 4$. Consider some minimal systems α and β in the surface Σ_g for which $d(K_\alpha, K_\beta) = 0$. Then there is a curve s and some full minimal system β' (potentially $\beta' = \beta$) in Σ_g such that*

$$
\min_{i=\alpha,\beta'}\{d(K_i,s)\} \ge 2,\ \max_{i=\alpha,\beta'}\{d(K_i,s)\} \ge k,\ d(K_\alpha,s)+d(K_{\beta'},s)-2 > n,
$$
$$
\text{and } \langle(\Sigma_g,\beta)\rangle_{\mathcal{C}}^{RT} = \langle(\Sigma_g,\beta')\rangle_{\mathcal{C}}^{RT}, \tag{2}
$$

where $n = \min\{d(K_\alpha, K_{\beta'}), 1\}$. Moreover, if $m = \max\{\|I_T(\alpha)\|, \|I_T(\beta)\|\}$, then $I_T(s)$ and $I_T(\beta')$ will have complexity $Nkc_2^{O(k^{c_1})}$ and are computed in similar time for any choice of Vafa's constant $N \in \mathbb{Z}^+$, where $c_1 = \log 3$ and $c_2 = O(gm|T|k)$.

Proof. We use Proposition 20 to find two curves, s_α and s_β, such that $d(K_\alpha, s_\alpha)$ and $d(K_\beta, s_\beta)$ are larger than k. Note that if s_α is diskbusting in K_β, we are done, as we can let $s = s_\alpha$, $\beta' = \beta$, and

$$
d(K_\alpha,s) \ge k,\ d(K_{\beta'},s) = d(K_\beta,s_\alpha) \ge 2,\ \text{and } d(K_\alpha,s)+d(K_{\beta'},s)-2 > 1.
$$

Therefore, assume $d(K_\beta, s_\alpha) < 2$.

By applying Theorem 15, we compute an intersection word for $\widetilde{\beta} = \tau_{s_\beta}^{N(k+3)}(\beta)$. Note that $\|I_T(\widetilde{\beta})\| = O(Nk\|I_T(s_\beta)\|^3) = O(Nk(gm|T|)^{243k^{c_1}}(4k)^{15k^{c_1}})$ and, by Theorem 9, $d(K_\beta, K_{\beta'}) \ge k$. Moreover, because β' was constructed by applying a power of Dehn twists multiple of N to β, by Theorem 11, $\langle(\Sigma_g,\beta)\rangle_{\mathcal{C}}^{RT} = \langle(\Sigma_g,\beta')\rangle_{\mathcal{C}}^{RT}$.

Pick any component $\widetilde{b}$ of $\widetilde{\beta}$. If $\widetilde{b}$ is diskbusting for K_α, then $s = \widetilde{b}$ and $\beta' = \beta$ have the desired properties. In particular, notice that

$$
d(K_\alpha,s) \ge 2,\ \ d(K_{\beta'},s) \ge d(K_\beta,K_{\widetilde{\beta}}) \ge k,\ \text{and } d(K_\alpha,s)+d(K_{\beta'},s)-2 > 1.
$$

If, however, $\widetilde{b}$ is not diskbusting for K_α, it means that $d(K_\alpha, K_{\widetilde{\beta}}) \leq d(K_\alpha, \widetilde{b}) < 2$. Letting $s = s_\alpha$ and $\beta' = \widetilde{\beta}$ gives

$$d(K_\alpha, s) \geq k, \;\; d(K_{\beta'}, s) \geq 2, \text{ and } d(K_\alpha, s) + d(K_{\beta'}, s) - 2 \geq k \geq 3 > d(K_\alpha, K_{\beta'}),$$

where the second inequality comes from $d(K_\beta, s_\alpha) < 2$ and $d(K_\beta, K_{\widetilde{\beta}}) \geq k \geq 4$ applied to Lemma 21. ◀

We can, once again, combine the above result with Theorem 9 to prove our main reduction.

▶ **Theorem 1.** *Let $(\Sigma_g, \alpha, \beta)$ be a Heegaard diagram of a closed 3-manifold M of complexity m to a triangulation T of Σ_g. Choose a modular category $\mathcal{C}$ with Vafa's constant N and fix an integer $k \geq 4$. Then there is a set of three Heegaard diagrams, computed in time $O_k(\mathrm{poly}(g, m, |T|, N))$ (of degree depending on k), representing manifolds with RT invariant over $\mathcal{C}$ equal to $\langle M \rangle_{\mathcal{C}}^{RT}$, one of them guaranteed to be hyperbolic and with no embedded incompressible orientable surface of genus at most $2k$.*

Proof. We start by using Lemma 18 to compute a stabilized splitting $(\Sigma_{g+1}, \alpha', \beta')$; note that $d(K_{\alpha'}, K_{\beta'}) = 0$. Construct two curves s_α and s_β as in the proof of Proposition 22. Let $\beta_1 = \beta'$ and $s_1 = s_\alpha$.

We proceed as in the proof of Proposition 22, computing an intersection word of a new minimal system $\widetilde{\beta} = \tau_{s_\beta}^{N(k+3)}(\beta')$ and defining $\beta_2 = \beta'$ and $\beta_3 = \widetilde{\beta}$, s_2 as any component of $\widetilde{\beta}$, and $s_3 = s_\alpha$. By Proposition 20, for at least one $1 \leq j \leq 3$

$$\min_{i=\alpha',\beta^j} \{d(K_i, s_j)\} \geq 2, \max_{i=\alpha',\beta^j} \{d(K_i, s_j)\} \geq k, d(K_{\alpha'}, s_j) + d(K_{\beta_j}, s_j) - 2 > n$$

where $n = \max\{d(K_{\alpha'}, K_{\beta_j}), 1\}$. Applying, once again, Theorems 15 and 9, we conclude that one of the splittings $(\Sigma_g, \alpha', \tau_{s_j}^{N(k+3)}(\beta_j))$ has a Hempel distance of, at least, k and an intersection word for $\tau_{s_j}^{N(k+3)}(\beta_j)$ can be found in time $O(Nk(\max\{\|I_T(\beta_j)\|, \|I_T(s_j)\|\})^3)$ $= O(N^4 k^4 (gm|T|)^{729k^{c_1}} (4k)^{45k^{c_1}})$. Theorem 7 finishes the proof. ◀

5 Computational reduction for quantum invariants

Theorem 1 gives a polynomial time algorithm to change a general closed 3-manifold into another manifold with very restricted topology without altering the RT invariant in the process. Therefore, the problems of either exactly computing or approximating the invariant of general 3-manifolds *reduce*, in a Cook-Turing sense, to the problems of exactly computing or approximating the invariant when the manifolds are assumed to have the properties of Theorem 1. Ultimately, this means that the *hardness of computation* is not altered in this restricted topology scenario.

We illustrate this reduction by showing that *value-distinguishing* approximations of the Reshetikhin–Turaev and the Turaev-Viro invariants are #P-hard, even for manifolds with the properties of Theorem 1, when we take $\mathcal{C}$ to be the category of representations of the quantum group $SO_r(3)$, for some prime $r \geq 5$ (in this case, one can use $N = 4r$ for Vafa's constant). We note that, by a value-distinguishing approximation, we mean the ability to determine whether the approximated quantity $c \in \mathbb{R}^+$ is $a > c$ or $b < c$ for any fixed $0 < a < b$ where we assume, as a premise, one of the two to hold. In particular, multiplicative approximations are value-distinguishing, although other less restrictive schemes also are [30].

5.1 Reshetikhin–Turaev invariant

For every Heegaard diagram $(\Sigma_g, \alpha, \beta)$, there is an orientation-preserving homeomorphism $\phi : \Sigma_g \to \Sigma_g$ such that $\beta = \phi(\alpha)$ [18, 56]. A lthough not unique, this map is well-defined on $\text{Mod}(\Sigma_g)$, so it can be described by a word on Lickorish generators. We do not differentiate between the notation of the homomorphism ϕ from its equivalence class in $\text{Mod}(\Sigma_g)$.

▶ **Theorem 23** ([3, 20]). *Consider the problem $\mathcal{P}$ of determining a value-distinguishing approximation of the $SO_r(3)$-RT invariant, $r \geq 5$ prime, of a manifold M, represented through a Heegaard splitting described by a word $\phi \in \text{Mod}(\Sigma_g)$ for some known $g \geq 2$. Then $\mathcal{P}$ is #P-hard in the sense of a Cook-Turing reduction.*

Before applying Theorem 1 to this result, we need to find an algorithm to convert the pair (Σ_g, ϕ) into a proper Heegaard diagram (Σ_g, β). This cannot be done through brute force computing $\beta = \phi(\alpha)$, since, as we saw in the proof of Proposition 20, it can lead to exponential bottlenecks. We fix the problem with the following lemma at the cost of potentially increasing the value of g. The proof of the next, as well as all other results of this section (except for Corollary 25), is found in the paper's full version.

▶ **Lemma 24.** *Consider a Heegaard splitting described by a word $\phi \in \text{Mod}(\Sigma_g)$ for some known $g \geq 2$. Then it is possible to compute, in time $O(\text{poly}(g, \phi))$, a Heegaard diagram $(\Sigma_{g'}, \beta)$ representing the same manifold, with β normal to a triangulation T of $\Sigma_{g'}$.*

▶ **Corollary 25.** *Fix a prime $r \geq 5$. Consider the problem $\mathcal{P}$ of, given a Heegaard diagram (Σ_g, β) of a closed 3-manifold M, returning a value-distinguishing approximation of its $SO_r(3)$-RT invariant if M has the properties of Theorem 1, otherwise remaining silent. Then $\mathcal{P}$ is #P-hard in the sense of a Cook-Turing reduction.*

Proof. Let $\mathcal{O}$ be an oracle machine that solves $\mathcal{P}$ and consider the problem $\mathcal{P}'$ of finding a value-distinguishing approximation of a general Heegaard splitting given as a pair (Σ_g, ϕ) for $\phi \in \text{Mod}(\Sigma_g)$. We will show that $\mathcal{O}$ solves P' with only a polynomial overhead. Because $\mathcal{O}$ solves $\mathcal{P}$ and, by Theorem 23, $\mathcal{P}'$ is #P-hard, then so is $\mathcal{P}$.

Let (Σ_g, ϕ) encode a Heegaard splitting of a manifold M, not necessarily with the properties of Theorem 1. Using Lemma 24, we transform (Σ_g, ϕ) into a Heegaard diagram $(\Sigma_{g'}, \beta)$ in polynomial time. Then apply the algorithm of Theorem 1 to this diagram, returning three new diagrams as output. We run the oracle $\mathcal{O}$ in parallel to these three diagrams, stopping the program whenever it halts for one of them. In the end, this gives, in polynomial time, a value-distinguishing approximation of $\langle M \rangle^{RT}_{SO_r(3)}$, concluding the proof. ◀

▶ Remark 26. We note that the sort of reduction assumed by the statement of Corollary 25 is related to what is often called in the complexity literature by a *semi-decision problem*, that is, an oracle that cannot return an incorrect answer, but may not halt for some inputs. Although weaker than the more common approach in which we assume that $\mathcal{P}$ *can* give incorrect approximation of the invariant if the input does not have the expected properties, this sort of oracle has also already been discussed for algorithm on 3-manifolds, e.g. see the Definition 1.3 in [35].

5.2 Turaev-Viro invariant

A compact 3-manifold can also be combinatorially described by a set of tetrahedra $\mathcal{T}$, together with rules on how to glue their triangular faces [10, 40]. This description is called a *triangulation of the 3-manifold*, but it should not be confused with triangulations of surfaces. Nonetheless, if M has a boundary, $\mathcal{T}$ naturally defines a (surface) triangulation for ∂M. For each $g \geq 1$, there exists a one-vertex triangulation of the handlebody of genus g [27].

The Turaev-Viro invariant (TV) is another quantum invariant for closed 3-manifolds. It is defined for spherical categories (which include modular categories) and is computed directly from a triangulation [7]. The Turaev-Walker theorem [53] states that, given a manifold M, $|\langle M\rangle_{\mathcal{C}}^{RT}|^2 = \langle M\rangle_{\mathcal{C}}^{TV}$, provided $\mathcal{C}$ is a modular category. We show that an approximation of the $SO_r(3)$-TV invariant can be used to compute an approximation of $SO_r(3)$-RT [3]. For such, we use the next theorem due to [21].

▶ **Theorem 27.** *Suppose β is a normal minimal system with respect to a one-vertex triangulation of the standard embedding of the genus g handlebody in $\mathbb{R}^3$, with $\|I_T(\beta)\| = m$. There is an algorithm to compute, in time* $\mathrm{poly}(m, g)$*, a triangulation of the 3-manifold of Heegaard diagram (Σ_g, β).*

▶ **Corollary 28.** *Fix a prime $r \geq 5$. Consider the problem $\mathcal{P}$ of, given a triangulation of a closed 3-manifold M, returning a value-distinguishing approximation of its $SO_r(3)$-TV invariant if M has the properties of Theorem 1, otherwise remaining silent. Then $\mathcal{P}$ is #P-hard in the sense of Cook-Turing reduction.*

References

1 Ian Agol, Joel Hass, and William Thurston. The computational complexity of knot genus and spanning area. *Transactions of the American Mathematical Society*, 358(9):3821–3850, 2006. arXiv:math/0205057.

2 D. Aharonov, V. Jones, and Z Landau. A polynomial quantum algorithm for approximating the Jones polynomial. *Algorithmica*, 55:395–421, 2009. doi:10.1007/s00453-008-9168-0.

3 Gorjan Alagic and Catharine Lo. Quantum invariants of 3-manifolds and NP vs #P. *Quantum Info. Comput.*, 17(1–2):125–146, February 2017. URL: https://arxiv.org/abs/1411.6049.

4 Itai Arad and Zeph Landau. Quantum computation and the evaluation of tensor networks. *SIAM Journal on Computing*, 39(7):3089–3121, 2010. doi:10.1137/080739379.

5 Rodrigo Arouca, Andrea Cappelli, and Hans Hansson. Quantum field theory anomalies in condensed matter physics. *SciPost Physics Lecture Notes*, September 2022. doi:10.21468/scipostphyslectnotes.62.

6 David Bachman, Ryan Derby-Talbot, and Eric Sedgwick. Computing Heegaard genus is NP-hard. *A Journey Through Discrete Mathematics: A Tribute to Jiří Matoušek*, pages 59–87, 2017. arXiv:1606.01553.

7 John Barrett and Bruce Westbury. Invariants of piecewise-linear 3-manifolds. *Transactions of the American Mathematical Society*, 348(10):3997–4022, 1996. URL: https://arxiv.org/abs/hep-th/9311155.

8 Mark C. Bell and Richard C. H. Webb. Polynomial-time algorithms for the curve graph. *arXiv: Geometric Topology*, 2016. URL: https://arxiv.org/abs/1609.09392.

9 Mark Christopher Bell. *Recognising mapping classes.* PhD thesis, University of Warwick, 2015. URL: https://wrap.warwick.ac.uk/id/eprint/77123/.

10 RH Bing. An alternative proof that 3-manifolds can be triangulated. *Annals of Mathematics*, 69(1):37–65, 1959. doi:10.2307/1970092.

11 Benjamin A. Burton. The HOMFLY-PT Polynomial is Fixed-Parameter Tractable. In Bettina Speckmann and Csaba D. Tóth, editors, *34th International Symposium on Computational Geometry (SoCG 2018)*, volume 99 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 18:1–18:14, Dagstuhl, Germany, 2018. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.SoCG.2018.18.

12 Benjamin A. Burton. The Next 350 Million Knots. In Sergio Cabello and Danny Z. Chen, editors, *36th International Symposium on Computational Geometry (SoCG 2020)*, volume 164 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 25:1–25:17, Dagstuhl, Germany, 2020. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.SoCG.2020.25.

13 Benjamin A. Burton, Clément Maria, and Jonathan Spreer. Algorithms and complexity for Turaev-Viro invariants. *Journal of Applied and Computational Topology*, 2(1-2):33–53, 2018. doi:10.1007/s41468-018-0016-2.

14 Qingtao Chen and Tian Yang. Volume conjectures for the Reshetikhin–Turaev and the Turaev–Viro invariants. *Quantum Topology*, 3:419–460, 2018. arXiv:1503.02547.

15 Colleen Delaney, Clément Maria, and Eric Samperton. An algorithm for Tambara-Yamagami quantum invariants of 3-manifolds, parameterized by the first Betti number, 2024. arXiv:2311.08514.

16 Pavel Etingof. On Vafa's theorem for tensor categories. *Mathematical Research Letters*, 9:651–657, 2002. arXiv:0207007.

17 Tatiana Evans. High distance Heegaard splittings of 3-manifolds. *Topology and its Applications*, 153(14):2631–2647, 2006. doi:10.1016/j.topol.2005.11.003.

18 Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 41. Princeton University Press, 2011.

19 Eduardo Fradkin. *Field theoretic aspects of condensed matter physics: An overview*, pages 27–131. Elsevier, 2024. doi:10.1016/b978-0-323-90800-9.00269-9.

20 M. Freedman, M. Larsen, and Z. Wang. The two-eigenvalue problem and density of Jones representation of braid groups. *Communications in Mathematical Physics*, 228:177–199, 2002. arXiv:0103200.

21 Alexander He, James Morgan, and Em K Thompson. An algorithm to construct one-vertex triangulations of Heegaard splittings. *arXiv preprint*, 2023. arXiv:2312.17556.

22 John Hempel. 3-manifolds as viewed from the curve complex. *Topology*, 40(3):631–657, 2001. arXiv:math/9712220.

23 Kristóf Huszár and Jonathan Spreer. 3-Manifold Triangulations with Small Treewidth. In Gill Barequet and Yusu Wang, editors, *35th International Symposium on Computational Geometry (SoCG 2019)*, volume 129 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 44:1–44:20, Dagstuhl, Germany, 2019. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.SoCG.2019.44.

24 Kristóf Huszár and Jonathan Spreer. On the Width of Complicated JSJ Decompositions. In Erin W. Chambers and Joachim Gudmundsson, editors, *39th International Symposium on Computational Geometry (SoCG 2023)*, volume 258 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 42:1–42:18, Dagstuhl, Germany, 2023. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.SoCG.2023.42.

25 Kristóf Huszár. On the pathwidth of hyperbolic 3-manifolds. *Computing in Geometry and Topology*, 1(1):1:1–1:19, February 2022. doi:10.57717/cgt.v1i1.4.

26 Ayako Ido, Yeonhee Jang, and Tsuyoshi Kobayashi. Heegaard splittings of distance exactly n. *Algebraic & geometric topology*, 14(3):1395–1411, 2014. arXiv:1210.7627.

27 William Jaco and J. Hyam Rubinstein. Layered-triangulations of 3-manifolds, 2006. arXiv:math/0603601.

28 Jesse Johnson. Non-uniqueness of high distance Heegaard splittings. *arXiv preprint*, 2013. arXiv:1308.4599.

29 A.Yu. Kitaev. Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1):2–30, 2003. doi:10.1016/S0003-4916(02)00018-0.

30 Greg Kuperberg. How hard is it to approximate the Jones Polynomial? *Theory Computing*, 11:183–219, 2009. arXiv:0908.0512.

31 Greg Kuperberg. Algorithmic homeomorphism of 3-manifolds as a corollary of geometrization. *Pacific Journal of Mathematics*, 301(1):189–241, 2019. arXiv:1508.06720.

32 William BR Lickorish. A finite set of generators for the homeotopy group of a 2-manifold. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 60, pages 769–778. Cambridge University Press, 1964. doi:10.1017/S030500410003824X.

33 Martin Lustig and Yoav Moriah. High distance Heegaard splittings via fat train tracks. *Topology and its Applications*, 156(6):1118–1129, 2009. arXiv:0706.0599.

34 J.A. Makowsky and J.P. Mariño. The parametrized complexity of knot polynomials. *Journal of Computer and System Sciences*, 67(4):742–756, 2003. Parameterized Computation and Complexity 2003. doi:10.1016/S0022-0000(03)00080-1.

35 Jason Fox Manning. Algorithmic detection and description of hyperbolic structures on closed 3–manifolds with solvable word problem. *Geometry & Topology*, 6(1):1–26, 2002. arXiv:math/0102154.

36 Clément Maria. Parameterized complexity of quantum knot invariants. In *Proceedings of the International Symposium on Computational Geometry, SoCG*, volume 189 of *LIPIcs*, pages 53:1–53:17, 2021. doi:10.4230/LIPIcs.SoCG.2021.53.

37 Clément Maria and Jessica S. Purcell. Treewidth, crushing and hyperbolic volume. *Algebraic & Geometric Topology*, 19:2625–2652, 2019. arXiv:1805.02357.

38 Clément Maria and Owen Rouillé. Computation of large asymptotics of 3-manifold quantum invariants. In *Proceedings of the Symposium on Algorithm Engineering and Experiments, ALENEX*, pages 193–206. SIAM, 2021. doi:10.1137/1.9781611976472.15.

39 Clément Maria and Jonathan Spreer. A polynomial-time algorithm to compute Turaev-Viro invariants tv(4,q) of 3-manifolds with bounded first Betti number. *Foundations of Computational Mathematics*, 20(5):1013–1034, 2020. arXiv:1607.02218.

40 Edwin E Moise. Affine structures in 3-manifolds: V. the triangulation theorem and hauptvermutung. *Annals of mathematics*, 56(1):96–114, 1952. doi:10.2307/1969769.

41 Lukas Müller and Lukas Woike. The Dehn twist action for quantum representations of mapping class groups. *arXiv preprint*, 2023. arXiv:2311.16020.

42 Ruifeng Qiu, Yanqing Zou, and Qilong Guo. The Heegaard distances cover all nonnegative integers. *Pacific Journal of Mathematics*, 275(1):231–255, 2015. arXiv:1302.5188.

43 Paolo Salvatore and Nathalie Wahl. Framed discs operads and the equivariant recognition principle, 2001. arXiv:math/0106242.

44 Eric Samperton. Topological quantum computation is hyperbolic. *Communications in Mathematical Physics*, 402:79–96, 2023. arXiv:2201.00857.

45 Nikolai Saveliev. *Lectures on the topology of 3-manifolds: an introduction to the Casson invariant.* Walter de Gruyter, 2011.

46 Marcus Schaefer, Eric Sedgwick, and Daniel Štefankovič. Algorithms for normal curves and surfaces. In *Computing and Combinatorics: 8th Annual International Conference, COCOON 2002 Singapore, August 15–17, 2002 Proceedings 8*, pages 370–380. Springer, 2002. doi:10.1007/3-540-45655-4_40.

47 Marcus Schaefer, Eric Sedgwick, and Daniel Stefankovic. Computing Dehn twists and geometric intersection numbers in polynomial time. In *CCCG*, volume 20, pages 111–114, 2008. URL: https://www.cs.rochester.edu/~stefanko/Publications/geometric.pdf.

48 Martin Scharlemann and Maggy Tomova. Alternate Heegaard genus bounds distance. *Geometry & Topology*, 10(1):593–617, 2006. arXiv:math/0501140.

49 Jennifer Schultens. *Introduction to 3-manifolds*, volume 151. American Mathematical Soc., 2014.

50 Joe Scull. The homeomorphism problem for hyperbolic manifolds I, 2021. arXiv:2108.00779.

51 Isadore Manuel Singer and John A Thorpe. *Lecture notes on elementary topology and geometry.* Springer, 2015.

52 Edith Nelson Starr. *Curves in handlebodies.* PhD thesis, University of California, Berkeley, 1992.

53 Vladimir G Turaev. *Quantum invariants of knots and 3-manifolds.* de Gruyter, 2010.

54 Cumrun Vafa. Toward classification of conformal theories. *Physics Letters B*, 206(3):421–426, 1988.

55 Éric Colin De Verdière and Francis Lazarus. Optimal pants decompositions and shortest homotopic cycles on an orientable surface. *Journal of the ACM (JACM)*, 54(4):18–es, 2007. doi:10.1007/978-3-540-24595-7_45.

56 Michael Yoshizawa. High distance Heegaard splittings via Dehn twists. *Algebraic & Geometric Topology*, 14(2):979–1004, 2014. arXiv:1212.1199.