Online Hitting Sets for Disks of Bounded Radii

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Abstract

We present algorithms for the online minimum hitting set problem in geometric range spaces: Given a set P of n points in the plane and a sequence of geometric objects that arrive one-by-one, we need to maintain a hitting set at all times. For disks of radii in the interval $[1, M]$, we present an $O(\log M \log n)$-competitive algorithm. This result generalizes from disks to positive homothets of any convex body in the plane with scaling factors in the interval $[1, M]$. As a main technical tool, we reduce the problem to the online hitting set problem for a finite subset of integer points and bottomless rectangles. Specifically, for a given $N > 1$, we present an $O(\log N)$-competitive algorithm for the variant where P is a subset of an $N \times N$ section of the integer lattice, and the geometric objects are bottomless rectangles.

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1 Introduction

In the general form of the *Hitting Set* problem, we are given a point set P and a collection of subsets $\mathcal{C} = \{S_1, \ldots, S_m\}$, and we need to find a subset $H \subset P$ (**hitting set**) of minimal size such that every set $S_i \in \mathcal{C}$ contains some point in H. In the ***Online*** *Hitting Set* problem, the set P is known in advance, but the subsets $S_1, S_2, \ldots$ in $\mathcal{C}$ arrive one at a time (without advance knowledge). We need to maintain a hitting set $H_i \subseteq P$ for the first i sets $\{S_1, \ldots, S_i\}$ such that $H_i \subseteq H_{i+1}$ for all $i \geq 1$ (that is, we can add new points to the hitting set, but we cannot delete any point). The study of the *Online Hitting Set* problem (which is dual to the *Online Set Cover* problem) was initiated by Alon et al. [2]. They designed a deterministic algorithm with competitive ratio $O(\log |P| \log |\mathcal{C}|)$ and obtained almost matching lower bound of $\Omega\left(\frac{\log |P| \log |\mathcal{C}|}{\log \log |P| + \log \log |\mathcal{C}|}\right)$.

Geometric Hitting Set. In the **geometric** *Hitting Set* problem, we have $P \subseteq \mathbb{R}^d$ for some constant dimension d, and the sets in $\mathcal{C}$ are geometric objects of some type: for example, balls, unit balls, simplices, axis-aligned cubes, or hyper-rectangles. Depending on whether P is finite or infinite, there are different versions of the problem. In this paper, we consider P to be a finite set of points in $\mathbb{R}^2$.

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Related Previous Work. When P is finite, Even and Smorodinsky [12] initiated the study of the geometric online *Hitting Set* problem for various geometric objects. They established an optimal competitive ratio of $\Theta(\log|P|)$ when the objects are intervals in $\mathbb{R}$, or half-planes or congruent disks in the plane. Later, Khan et al. [14] investigated this problem for a finite set of integer points $P \subseteq [0,N)^2 \cap \mathbb{Z}^2$ and a collection $\mathcal{C}$ of axis-aligned squares $S \subseteq [0,N)^2$ with integer coordinates for $N > 0$. They developed an $O(\log N)$-competitive algorithm for this variant. They also established a randomized lower bound of $\Omega(\log|P|)$, where $P \subset \mathbb{R}^2$ consists of finitely many points and $\mathcal{C}$ consists of translates of an axis-aligned square. Recently, De et al. [6] considered the variant when P is set of n points in $\mathbb{R}^2$ and $\mathcal{C}$ consists of homothetic copies of a regular k-gon (for $k \geq 4$) with scaling factors in the interval $[1,M]$, and designed an $O(k^2 \log M \log n)$-competitive randomized algorithm. Even though a disk can be approximated by a regular k-gon as $k \to \infty$, this does not imply any competitive algorithm for disks with radii in the interval $[1,M]$.

Our results and Technical Contribution. We study the *Online Hitting Set* problem when P is set of n points in $\mathbb{R}^2$. Table 1 summarizes the existing results and the results of the current paper.

Table 1 Summary of known and new results for the geometric *Online Hitting Set* problem where $|P| = n$ is finite. (#) indicates randomized results. Our results are listed in the last three lines.

Points	Objects	Lower Bound	Upper Bound
$P \subset \mathbb{R}$	Intervals in $\mathbb{R}$	$\Omega(\log n)$ [12]	$O(\log n)$ [12]
$P \subset \mathbb{R}^2$	Half-planes and translates of a disk in $\mathbb{R}^2$	$\Omega(\log n)$ [12]	$O(\log n)$ [12]
$P \subseteq [0,N)^2 \cap \mathbb{Z}^2$	Axis-aligned squares in $[0,N)^2 \cap \mathbb{Z}^2$ with integer coordinates	$\Omega(\log n)$ [14] (#)	$O(\log N)$ [14]
$P \subset \mathbb{R}$	Homothetic copies of a regular k-gon ($k \geq 4$) with scaling factors in the interval $[1,M]$	$\Omega(\log n)$ [14] (#)	$O(k^2 \log M \log n)$ [6] (#)
$P \subseteq [0,N)^2 \cap \mathbb{Z}^2$	Bottomless rectangles (for definition, see Section 2.1)	$\Omega(\log n)$ [12]	$O(\log N)$ [Theorem 1]
$P \subset \mathbb{R}^2$	Disks having radii in the interval $[1,M]$	$\Omega(\log n)$ [12]	$O(\log M \log n)$ [Theorem 12]
$P \subset \mathbb{R}^2$	Positive homothets of an arbitrary convex body in $\mathbb{R}^2$ with scaling factors in the interval $[1,M]$	$\Omega(\log n)$ [14] (#)	$O(\log M \log n)$ [Theorem 14]

We now present our contributions and briefly discuss the technical ideas involved.

Bottomless Rectangles in $[0,N]^2$. We present an $O(\log N)$-competitive deterministic algorithm for the geometric *Online Hitting Set* problem, where $P \subset [0,N)^2 \cap \mathbb{Z}^2$, and $\mathcal{C}$ is a sequence of bottomless rectangles of the form $[a,b) \times [0,c)$ arriving one by one (Theorem 1 in Section 2). When a bottomless rectangle $[a,b) \times [0,c)$ arrives, our algorithm chooses hitting points guided by the **canonical partition** of the interval $[a,b]$ (see Section 2 for a definition). For each point p in an offline optimum, this structured canonical partition ensures that $O(\log N)$ points are sufficient to hit all the incoming rectangles in $[0,N]^2 \cap \mathbb{Z}^2$ that are hit by p. We prove that our algorithm is $O(\log N)$-competitive for a broader class of objects– sets $S \subset [a,b) \times \mathbb{R}$ with **lowest-point property** (see Section 2.2 for a definition).

Disks with Radii in $[1, M]$. Our main result is a deterministic $O(\log M \log n)$-competitive *Online Hitting Set* algorithm for an arbitrary set P of n points in the plane, and a sequence of disks of radii in the interval $[1, M]$ (Theorem 12 in Section 4). Previously, an $O(\log n)$-competitive algorithm was known only for congruent disks [12]. In particular, our result is the first $O(\log n)$-competitive algorithm that works for disks of radii in the interval $[1, 1+\varepsilon]$ for any constant $\varepsilon > 0$ (Corollary 13).

However, a finite set of disks in the plane do not necessarily have the lowest-point property. We reduce the problem to objects with the lowest-point property in two steps. First, we consider a restricted version, the **line-separated setting** (Section 3), where the centers of disks in $\mathcal{C}$ lie on one side of a line (w.l.o.g., the x- or y-axis), while P lies on the other side. We use the concept of **disk hull** for a point set (introduced by Dumitrescu et al. [10]), which generalizes the notion of convex hulls and α-hulls. Among other important properties, the boundary of the disk hull is monotone w.r.t. the separating line. Using these properties, we reduce the *Hitting Set* problem in the line-separated setting to objects with the lowest-point property, and obtain an $O(\log n)$-competitive algorithm in the line-separated setting (Theorem 9 in Section 3).

In general, there is no restriction on the location of the points in P and the centers of disks. We reduce the general problem to the line-separated setting as follows: We partition the disks of radii in the interval $[1, M]$ into $O(\log M)$ layers, ensuring that the ratio of radii of disks in each layer is bounded by at most 2. For each layer, our algorithm maintains a tiling of the plane into axis-aligned squares such that (a) any disk of a given layer contains the entire tile that contains the disk center, and (b) each disk intersects only $O(1)$ tiles. Our algorithm simultaneously runs several invocations of the line-separating algorithm (one for each directed grid line). When a disk arrives, our algorithm inserts it into all relevant invocations of the line-separating algorithms; we show that only $O(1)$ invocations are relevant. In the competitive analysis, we show that for each point p in an offline optimum solution, our algorithm uses $O(\log n)$ hitting points for the disks in each layer that contain p. Since there are $O(\log M)$ layers, our algorithm is $O(\log M \log n)$-competitive.

Homothets of a Convex Body with Diameters in $[1, M]$. We generalize our main result from disks to positive homothets of any convex body in the plane, where the radii in the interval $[1, M]$ are replaced by scaling factors in the interval $[1, M]$ (Theorem 14 in Section 5). Our online algorithm is based on a two-stage approach, similar to the case of disks, and it is $O(\log M \log n)$-competitive. The key technical difficulty arises from the geometric differences between a disk and a general convex body. It is easy to extend the concept of a disk hull to hulls for homothetic convex bodies. However, unlike for disks, the boundary of the hull is not necessarily x- or y-monotone: We show that it is monotone w.r.t. some carefully chosen directions. To generalize a layered decomposition of axis-parallel lines, we need **two** directions in which the hull is monotone, the two directions must be far apart (in the space of directions), to create a tiling with properties (a) and (b) above. We call a pair of directions satisfying these requirements a **good pair** of directions. We use a careful geometric argument, which heavily relies on convexity, a suitable affine transformation, and the variational method (i.e., the intermediate value theorem) to prove that every convex body in the plane admits a good pair of directions.

1.1 Further Related Work

When the point set P is infinite, one may further distinguish between the **continuous** setting where $P = \mathbb{R}^d$ (also known as the **piercing problem**) and the **discrete** setting where P is a discrete subset of $\mathbb{R}^d$ (for example, $P = \mathbb{Z}^d$).

Continuous Setting. In the geometric setting, the duality between the *Hitting Set* problem and the *Set Cover* problem only holds when the objects are translates of a convex body [5, Theorem 2]. Hence the results obtained for the *Set Cover* problem for translates of a convex body also hold for the *Hitting Set* problem. Charikar et al. [3] studied the *Online Set Cover* problem for translates of a ball. They proposed an algorithm with a competitive ratio $O(2^d d \log d)$. They also proved $\Omega(\log d/\log\log\log d)$ as the deterministic lower bound of the competitive ratio for this problem. Dumitrescu et al. [9] improved the bounds on the competitive ratio for translates of a ball, establishing an upper bound of $O(1.321^d)$ and a lower bound of $\Omega(d+1)$. For translates of a centrally symmetric convex body, they proved that the competitive ratio of every deterministic algorithm is at least $I(s)$, where $I(s)$ is the illumination number of the object s[1]. For translates of an axis-aligned hypercube in $\mathbb{R}^d$, Dumitrescu and Tóth [11] proved that the competitive ratio of any deterministic algorithm for *Online Set Cover* is at least 2^d. Later, De et al. [5] studied the *Online Hitting Set* problem for α-fat objects in $\mathbb{R}^d$ with diameters in $[1, M]$ and designed a deterministic algorithm with competitive ratio $O\left((2+\frac{2}{\alpha})^d \log M\right)$. For hitting axis-aligned homothetic hypercubes with side lengths in $[1, M]$, they gave a deterministic algorithm with competitive ratio at most $3^d\lceil \log_2 M\rceil + 2^d$. They also proved a $\Omega(d \log M + 2^d)$ lower bound for the problem of hitting homothetic hypercubes in $\mathbb{R}^d$, with side lengths in the interval $[1, M]$.

Discrete Setting. De and Singh [7, 8] studied a variant of this problem where $P = \mathbb{Z}^d$ and $\mathcal{C}$ consists of translates of a ball or an axis-aligned hypercube in $\mathbb{R}^d$. For translates of an axis-aligned hypercube, they showed that there is a randomized algorithm with an expected competitive ratio of $O(d^2)$ and also proved that every deterministic algorithm has a competitive ratio of at least $d+1$. For translates of a ball in $\mathbb{R}^d$, they proposed a deterministic algorithm having a competitive ratio of $O(d^4)$ and proved that every deterministic algorithm has a competitive ratio of at least $d+1$, for $d \leq 3$. Recently, Alefkhani et al. [1] considered the variant where $P = (0, N)^d \cap \mathbb{Z}^d$ and $\mathcal{C}$ is a family of α-fat objects in $(0, N)^d$, for some constant $\alpha > 0$. They proposed a deterministic algorithm with a competitive ratio of at most $(\frac{4}{\alpha}+1)^{2d} \log N$, and proved that the competitive ratio of every deterministic algorithm is $\Omega\left(\frac{\log N}{1+\log\alpha}\right)$. Very recently, De et al. [6] improved both the upper and lower bounds of Alefkhani et al. [1]. They considered the case where $P = \mathbb{Z}^d$ and $\mathcal{C}$ is a family of α-fat objects having diameters in $[1, M]$, for some constant $\alpha > 0$. They proposed a deterministic algorithm with competitive ratio $O((\frac{2}{\alpha})^d \log M)$, and established that the competitive ratio of any randomized algorithm is $\Omega(d \log M)$.

2 Bottomless Rectangles and Integer Points

We present an $O(\log N)$-competitive algorithm for the *Online Hitting Set* problem where P is a subset of an $N \times N$ section of the integer lattice, and the objects are *bottomless rectangles* (Section 2.1); and then generalize the algorithm for the same point set but with objects that have the *lowest-point property* (Section 2.2).

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1 The **illumination number** of an object s, denoted by $I(s)$, is the minimum number of smaller homothetic copies of s (λs, where $\lambda \in (0, 1)$) whose union contains s.

2.1 Bottomless Rectangles

In this section we present an $O(\log N)$-competitive algorithm for the *Online Hitting Set* problem where P is a subset of the integer lattice with nonnegative coordinates less than N, that is, $P \subset [0,N)^2 \cap \mathbb{Z}^2$; and the objects are bottomless rectangles. **Bottomless rectangles** are of the form $r_i = [a_i, b_i) \times [0, c_i)$, where $0 \le a_i < b_i \le N$ and $0 \le c_i \le N$. Note that there are only $O(N^3)$ combinatorially different rectangles w.r.t. P, so the general result by Alon et al. [2] gives an algorithm for the online hitting set with competitive ratio $O(\log^2 N)$. In this section, we present an $O(\log N)$-competitive algorithm, which is the best possible (a matching lower bound follows from the lower bound for the *Online Hitting Set* problem for intervals in one-dimension [12]).

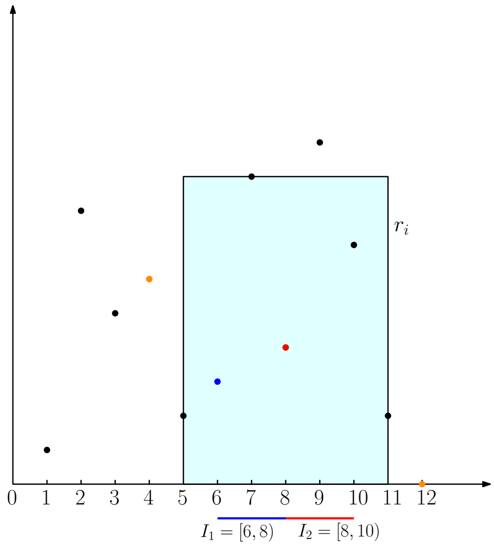
Preliminaries. We need some preparation before we can present the algorithm. We may assume w.l.o.g. that N is a power of 2, and every bottomless rectangle $r_i = [a_i, b_i) \times [0, c_i)$ is given with integer parameters a_i, b_i, and c_i. An interval I is **canonical** if it is of the form $I = \big[q2^j, (q+1)2^j\big)$ for some integers $q, j \ge 0$. For a canonical interval $I = \big[q2^j, (q+1)2^j\big)$, we also define the **left neighbor** $L(I) = \big[(q-1)2^j, q2^j\big)$ and the **right neighbor** $R(I) = \big[(q+1)2^j, (q+2)2^j\big)$. For every canonical interval I, if $(I \times [0,N)) \cap P \neq \emptyset$, then let $p(I)$ denote a **lowest-point** in $(I \times [0,N)) \cap P \neq \emptyset$ (that is, a point with minimum y-coordinate; ties are broken arbitrarily). If $(I \times [0,N)) \cap P = \emptyset$, then $p(I)$ is undefined.

For every interval $[a,b)$ with nonnegative integer endpoints, we define a **canonical partition**, i.e., a partition of $[a,b)$ into canonical intervals. This partition is standard – we walk through some of the technical details because we need them for our algorithm and its analysis. Let $j \ge 0$ be the largest integer such that $q2^j \in (a,b)$, for some $q \in \mathbb{Z}$. (Note that $q \in \mathbb{Z}$ is unique. Indeed, suppose that q is not unique, say $q2^j, (q+1)2^j \in (a,b)$. Since q or $q+1$ is even, then $q/2$ or $(q+1)/2$ is an integer. Now, we have $\frac{q}{2}2^{j+1}$ or $\frac{q+1}{2}\,2^{j+1} \in (a,b)$, which contradicts the maximality of j.) We call the integer $s_{[a,b)} := q2^j$ the **splitting point** of $[a,b)$. We can partition a given interval $[a,b)$ into canonical intervals as follows. If $[a,b)$ is not canonical, find its splitting point $s = s_{[a,b)}$, partition it into two intervals $[a,b) = [a,s) \cup [s,b)$, and recurse on $[a,s)$ and $[s,b)$. For example, the splitting point of interval $[5,11)$ is 8, and its canonical partition is $[5,11) = [5,6) \cup [6,8) \cup [8,10) \cup [10,11)$; see Figure 1a) for an illustration.

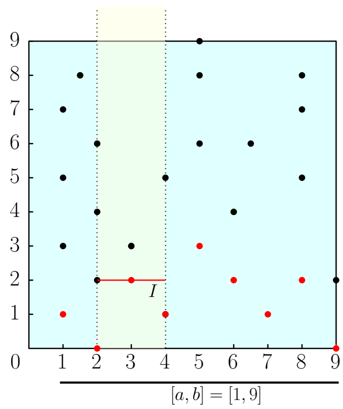
Note also that in the canonical partition of $[a,s)$ (resp., $[s,b)$), there is at most one interval of each size, where the possible sizes are powers of 2 between 1 and $s-a$ (resp., $b-s$). Specifically, if I is in the canonical partition of $[a,s)$, then its left neighbor $L(I)$ is not contained in $[a,b)$, consequently $a \in \overline{L(I)}$, where $\overline{L(I)}$ is the closure of $L(I)$. Similarly, if I is in the canonical partition of $[s,b)$, then $b \in R(I)$.

Online algorithm ALG for bottomless rectangles. We can now present our online algorithm. We maintain a hitting set $H_i \subseteq P$, which is initially empty: $H_0 = \emptyset$. When the i-th bottomless rectangle $r_i = [a_i, b_i) \times [0, c_i)$ arrives, initialize $H_i := H_{i-1}$. If $r_i \cap H_i \neq \emptyset$, then do not add any new points to H_i. Otherwise, we may assume that $r_i \cap H_i = \emptyset$. Compute the splitting point s_i of $[a_i, b_i)$, and the canonical partitions $\mathcal{A}_i$ and $\mathcal{B}_i$ of $[a_i, s_i)$ and $[s_i, b_i)$, respectively. If $([a_i, s_i) \times [0, c_i)) \cap P \neq \emptyset$, then find the largest canonical interval $I \in \mathcal{A}_i$ such that $p(I) \in r_i$, and put $H_i := H_i \cup \{p(I)\}$. Similarly, if $([s_i, b_i) \times [0, c_i)) \cap P \neq \emptyset$, then find the largest interval $I \in \mathcal{B}_i$ such that $p(I) \in r_i$, and put $H_i := H_i \cup \{p(I)\}$. Overall, we add at most two new points to H_i in step i.

Competitive analysis. We now prove that ALG is $O(\log N)$-competitive.



(a)



(b)

Figure 1 (a) When the ith bottomless rectangle $r_i = [5, 11) \times [0, c_i)$ arrives, suppose that the hitting set H_i contains the orange points, and $r_i \cap H_i = \emptyset$. The splitting point of $[5, 11)$ is 8, with canonical partitions $\mathcal{A}_i = [5, 6) \cup [6, 8)$ and $\mathcal{B}_i = [8, 10) \cup [10, 11)$, respectively. Here, $I_1 = [6, 8) \subset \mathcal{A}_i$ and $I_2 = [8, 10) \subset \mathcal{B}_i$ are the largest canonical intervals in $\mathcal{A}_i$ and $\mathcal{B}_i$, respectively. The blue (resp., red) point is the lowest-point in $I_1 \times [0, N) \cap P$ (resp., $I_2 \times [0, N) \cap P$). We add both points to H_i. (b) The black and red colored points denote the set P, while the red colored points denote the set S. The yellow strip denotes $I \times \mathbb{R}$.

▶ **Theorem 1.** *For the Online Hitting Set problem for a point set $P \subseteq [0, N)^2 \cap \mathbb{Z}^2$ and a sequence of bottomless rectangles, the online algorithm* ALG *has competitive ratio $O(\log N)$.*

Proof. Let $\mathcal{C}$ be a sequence of bottomless rectangles. Let H and OPT be the hitting set returned by the online algorithm ALG and an (offline) minimum hitting set of $\mathcal{C}$, respectively. For a point $p \in$ OPT, let $\mathcal{C}_p$ be the subsequence of bottomless rectangles that contain p. It is enough to show that for every $p \in$ OPT, our algorithm adds $O(\log N)$ points to H in response to the objects in $\mathcal{C}_p$.

Let $p \in$ OPT, with coordinates $p = (p_x, p_y)$; and let $r_1, \ldots, r_m$ be a sequence of bottomless rectangles in $\mathcal{C}_p$ for which our algorithm adds new points to the hitting set. We show that $m = O(\log N)$. We can distinguish between two types of rectangles $r_i = [a_i, b_i) \times [0, c_i)$ depending on whether the x-coordinate p_x of p is on the left or right of the splitting point s_i of $[a_i, b_i)$: namely, $p_x < s_i$ or $s_i \leq p_x$. We analyze the two types separately (the two cases are analogous).

Assume w.l.o.g. that $p_x < s_i$ for $i = 1, \ldots, m$. This means that $p \in [a_i, s_i) \times [0, c_i)$, and so ALG adds the hitting point $p(I)$ for exactly one interval $I \in \mathcal{A}_i$. Suppose that the algorithm adds the hitting point $p(I)$ for $I \in \mathcal{A}_i$. Then I is the largest (hence rightmost) canonical interval in $\mathcal{A}_i$ such that $(I \times [0, c_i)) \cap P \neq \emptyset$. Recall that $a_i \in \overline{L(I)}$, where $L(I)$ is the left neighbor of the canonical interval I. This implies that $p_x \in L(I) \cup I$. That is, either I or its left neighbor $L(I)$ contains p_x. Note that p_x is contained in $\log N$ canonical intervals (one for each possible size), and each of these canonical intervals has a unique right neighbor. Consequently, I is one of at most $2 \log N$ canonical intervals under the assumption that $p_x < s_i$ for all $i = 1, \ldots, m$. This proves that $m \leq 4 \log N$. ◀

2.2 Objects with the lowest-point property

In this section, we generalize Theorem 1 to a broader class of objects. Similarly to Section 2.1, let $P \subseteq [0,N)^2 \cap \mathbb{Z}^2$. For a set $S \subseteq P$, the span of S, denoted $\mathrm{span}(S)$ is the smallest interval $[a,b)$ with integer endpoints $a,b \in \mathbb{Z}$ such that $S \subset [a,b) \times \mathbb{R}$. An object $S \subseteq P$ has the **lowest-point property** if for every point $s = (s_x, s_y)$ in S and every interval $I \subset \mathrm{span}(S)$ that contains s_x, the object S contains all points in $P \cap (I \times \mathbb{R})$ with the minimum y-coordinates. For an illustration of set S see Figure 1b. Note, in particular, that every bottomless rectangle $r_i = [a_i, b_i) \times [0, c_i)$ has the lowest-point property: Indeed, if $s_x \in I \subset [a_i, b_i)$, then $I \times [0, s_y] \subset r_i$.

Our online hitting set algorithm and its analysis readily generalize when the objects have the lowest-point property. Let $\mathcal{C} = (S_1, \ldots, S_m)$ be a sequence of objects with the lowest-point property.

Online algorithm ALG_0 for objects with the lowest-point property. We maintain a hitting set $H_i \subseteq P$, which is initially empty: $H_0 = \emptyset$. When set S_i arrives, initialize $H_i := H_{i-1}$. If $S_i \cap H_i \neq \emptyset$, then do not add any new points to H_i. Suppose that $S_i \cap H_i = \emptyset$. Let $[a_i, b_i) = \mathrm{span}(S_i)$. Compute the splitting point s_i of $[a_i, b_i)$, and the canonical partitions $\mathcal{A}_i$ and $\mathcal{B}_i$ of $[a_i, s_i)$ and $[s_i, b_i)$, respectively. If $([a_i, s_i) \times \mathbb{R}) \cap S_i \cap P \neq \emptyset$, then find the largest interval $I \in \mathcal{A}_i$ such that $p(I) \in S_i$, and put $H_i := H_i \cup \{p(I)\}$. Similarly, if $([s_i, b_i) \times \mathbb{R}) \cap S_i \cap P \neq \emptyset$, then find the largest interval $I \in \mathcal{B}_i$ such that $p(I) \in S_i$, and put $H_i := H_i \cup \{p(I)\}$. Overall, we add at most two new points to H_i in step i.

Correctness and competitive analysis. When ALG_0 adds a points $p(I)$ to H_i in step i, the lowest-point property ensures that $p(I) \in S_i$. Therefore, ALG_0 maintains that H_i is a hitting set for $\{S_1, \ldots, S_i\}$, proving the correctness of ALG_0. We now show that ALG_0 is $O(\log N)$-competitive.

▶ **Theorem 2.** *For the Online Hitting Set problem for a point set $P \subset [0,N]^2 \cap \mathbb{Z}^2$ and a sequence $\mathcal{C} = (S_1, \ldots, S_m)$ of objects with the lowest-point property, algorithm ALG_0 has competitive ratio $O(\log N)$.*

Proof. Let $\mathcal{C}$ be a sequence of objects with the lowest-point property. Let H and OPT be the hitting set returned by the online algorithm ALG_0 and an (offline) minimum hitting set of $\mathcal{C}$, respectively. For each $p \in \mathsf{OPT}$, let $\mathcal{C}_p$ the subsequence of sets in $\mathcal{C}$ that contain p. It is enough to show that for every $p \in \mathsf{OPT}$, our algorithm adds $O(\log N)$ points to H in response to the objects in $\mathcal{C}_p$.

Let $p \in \mathsf{OPT}$, with coordinates $p = (p_x, p_y)$; and let $S_1, \ldots, S_m$ be a sequence of sets in $\mathcal{C}_p$ for which our algorithm adds new points to the hitting set. We show that $m = O(\log N)$. We can distinguish between two types of sets S_i depending on whether the x-coordinate p_x of p is to the left or right of the splitting point s_i: namely, $p_x < s_i$ or $s_i \leq p_x$. We analyze the two types separately (the two cases are analogous).

Assume w.l.o.g. that $p_x < s_i$ for $i = 1, \ldots, m$. This means that $p \in [a_i, s_i) \times \mathbb{R}$, and so ALG adds the hitting point $p(I)$ for exactly one interval $I \in \mathcal{A}_i$. Suppose that the algorithm adds the hitting point $p(I)$ for $I \in \mathcal{A}_i$. Then I is the largest (hence rightmost) canonical interval in $\mathcal{A}_i$ such that $(I \times \mathbb{R}) \cap P \neq \emptyset$. Recall that $a_i \in L(I)$, where $L(I)$ is the left neighbor of the canonical interval I. This implies that $p_x \in L(I) \cup I$. That is, either I or its left neighbor $L(I)$ contains p_x. Note that p_x is contained in $\log N$ canonical intervals (one for each possible size), and each of these canonical intervals has a unique right neighbor. Consequently, I is one of at most $2 \log N$ canonical intervals under the assumption that $p_x < s_i$ for all $i = 1, \ldots, m$. This proves that $m \leq 4 \log N$. ◀

3 Disks in the Plane: Separated Setting

In this section, we consider the *Online Hitting Set* problem in the plane, where P is a finite set above the x-axis (given in advance); and $\mathcal{C}$ consists of disks of arbitrary radii with centers located on or below the x-axis (arriving one-by-one). Note that the disks in $\mathcal{C}$ do not necessarily have the lowest-point property; see Figure 2.

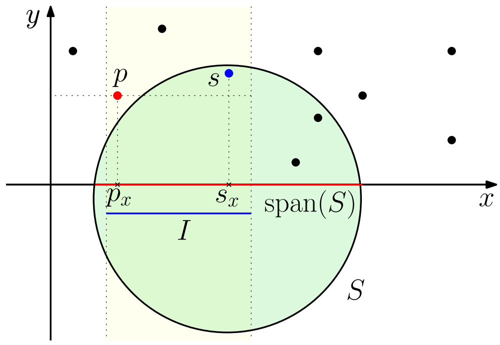


Figure 2 Disk S with center below the x-axis does not have the lowest-point property: We have $s \in S$, and the interval $I \subset \text{span}(S)$ contains s_x, but S does not contain the point $p \in P$ with minimum y-coordinate in the strip $I \times \mathbb{R}$.

Disk hulls for a point set w.r.t. disks and its properties. The unit disk hull of a point set was introduced by Dumitrescu et al. [10] as an analogue of the convex hull. Recall that the convex hull $\text{conv}(P)$ of a point set $P \subset \mathbb{R}^2$ is the smallest convex set in the plane that contains P. Equivalently, it is the intersection of all closed half-planes that contain P; it can be computed by the classical "rotating calipers" algorithm, where we continuously rotate a line ℓ around P while P remains in one closed half-plane bounded by ℓ. Intuitively, we obtain the unit disk hull of P by rolling a unit disk, with center on or below the x-axis, around P. We generalize this notion to disks of any fixed radius $t > 0$.

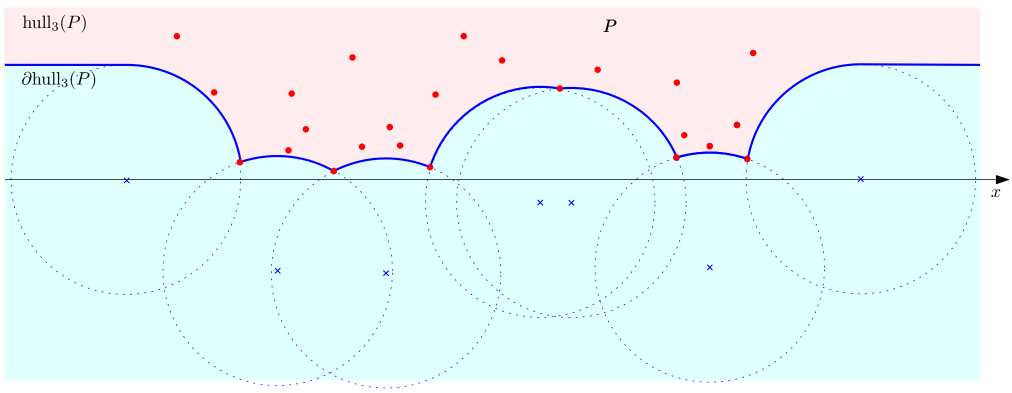


Figure 3 A point set P (red) and region $\text{hull}_3(P)$ (pink). The boundary $\partial\text{hull}_3(P)$ is composed of horizontal lines and circular arcs.

▶ **Definition 3.** *Let $P \subset \mathbb{R}^2$ be a finite set of points above the x-axis and let $t > 0$. Let $\mathcal{D}_t$ be the set of all disks of radius t with centers on or below the x-axis. Let $M_t(P)$ be the union of all disks $D \in \mathcal{D}_t$ such that $P \cap \text{int}(D) = \emptyset$. Now, we define the **t-hull** of P as $\text{hull}_t(P) = \mathbb{R}^2 \setminus \text{int}(M_t(P))$. The boundary of $\text{hull}_t(P)$ is denoted by $\partial\text{hull}_t(P)$; for an illustration see Figure 3.*

Dumitrescu et al. [10, Lemma 4] proved that $\partial\text{hull}_t(P)$ is x-monotone[2] for any $t > 0$, and established other properties, which were used by Conroy and Tóth [4], as well.

▶ **Lemma 4** (Dumitrescu et al. [10]). *For a finite set $P \subset \mathbb{R}^2$ above the x-axis and $t > 0$, the following holds:*
1. *$\partial\text{hull}_t(P)$ lies above the x-axis;*
2. *every vertical line intersects $\partial\text{hull}_t(P)$ in one point, thus $\partial\text{hull}_t(P)$ is an x-monotone curve;*
3. *for every disk $D \in \mathcal{D}_t$, the intersection $D \cap (\partial\text{hull}_t(P))$ is connected (possibly empty);*
4. *for every disk $D \in \mathcal{D}_t$, if $P \cap D \neq \emptyset$, then $P \cap D$ contains a point in $\partial\text{hull}_t(P)$.*

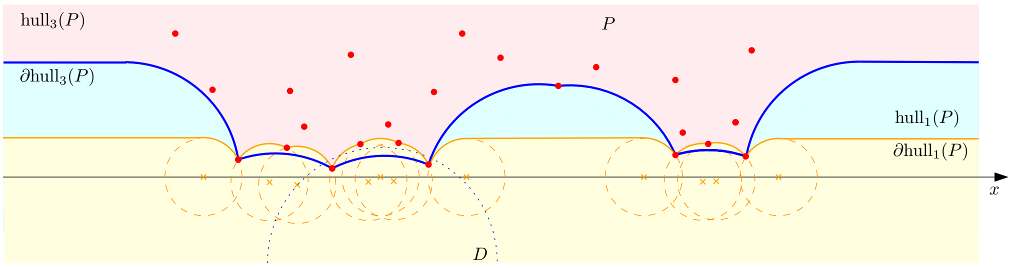


Figure 4 A point set P (red), $\text{hull}_3(P)$ (pink), and $\text{hull}_1(P)$ (light blue or pink). A disk $D \in \mathcal{D}_3$ of radius 3 (dotted blue), where the intersection $D \cap (\partial\text{hull}_1(P))$ has two components.

Since we consider the case of disks with bounded radii, for our purposes, we need to compare two disk hulls for the same point set P w.r.t. different radii; see Figure 4. We start with an easy observation.

▶ **Lemma 5.** *Let γ_1 and γ_2 be circular arcs lying entirely above the x-axis, such that γ_1 and γ_2 are arcs of circles C_1 and C_2, resp., of radii r_1 and r_2, with centers on or below the x-axis.*
1. *Then both γ_1 and γ_2 are x-monotone and concave curves.*
2. *Furthermore, if points $p_1, p_2 \in \mathbb{R}^2$ are contained in both γ_1 and γ_2, and $r_1 < r_2$, then γ_1 lies above γ_2 (i.e., for every vertical line L that separates p_1 and p_2, point $\gamma_1 \cap L$ lies above point $\gamma_2 \cap L$).*

Proof. (1) For every $i \in \{1, 2\}$, the center of C_i is below the x-axis, and so the leftmost and rightmost points of C_1 are also below the x-axis. The leftmost and rightmost points partition C_i into two halfcircles, one above the center and one below the center. Both halfcircles are x-monotone: The lower halfcircle is convex curve and the upper halfcircle is concave. Since γ_i lies entirely above the x-axis, it is contained in the upper halfcirlce, which is x-monotone and concave.
(2) The locus of centers of circles that contain both p_1 and p_2 is the orthogonal bisector of the line segment p_1p_2, that we denote by $(p_1p_2)^\perp$. Note that p_1p_2 is not vertical (or else $(p_1p_2)^\perp$ would be a horizontal line above the x-axis, and the centers of C_1 and C_2 would also be above the x-axis). As the center a circle containing p_1 and p_2 continuously moves from the center of C_1 down to $y = -\infty$, the circular arc between p_1 and p_2 deforms continuously from γ_1 to the line segment p_1p_2. Since γ_1 concave, it lies above the segment p_1p_2. Since $r_1 < r_2$, the arc γ_2 lies between the arc γ_1 and the segment p_1 and p_2. Consequently, then γ_2 lies below γ_1, as claimed. ◀

[2] A curve in the plane is x-monotone if every vertical line intersects it at most once.

▶ **Lemma 6.** *For every finite set $P \subset \mathbb{R}^2$ above the x-axis, the following holds:*

1. *If $0 < s < t$, then for every disk $D \in \mathcal{D}_s$ of radius s, the intersection $D \cap (\partial\text{hull}_t(P))$ is connected (possibly empty).*
2. *Suppose that $p \in P$ lies on the curve $\partial\text{hull}_t(P)$ for some $t > 0$. Then there is a radius $r_p \in (0, t)$ such that p is also on $\partial\text{hull}_s(P)$ for all $s \in [r_p, t]$, but p is below $\partial\text{hull}_s(P)$ for all $s \in [0, r_p)$.*

Proof. (1) Let $D \in \mathcal{D}_s$. Suppose, to the contrary, that the intersection $D \cap (\partial\text{hull}_t(P))$ has two or more components. By Lemma 4(2), the x-coordinates of the components form disjoint intervals, and the components have a natural left-to-right ordering. Let q_1 be the rightmost point in the first component, and let q_2 be the leftmost point in the second component. Clearly $q_1, q_2 \in \partial D$. Let q' be an arbitrary point in $\partial\text{hull}(A)$ between q_1 and q_2. Then q' lies on the boundary of some disk D' of radius t whose center is below the x-axis, and whose interior is disjoint from P. In particular, neither q_1 nor q_2 is in the interior of D'. Since the center of D' is below the x-axis, $\partial D'$ contains two interior-disjoint circular arcs between q and the x-axis; and both arcs must cross ∂D. We have found two intersection points $p_1, p_2 \in \partial D \cap \partial D'$ above the x-axis. Furthermore, between p_1 and p_2, the circular arc ∂D lies above the circular arc $\partial D'$, contradicting Lemma 5(2). This completes the proof of Property 1.
(2) Consider a point $p \in P$ that lies on the curve $\partial\text{hull}_t(P)$ for some $t > 0$. Then there exists a disk $D \in \mathcal{D}_t$ of radius t centered at some point c below the x-axis such that $p \in \partial D$. Let c_1 be the intersection point of the x-axis the line cp, and c_2 the orthogonal projection of p to the x-axis. We describe two continuous motions, where the disk D continuously changes while p is in the circle ∂D and there is no point in P in the interior of D: First, a central dilation from center p continuously moves D to a disk D_1 centered at c_1. Second, the center of D moves from c_1 towards c_2 continuously until its center reaches c_2 or a point c_3 where ∂D contains both p and another point $p' \in P$. Let r_p be radius of D at that time. The continuous motion shows that $p \in \partial\text{hull}_s(P)$ for all $s \in [r_p, t]$, but it is not in $\partial\text{hull}_s(P)$ for all $s < r_p$. ◀

Note that Lemma 6(1) is not symmetric for $s < t$: For a disk $D \in \mathcal{D}_t$ of radius t, the intersection $D \cap (\partial\text{hull}_s(P))$ is not necessarily connected; see Figure 4 for an example.

Reduction. We can reduce the *Online Hitting Set* problem for a finite set $P \subset \mathbb{R}^2$ and disks of bounded radii in the separated setting, to the *Online Hitting Set* problem for a finite subset of integer points and objects with the lowest-point property. We achieve the reduction in two steps:
(1) We choose a subset $Q \subseteq P$ of points that are relevant for a hitting set (Lemma 7); and
(2) we map the points in P into a set of integer points $P' \subset [0, n]^2 \cap \mathbb{Z}^2$ (Lemma 8).

For a finite point set P in the plane above the x-axis, let $Q = Q(P)$ be the set of points $p \in P$ such that $p \in \partial\text{hull}_t(P)$ for some $t > 0$.

▶ **Lemma 7.** *For a finite point set P in the plane above the x-axis, $Q = Q(P)$ has the following property: For every disk D centered below the x-axis, if $D \cap P \neq \emptyset$, then $D \cap Q \neq \emptyset$.*

Proof. Let D be a disk of radius $t > 0$ centered below the x-axis. By Lemma 4(4), $D \cap P$ contains a point in $\partial\text{hull}_t(P)$. By the definition of Q, this point is in Q. ◀

We may assume that the points in P have distinct x-coordinates (if two or more points in P have the same x-coordinate, w.l.o.g. a minimum hitting set would contain only the point with the smallest y-coordinate). Sort P by increasing x-coordinates such that $P = \{p_0, \ldots, p_{n-1}\}$.

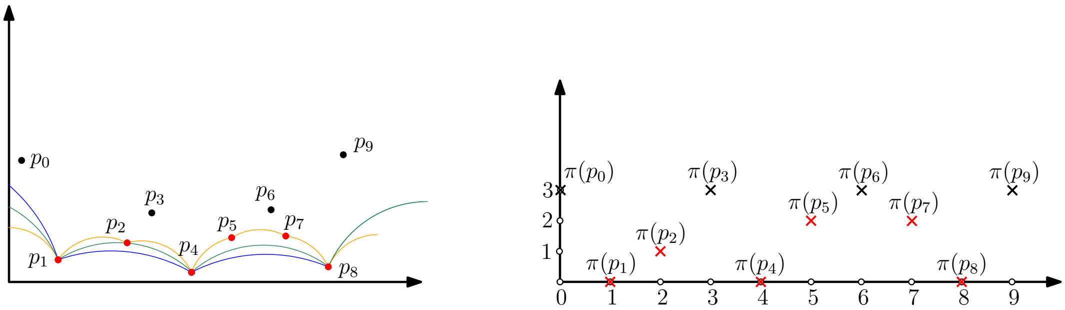


Figure 5 Description of the function π. Left: $\partial\text{hull}_1(P)$ is orange, $\partial\text{hull}_2(P)$ is green, and $\partial\text{hull}_3(P)$ is blue. Right: The grid points $\pi(p_0), \ldots, \pi(p_9)$ corresponding to $p_0, \ldots, p_9$.

For every point $q \in Q$, let $t(q) > 0$ be the maximum radius such that $q \in \partial\text{hull}_{t(q)}(P)$. Consider the set of radii $T = \{t(q) : q \in Q\}$. Sort the radii in T in decreasing order as $t_0 > t_1 > \ldots > t_{|T|-1}$. We can now define the function $\pi : P \to [0, n)^2 \cap \mathbb{Z}^2$. For every $p_i \in Q$, let $\pi(p_i) = (i, j)$ if and only if $t(p_i) = t_j$, that is, the first coordinate of $\pi(p_i)$ corresponds to the index i of p_i (the x-order of all points in Q), and the second coordinate of $\pi(p_i)$ corresponds to index j of the radius $t_j = t(p_i)$. For every $p_i \in P \setminus Q$, let $\pi(p_i) = (i, |T|)$; see Figure 5 for an illustration. Finally, let $P' = \pi(P) = \{\pi(p_i) : p_i \in P\}$ and $Q' = \pi(Q) = \{\pi(p_i) : p_i \in Q\}$. Note that the points in $P' \setminus Q'$ lie above all points in Q'. Since π is injective, then it is a bijection between P and P'. Note also that $|T| \le |Q| \le |P| = n$, consequently $P' \subset [0, n]^2 \cap \mathbb{Z}^2$.

▶ **Lemma 8.** *For a set P of n points in the plane above the x-axis and for every disk D centered below the x-axis, the set $\pi(D \cap P)$ has the lowest-point property.*

Proof. Let D be a disk centered below the x-axis. We rephrase the lowest-point property in terms of $D \cap P$. Recall that the points in P are sorted by x-coordinates. Suppose that $s = (s_x, s_y)$ is in $D \cap P$ and $s_x \in I \subset \text{span}(D \cap P)$. Consider the point sets $P(I) := \{p = (p_x, p_y) \in D \cap P : p_x \in I\}$. By Lemma 7, we know that $D \cap Q \neq \emptyset$; let t be the largest radius in T such that $Q(I) \cap \partial\text{hull}_t(P) \neq \emptyset$. We need to show that D contains all points in $P(I) \cap \partial\text{hull}_t(P)$.

Let q_{left} and $q_{\text{right}} \in P(I)$, resp., be the leftmost and rightmost points in $P(I) \cap Q$; and let L_{left} and L_{right} be the vertical lines through q_{left} and q_{right}. By the definition of Q, we have $q_{\text{left}} \in \partial\text{hull}_{t(q_{\text{left}})}(P)$ and $q_{\text{right}} \in \partial\text{hull}_{t(q_{\text{right}})}(P)$, and $t \ge \max\{t(q_{\text{left}}), t(q_{\text{right}})\}$ by the definition of t. Consequently, the intersection point $\ell := L_{\text{left}} \cap \partial\text{hull}_t(P)$ lies at or below q_{left}, the intersection point $r := L_{\text{right}} \cap \partial\text{hull}_t(P)$ lies at or below q_{right}. Since $q_{\text{left}}, q_{\text{right}} \in D$, then D contains both ℓ and r. We know that $\partial\text{hull}_t(P)$ is an x-monotone curve by Lemma 4(2), and $D \cap \partial\text{hull}_t(P)$ is connected by Lemma 6(2). Since D contains both ℓ and r, then D contains the sub-curve of $\partial\text{hull}_t(P)$ between r and ℓ. Since all points in $P(I)$ are between the vertical lines L_{left} and L_{right}, then D contains all points in $P(I) \cap \partial\text{hull}_t(P)$, as required. ◀

Online algorithm for disks in the separated setting. We can now complete the reduction.

▶ **Theorem 9.** *For the Online Hitting Set problem for a set $P \subset \mathbb{R}^2$ of n points above the x-axis and disks centered below the x-axis, there is an $O(\log n)$-competitive algorithm.*

Proof. We are given a set $P \subset \mathbb{R}^2$ of n points above the x-axis, and we receive a sequence $\mathcal{C} = (D_1, \ldots, D_m)$ of disks centered on or below the x-axis in an online fashion. Let $\mathsf{OPT} \subseteq P$ be a minimum hitting set for $\mathcal{C}$.

Initially, we compute the set $P' \subset [0,n]^2 \cap \mathbb{Z}^2$ as defined above Lemma 8. When a disk D_i arrives, we compute the set $S_i = \pi(D_i \cap P)$, which has the lowest-point property by Lemma 8. The bijection π maps OPT to a set $\mathsf{OPT}' = \pi(\mathsf{OPT}) \subseteq P'$, where $|\mathsf{OPT}| = |\mathsf{OPT}'|$. Here, OPT' is a hitting set for the sets $\mathcal{C}' = (S_1, \dots, S_m)$.

We run the online algorithm ALG_0 described in Section 2.2 for the point set P' and the sequence $\mathcal{C}'$ of sets. By Theorem 1, ALG returns a hitting set $H' \subseteq P'$ of size $|\mathsf{OPT}'| \cdot O(\log n)$. By Lemma 7, $H = \pi^{-1}(H') \subset P$ is a hitting set for $\mathcal{C}$, and its size is bounded by $|H| = |H'| \leq |\mathsf{OPT}'| \cdot O(\log n) = |\mathsf{OPT}| \cdot O(\log n)$, as required. ◀

4 Disks of Bounded Radii: General Setting

In this section, we consider the *Online Hitting Set* problem, where P is a finite set (given in advance) in the plane; and the objects are disks with radii in the interval $[1, M)$, where $M > 1$ is a constant.

Distinguishing layers of disks, according to their radii. We partition the disks of radii in the interval $[1, M)$ into $\lfloor \log M \rfloor + 1$ layers as follows: for each $j \in \{0, 1, \dots, \lfloor \log M \rfloor\}$, let **layer** L_j be the set of disks of radii in the interval $[2^j, 2^{j+1})$. The index of each layer L_j is denoted by j.

Tiling of the plane for each layer index j. For every $j \in \{0, 1, \dots, \lfloor \log M \rfloor\}$, let $\Lambda_j = \{\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 : (\alpha_1, \alpha_2) \in \mathbb{Z}^2\}$ be a two-dimensional lattice spanned by vectors $\mathbf{v}_1 = 2^{j-1/2}\mathbf{e}_1$ and $\mathbf{v}_2 = 2^{j-1/2}\mathbf{e}_2$, where $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ are the standard basis vectors. Let $\tau_j = \left[0, 2^{j-1/2}\right]^2$ be a square of side length $2^{j-1/2}$ with lower-left corner at the origin. Translates of τ_j (**tiles**), with translation vectors in the lattice Λ_j, form the **tiling** $\mathcal{T}_j$. Let $\mathcal{L}_j$ denote the set of axis-parallel lines spanned by the sides of the tiles in $\mathcal{T}_j$.

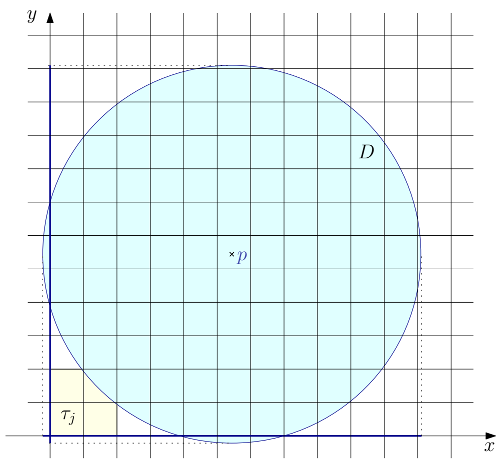


Figure 6 A section of the tiling $\mathcal{T}_j$, the tile τ_j of side length $2^{j-1/2}$, and a disk D of radius 2^{j+2}.

We observe two key properties of the construction of layers and the tilings.

▶ **Observation 10.** *For every $j \in \{0, 1, \dots, \lfloor \log M \rfloor\}$, if $\sigma \in L_j$ and the center of σ is in a tile $\tau \in \mathcal{T}_j$, then $\tau \subset \sigma$; see Figure 6.*

Proof. Since $\sigma \in L_j$, the radius of the disk σ is in at least 2^j. The tile τ is a translate of $\tau_j = [0, 2^{j-1/2}]^2$, and so its diameter is $\sqrt{2} \cdot 2^{j-1/2} = 2^j$. If the center c of σ is in S, then every $p \in \tau$ is within distance 2^j from c, which implies that $\tau \subset \sigma$. ◀

▶ **Observation 11.** *For every $j \in \{0, 1, \ldots, \lfloor \log M \rfloor\}$, every disk D of radius at most 2^{j+2} intersects at most 24 lines in $\mathcal{L}_j$: at most 12 horizontal and 12 vertical lines.*

Proof. Let D be a disk of radius 2^{j+2}; see Figure 6. The orthogonal projection of D to the x-axis (resp., y-axis) is an interval of length at most 2^{j+3}. Since the distance between any two consecutive vertical (resp., horizontal) lines in $\mathcal{L}_j$ is $2^{j-1/2}$, then D intersects at most $\lceil 2^{j+3}/2^{j-1/2} \rceil = \lceil 2^{7/2} \rceil = 12$ horizontal and at most 12 vertical lines in $\mathcal{L}_j$. ◀

Subproblem for a directed line L. For a directed line L, we denote by L^- and L^+ the closed half-plane on the left and right of L, respectively. Given a directed line L and the input $(P, \mathcal{C})$ of the *Online Hitting Set* problem, where P is a set of points, and $\mathcal{C}$ is a sequence of disks in the plane, we define a subproblem $(P_L, \mathcal{C}_L)$ as follows: Let $P_L = P \cap L^-$, and let $\mathcal{C}_L$ be the subsequence of disks $\sigma_i \in \mathcal{C}$ such that the center of σ_i is in L^+ and σ_i contains at least one point in P_L. Now for each subproblem $(P_L, \mathcal{C}_L)$, we can run the online algorithm ALG_0 described in Theorem 9, which was developed for the separated setting in Section 3. Let $\mathsf{ALG}_0(L)$ denote the online algorithm, where we run the online algorithm ALG_0 on the subproblem $(P_L, \mathcal{C}_L)$.

Online algorithm. We can now present our online algorithm ALG. In the current algorithm, we use the online algorithm $\mathsf{ALG}_0(L)$ as a subroutine. For each $j \in \mathbb{N} \cup \{0\}$, let layer L_j be the set of disks of radii in the interval $[2^j, 2^{j+1})$. The algorithm maintains a hitting set $H \subseteq P$ for the disks presented so far. Upon the arrival of a new disk σ with radius r, if it is already hit by a point in H, then do nothing. Otherwise, proceed as follows.

- First, find the layer L_j, where $j = \lfloor \log r \rfloor$, in which σ belongs.
- Find the tile $\tau \in \mathcal{T}_j$ that contains the center of σ.
  - If $P \cap \tau \neq \emptyset$, then choose an arbitrary point $p \in P \cap \tau$ and add it to H.
  - Otherwise, for every line $L \in \mathcal{L}_j$ that intersects σ, direct L such that L^+ contains the center of σ, feed the disk σ to the online algorithm $\mathsf{ALG}_0(L)$, and add any new hitting point chosen by $\mathsf{ALG}_0(L)$ to H.

Competitive analysis. We now prove that ALG is $O(\log M \log n)$-competitive.

▶ **Theorem 12.** *For the Online Hitting Set problem for a set P of n points in the plane and a sequence $\mathcal{C} = (\sigma_1, \ldots, \sigma_m)$ of disks of radii in the interval $[1, M]$, the online algorithm ALG has competitive ratio $O(\log M \log n)$.*

Proof. Let $\mathcal{C}$ be a sequence of disks. For each $j \in \{0, 1, \ldots, \lfloor \log M \rfloor\}$, let $\mathcal{C}^j$ be the collection of disks in $\mathcal{C}$ with radii in the interval $\left[2^j, 2^{j+1}\right)$. Let H and OPT, resp., be the hitting set returned by the online algorithm ALG and an (offline) minimum hitting set for $\mathcal{C}$. For every point $p \in \mathsf{OPT}$, let $\mathcal{C}_p$ be the set of disks in $\mathcal{C}$ containing p. For each $j \in \{0, 1, \ldots, \lfloor \log M \rfloor\}$, let $\mathcal{C}_p^j$ be the set of disks in $\mathcal{C}^j$ containing p, i.e., $\mathcal{C}_p^j = \mathcal{C}^j \cap \mathcal{C}_p$. Let $H_p^j \subseteq H$ be the set of points that ALG adds to H in response to hit objects in $\mathcal{C}_p^j$. It is enough to show that for every $j \in \{0, 1, \ldots, \lfloor \log M \rfloor\}$ and $p \in \mathsf{OPT}$, we have $|H_p^j| \leq O(\log n)$.

Let τ be the tile in $\mathcal{T}_j$ that contains p, and let ${\mathcal{C}'}_p^j \subseteq \mathcal{C}_p^j$ be the subset of disks whose centers are located in τ. To hit the first disk $\sigma \in {\mathcal{C}'}_p^j$, our algorithm adds a point from $P \cap \tau$ to H. By Observation 10, any point in $P \cap \tau$ hits σ, as well as any subsequent disks in ${\mathcal{C}'}_p^j$. Our algorithm adds at most 1 point to H to hit all the disks in ${\mathcal{C}'}_p^j$.

It remains to bound the number of points our algorithm adds for disks in $\mathcal{C}_p^j \setminus \mathcal{C}'^j_p$. Notice that a disk D_0 centered at p of radius 2^{j+1} contains all the centers of the disks in $\mathcal{C}_p^j \setminus \mathcal{C}'^j_p$. By the triangle inequality, a disk D centered at p of radius 2^{j+2} contains all disks in $\mathcal{C}_p^j \setminus \mathcal{C}'^j_p$. For any disk $\sigma \in \mathcal{C}_p^j \setminus \mathcal{C}'^j_p$, our algorithm uses algorithm $\mathsf{ALG}_0(L)$ for a line $L \in \mathcal{L}_j$, directed such that L^+ contains the center of σ. According to Observation 11, the disk D intersects at most 24 lines in $\mathcal{L}_j$. However, depending on the location of the center of σ, each line may be used in either direction for $\mathsf{ALG}_0(L)$. As a result, for all disks in $\mathcal{C}_p^j \setminus \mathcal{C}'^j_p$, algorithm $\mathsf{ALG}_0(L)$ is called with at most 48 directed lines L.

For each directed line L, the online algorithm $\mathsf{ALG}_0(L)$ maintains a hitting set $H(L)$ for the disks fed into this algorithm. For the point p, let $H_p^j(L)$ denote the set of points that algorithm $\mathsf{ALG}_0(L)$ adds to $H(L)$ in response to a disk in $\mathcal{C}_p^j \setminus \mathcal{C}'^j_p$ that it receives as input. By Theorem 9, we have $|H_p^j(L)| \le O(\log |\mathcal{C}_p^j \setminus \mathcal{C}'^j_p|) \le O(\log n)$ for every directed line L. This yields $|H_p^j| \le 1 + 48 \cdot O(\log n) = O(\log n)$, as required.

By construction, we have $H = \bigcup_{j=0}^{\lfloor \log M \rfloor} \bigcup_{p \in \mathsf{OPT}} H_p^j$. We have shown that $|H_p^j| = O(\log n)$, for all $j \in \{0, 1, \dots, \lfloor \log M \rfloor\}$ and $p \in \mathsf{OPT}$. Consequently, we obtain $|H| \le \sum_{j=0}^{\lfloor \log M \rfloor} \sum_{p \in \mathsf{OPT}} O(\log n) = (\lfloor \log M \rfloor + 1)\, |\mathsf{OPT}|\, O(\log n) = O(\log M \log n)|\mathsf{OPT}|$. ◀

For disks of radii in $[1, 1+\varepsilon]$ where $\varepsilon > 0$ is constant, Theorem 12 implies the following.

▶ **Corollary 13.** *For the Online Hitting Set problem for a set P of n points in the plane and a sequence $\mathcal{C} = (\sigma_1, \dots, \sigma_m)$ of disks of radii in the interval $[1, 1+\varepsilon]$, where $\varepsilon > 0$ is a constant, the online algorithm ALG is $O(\log n)$-competitive.*

5 Generalization to Positive Homothets of a Convex Body

In this section, we generalize Theorem 12 for positive homothets of an arbitrary convex body C in the plane. A set $C \subset \mathbb{R}^2$ is a **convex body** if it is convex and has a nonempty interior; and it is **centrally symmetric** (w.r.t. the origin) if $C = -C$, where $-C = \{-p : p \in C\}$.

The key components of our $O(\log n)$-competitive algorithm for disks of comparable sizes were an $O(\log n)$-competitive online algorithm in the line-separated setting and a grid tiling that allowed a reduction to the line-separated setting. Specifically, Observation 10 and Observation 11 formulate the two essential properties of a tiling: If a center of disk σ lies in a tile τ, then $\tau \subset \sigma$ (Observation 10); and every disk intersects $O(1)$ grid lines (Observation 11).

We state the main result of this section here. For a detailed explanation and the complete proof, please refer to the full version of the paper.

▶ **Theorem 14** ($\star$). *Given any convex body $\sigma \subset \mathbb{R}^2$ and a parameter $M \ge 1$, there is an online algorithm with a competitive ratio of $O(\log M \log n)$ for the Online Hitting Set problem for a set P of n points in the plane and a sequence $\mathcal{C} = (\sigma_1, \dots, \sigma_m)$ of positive homothets $\sigma_i = a_i \sigma + b_i$, where $a_i \in [1, M]$.*

For positive homothets of a convex object with scaling factor in $[1, 1+\varepsilon]$, where $\varepsilon > 0$ is a constant, Theorem 14, implies the following.

▶ **Corollary 15.** *Given any convex body $\sigma \subset \mathbb{R}^2$ and constant $\varepsilon > 0$, there is an online algorithm of competitive ratio $O(\log n)$ for the Online Hitting Set problem for a set P of n points in the plane and a sequence $\mathcal{C} = (\sigma_1, \dots, \sigma_m)$ of positive homothets $\sigma_i = a_i \sigma + b_i$, where $a_i \in [1, 1+\varepsilon]$.*

A good pair of lines. The key technical tool for the proof of Theorem 14 is Definition 16. Given a convex body C, we first consider an inscribed triangle of the maximum area (see [13]). We then apply an area-preserving (unary) affine transformation to transform C so that this inscribed triangle of the maximum area becomes an equilateral triangle. (This is similar to mapping the minimum enclosing ellipse of C into a circle, or assuming that C is "fat" after a suitable affine transformation.) We may further assume, by scaling, that the inscribed circle of this triangle has a unit diameter.

▶ **Definition 16.** *Let C be a convex body in the plane such that an inscribed triangle of the maximum area is an equilateral triangle T_{in}, and the circle inscribed in T_{in} is a circle of a unit diameter. A pair of lines $\{\ell_1, \ell_2\}$ is a **good pair for C** if they satisfy the following properties:*

1. *The angle between the two lines is bounded from below by $\angle(\ell_1, \ell_2) \geq \pi/15$.*
2. *For $i \in \{1, 2\}$, there exist points $p_i, q_i \in \partial C$ such that the two lines tangent to C parallel to ℓ_i contain p_i and q_i, respectively; furthermore, C contains the disk $B(x, \frac{1}{50})$ of diameter $\frac{1}{25}$ centered at the intersection point $x = p_1q_1 \cap p_2q_2$.*

In the full version, we prove that every convex body C specified in Definition 16 admits a good pair of lines, which can be computed in polynomial time if C is a convex polygon. No attempts were made to optimize the constants $\pi/15$ and $\frac{1}{50}$ in Definition 16.

6 Conclusions and Open Problems

We revisited the *Online Hitting Set* problem for a set of n points in the plane and geometric objects that arrive in an online fashion, such as disks or homothets of a convex body of comparable sizes, or bottomless rectangles in the plane. In all these cases, we designed $O(\log n)$-competitive online algorithms, which is the best possible. It remains an open problem whether our results generalize to 3- or higher dimensions. In fact, no $O(\log n)$-competitive algorithm is currently known for simple geometric objects in 3-space, for example, a set of n points and a sequence of unit balls in $\mathbb{R}^3$; or a set of n points $P \subset [0, n)^3 \cap \mathbb{Z}^3$ and a sequence of axis-aligned cubes in $\mathbb{R}^3$.

Our results provide further evidence that there may exist $O(\log n)$-competitive algorithms for the *Online Hitting Set* problem for n points in $\mathbb{R}^d$ and any sequence of objects $\mathcal{C}$ of bounded VC-dimension – an open problem raised by Even and Smorodinsky [12]; see also [14]. This problem remains open: The current best lower and upper bounds are $\Omega(\log n)$ and $O(\log^2 n)$ [2]. No better bounds are known even in some of the most common geometric range spaces, for example, when P is a subset of the grid $[0, n)^2 \cap \mathbb{Z}^2$ and $\mathcal{C}$ is a sequence of axis-aligned rectangles in the plane; or when P is a set of n points in the plane, and $\mathcal{C}$ is a sequence of disks of arbitrary radii.

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