Covering Simple Orthogonal Polygons with Rectangles

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Abstract

We study the problem of Covering Orthogonal Polygons with Rectangles, focusing on three variants: covering the interior, the boundary, and the corners. While previous work provided constant-factor approximation algorithms for these problems, significant improvements had not been achieved for over two decades. The main contribution of this work is the development of a Polynomial Time Approximation Scheme (PTAS) for both the Boundary Cover and Corner Cover problems on simple polygons, using a local search algorithm. Our work advances the state of the art, improving upon the previous best-known 4-approximation for the Boundary Cover and 2-approximation for the Corner Cover problems.

The technical core of our work lies in proving the existence of planar support graphs for certain geometric hypergraphs defined by the polygon and its containment-maximal rectangles. This structural insight enables the application of the local search framework to achieve the PTAS results. We also demonstrate the limitations of this approach by constructing instances where local search fails for the Interior Cover and certain dual problems, such as the Maximum Antirectangle and Hitting Set problems. Additionally, the methods yield a PTAS for a special case of the Discrete Independent Set problem for rectangles. These results not only settle longstanding open questions but also introduce new techniques that may be of independent interest within computational geometry.

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1 Introduction

Covering orthogonal polygons with the minimum number of rectangles is a fundamental problem in computational geometry, with direct applications in image compression [37], satellite image mosaic selection [46], and access control lists in network routers [4]. Given a polygon whose sides are all axis-aligned, the goal is to cover it exactly with the smallest set of axis-parallel rectangles. This problem, also known as the rectilinear picture compression problem, has been known to be NP-hard for decades, even for polygons without holes [23].

The best-known polynomial time approximation algorithm independent of the number of holes in the polygon has an approximation ratio of $O(\sqrt{\log n})$ due to Kumar and Ramesh [39]. Franzblau gave a $(2 + h)$-approximation algorithm which implies a 2-factor approximation for simple polygons [27], where h is the number of holes.

Variants of the problem have also been studied. The Boundary Cover problem asks for the minimum number of rectangles to cover the boundary of the polygon, while the Corner Cover problem focuses on covering just the corners. In 1981, Chaiken, Kleitman, Saks, and Shearer [15] studied their respective duality gaps, whereas Culberson and Reckhow [23] proved that even the boundary cover problem is NP-hard for hole-free polygons. Furthermore, the problem was proved to be APX-hard when we allow holes in the polygon for the interior cover as well as the boundary cover problem by Berman and DasGupta [11].

Although several articles have been published over the years, especially from the perspective of applications [46, 37] and experiments [31, 34], researchers have failed to make any theoretical improvements for the last two decades.

In this work, we settle the status of the Boundary Cover problem by showing a Polynomial Time Approximation Scheme (PTAS) using Local Search for simple orthogonal polygons. To our knowledge, the best-known approximation factor is 4 since 1997 [11]. See Table 1 for an overview of the known results and our contributions.

Table 1 State of the art for the studied variants of the Polygon Covering problem, where our results are highlighted in bold.

	Without holes	Possibly with holes
Corner	2-apx [11], **PTAS**	NP-h [11], 4-apx [11]
Boundary	NP-h [23], **PTAS**	NP-h [22], 4-apx [11], APX-h [11]
Interior	NP-h [23], 2-apx [27]	NP-h [28], $O(\sqrt{\log n})$-apx [39], APX-h [11]

1.1 Our Contribution

We prove that Local Search yields a PTAS for the Boundary Cover problem on simple orthogonal polygons, improving the previous best-known 4-approximation [11]. We show that the same Local Search approach also gives a PTAS for the Corner Cover problem on simple polygons, surpassing the earlier 2-approximation [11]. Our main technical advance is to show that, for any simple orthogonal polygon, the hypergraph defined by its containment-maximal rectangles and boundary points admits a planar support graph. This structural property is the key to enabling the PTAS results. As a by-product, we obtain a PTAS for a special case of the Discrete Independent Set problem for rectangles, where the rectangles are containment-maximal with respect to a fixed simple orthogonal polygon and the point set lies on the boundary. We also establish the limitations of this Local Search framework: for the Interior Cover problem and for dual problems like Maximum Antirectangle and Hitting Set (with points restricted to the boundary), we construct instances where the Local Search solution size can be arbitrarily far from that of the optimal solution.

In the following, we define a few terms and state our results. Given a simple orthogonal polygon P and a family of containment-maximal rectangles $\mathcal{R}$ contained in P, for every point $p \in \partial P$ we define a hyperedge $\mathcal{R}_p := \{R \in \mathcal{R} \mid p \in R\}$. Thus we define a hypergraph

$(\mathcal{R}, \{\mathcal{R}_p\}_{p\in\partial P})$ where the vertex set is $\mathcal{R}$ and the hyperedges are $\mathcal{R}_p$ for every $p \in \partial P$. For the sake of brevity, we shall denote $(\mathcal{R}, \{\mathcal{R}_p\}_{p\in\partial P})$ as $(\mathcal{R}, \partial P)$. We denote its dual hypergraph by $(\partial P, \mathcal{R})$. Clearly, the Boundary Cover problem is the Set Cover problem on the class of hypergraphs $(\mathcal{R}, \partial P)$, i.e., covering ∂P with a minimum sized subset of $\mathcal{R}$.

Below we state our result on the Boundary Cover problem which improves upon the 4-approximation algorithm by Berman and DasGupta [11].

▶ **Theorem 1.** *Local Search yields a PTAS for the Boundary Cover problem for simple orthogonal polygons.*

The same Local Search algorithm yields a PTAS for the Corner Cover problem improving upon their 2-approximation algorithm for simple polygons [11]. However, it is not known whether the Corner Cover problem is NP-hard when the polygons are simple.

▶ **Theorem 2.** *Local Search yields a PTAS for the Corner Cover problem for simple orthogonal polygons.*

The Maximum Independent Set of Rectangles problem has garnered a lot of attention in the past 25 years [3, 16]. In the recent past, Mitchell made a breakthrough [40] by proving a 10-approximation algorithm which was subsequently improved to $2+\epsilon$ by Gálvez, Khan, Mari, Mömke, Reddy, and Wiese [30]. There is a QPTAS known due to Adamaszek and Wiese [2], and the open question is to find a PTAS. The discrete version of the problem, the Maximum Discrete Independent Set of Rectangles (MDISR) has also been studied where apart from the family of rectangles there is also a point set as part of the input. Here two rectangles are said to have a discrete intersection if their (continuous) intersection also contains an input point. For this problem, the only known result is an $O(\log n)$-approximation algorithm by Ene, Har-Peled, and Raichel [26]. A PTAS is known via Local Search for this discrete version if we replace rectangles with non-piercing regions[1] [10].

The Maximum Independent Set problem on the hypergraph $(\mathcal{R}, \partial P)$ translates to a special class of the MDISR problem where the rectangles are containment-maximal rectangles with respect to some fixed input simple orthogonal polygon P and the input point set is a subset of the boundary of P.

▶ **Theorem 3.** *Local Search yields a PTAS for the Maximum Independent Set problem on the geometric hypergraph* $(\mathcal{R}, \partial P)$.

In order to show the PTAS, our main technical contribution is to show the existence of *planar support* for the above hypergraphs defined on the boundary cover problem instance as stated below. Informally, given a hypergraph $H = (V, \mathcal{F})$, we say $G = (V, E)$ is a support graph of H if for every hyperedge $F \in \mathcal{F}$, the induced subgraph $G[F]$ is connected. In addition, if G is planar we call it a planar support. See Definition 9 for a formal definition.

▶ **Theorem 4** (Planar Support for Boundary Cover)**.** *Given a simple orthogonal polygon P and a set of containment-maximal rectangles $\mathcal{R}$ with respect to P, there exists a planar support graph for* $(\mathcal{R}, \partial P)$.

In order to prove this theorem, we observe many novel properties of containment-maximal rectangles, which could be of independent interest and useful elsewhere. As a corollary of this theorem, it immediately follows that Local Search yields a PTAS for the boundary

[1] A family of geometric objects is called non-piercing if for every pair of intersecting objects A and B, both $A \setminus B$ and $B \setminus A$ are simply connected regions. Observe that rectangles are piercing regions.

cover problem and the corner cover problem by applying a result by Mustafa and Ray [41]. Similarly, Theorem 4 also implies Theorem 3 from a result by Chan and Har-Peled [20]. These Local Search results by Chan and Har-Peled [20], and Mustafa and Ray [41] were independently devised and constitute the above-mentioned Local Search framework. Please refer to Subsection 2.1 for more on this framework. For the rest of this article, the reader may alternatively consider the framework as a black box, and focus on the proof of Theorem 4.

On the negative side, we construct interior cover problem instances whose minimal support graphs contain arbitrarily large bicliques, thus implying that a PTAS cannot be attained using the above Local Search framework.

▶ **Theorem 5.** *For every $r \geq 3$, there exists simple orthogonal polygon P and an interior cover $\mathcal{R}$ such that every arbitrary support graph of $(\mathcal{R}, P)$ contains $K_{r,r}$.*

The Maximum Antirectangle problem restricted to the boundary of the polygon is the same as the Maximum Independent Set problem for the dual hypergraph $(\partial P, \mathcal{R})$, i.e., find a maximum sized subset $P' \subset \partial P$ such that for every $R \in \mathcal{R}$, $|R \cap P'| \leq 1$. We give a negative result for this problem (also known as the maximum hidden set problem) by showing the existence of instances such that the dual intersection graphs have arbitrarily large bicliques, thus implying Local Search solution can be arbitrarily smaller than the optimum solution.

▶ **Theorem 6.** *There exist simple polygons for which the local optimum solution size can be arbitrarily smaller than that of the global optimum solution for the Maximum Antirectangle problem.*

To the best of our knowledge, this problem has only been studied in the context of obtaining duality gaps. However, for the more general problem when the point set and the family of rectangles can be arbitrary, there is an $O(n^{0.368})$-approximation algorithm known due to Chan [17].

The exact same construction as above also implies the same for the Hitting Set problem for the hypergraph $(\partial P, \mathcal{R})$.

▶ **Theorem 7.** *There exist simple polygons for which the local optimum solution size can be arbitrarily larger than that of the global optimum solution for the Minimum Hitting Set problem for $(\partial P, \mathcal{R})$.*

Again we are unaware of any Hitting Set result for this restricted hypergraph. However, for general rectangle hypergraphs the problem enjoys an $O(\log \log OPT)$-approximation algorithm via ϵ-nets [5]. There are also matching lower bounds on the size of ϵ-nets due to Pach and Tardos [42] which implies one cannot hope to improve the approximation factor using ϵ-nets or by using the natural set cover LP relaxation [12]. The problem is also known to be APX-hard due to Chan and Grant [18] even when the boundary of every pair of rectangles intersects 0 or 4 times.

Please refer to Section 2 for the precise definitions of the above problems.

1.1.1 Technical Overview and Organization

There are broadly two parts to our proof of the existence of the planar supports for hypergraphs defined on the boundary cover instances. If we think of the existence of planar supports as the desired property then we prove that this property is *hereditary*, i.e., for a fixed simple orthogonal polygon P, if a family of rectangles satisfies the property then so does any of its sub-families (Lemma 31 in Section 4). We also prove that the *complete* families of containment-maximal rectangles have this property (Lemma 33 in Section 5). By complete we mean the set of all possible containment-maximal rectangles with respect to P.

The central idea in proving the hereditary property is that if a family has a star graph as its support then the family without the rectangle corresponding to the center of the star, still has a planar support. See Figure 1 for an example and Section 3 for the technical details.

We prove the planar support property for complete family of rectangles by induction on some appropriate complexity of the polygon. Given an orthogonal polygon P, we prove that if the statement is true for a complete family with respect to a *smaller* polygon, then it is also true for the complete family with respect to P. The formal statement can be found in Section 5.

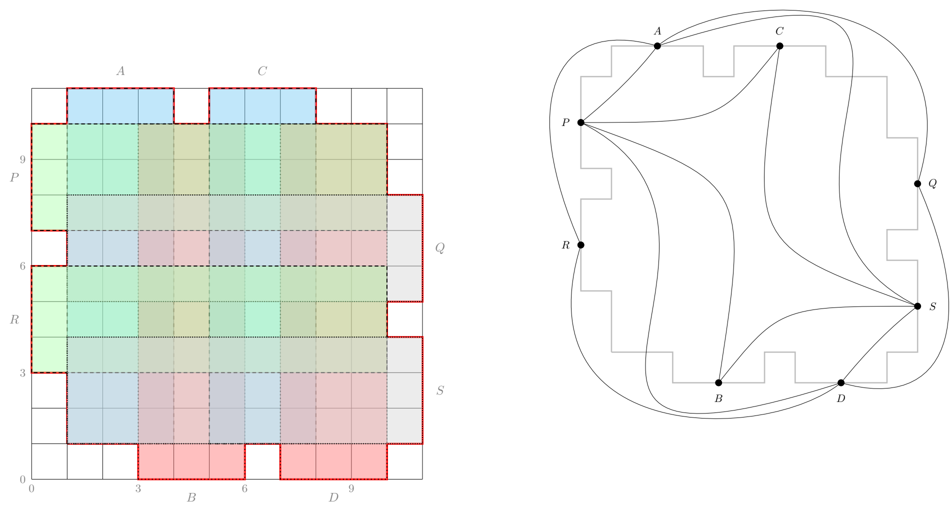


Figure 1 On the left, the boundary of polygon P is drawn with thick red lines on an integer grid. $\mathcal{R} = \{A, B, C, D, P, Q, R, S\}$ where $A = [1, 4] \times [1, 11]$, $B = [3, 6] \times [0, 10]$, $C = [5, 8] \times [1, 11]$, $D = [7, 10] \times [0, 10]$, $P = [0, 10] \times [7, 10]$, $Q = [1, 11] \times [5, 8]$, $R = [0, 10] \times [3, 6]$, $S = [1, 11] \times [1, 4]$. A support graph for the boundary cover instance $\mathcal{R}$ with respect to P is drawn on the right. Let $R_c = [1, 10] \times [1, 10]$. The minimal support graph for the boundary cover instance $\mathcal{R} \cup \{R_c\}$ is a star graph with R_c as the center of the star.

1.2 Related Work

Given an orthogonal polygon P, define a graph $G = (V, E)$ where V is a finite point set in P, and $(p, q) \in E(G)$ if there exists an axis-parallel rectangle $R \subseteq P$ such that $p, q \in R$. Observe that a polygon covering corresponds to a clique cover in G if V is sufficiently dense in P[2]. A natural lower bound is the size of an independent set in G, which is also called an antirectangle. The minimum clique cover size is denoted by θ and the maximum antirectangle (independent set) size is denoted by α. As mentioned by Chaiken et al. [15], Chvátal asked whether $\theta \stackrel{?}{=} \alpha$, which was disproved by Szemerédi by constructing a polygon with $\theta = 8$ and $\alpha = 7$. Chung managed to attain the same bound using a simple polygon. Erdős asked whether there exists a constant $c > 1$ such that $\theta \leq c \cdot \alpha$. The only known upper bound is $\theta = O(\alpha \log \alpha)$ due to Franzblau [27].

[2] E.g., extend all the sides of P to form a grid. Define V to be all the grid points that are in P.

Covering general polygons (not-necessarily orthogonal) by minimum number of convex polygons has also been studied and it is known as the Convex Cover problem and its dual problem is known as the Hidden Set problem. There has been recent progress made by Browne, Kasthurirangan, Mitchell, and Polishchuk [13], where they proved approximation ratios of 6 and 8 for the covering and its dual problem, respectively. On the other hand Abrahamsen proved that the Convex Corner problem is $\exists\mathbb{R}$-complete [1].

Even more generally, Haussler and Welzl proved that any hypergraph with VC-dimension d admits ϵ-nets of size $O(\frac{d}{\epsilon}\log\frac{d}{\epsilon})$ [33]. Using the multiplicative weight update method by Brönnimann and Goodrich [12], this implies an $O(\log OPT)$-approximation for the corresponding covering problem. For general rectangle families, there are also matching lower bounds of $O(1/\epsilon \log 1/\epsilon)$ [42]. Also see [19, 21, 47] that studies other measures of complexity of geometric hypergraphs like union complexity and shallow cell complexity which have an effect on the approximation factors of the corresponding covering problem.

Also, the authors in [4] studied the Rectangle Rule List Minimization problem, which is a generalization of the Orthogonal Polygon Covering problem.

2 Preliminaries

All polygons are assumed to be orthogonal, i.e., their sides are parallel to either the vertical or horizontal axes. A polygon is said to be simple if it has no holes, i.e., it is simply connected. Given a simple orthogonal polygon P, we denote ∂P to be the boundary of P. All rectangles are assumed to be axis-parallel. A rectangle $R \subseteq P$ is called containment-maximal with respect to a polygon P if R is not contained in a rectangle $R' \subseteq P$ of larger area. Henceforth, we shall use the word "maximal" to denote containment-maximal for the sake of brevity. Observe, as a consequence of maximality, all four sides of R have non-empty intersection with ∂P. We denote $\mathcal{R}^K(P)$ to be the set of all maximal rectangles with respect to P, also referred as the complete family of rectangles with respect to P.

▶ **Definition 8** (Blocker of a Maximal Rectangle). *Given an orthogonal polygon P and a maximal rectangle $R \subseteq P$, any point $p \in \partial P$ is called a blocker of R if $p \in R$ and p is not a corner of R. If p intersects the top (resp. bottom, left, right) side of R then p is called a top (resp. bottom, left, right) blocker of R.*

For the sake of brevity, we shall sometimes call a side s of P to be a blocker of R if $s \cap R \neq \emptyset$.

▶ **Definition 9** (Support graph of a hypergraph). *Given a hypergraph $H = (V, \mathcal{F})$ where V is the vertex set and $\mathcal{F} \subseteq 2^V$ is a family of subsets of V, we say $G = (V, E)$ is a support graph of H if for every $F \in \mathcal{F}$, the subgraph induced on F, $G[F]$ is connected.*

See Figure 1 for an example of a family of maximal rectangles $\mathcal{R}$ with respect to polygon P on the left, and a planar support graph for the hypergraph $(\mathcal{R}, \partial P)$ to its right. Observe, that the intersection graph of the rectangles is always a valid support graph, which may not be planar.

Interior Cover problem
Input: An orthogonal polygon P in the plane.
Question: A minimum sized set of rectangles $\mathcal{R}$ such that

$$\bigcup_{R_i \in \mathcal{R}} R_i = P.$$

Boundary Cover problem
Input: An orthogonal polygon P in the plane.
Question: A minimum sized set of rectangles $\mathcal{R}$ such that

$$\partial P \subseteq \bigcup_{\substack{R_i \in \mathcal{R} \\ R_i \subseteq P}} R_i.$$

Corner Cover problem
Input: An orthogonal polygon P in the plane.
Question: A minimum sized set of rectangles $\mathcal{R}$ such that

$$\mathsf{corner}P \subseteq \bigcup_{\substack{R_i \in \mathcal{R} \\ R_i \subseteq P}} R_i$$

where $\mathsf{corner}P$ denotes the set of corners of P.

Maximum Independent Set problem for $(\mathcal{R}, \partial P)$
Input: An orthogonal polygon P in the plane and a set of containment-maximal rectangles $\mathcal{R}$ with respect to P.
Question: A maximum sized subset of rectangles $\mathcal{R}' \subseteq \mathcal{R}$ such that for every $R_i, R_j \in \mathcal{R}'$, there does not exist a point $p \in \partial P$ where $p \in R_i \cap R_j$.

Maximum Antirectangle problem
Input: An orthogonal polygon P in the plane.
Question: A maximum sized set of points $Q \subset P$ such that for every $p, q \in Q$, there does not exist a rectangle $R \subseteq P$ where $p, q \in R$.

Minimum Hitting Set problem for $(\partial P, \mathcal{R})$
Input: An orthogonal polygon P in the plane and a set of containment-maximal rectangles $\mathcal{R}$ with respect to P.
Question: A minimum sized point set $P' \subset \partial P$ such that for every $R \in \mathcal{R}$, $R \cap P' \neq \emptyset$.

2.1 Local Search Framework

Local search has been a very useful heuristic for decades, and has been proved to give good approximation ratios, particularly for clustering and geometric optimization algorithm [6, 29]. In this work, we will consider the particular Local Search framework introduced by Mustafa and Ray for covering problems in [41]. Almost the same framework works for packing problems, which was introduced around the same time by Chan and Har-Peled in [20]. Over the years, this framework has been used for a variety of packing and covering problems [7, 8, 9, 10, 32, 38, 43, 44]. As a consequence of this framework, it is sufficient to show the existence of a sparse support graph for a hypergraph: in order to prove that Local Search yields a PTAS for the corresponding covering problem. Here by sparse, we mean the graph has strongly sublinear separators, which is a necessary condition for planarity.

The study of planar supports predates the Local Search framework in question. It has been studied in the context of hypergraph drawing. See [14, 35, 36], and Section 2 in [44] for a short survey. Durocher and Fraser [25] were arguably the first to make the connection and use the word “support” in the context of Local Search for covering problems.

Below, we state an abstraction of the result of Mustafa and Ray [41].

▶ **Lemma 10.** *Given a hypergraph $H = (V, \mathcal{F})$ where V is the vertex set and $\mathcal{F} \subseteq 2^V$ is a family of subsets of V, we want to compute the minimum sized $V' \subseteq V$ such that for every $F \in \mathcal{F}$, $V' \cap F \neq \emptyset$. If there exists a planar support of H, then the Local Search algorithm computes a $(1+\epsilon)$-approximate solution in time $n^{O(1/\epsilon^2)}$, for $\epsilon > 0$.*

Conversely, we state an abstraction of the result of Chan and Har-Peled [20]. Before that we define the dual intersection graph G of a subset $U \subseteq V$ with respect to hypergraph $H = (V, \mathcal{F})$. The vertex set is U, and for every $v, w \in U$, $(v, w) \in E(G)$ if there exists $F \in \mathcal{F}$ such that $v, w \in F$.

▶ **Lemma 11.** *Given a hypergraph $H = (V, \mathcal{F})$ where V is the vertex set and $\mathcal{F} \subseteq 2^V$ is a family of subsets of V, we want to compute the maximum sized $V' \subseteq V$ such that for every $F \in \mathcal{F}$, $|V' \cap F| \leq 1$. Let $A \subseteq V$ be an optimum solution and $B \subseteq V$ be the solution returned by the Local Search algorithm. If the dual intersection graph of $A \cup B$ with respect to H is planar, then the Local Search algorithm computes a $(1+\epsilon)$-approximate solution in time $n^{O(1/\epsilon^2)}$, for $\epsilon > 0$.*

Lastly, see below for the Local Search algorithm for covering the boundary of the input simple polygon P, where k is the Local Search parameter which is set to c/ϵ^2 for some appropriate constant c.

Algorithm 1 Local Search Algorithm to cover the boundary of P.

Require: A simple polygon $P \subset \mathbb{R}^2$, $\mathcal{R}^K(P)$ is the complete family wrt P, and $k \in \mathbb{N}$.
Ensure: $\mathcal{R} = \{R_1, \ldots, R_m\}$ such that $\partial P \subseteq \bigcup \mathcal{R}$.

$S \leftarrow \mathcal{R}^K(P)$
while $\exists X \subseteq S, \exists Y \subseteq (\mathcal{R}^K(P) \setminus S)$ such that $|X| \leq k$, $|Y| < |X|$, and $(S \setminus X) \cup Y$ covers ∂P **do**
  $S \leftarrow (S \setminus X) \cup Y$
end while
return S

2.2 Intersection Patterns

Given a simple orthogonal polygon P, if two maximal rectangles R and R' with respect to P have a non-empty intersection then they intersect in one of the following ways:

1. **Corner Intersection.** R has exactly one corner of R' in its interior and vice versa.
2. **Piercing Intersection.** The two vertical (resp. horizontal) sides of R intersect the two horizontal (resp. vertical) sides of R'.

Furthermore, if R and R' have a piercing intersection and one of the sides of R is contained in one of the sides of R' then we say that the two rectangles are *aligned* to each other. See Figure 2.

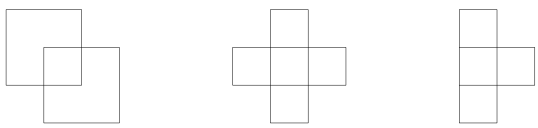


Figure 2 The left pair has a corner intersection. The middle and the right have a piercing intersection; in which the right pair are also aligned to each other.

▶ **Definition 12.** *Given a simple orthogonal polygon P and a set of maximal rectangles $\mathcal{R}$ with respect to P, define a binary relation $(\mathcal{R}, \prec)$. If for every $R, R' \in \mathcal{R}$ such that R and R' have a piercing intersection and the two vertical sides of R intersect the two horizontal sides of R' then we say $R \prec R'$.*

Observe that $(\mathcal{R}, \prec)$ is irreflexive, asymmetric, and transitive and hence $\prec$ defines a strict partial order on $\mathcal{R}$. More informally, we say $R \prec R'$ if they have a piercing intersection and the width of R is less than that of R'.

2.3 Left-Right Coloring

In this section, we recall a few definitions and results from a series of work by de Fraysseix, Ossona de Mendez, and Rosenstiehl [24]. They presented the Left-Right Coloring technique that can be used to test whether a graph is planar. Fix some arbitrary vertex in the graph as the root and traverse in a depth-first way. Thus we get a DFS tree, whose edges are subset of the given graph. Left-Right Coloring is a coloring (partitioning) of the cotree edges (the edges in the graph that are not in the DFS tree) such that if such a coloring is possible then cotree edges can be drawn without crossing each other, thus implying the given graph is planar.

▶ **Definition 13** (DFS-orientation). *Given an undirected, connected graph $G = (V, E)$ and a fixed vertex $r \in V$, its DFS-orientation is a directed graph $G' = (V, T \uplus B)$ where T is the set of the tree edges obtained by depth-first traversal of G starting from r, and $B = E \setminus T$ is the set of cotree edges. For every tree edge $(u, v) \in E$, if u is traversed before v in the depth-first traversal then there is a directed edge (u, v) in G'. Conversely, for every cotree edge $(u, v) \in E$, if u is traversed before v in the depth-first traversal then there is a directed edge (v, u) in G'.*

Thus, T defines a partial order on V where the root of T is the minimum vertex with respect to this partial order. In the following, terms like *high* and *low* are used with respect to the same partial order. Given a tree edge $e = (u, v)$, *return edges* of e are the cotree edges $e' = (v', u')$ such that v' is a descendant of v or $v = v'$, and u' is an ancestor of u that is different from u. Define a mapping $low : E \to V$, where for every $e \in E$, $low(e)$ is the minimum vertex with respect to the partial order that can be reached by a directed path starting with e and containing at most one cotree edge.

▶ **Definition 14** (Left-Right Coloring). *Let $G' = (V, T \uplus B)$ be a DFS-oriented graph. A bipartition of its cotree edges $B = L \uplus R$ into two classes, referred to as left and right, is called left-right coloring, if for every $v \in V$ and outgoing edges e_i and e_j from v, the following two conditions are satisfied.*

- *All return edges of e_i ending strictly higher than $low(e_j)$ belong to one class, and*
- *All return edges of e_j ending strictly higher than $low(e_i)$ belong to the other class.*

▶ **Lemma 15** ([24]). *A finite graph G is planar if and only if there exists a Left-Right coloring of the cotree edges with respect to a depth-first-tree T on G.*

3 Family of Rectangles with a Kernel

The main results of this section are Lemmata 18 and 30, which in turn are used to prove Lemma 31 in Section 4. In the following, we define a kernel of a rectangle family. Informally, Lemma 18 says that if a rectangle family has a star graph as its support, then the rectangle corresponding to the center of the star belongs to the kernel. Lemma 30 states that a family of rectangles without its kernel still has a planar support.

▶ **Definition 16** (Kernel of a Family). *Given a family of maximal rectangles $\mathcal{R}$ with respect to a simple polygon P, the kernel of $(\mathcal{R}, \partial P)$ is defined as follows.*

$$ker(\mathcal{R}) := \bigcap_{\substack{p \in \partial P \\ |\mathcal{R}_p| \geq 2}} \mathcal{R}_p$$

where $\mathcal{R}_p := \{R \in \mathcal{R} \mid p \in R\}$.

A family $(\mathcal{R}, \partial P)$ with non-empty kernel $ker(\mathcal{R})$ is called *proper* if for every $R \in \mathcal{R}$ there exists $p \in \partial P$ such that $ker(\mathcal{R}) \cup \{R\} \subseteq \mathcal{R}_p$. See Figure 3 for an example of a family that is not proper.

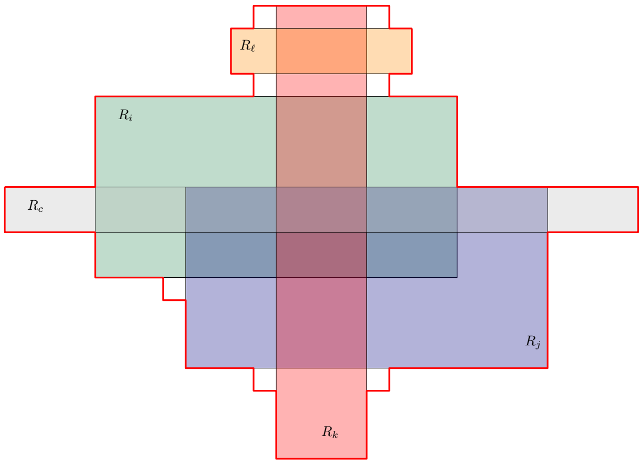


Figure 3 $(\{R_i, R_j, R_k, R_\ell, R_c\}, \partial P)$ has a non-empty kernel $\{R_c\}$ but it is not proper. Whereas, $(\{R_i, R_j, R_c\}, \partial P)$ is proper and has the same kernel. Note that ∂P is drawn with red lines.

▶ **Observation 17.** *Every proper family $(\mathcal{R}, \partial P)$ with a non-empty kernel has a support that is a star graph with the center vertex corresponding to a kernel rectangle.*

In the following, we show that the converse is also true.

▶ **Lemma 18.** *Given a family of maximal rectangles $\mathcal{R}$ with respect to a simple orthogonal polygon P. If there exists a minimal support graph G of $(\mathcal{R}, \partial P)$ such that G is a star graph then the rectangle corresponding to the center of G belongs to $ker(\mathcal{R})$ and $\mathcal{R}$ is proper.*

Proof. Let R_c be the center of G. As G is a minimal support graph of $(\mathcal{R}, \partial P)$, for every $p \in \partial P$ such that $|\mathcal{R}_p| \geq 2$, $R_c \in \mathcal{R}_p$. If there exists $p \in \partial P$ such that $|\mathcal{R}_p| \geq 2$ and $R_c \notin \mathcal{R}_p$ then p is not supported by G and hence a contradiction. This implies $R_c \in ker(\mathcal{R})$.

$\mathcal{R}$ is proper because otherwise there are isolated vertices in a minimal support. ◀

Throughout this section we assume $\mathcal{R}$ to be proper and have a non-empty kernel. Without loss of generality, we consider $\mathcal{R}$ to have a unique kernel rectangle, i.e., $ker(\mathcal{R}) = \{R_c\}$. From Definition 12, we can partition the family without its kernel rectangle R_c into the following subsets. The set of rectangles that have a corner intersection with R_c that we denote by $\mathcal{R}_N$. The remaining rectangles in $\mathcal{R}$ have a piercing intersection with R_c. Among the ones having a piercing intersection, we call the rectangles $R \in \mathcal{R}$ to be vertical rectangles if $R \prec R_c$, i.e., $\mathcal{R}_V := \{R \in \mathcal{R} \mid R \prec R_c\}$. Likewise, we call the rectangles $R \in \mathcal{R}$ to be horizontal rectangles if $R_c \prec R$, i.e., $\mathcal{R}_H := \{R \in \mathcal{R} \mid R_c \prec R\}$. Thus, $\mathcal{R} := \mathcal{R}_N \uplus \mathcal{R}_V \uplus \mathcal{R}_H \uplus \{R_c\}$.

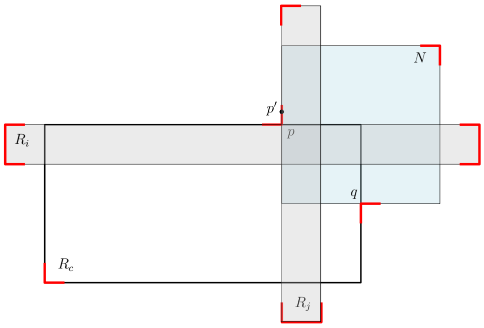


Figure 4 Let $\mathcal{R}$ be a proper family with R_c as the kernel rectangle. The above figure shows parts of ∂P in bold red. Corner intersection $N \cap R_c \cap \partial P = \{p, q\}$. If R_j belongs to $\mathcal{R}$ then p' witnesses the intersection $N \cap R_j \cap \partial P$ but it is outside R_c, which implies R_c is not a kernel. However, R_i can be part of $\mathcal{R}$ without contradicting the fact that R_c is the kernel rectangle.

▶ **Lemma 19.** *Let $N \in \mathcal{R}_N$ such that it contains the top-right corner of R_c. If there exists $R \in \mathcal{R}$ such that $R \cap N \cap \partial P \neq \emptyset$, then R is either aligned to the top side and/or the right side of R_c.*

Proof. Given N, a corner rectangle, there exists two points p and q where $\partial N \cap \partial R_c \neq \emptyset$. See Figure 4. As $N \in \mathcal{R}$ and $\mathcal{R}$ is proper, at least one or both p and q belongs to ∂P. If there exists $R \in \mathcal{R} \setminus \{R_c\}$ such that $R \cap N \cap R_c \cap \partial P \neq \emptyset$ then either p or q or both are contained in R. Without loss of generality, assume that $p \in R \cap \partial P$. Then observe that the top side of R is aligned with the top side of R_c. The left side of R cannot be aligned with the left side of N, because otherwise there exists a point $p' \in \partial P$ such that $p' \in (N \cap R \cap \partial P) \setminus R_c$, which implies R_c is not a kernel rectangle. See Figure 4. Similarly, if $q \in R \cap \partial P$ then the right side of R is aligned with the right side of R_c. ◀

The following observation can be made from Lemma 19.

▶ **Observation 20.** *For $R_i, R_j \in \mathcal{R}_N$, $R_i \cap R_j \cap \partial P = \emptyset$.*

▶ **Lemma 21.** *Given a proper family $\mathcal{R}$ with kernel rectangle R_c, there are at most 2 corner rectangles, N_i and N_j containing a fixed corner of R_c. Furthermore, N_i and N_j pierce each other in a non-aligned way.*

Proof. Let N_i and N_j be two corner rectangles, i.e., $N_i, N_j \in \mathcal{R}_N$ and they contain the top-right corner of R_c. We do a case analysis below and conclude that N_i and N_j can only have a piercing intersection in a non-aligned way.

1. Assume N_i and N_j have a corner intersection between themselves. Without loss of generality, further assume that the left side of N_i is to the left of that of N_j. There are two subcases. First, the top side of N_i is above that of N_j. Second, the top side of N_i is below that of N_j.
   In the first subcase, the left side of N_j is completely in the interior of N_i and R_c, which implies that N_j is not maximal. Hence, contradiction (see Figure 5, part *a*). In the second subcase, $N_j \cap R_c \cap \partial P = \emptyset$ which implies $\mathcal{R}$ is not a proper family, hence a contradiction again (see Figure 5, part *b*).
2. Assume N_i and N_j have a piercing intersection. The first subcase is that they are aligned to each other and the second subcase is that they are not.

Without loss of generality, assume one of the sides of N_j, s_j is aligned to s_i, the corresponding side of N_i and $s_j \subset s_i$. As N_j is maximal, $s_j \cap \partial P \neq \emptyset$. This also implies that $s_i \cap s_j \cap \partial P \neq \emptyset$. Then it contradicts Observation 20 (see Figure 5, part c).

For an example of N_i and N_j having a non-aligned piercing intersection, see part d of Figure 5.

Next assume, that there is a third rectangle N_k that also contains the top-right corner of R_c and $N_i \prec N_k \prec N_j$. This implies $N_k \cap R_c \cap \partial P = \emptyset$, which means $\mathcal{R}$ is not proper and hence contradiction (see Figure 5, part d). ◀

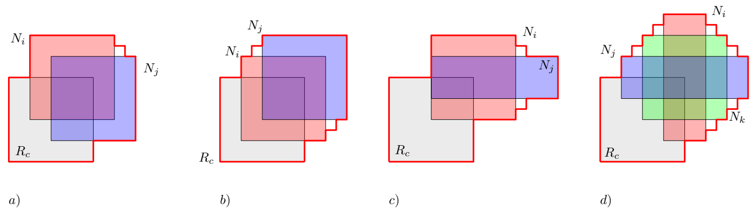


Figure 5 The subfigures are to support the various subcases of the proof of Lemma 21. ∂P is drawn in thick red. The fill colors of rectangles R_c, N_i, N_j, N_k are gray, red, blue, and green, respectively.

As a consequence of Lemma 21, we know that there are at most 8 corner rectangles in $\mathcal{R}$, at most 2 per corner of R_c. If there is a unique corner rectangle containing the top-right corner of R_c, we shall denote it by $N_{\nearrow}$. If there are two of them then we shall denote them by $N^v_{\nearrow}$ and $N^h_{\nearrow}$, where $N^v_{\nearrow} \prec N^h_{\nearrow}$. In general, a corner rectangle is denoted by N^a_b, where $a \in \{v, h, \emptyset\}$ and $b \in \{\swarrow, \nwarrow, \nearrow, \searrow\}$.

3.1 Vertical and Horizontal Rectangles

We define the subset of maximally vertical rectangles with respect to the partial order $\prec$ as $\mathcal{R}^*_V := \max\{R \in \mathcal{R}_V\}$. Similarly, we define the subset of minimally horizontal rectangles $\mathcal{R}^*_H := \min\{R \in \mathcal{R}_H\}$.

▶ **Lemma 22.** *There does not exist 3 rectangles $R_i, R_j, R_k \in \mathcal{R}^*_V$ (resp. $\mathcal{R}^*_H$) such that each one has a corner intersection with the other two in ∂P, i.e., $R_i \cap R_j \cap \partial P \neq \emptyset$, $R_j \cap R_k \cap \partial P \neq \emptyset$, and $R_i \cap R_k \cap \partial P \neq \emptyset$.*

Proof. Assume there exist 3 rectangles, $R_i, R_j, R_k \in \mathcal{R}$ such that $R_i \cap R_j \cap \partial P \neq \emptyset$, $R_j \cap R_k \cap \partial P \neq \emptyset$, and $R_i \cap R_k \cap \partial P \neq \emptyset$. Combinatorially, triple of rectangles can be have pairwise corner intersection in two ways as shown in Figure 6. If the triple intersect as shown on the left then $R_i \cap R_k \cap \partial P = \emptyset$ which is a contradiction. If the triple intersect as shown on the right then the bottom side of R_j and the right side of R_i are completely in the interior of the union of the other two. This implies R_i and R_j are not maximal which is a contradiction. ◀

Lemma 22 implies that we can linearly order the rectangles in $\mathcal{R}^*_V$ (resp. $\mathcal{R}^*_H$) as shown below.

▶ **Definition 23.** *Define a linear ordering $<_v$ on the rectangles in $\mathcal{R}^*_V$, such that $R <_v R'$ if the left side of R is to the left of that of R', for every $R, R' \in \mathcal{R}^*_V$.*

Let $\mathcal{R}^*_V = \{V_1, \dots, V_k\}$, where $V_1 <_v \cdots <_v V_k$.

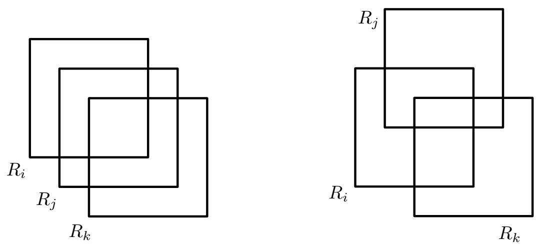


Figure 6 Three rectangles can intersect in two ways such that every pair has a corner intersection.

▶ **Definition 24.** *Define a linear ordering $<_h$ on the rectangles in $\mathcal{R}_H^*$, such that $R <_h R'$ if the bottom side of R is below that of R', for every $R, R' \in \mathcal{R}_H^*$.*

Let $\mathcal{R}_H^* = \{H_1, \dots, H_\ell\}$, where $H_1 <_h \cdots <_h H_\ell$.

▶ **Lemma 25.** *Let $R_i, R_j \in \mathcal{R}_V^*$ such that $R_i \cap R_j \cap \partial P \neq \emptyset$. There does not exist $R_k \in \mathcal{R}_V$ such that $R_k \prec R_i$ and $R_k \prec R_j$.*

Proof. Without loss of generality, assume $R_i <_v R_j$. For the sake of contradiction, lets assume such an R_k exists. Then R_k should be within the left side of R_j and the right side of R_i in order to have a piercing intersection with them. If R_k is aligned to R_i then they have an intersection with ∂P outside R_c, which means R_c is not a kernel rectangle, hence a contradiction (see the left half of Figure 7). Same is the case if it is aligned to R_j. If it is aligned to neither R_i nor R_j, then $R_k \cap R_c \cap \partial P = \emptyset$, which means $\mathcal{R}$ is not proper, hence a contradiction again (see the right half of Figure 7). ◀

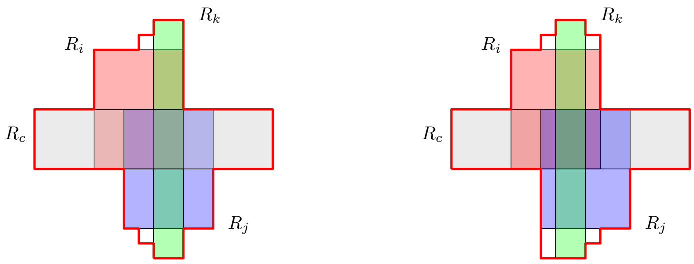


Figure 7 The above subfigures are to support Lemma 25.

▶ **Lemma 26.** *Let $\mathcal{R}_V^{\prec}(V_i) := \{R \in \mathcal{R}_V \mid R \prec V_i\}$, for $V_i \in \mathcal{R}_V^*$. There exists a support graph of $(\mathcal{R}_V^{\prec}(V_i) \cup \{V_i\}, \partial P)$ that is a star graph with V_i as the center.*

Proof. Lemma 25 implies that V_i is a kernel rectangle of $(\mathcal{R}_V^{\prec}(V_i) \cup \{V_i\}, \partial P)$. Observation 17 implies that the support graph of $(\mathcal{R}_V^{\prec}(V_i) \cup \{V_i\}, \partial P)$ is a star graph with V_i as the center. ◀

Same arguments hold for $H_j \in \mathcal{R}_H^*$.

We refer V_1, V_k, H_1, and H_ℓ to be terminal rectangles.

If V_1 and R_c have a common left blocker, then we call V_1 to be left-aligned. Likewise, if V_k and R_c have a common right blocker, then we call V_k to be right-aligned. Conversely, if H_1 and R_c have a common top blocker, then we call H_1 to be top-aligned. Finally, if H_ℓ and R_c have a common bottom blocker, then we call H_ℓ to be bottom-aligned. Note that a terminal rectangle may or may not be aligned.

3.2 Intersection between Corner and Non-corner Rectangles

From Lemma 19 we know that any vertical or horizontal rectangle having an intersection with $N^a_\nearrow$ at ∂P, should be top-aligned and/or right-aligned to R_c, where $a \in \{v, h, \emptyset\}$.

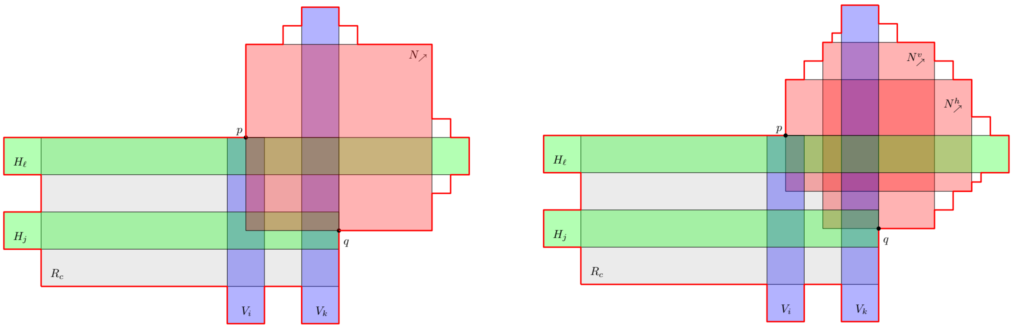
▶ **Lemma 27.** *There are at most 2 vertical rectangles in $\mathcal{R}^*_V$ and 2 horizontal rectangles in $\mathcal{R}^*_H$ intersecting a corner rectangle at ∂P. Among these up to 4 rectangles, there are at most 1 vertical terminal rectangle, and at most 1 horizontal terminal rectangle. Furthermore, for a fixed corner of R_c, if there are two corner rectangles, then there are at most 1 vertical and 1 horizontal rectangle per corner rectangle.*

Proof. Without loss of generality, fix the top-right corner of R_c. If there are corner rectangles, then either there is a single corner rectangle or there are two of them.

For the first case, the corner rectangle is $N_\nearrow$ and there are possibly 4 rectangles $V_i, V_k \in \mathcal{R}^*_V$ and $H_j, H_\ell \in \mathcal{R}^*_H$, where $1 \leq i < k$ and $1 \leq j < \ell$ such that they intersect $N_\nearrow$ at ∂P. Recall that V_k and H_ℓ are terminal rectangles (see Figure 8a).

Similar is the case, if there are two corner rectangles, $N^v_\nearrow$ and $N^h_\nearrow$. $N^v_\nearrow$ has an intersection with V_i and H_ℓ at a boundary point p, whereas, $N^h_\nearrow$ has an intersection with V_k and H_j at another boundary point q (see Figure 8b). ◀

See Figure 9 for an illustration of all corners having two corner rectangles each, and also Figure 10 for the corresponding support graph.



(a) $N_\nearrow$ is the unique corner rectangle containing the top-right corner of R_c. Boundary points $p \in V_i \cap H_\ell \cap N_\nearrow \cap \partial P$ and $q \in V_k \cap H_j \cap N_\nearrow \cap \partial P$.

(b) $N^v_\nearrow$ and $N^h_\nearrow$ are the two corner rectangles containing the top-right corner of R_c. Boundary points $p \in V_i \cap H_\ell \cap N^h_\nearrow \cap \partial P$ and $q \in V_k \cap H_j \cap N^v_\nearrow \cap \partial P$.

Figure 8 The intersection patterns between corner and non-corner rectangles.

We make the following observation.

▶ **Lemma 28.** *If there are two corner rectangles N^v_b, N^h_b for a fixed corner b of R_c, where $b = \{\nearrow, \nwarrow, \searrow, \swarrow\}$, then there exists a support graph such that degree of both these corner rectangles in this graph is at most 2.*

Proof. Fix $b = \nearrow$ and consider $N^v_\nearrow$. From Lemma 27, and Figures 8b and 9, we know there exists at most one maximally horizontal rectangle H_j, and similarly at most one maximally vertical rectangle V_k, such that there is a unique point $p \in N^v_\nearrow \cap H_j \cap V_k \cap \partial P \neq \emptyset$. Recall, from Lemma 26, there exists a support graph such that non-maximally horizontal (resp. vertical) rectangles are adjacent to a unique maximally horizontal (resp. vertical) rectangle. Thus, in a minimal support graph, $N^v_\nearrow$ has degree at most two. Similar arguments hold for $N^h_\nearrow$, and also for other values of b. ◀

▶ Remark 29. For a given corner $b \in \{\nearrow, \nwarrow, \searrow, \swarrow\}$, we shall ignore all pairs of corner rectangles N_b^v and N_b^h, if they exist, from our following graph drawing algorithm. Suppose v is a degree-2 vertex, incident to edges (u, v) and (v, w). Instead of drawing the pair of edges, one can draw an edge (u, w) and place v anywhere in the interior of the drawing of this edge. On the other hand, if v is a degree-1 vertex and (u, v) is the only edge incident to it, then this edge can be drawn arbitrarily close to u without intersecting any other edges.

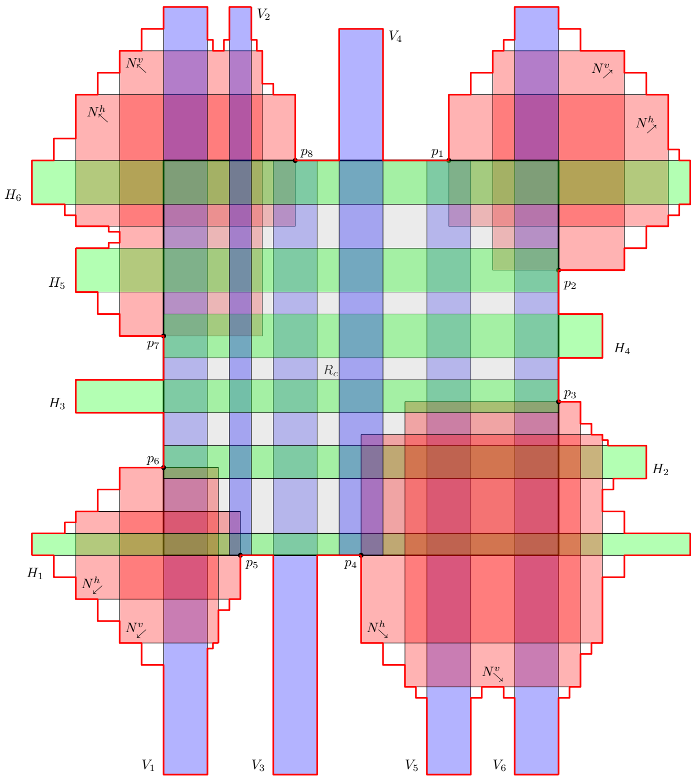


Figure 9 This figure is an extension of Figure 8b to all the 4 corners of R_c. See Figure 10 for the corresponding support graph.

3.3 Algorithm to Draw Planar Support for $\mathcal{R} \setminus ker(\mathcal{R})$

This section is an extension of Section 3. In the following we give a simple algorithm to draw a planar support of $(\mathcal{R} \setminus ker(\mathcal{R}), \partial P)$. We put an edge between the left-aligned (resp. right-aligned) rectangle, if it exists, and every horizontal rectangle. Similarly, we put an edge between the top-aligned (resp. bottom-aligned) rectangle, if it exists, and every vertical rectangle. Without loss of generality, assume that all the aligned rectangles exist.

3.3.1 Placing the Vertices

Draw a circle and place the terminal vertices V_1, H_1, V_k, H_ℓ on it in this order i.e., alternating between the horizontal and vertical rectangles as shown in Figure 11a where the circle is drawn in bold. Place the vertices corresponding to the corner rectangle between the terminal rectangles, i.e., $N_\swarrow$ between V_1 and H_1, $N_\searrow$ between V_k and H_1, $N_\nearrow$ between V_k and H_ℓ, and $N_\nwarrow$ between V_1 and H_ℓ.

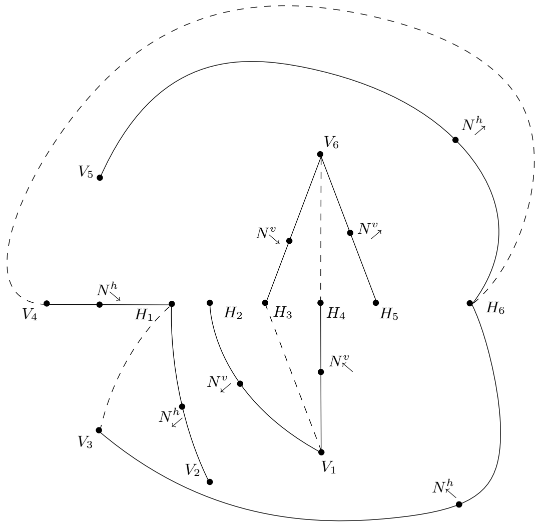
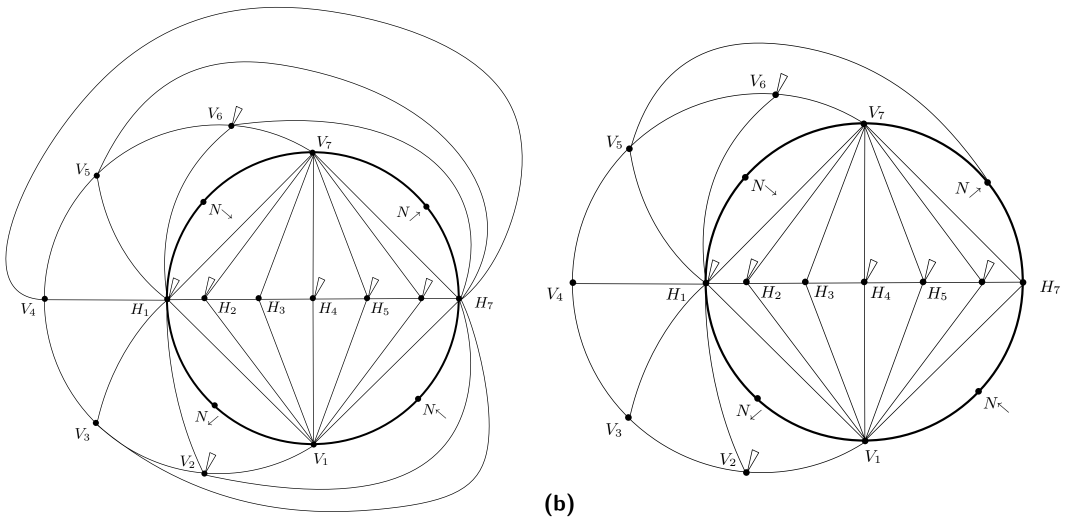


Figure 10 This figure shows the drawing of the support graph of the arrangement shown in Figure 9. The edges drawn in solid lines are to support rectangles intersecting at p_i, $1 \leq i \leq 8$. Whereas the edges that do not involve corner rectangles are drawn in dashed lines.



(a)

Figure 11 **(a)** Schema of a planar support graph for $(\mathcal{R} \setminus ker(\mathcal{R}), \partial P)$. $\mathcal{R}_V^* = \{V_1, \ldots, V_7\}$, $\mathcal{R}_H^* = \{H_1, \ldots, H_7\}$, $\mathcal{R}_N = \{N_\swarrow, N_\nwarrow, N_\nearrow, N_\searrow\}$. For every $V_i \in \mathcal{R}_V^*$, if there exists $R \prec V_i$, the vertex corresponding to R is placed in the private triangular region attached to V_i. Likewise for vertices in $\mathcal{R}_H^*$. **(b)** Schema of a planar support graph. $\mathcal{R}_V^* = \{V_1, \ldots, V_7\}$, $\mathcal{R}_H^* = \{H_1, \ldots, H_7\}$, $\mathcal{R}_N = \{N_\swarrow, N_\nwarrow, N_\nearrow, N_\searrow\}$. In comparison to Figure 11a, here H_7 is not top-aligned and $V_5 \cap N_\nearrow \cap \partial P \neq \emptyset$, hence there is the edge $(V_5, N_\nearrow)$.

Define a line joining H_1 and H_ℓ in the inner face of the cycle. Place the vertices corresponding to the other horizontal rectangles $H_j \in \mathcal{R}^*_H \setminus \{H_1, H_\ell\}$, on this line obeying the linear order defined in Definition 23. Similarly, define a line joining V_1 and V_k in the outer face of the cycle. Place the vertices corresponding to the other vertical rectangles $V_i \in \mathcal{R}^*_V \setminus \{V_1, V_k\}$, on this line obeying the linear order defined in Definition 24.

For every $V_i \in \mathcal{R}^*_V$, if there exists R such that $R \prec V_i$, then define a *private* region in the plane (shown by a triangle with a vertex at V_i in Figure 11a). Place the vertices corresponding to all such rectangles in the V_i's private region. Similarly define private regions for $H_j \in \mathcal{R}^*_H$, if required, and place all vertices corresponding to rectangles R in the region, where $H_j \prec R$.

3.3.2 Drawing the Edges

1. **($\mathcal{R}^*_V \times \mathcal{R}^*_H$).** If V_1 is left-aligned with R_c then draw an edge (V_1, H_j), for every $H_j \in \mathcal{R}^*_H$ as shown in Figure 11a. Similarly, if V_k is right-aligned then draw an edge (V_k, H_j), for every $H_j \in \mathcal{R}^*_H$. Conversely, if H_1 is bottom-aligned then draw an edge (V_i, H_1), for every $V_i \in \mathcal{R}^*_V$. Likewise, if H_ℓ is top-aligned then draw an edge (V_i, H_ℓ), for every $V_i \in \mathcal{R}^*_V$.
2. **($\mathcal{R}_V \times \mathcal{R}_V$).** If $V_i \cap V_{i+1} \cap \partial P \neq \emptyset$ then draw an edge (V_i, V_{i+1}) for every $V_i \in \mathcal{R}^*_V$ where $1 \leq i < k$, along the line where the V_i's are placed, as shown in Figure 11a. Draw an edge (R, V_i) inside V_i's private region, for every $V_i \in \mathcal{R}^*_V$ and $R \prec V_i$.
3. **($\mathcal{R}_H \times \mathcal{R}_H$).** If $H_j \cap H_{j+1} \cap \partial P \neq \emptyset$ then draw an edge (H_j, H_{j+1}) for every $H_i \in \mathcal{R}^*_H$ where $1 \leq j < \ell$, along the line where the H_j's are placed, as shown in Figure 11a. Draw an edge (R, H_j) inside H_j's private region, for every $H_j \in \mathcal{R}^*_H$ and $H_j \prec R$.
4. **($\mathcal{R}_N \times (\mathcal{R}^*_V \cup \mathcal{R}^*_H)$).** Lemma 19 implies that if $N_\nearrow \cap R \cap \partial P \neq \emptyset$ where $R \in \mathcal{R}^*_V \cup \mathcal{R}^*_H$ then R is either top-aligned and/or right-aligned to R_c. If R is top-aligned to R_c and $R \in \mathcal{R}^*_H$ then $R = H_\ell$. Draw the edge $(N_\nearrow, H_\ell)$. If there exists $R' \in \mathcal{R}^*_V$ such that $R' \cap N_\nearrow \cap \partial P \neq \emptyset$ and R' is top-aligned with R_c then draw the edge $(N_\nearrow, R')$ if $N_\nearrow \cap H_\ell \cap \partial P = \emptyset$, otherwise do not draw the edge $(N_\nearrow, R')$. See Figure 11b. The case where R is top-aligned is identical and we accordingly draw the edges. Finally, we iterate over the other corner rectangles in $\mathcal{R}_N$ and do a similar case analysis.

▶ **Lemma 30.** *Given a family $(\mathcal{R}, \partial P)$ such that $ker(\mathcal{R}) \neq \emptyset$, then $(\mathcal{R} \setminus ker(\mathcal{R}), \partial P)$ has a planar support.*

Proof. It is easy to see with the help of Figures 11a and 11b that the resultant graph G is planar. We claim that G is also a support graph. For $p \in \partial P$ consider $\mathcal{R}_p \setminus \{R_c\}$. Consider $R, R' \in \mathcal{R}_p \setminus \{R_c\}$. We claim R and R' are connected in G.

1. If $R, R' \in \mathcal{R}_V$ then let $V, V' \in \mathcal{R}^*_V$ such that $R \preceq V$ (i.e., $R \prec V$ or $R = V$) and $R' \preceq V'$. From Lemma 26, (R, V) and (R', V') are edges in G. This also means V and V' must be consecutive in the ordering $<_v$ and hence they are adjacent in G. Thus, R and R' are connected in G. Same is the case for $R, R' \in \mathcal{R}_H$.
2. If $R \in \mathcal{R}_V$ and $R' \in \mathcal{R}_H$. Let $V \in \mathcal{R}^*_V$ and $H \in \mathcal{R}^*_H$ such that $R \prec V$ and $H \prec R'$. Then observe that either V is a left/right aligned terminal rectangle or H is a top/bottom aligned terminal rectangle, or both. In either cases, (H, V) is an edge in G. Hence, R and R' are connected in G.
3. If $R \in \mathcal{R}_N, R' \in \mathcal{R}_V$. Without loss of generality, let $R = N_\nearrow$ and let $V \in \mathcal{R}^*_V$ and $R \preceq V$. From Lemma 19 there are two cases, either V is top-aligned with R_c or right-aligned with R_c. In either cases, there is an edge drawn between V and $N_\nearrow$. For the former case, see Figure 11b. For the latter case, see Figure 11a. Thus, R and $N_\nearrow$ are connected. The other cases involving $\mathcal{R}_N$ and $\mathcal{R}_H$ are identical. ◀

4 Hereditary Property of Planar Boundary Supports

In this section, we prove Lemma 31, i.e., the property of the boundary supports being planar is hereditary, which in turn is used to prove Theorem 4. To that end, we need Lemmata 15, 18, and 30.

▶ **Lemma 31** (Hereditary Property). *Given a simple orthogonal polygon P and a set of maximal rectangles $\mathcal{R}$ with respect to P. If $(\mathcal{R}, \partial P)$ has a planar support then for every $\mathcal{R}' \subset \mathcal{R}$, $(\mathcal{R}', \partial P)$ also has a planar support.*

Proof. We are given a planar support graph G for $(\mathcal{R}, \partial P)$. We claim that for every $R \in \mathcal{R}$, there exists a planar support graph G' for $(\mathcal{R} \setminus \{R\}, \partial P)$. If this claim is true, then for every $\mathcal{R}' \subset \mathcal{R}$, $(\mathcal{R}', \partial P)$ also has a planar support.

Fix some $R \in \mathcal{R}$ and let v be the vertex in G corresponding to R. In short, we denote $R = R(v)$. Recall the definition of left-right coloring (Subsection 2.3). Consider a DFS tree T defined on G rooted at v. Let w_i be a child of v in T, for every $i \in [t]$. We denote $T(w_i)$ to be the subtree rooted at w_i in T. Let $(x_{i,j}, v)$ be a cotree edge such that it points from $x_{i,j}$ to v, where $x_{i,j} \in T(w_i)$.

From Lemma 15, we know that if G is planar, then there exists a left-right coloring of the cotree edges. We do the following *shortcutting* operation on the cotree edges $(x_{i,j}, v)$. We replace the edge $(x_{i,j}, v)$ by $(x_{i,j}, w_i)$ such that the color class of the cotree edge, in the aforesaid left-right coloring, is same as before after doing the shortcutting. See Figure 12. From Lemma 15, it follows that planarity is preserved. Let E_1 be the set of edges that we get after the shortcutting operation.

For every $p \in \partial P$, if $\mathcal{R}_p$ lies in some subtree $T(w_i)$, for some $i \in [t]$, then $\mathcal{R}_p$ is supported by this shortcutting. If the vertices corresponding to the rectangles in $\mathcal{R}_p$ are distributed across the subtrees rooted at multiple children of v then observe the following.

▶ **Observation 32.** *If there exist vertices $x, y \in \mathcal{R}_p$ such that $x \in T(w_i)$ and $y \in T(w_j)$, for $i, j \in [t]$ and $i \neq j$, then $w_i, w_j \in \mathcal{R}_p$.*

This is true because otherwise point p is not supported by G. Therefore, we can ignore the descendants of w_i, for every $i \in [t]$ and only put edges among the w_i's in order to support $\mathcal{R}_p$.

Now we can use our results from Section 3. Observe that $G[\{w_1, \dots, w_t, v\}]$ is a star centered at v. From Lemma 18, R belongs to the kernel of $(\{R(w_1), \dots, R(w_t), R\}, \partial P)$, where $R = R(v)$. Furthermore, from Lemma 30, $(\{R(w_1), \dots, R(w_t)\}, \partial P)$ has a planar support. Let E_2 be the edge set of this support (drawn in red in Figure 12).

Thus, $G' := (\mathcal{R} \setminus \{R\}, E_1 \cup E_2)$ is a support of $(\mathcal{R} \setminus \{R\}, \partial P)$. G' is also planar because an edge in E_1 and that in E_2 can be drawn without crossing each other. As for every distinct $i, j \in [t]$, edges in $E(T(w_i))$ can be drawn in their private regions which are disjoint to that of $E(T(w_j))$. Finally, the edges in E_2 can be drawn outside the union of these private regions. ◀

5 Complete Family of Maximal Rectangles

In this section, we state Lemma 33 and give a proof sketch. This lemma is used to prove Theorem 4. Refer to [45] for the full proof of the lemma. Recall that for a simple orthogonal polygon P, $\mathcal{R}^K(P)$ is the set of all maximal rectangles with respect to P.

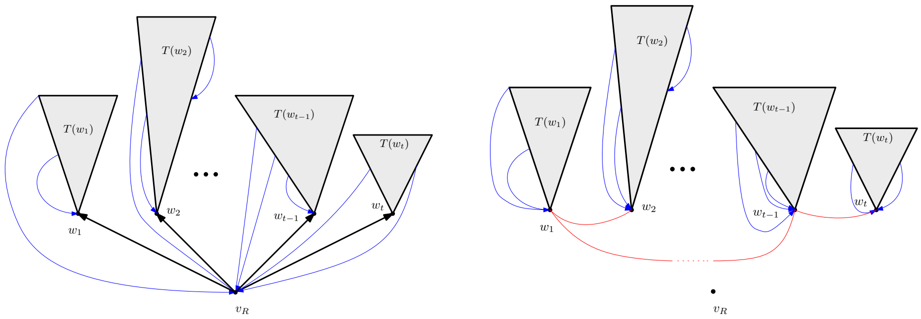


Figure 12 On the left, there is a schematic diagram of G, the support graph of $(\mathcal{R}, \partial P)$. On the right, there is a schematic diagram of G', the support graph of $(\mathcal{R} \setminus \{R\}, \partial P)$. G' is obtained from G by shortcutting the cotree edges (drawn in blue), incoming to v, and adding the edges in E_2 (red edges among w_j's) such that $(\{R(w_1), \dots, R(w_t)\}, \partial P)$ has a planar support.

▶ **Lemma 33.** *$(\mathcal{R}^K(P), \partial P)$ has a planar support.*

Proof Sketch. Given an orthogonal polygon P, we show that if the statement is true for a subpolygon $P' \subset P$, where $P \setminus P'$ is a rectangle, then it also true for P. Thus, inductively one can argue that if $G_{P'}$ is a planar support for $(\mathcal{R}^K(P'), \partial P')$ then there exists a planar support G_P for $(\mathcal{R}^K(P), \partial P)$. In particular, given $G_{P'}$ as an input, we give an algorithm to draw G_P. We make a number of observations between the interplay of the rectangle families in P and P', and their corresponding support graphs. We observe that the *new* rectangles that are part of P but not in P' have a unique rectangle in P' that share common blockers in one or two sides; let's call them the *old* rectangles in P'. Owing to which, we need to put an edge between every such new-old pair in G_P. We further observe that in any support graph for P (P' resp.) these new (old resp.) rectangles induce a path in G_P ($G_{P'}$ resp.). Lastly, we need to remove a set of vertices for which an old vertex acts as a cut vertex in $G_{P'}$ and make them adjacent to their new counterpart. We carefully argue that these sequence of operations ensure that the resultant graph G_P is a planar support for $(\mathcal{R}^K(P), \partial P)$. ◀

6 Planar Support Graphs and its Implications

Finally we present our main result.

▶ **Theorem 4** (Planar Support for Boundary Cover). *Given a simple orthogonal polygon P and a set of containment-maximal rectangles $\mathcal{R}$ with respect to P, there exists a planar support graph for $(\mathcal{R}, \partial P)$.*

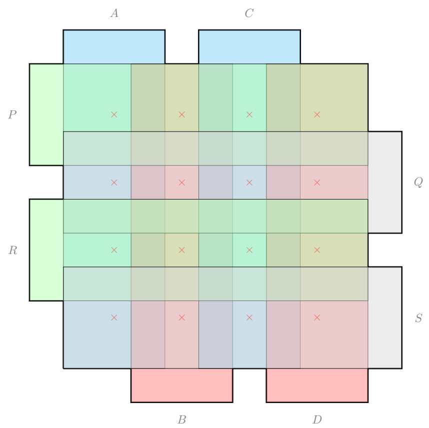
Proof. From Lemma 33 it follows that $(\mathcal{R}^K(P), \partial P)$ has a planar support. By applying Lemma 31 we prove that $(\mathcal{R}, \partial P)$ has a planar support, as $\mathcal{R} \subseteq \mathcal{R}^K(P)$. ◀

Theorem 4 and Lemma 10 together implies Theorems 1 and 2.

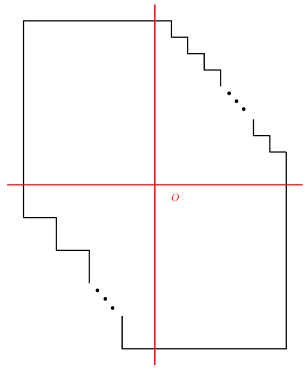
▶ **Theorem 1.** *Local Search yields a PTAS for the Boundary Cover problem for simple orthogonal polygons.*

▶ **Theorem 2.** *Local Search yields a PTAS for the Corner Cover problem for simple orthogonal polygons.*

Similarly, Theorem 4 and Lemma 11 together implies Theorem 3.



(a) A simple polygon P with $\mathcal{R} = \{A, B, C, D, P, Q, R, S\}$. The minimal support graph of $(\mathcal{R}, P)$ is $K_{4,4}$.



(b) A schematic diagram of a polygon whose top-right quadrant has r corners and its bottom-left quadrant has s corners.

Figure 13 Instances that lead to (arbitrarily large) bicliques in the support graphs of the Interior Cover (left) and the Antirectangle problem (right).

▶ **Theorem 3.** *Local Search yields a PTAS for the Maximum Independent Set problem on the geometric hypergraph* $(\mathcal{R}, \partial P)$.

Proof. From Lemma 11, we know that if the intersection graph of the union of the optimum and the Local Search solution is planar, then Theorem 3 follows.

As both optimum and Local Search solutions are independent sets, every point in ∂P is contained in at most 2 rectangles. Thus, the support graph of their union is the intersection graph. Therefore, from Theorem 4 the planarity of the intersection graph of their union follows. Hence proved. ◀

7 Negative Results

We have the following negative results.

▶ **Theorem 5.** *For every* $r \geq 3$*, there exists simple orthogonal polygon* P *and an interior cover* $\mathcal{R}$ *such that every arbitrary support graph of* $(\mathcal{R}, P)$ *contains* $K_{r,r}$.

Proof. See Figure 13a. For $r \geq 3$, one can construct a simple polygon P and define a set of maximal rectangles as shown in Figure 13a such that the minimal interior support graph is $K_{r,r}$. ◀

Recall the visibility graph that we defined in Section 1.2.

▶ **Lemma 34.** *For every* $r, s \geq 3$*, there exists a simple polygon* P *and a finite point set* $Q \subset P$ *such that* Q *is a union of two antirectangles and the visibility graph defined on* Q *is* $K_{r,s}$.

Proof. See Figure 13b. Observe that the set of convex corners in the top-right (resp. bottom-left) quadrant forms an antirectangle. Also, observe that for every pair of corner, one from the top-right quadrant and the other from the bottom-left quadrant, there exist a rectangle containing them and in P. ◀

Lemma 34 implies the following.

▶ **Theorem 6.** *There exist simple polygons for which the local optimum solution size can be arbitrarily smaller than that of the global optimum solution for the Maximum Antirectangle problem.*

Proof. Consider Figure 13b. Assume $r = \Omega(n)$ and $s = O(\log n)$. The global optimum solution size is $\geq r$, one such solution is picking all the convex corners of the polygon in the top-right quadrant. If the initial feasible solution for the Local Search algorithm is the set of corners in the bottom-left quadrant, then it can never improve its solution and that would be the local optimum solution. ◀

▶ **Theorem 7.** *There exist simple polygons for which the local optimum solution size can be arbitrarily larger than that of the global optimum solution for the Minimum Hitting Set problem for $(\partial P, \mathcal{R})$.*

Proof. See Figure 13b. Observe that the set of convex corners in the top-right (resp. bottom-left) quadrant forms a hitting set set. As in proof of Theorem 6, assume $r = \Omega(n)$ and $s = O(\log n)$. The global optimum solution size is $\leq s$, one such solution is picking all the convex corners of the polygon in the bottom-left quadrant. If the initial feasible solution for the Local Search algorithm is the set of corners in the top-right quadrant, then it can never improve its solution and that would be the local optimum solution. ◀

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