Avoiding Range via Turan-Type Bounds

Neha Kuntewar
Department of Computer Science and Engineering, IIT Madras, Chennai, India

Jayalal Sarma
Department of Computer Science and Engineering, IIT Madras, Chennai, India

Abstract

Given a circuit $C : \{0,1\}^n \to \{0,1\}^m$ from a circuit class $\mathcal{C}$, with $m > n$, finding a $y \in \{0,1\}^m$ such that $\forall x \in \{0,1\}^n$, $C(x) \neq y$, is the *range avoidance problem* (denoted by $\mathcal{C}$-Avoid). Deterministic polynomial time algorithms (even with access to NP oracles) solving this problem are known to imply explicit constructions of various pseudorandom objects like hard Boolean functions, linear codes, PRGs etc.

Deterministic polynomial time algorithms are known for NC^0_2-Avoid when $m > n$, and for NC^0_3-Avoid when $m \geq \frac{n^2}{\log n}$, where NC^0_k is the class of circuits with bounded fan-in which have constant depth and the output depends on at most k of the input bits. On the other hand, it is also known that NC^0_3-Avoid when $m = n + O\left(n^{2/3}\right)$ is at least as hard as explicit construction of rigid matrices. In fact, algorithms for solving range avoidance for even NC^0_4 circuits imply new circuit lower bounds.

In this paper, we propose a new approach to solving the range avoidance problem via hypergraphs. We formulate the problem in terms of Turan-type problems in hypergraphs of the following kind: for a fixed k-uniform hypergraph H, what is the maximum number of edges that can exist in H_C, which does not have a sub-hypergraph isomorphic to H? We show the following:

- We first demonstrate the applicability of this approach by showing alternate proofs of some of the known results for the range avoidance problem using this framework.
- We then use our approach to show (using several different hypergraph structures for which Turan-type bounds are known in the literature) that there is a constant c such that Monotone-NC^0_3-Avoid can be solved in deterministic polynomial time when $m > cn^2$.
- To improve the stretch constraint to linear, more precisely, to $m > n$, we show a new Turan-type theorem for a hypergraph structure (which we call the *loose $\mathcal{X}_{2\ell}$-cycles*). More specifically, we prove that any connected 3-uniform linear hypergraph with $m > n$ edges must contain a loose $\mathcal{X}_{2\ell}$ cycle. This may be of independent interest.
- Using this, we show that Monotone-NC^0_3-Avoid can be solved in deterministic polynomial time when $m > n$, thus improving the known bounds of NC^0_3-Avoid for the case of monotone circuits. In contrast, we note that efficient algorithms for solving Monotone-NC^0_6-Avoid, already imply explicit constructions for rigid matrices.
- We also generalise our argument to solve the special case of range avoidance for NC^0_k where each output function computed by the circuit is the majority function on its inputs, where $m > n^2$.

2012 ACM Subject Classification Theory of computation → Circuit complexity

Keywords and phrases circuit lower bounds, explicit constructions, range avoidance, linear hypergraphs, Turán number of hypergraphs

Digital Object Identifier 10.4230/LIPIcs.APPROX/RANDOM.2025.62

Category RANDOM

Related Version *Full Version*: https://arxiv.org/abs/2503.17114v2 [16]

Funding *Neha Kuntewar*: This work was supported by Prime Minister's Research Fellowship, Govt. of India.

Acknowledgements The authors would like to thank the anonymous reviewers for their insightful comments, which led to improvement in the presentation of the paper.

1 Introduction

Let $C : \{0,1\}^n \to \{0,1\}^m$ be a Boolean circuit with $\{\wedge, \vee, \neg\}$ gates, with $m > n$. The range of the function represented by the circuit : $\mathsf{Range}(C) = \{C(x) \mid x \in \{0,1\}^n\}$. Clearly, $\exists y \in \{0,1\}^m$ such that $y \notin \mathsf{Range}(C)$. The range avoidance problem (denoted by Avoid) asks, given a circuit C, with $m > n$, find a $y \notin \mathsf{Range}(C)$.

The Avoid problem (introduced by [14]) has been shown to have connections to some of the central research questions in circuit lower bounds and pseudorandomness. In particular, even an $\mathsf{FP}^{\mathsf{NP}}$ algorithm for Avoid is known to imply new circuit lower bounds [15] and new constructions of many other pseudorandom objects [15].

On the algorithms side, Avoid has a trivial $\mathsf{ZPP}^{\mathsf{NP}}$ algorithm. Indeed, given a circuit $C : \{0,1\}^n \to \{0,1\}^m$ with $m > n$, choose $y \in \{0,1\}^m$ and use the NP oracle to check if $\exists x \in \{0,1\}^n$ such that $C(x) = y$. Since $m > n$, there are at least $\frac{1}{2}$ fraction of ys which are outside $\mathsf{Range}(C)$, and hence the algorithm succeeds with at least $\frac{1}{2}$ probability. Designing a deterministic polynomial time algorithm, with access to an NP oracle ($\mathsf{FP}^{\mathsf{NP}}$ algorithm) to solve Avoid is a central open problem. Recently, [2] obtained the first single-valued $\mathsf{FS_2P}$ algorithm for Avoid which works infinitely often. Subsequently, [17] gave an improved $\mathsf{FS_2P}$ algorithm that works for all n, thus establishing explicit functions in $\mathsf{S_2E}$ requiring maximum circuit complexity. [2] showed an unconditional zero-error pseudodeterministic algorithm with an NP oracle and one bit of advice that solves Avoid for infinitely many inputs. These results imply pseudo-deterministic constructions for Ramsey graphs, rigid matrices, pseudo-random generators etc.(See [2]). [11] shows that if there is a deterministic polynomial time algorithm for Avoid then either $\mathsf{NP} = \mathsf{coNP}$ or there does not exist JLS secure iO.

Given the central nature of the problem, it is also meaningful to consider simpler versions first: consider $\mathcal{C}$-Avoid to be the restricted version of Avoid where the circuit is guaranteed to be from the class $\mathcal{C}$. *For which classes of circuits $\mathcal{C}$ do we have an efficient (or $\mathsf{FP}^{\mathsf{NP}}$) algorithm for $\mathcal{C}$-Avoid?* On this frontier, Ren, Santhanam and Wang [18] showed that an $\mathsf{FP}^{\mathsf{NP}}$ algorithm for $\mathcal{C}$-Avoid implies breakthrough lower bounds even when $\mathcal{C}$ is restricted to weaker circuit models such as AC^0 and NC^1 circuits. To go down even further, for every constant k, consider the restricted class of circuits NC^0_k where the depth of the circuit is constant and each output bit depends on at most k of the input bits. In a surprising result, Ren, Santhanam and Wang [18] showed that an $\mathsf{FP}^{\mathsf{NP}}$ algorithm for NC^0_4-Avoid implies $\mathsf{FP}^{\mathsf{NP}}$ algorithms for NC^1-Avoid. Additionally, this will also imply new circuit lower bounds - that there is a family of functions in E^{NP} that requires circuits of depth at least $\Omega(n^{1-\epsilon})$.

Complementing the above, [18] also exhibited $\mathsf{FP}^{\mathsf{NP}}$ algorithm for Avoid, when $\mathcal{C}$ is restricted to De Morgan formulas of size s with $m > n^{\omega(\sqrt{s}\log s)}$. At the low-end regime, [8] showed a polynomial time algorithm for NC^0_2 class, where each output bit depends on at most 2 input bits. They show a general template for obtaining $\mathsf{FP}^{\mathsf{NP}}$ algorithms for restricted classes via hitting set constructions. In particular, they give $\mathsf{FP}^{\mathsf{NP}}$ algorithms for NC^0_k circuits, de Morgan formulas, CNF or DNF, provided the stretch is large enough. En route, they also give a general method for obtaining the hitting set in polynomial time using the approximation degree of polynomials. Complementing this further, [6] designed deterministic polynomial time algorithms for all NC^0_k-Avoid for $m \geq \frac{n^{k-1}}{\log n}$. For $k = 3$, which was the frontier beyond [8], this requires $m \geq \frac{n^2}{\log n}$. They also showed the reason for the lack of progress for NC^0_3-Avoid by proving that a deterministic polynomial time algorithm for NC^0_3-Avoid, where $m = n + O(n^{2/3})$ would imply explicit construction of rigid matrices in deterministic polynomial time. This demonstrates the importance of the stretch function in the context of Avoid problem.

Our Results

In this paper, we propose a new approach to the range avoidance problem for NC^0_k circuits via Turan-type extremal problems in hypergraphs. For an NC^0_k circuit C, for each function $f_i \in C$ where $i \in [m]$, let $I(f_i)$ denote the set of input variables that f_i depends on. Let H_C denote the hypergraph defined as follows: Let V be the set of inputs in C. For $1 \leq i \leq m$, define a hyperedge e_i as $\{x_j \mid j \in [n], x_j \in I(f_i)\}$. Thus the multiset $E = \{\{e_i \mid i \in [m]\}\}$ has exactly m elements, each of size at most k. Let $\mathcal{F}$ be the family of functions that appear in the circuit C, and let $\phi : E \to \mathcal{F}$ be a labelling of the edges of the hypergraph H with the corresponding function. Without loss of generality, by assigning colors to Boolean bits, say red R for 0, and blue B for 1, we can interpret these functions as functions from $\{R, B\}^k \to \{R, B\}$, which induces a color to the hyperedge given any 2-coloring of the vertices of the hypergraph. A 2-coloring of the hyperedges of H is said to be a *ϕ-coloring* if there is a vertex coloring which induces this edge coloring via ϕ. With this notation, we state the following theorem, which essentially follows from the above definition (see section 3).

▶ **Theorem 1.** *Let $C : \{0,1\}^n \to \{0,1\}^m$ be a circuit and H_C be the corresponding hypergraph and let ϕ be the labelling function. Suppose H_C contains a sub-hypergraph H and an edge-coloring of H which is not a ϕ-coloring, where both the subhypergraph and the edge-coloring can be found in polynomial time. Then, there is a polynomial time algorithm to solve* Avoid *for C.*

The main idea of the above proof is that it suffices to identify a sub-hypergraph H in H_C such that there is an edge-coloring of H which is not ϕ-coloring on H. If we can ascertain the existence of a copy of $\mathcal{H}$' by extremal hypergraph theory, then it can be leveraged to solve the Avoid problem. In particular, this formulates range avoidance problem in terms of Turan-type extremal problems in hypergraphs of the following kind - *for a fixed family of hypergraphs $\mathcal{H}$, what is the maximum number of edges that can exist in an n-vertex hypergraph that avoids any member of the family H as an (induced) subgraph*? This is particularly well-studied for k-uniform hypergraphs and is denoted by $\mathsf{ex}_k(n, \mathcal{H})$. Notice that the family $\mathcal{H}$ may critically depend on the set of functions $\mathcal{F}$, and the mapping ϕ, and hence on the circuit C. A natural question is, is there a family of hypergraphs $\mathcal{H}$ that works for all circuits in the circuit class[1] for which we are interested to solve Avoid problem.

We first demonstrate immediate, simple applications of this framework for designing algorithms for Avoid in restricted settings. As mentioned above, [8] designed a simple iterative algorithm for solving Avoid when the input is restricted to circuits where each output function depends on at most 2 input bits. We show that the same special case can also be solved using our approach. Thus, as a warm-up, we derive an alternative proof of the following theorem (originally due to [8]) using our framework.

▶ **Theorem 2.** *There is a deterministic polynomial time algorithm for* NC^0_2-Avoid *when $m > n$. Using the same framework, there is a polynomial time algorithm for solving* {And$_k$, Or $_k$}-Avoid*, for a fixed k.*

As a second demonstration of our framework, we use tools from extremal graph theory to provide deterministic polynomial time algorithms for Avoid when $\mathcal{F}$ contains only $\wedge$ and $\vee$ functions that depend on exactly k input bits. We remark that such powerful tools are not

[1] We remark that while it may not be easy to find the hypergraph structure from the given circuit in general, for the important special cases of the problem mentioned in the previous discussion, just the exhaustive search based on the circuit structure will yield efficient algorithms to find the hypergraph structure.

needed to solve this special case, as the iterative algorithmic idea due to [8] can be extended to this case as well. Nevertheless, we argue that it serves as a demonstration of our technique itself. We state both of these applications as the following proposition.

▶ **Proposition 3.** *There is a deterministic polynomial time algorithm for* NC^0_2-Avoid *when* $m > n$. *Using the same framework, there is a polynomial time algorithm for solving* $\{\text{And}_k, \text{Or}_k\}$-Avoid, *for a fixed* k.

We now demonstrate the main application of this framework. Towards describing the setting of our application, we study the complexity of Avoid in the restricted case of when the circuit is monotone. A first observation is that by applying De Morgan's law we can reduce the Avoid (when $m > 2n$) in polynomial time to $\mathcal{C}$-Avoid where $\mathcal{C}$ is restricted to monotone circuits, where reduction preserves the depth of the circuit and at most doubles the size of the circuit. Hence, solving Avoid even for monotone circuits implies breakthrough circuit lower bounds. We provide the details of the following proposition in the full version [16].

▶ **Proposition 4.** *If* $m > 2n$, Avoid *reduces to* Monotone-Avoid *in polynomial time.*

Therefore, we can restrict our attention to solving the range avoidance problem for the monotone circuits. We also observe the following corollary for the case of NC^0_k circuits, which are the current boundary for polynomial time algorithms solving Avoid. For any $k > 0$, if $m > 2n$, NC^0_k-Avoid reduces to Monotone-NC^0_{2k}-Avoid in deterministic polynomial time. In particular, when $m > 2n$, NC^0_3-Avoid reduces to Monotone-NC^0_6-Avoid in deterministic polynomial time. In the remaining part of the paper, we use the framework in Theorem 1 to show polynomial time algorithms for Monotone-NC^0_3-Avoid and related problems.

Deterministic Polynomial Time Algorithm for Monotone-NC^0_3-Avoid with Quadratic stretch

We now show the main technical application of the framework in the case when $\mathcal{F}$ contains only MAJ function on k inputs, with the additional constraint that two functions should depend on at most one common input variable. As per the above formulation, this makes the hypergraph to be linear and k-uniform. In the setting of k-uniform linear hypergraphs, using Turan-type results for specifically designed graphs in $\mathcal{H}$, we can use Theorem 1 for deriving polynomial time algorithms for solving Avoid for specific classes of circuits.

Turan-type extremal problems for hypergraphs were introduced by [1] and bounds are known (see [12] for a survey) for $\mathsf{ex}_k(n, \mathcal{H})$ for a few hypergraphs $\mathcal{H}$. Bounds are known for when $\mathcal{H}$ is a $k \times k$ grid [5], wickets [19], Fano plane [13] (see section 2 for a brief overview of Turan-type extremal problems, and exact definition of these hypergraphs). We exhibit several different fixed hypergraphs H and use them to provide different proofs of the following algorithmic upper bound

▶ **Theorem 5.** Monotone-NC^0_3-Avoid *when* $m > cn^2$ *for any constant* c *can be solved in deterministic polynomial time.*

Our proof relies on a reduction of the problem to a more restricted case of Monotone-NC^0_3-Avoid where each individual output function is the majority of three input bits. We call this version MAJ_3-Avoid. Notice that for any two functions, there can be at most two input bits that both of them can depend on. By using a combinatorial argument on the circuit, we show how to reduce MAJ_3-Avoid to the case where two functions can depend on at most one common input. We denote this version as One-Intersect-MAJ_3-Avoid. As mentioned above, using Theorem 1 along with the fact that H_C is k-uniform, and $\mathcal{F} = \{\text{MAJ}_3\}$, we

design polynomial time algorithms for One-Intersect-MAJ$_3$-Avoid where each function is majority on three inputs and two different functions depend on at most one common input bit. We exhibit several different hypergraphs H - wickets (Lemma 18), $k \times k$ grid (Lemma 20), and several other structures k-cage, weak Fano plane, (k, ℓ)-butterfly, (k, ℓ)-odd kite (see full version [16]) which can be also used to derive polynomial time algorithms for One-Intersect-MAJ$_3$-Avoid. Extremal bounds are known for $\mathsf{ex}_k(n, \mathcal{H})$ only when $\mathcal{H}$ is a weak Fano plane, 3×3 grid and the wicket. The best known bounds for $\mathsf{ex}_k(n, \mathcal{H})$ among these are when $\mathcal{H}$ is the hypergraph called a wicket (see Section 2 for a definition), and hence we use it in the proof of Theorem 5.

Observing that the above reduction to the monotone case also doubles the number of input bits on which each function depends, for NC^0_k-Avoid with $m > 2n$, this implies a reduction to $\mathsf{Mon\text{-}NC}^0_{2k}$-Avoid. We use Theorem 1, with $\mathcal{H}$ as the $k \times k$ grid (see (Lemma 20)), we show that:

▶ **Theorem 6.** *There is a deterministic polynomial time algorithm for* One-Intersect-MAJ$_k$-Avoid *when* $m > n^2$.

However, unlike the case when $k = 3$, it is unclear how to reduce Monotone-NC^0_k-Avoid to MAJ$_k$-Avoid, and then further to One-Intersect-MAJ$_k$-Avoid. Indeed, designing polynomial time algorithms for Monotone-NC^0_6-Avoid itself with $m = n + O(n^{2/3})$ already leads to explicit construction of rigid matrices[6], which is an important problem in the area.

Deterministic Polynomial Time Algorithm for Monotone-NC^0_3-Avoid with Linear Stretch

Deterministic polynomial time algorithms are known[6] for NC^0_3-Avoid when $m \geq \frac{n^2}{\log n}$ and as mentioned above, improving the stretch constraint to $m = n + O(n^{2/3})$ would imply explicit construction of rigid matrices. We next aim to improve the stretch requirement in the above theorem to linear (in fact, to just n), thus improving the known bounds for the case of Monotone-NC^0_3-Avoid.

Towards this, notice that the above argument for Theorem 5 uses the bounds for the Turan number from the literature in a black-box manner. In fact, the quadratic constraints on the stretch function m that we have imposed in Theorem 5 can be relaxed by using stricter variants of the *power-bound conjecture* in the context of Turan numbers of linear hypergraphs. In particular, [7] conjectures that there exists an ϵ such that any linear hypergraph on n vertices having more than $\omega(n^{2-\epsilon})$ edges must contain a $(u, u-4)$-hypergraph[2] as a subgraph. The specific hypergraph of wicket is an example of a $(9, 5)$-hypergraph. However, there are $(9, 5)$-hypergraphs which are not suitable for our purpose. Hence, we need a stronger variant of this conjecture, which insists on having a wicket as a subgraph instead of just $(9, 5)$-subgraphs.

Specifically, in the case of wickets, which we critically use in our argument for Theorem 5, [4, 19] conjectured that the Turan number for 3-uniform hypergraphs avoiding wickets is $n^{2-o(1)}$, and this will directly improve the constraint on m as $m = n^{2-o(1)}$ for Theorem 5 as well.

To go beyond the limitations posed by the above structures for which Turan number bounds are classically studied, we define a new notion of cycles called $\mathcal{X}_{2\ell}$-cycles in 3-uniform linear hypergraphs. We first show a new Turan-type theorem for such cycles for connected hypergraphs, which might be of independent interest.

[2] A $(u, u-4)$-hypergraph in this context is a 3-uniform linear hypergraph which has $u-4$ edges spanning u vertices.

▶ **Theorem 7.** *Any connected 3-uniform linear hypergraph with $m > n$ edges must contain a loose $\mathcal{X}_{2\ell}$ cycle.*

In the context of One-Intersect-MAJ$_3$-Avoid where $m > n$, using the above theorem, we show that the corresponding hypergraph H_C contains a loose $\mathcal{X}_{2\ell}$ cycle. Using the framework of Theorem 1, where we show (Lemma 26) there exists an edge-coloring of $\mathcal{X}_{2\ell}$ which is not MAJ-coloring. In addition, we note that although the cycle is not of fixed size, we can find it in H_C in polynomial time. This gives us the following theorem.

▶ **Theorem 8** (Main Theorem). *For $m > n$,* Monotone-NC^0_3-Avoid *can be solved in deterministic polynomial time.*

Thus, while the applicability of Theorem 1 seems to impose constraints such as k-uniformity and one-intersection to the cases of Avoid that they can be used to solve, the above theorem indicates that along with other combinatorial reductions to such special cases, it can still lead to useful bounds for the more general problem.

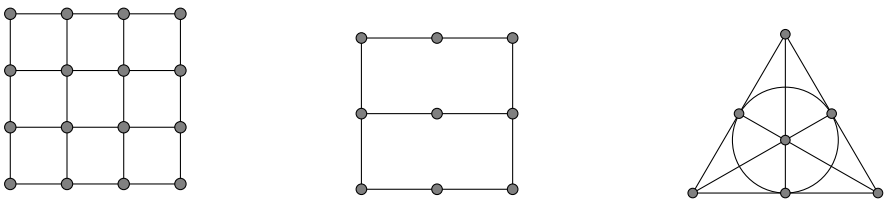
2 Preliminaries

We study the range avoidance problem for restricted circuit classes. $\mathcal{C}$-Avoid is the following problem: Given a multi-output circuit $C : \{0,1\}^n \to \{0,1\}^m$ such that $m > n$ where each output function can be computed by a circuit in class $\mathcal{C}$, find a $y \in \{0,1\}^m$ which is outside the range of C.

Hypergraphs

We collect the preliminaries from the theory of hypergraphs that we use in the paper. We will work with hypergraphs $G(V, E)$ where $E \subseteq 2^V$. A hypergraph is linear if two edges intersect in at most one vertex - $\forall e_1, e_2 \in E$, $|e_1 \cap e_2| \leq 1$. A hypergraph is said to be k-uniform if every edge has exactly k elements from V - $\forall e \in E, |e| = k$. We will be working with k-uniform linear hypergraphs, and in particular with $k = 3$. We will define some of the hypergraphs that are used in the paper.

A $k \times k$ grid in a hypergraph is a set of k^2 vertices $\{v_{11}, v_{12}, \ldots v_{kk}\}$ such that there are edges $r_1, r_2, \ldots r_k, c_1, c_2, \ldots c_k \in E$, corresponding to the vertices in the rows and columns respectively, when the vertices are arranged in the row-major order (See Figure 1(a)). A particular special hypergraph (for $k = 3$) is called a *wicket* is the 3×3 grid with one edge removed (See Figure 1(b)).



(a) (b) (c)

Figure 1 (a) 4×4 grid (b) A wicket (c) Fano plane.

A Berge path is defined as $v_1 e_1 v_2 \ldots v_k e_k v_{k+1}$ where each edge e_i contains vertices v_i, v_{i+1}. A Berge path is said to be even if k is even and odd otherwise. We say a hypergraph is connected if there exists a Berge path between any two pairs of vertices.

Turan-type Problems in Hypergraphs

One of the classical extremal Turan-type problems introduced in the context of hypergraphs by [1] is the following: *for a fixed set of k-uniform hypergraphs $\mathcal{H}$, what is the maximum number of edges that a k-uniform linear hypergraph can have, if it does not contain a subgraph isomorphic to any of the hypergraphs in the collection $\mathcal{H}$ of hypergraphs.* This number is denoted by $\mathsf{ex}_k(n, \mathcal{H})$. The Turan density of $\mathcal{H}$ is denoted by $\pi(\mathcal{H}) = \lim_{n\to\infty} \left(\binom{n}{k}^{-1} \mathsf{ex}_k(n, \mathcal{H})\right)$. It is known that a k-uniform hypergraph $\mathcal{H}$ has $\pi(\mathcal{H}) = 0$ (also called *degenerate*) if and only if it is k-partite.

The special 3-uniform hypergraph called *wicket*, denoted by W (see figure 1(b)), forms an important step in our argument. We will need the following result about 3-uniform linear hypergraphs due to Solymosi [19].

▶ **Proposition 9** ([19]). *If a 3-uniform linear hypergraph does not contain a wicket, then the number of hyperedges is bounded by $o(n^2)$.*

A slightly weaker result was proven by [10] where they showed that $\mathsf{ex}_3(n, W) \leq \frac{(1-c)n^2}{6}$. Another standard hypergraph that we will be using is the *Fano plane*. A Fano plane F is a 3-uniform linear hypergarph which is isomorphic to the hypergraph $H(V, E)$ with vertex set $V = [7]$ and edge set $E = \{\{1,2,3\},\{3,4,5\},\{1,5,6\},\{3,6,7\},\{2,5,7\},\{1,4,7\},\{2,4,6\}\}$. The following result shows a bound on $\mathsf{ex}_3(n, F)$.

▶ **Proposition 10** ([13]). *If a 3-uniform linear hypergraph contains more than $\binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}$ then it must contain a Fano plane as a sub-hypergraph.*

Coming to k-uniform hypergraphs, we use the following result about $k \times k$ grid G_k, which exhibits a bound for $\mathsf{ex}_k(n, \{G_k\})$.

▶ **Proposition 11** ([5]). *Let H be a k-uniform linear hypergraph with $m > \frac{n(n-1)}{k(k-1)}$ edges. Then H contains a $k \times k$ grid G_k as a sub-hypergraph.*

Cycles in Hypergraphs

Various notions of cycles have been explored in the hypergraph setting. [3] consider the notion of linear cycles of length ℓ (denoted by C_ℓ) which is defined as set of ℓ edges such that each pair of adjacent edges e_i, e_{i+1} (modulo ℓ) intersect in exactly one vertex and each pair of non-adjacent edges are disjoint. [3] show that $\mathsf{ex}_k(n, C_{2\ell}) \leq c_{k,\ell} n^{1+\frac{1}{\ell}}$ and $\mathsf{ex}_k(n, C_{2\ell+1}) \leq c'_{k,\ell} n^{1+\frac{1}{\ell}}$. Another notion of cycles which is well-studied is that of *Berge cycle of length ℓ*, denoted by $\mathcal{B}_\ell$ which consists of ℓ distinct edges such that each hyperedge e_i contains vertices v_i, v_{i+1} where $1 \leq i < \ell$ and the edge e_ℓ contains vertices v_1, v_ℓ where $v_1, \ldots v_\ell$ are distinct vertices. [9] show that $\mathsf{ex}_k(n, \mathcal{B}_{2\ell}) = d_{k,\ell} n^{1+\frac{1}{\ell}}$ and $\mathsf{ex}_k(n, \mathcal{B}_{2\ell+1}) = d'_{k,\ell} n^{1+\frac{1}{\ell}}$.

$\mathcal{X}_{2\ell}$-cycles and loose $\mathcal{X}_{2\ell}$-cycles

We define a new notion of cycles called $\mathcal{X}_{2\ell}$-cycles in 3-uniform linear hypergraphs. We define $\mathcal{X}_{2\ell}$ to be a relaxation of $\mathcal{B}_{2\ell}$ where $\exists e_i, e_j$ such that $|e_i \cap e_j| = 1$, $i + 1 \equiv j \mod 2$ and $|i - j| > 2$. We call the (e_i, e_j)-pair a χ-structure in the cycle. Alternatively, we can view $\mathcal{X}_{2\ell}$ using the lens of the Berge path. $\mathcal{X}_{2\ell}$ can be thought as $\mathcal{B}_{2\ell_1} \cup \mathcal{B}_{2\ell_2} \cup \chi$ where $\mathcal{B}_{2\ell_1} = u_1 e_1 u_2 \ldots u_{2\ell_1} e_{2\ell_1} u_{2\ell_1+1}$, $\mathcal{B}_{2\ell_2} = v_1 e'_1 v_2 \ldots v_{2\ell_2} e'_{2\ell_2} v_{2\ell_2+1}$, $\chi = (e, e')$ where $e, e' \notin \{e_1, \ldots e_{2\ell_1}, e'_1, \ldots e'_{2\ell_2}\}$ and $e = \{u_1, 2u_{\ell_1}, w\}, e' = \{v_1, 2v_{\ell_2}, w\}$. If we relax the Berge paths to Berge walks (denoted by $\mathcal{P}_\ell$) by allowing repetition of edges and vertices, we get a loose $\mathcal{X}_{2\ell}$ cycle. In section 4, we shall prove extremal bounds on this structure and use it to solve range avoidance for restricted circuit classes.

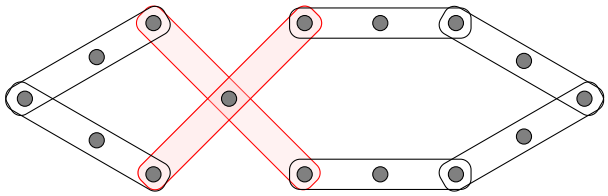


Figure 2 A χ_8 cycle.

3 Range Avoidance for NC^0_k via Hypergraphs

In this section, we describe the main technical tool of the paper by formulating the range avoidance problem as a way of avoiding certain hypergraphs, leading to a Turan-type formulation of the problem.

We recall the notation from the introduction: for an NC^0_k circuit $C = (f_i)_{i\in[m]} : \{0,1\}^n \to \{0,1\}^m$, let $\mathcal{H}_C$ denote the hypergraph defined as follows: The set of vertices is $[n]$. For $1 \le i \le m$, define a hyperedge e_i as $\{x_j \mid j \in [n], x_j \in I(f_i)\}$. Thus $E = \{\{e_i \mid i \in [m]\}\}$ has exactly m edges, each of size at most k. Let $\mathcal{F}$ be the family of functions that appear in the circuit C, and let $\phi : E \to \mathcal{F}$ be a labelling of the edges of the hypergraph H with the corresponding function. A 2-coloring of the hyperedges of $\mathcal{H}$, is said to be a *ϕ-coloring* if there is a vertex coloring which induces this edge coloring via ϕ. We argue the following:

▶ **Theorem 1.** *Let $C : \{0,1\}^n \to \{0,1\}^m$ be a circuit and H_C be the corresponding hypergraph and let ϕ be the labelling function. Suppose H_C contains a sub-hypergraph H and an edge-coloring of H which is not a ϕ-coloring, where both the subhypergraph and the edge-coloring can be found in polynomial time. Then, there is a polynomial time algorithm to solve* Avoid *for C.*

Proof. Let H_C be the hypergraph corresponding to the circuit C. Let H be a sub-hypergraph of fixed size ℓ in H_C. We note that since H is of fixed size, we can exhaustively search for a copy of H in H_C in polynomial time. Let $\Gamma : E(H) \to \{R, B\}$ be an edge-coloring of H which is not a ϕ-coloring. Let C' be the circuit corresponding to the hypergraph H. Let $y = y_1 y_2 \dots y_m$ be defined as follows:

$$y_i = \begin{cases} *, & \text{if } e_i \notin E(H) \\ 0, & \text{if } e_i \in E(H) \text{ and } \Gamma(e_i) = R \\ 1, & \text{otherwise} \end{cases}$$

where $*$ denotes that y_i can take any value in $\{0,1\}$. We will show that $y \notin \mathsf{Range}(C)$. Let $\Delta : \{R, B\} \to \{0,1\}$ be a function such that $\Delta(R) = 0, \Delta(B) = 1$. For the sake of brevity, let $\Delta(x) = \Delta(x_1, \dots x_n) = \Delta(x_1)\Delta(x_2)\dots\Delta(x_n)$. It suffices to show the following claim.

▷ Claim 12. $y \in \mathsf{Range}(C)$ if and only if $\Delta(y)$ is a valid ϕ-coloring.

Proof. Notice that $y \in \mathsf{Range}(C)$ if and only if $\exists x \in \{0,1\}^n$ such that $C(x) = y$. This is equivalent to $\exists \Pi : V \to \{R, B\}$ such that the corresponding $\Gamma : E \to \{R, B\}$ is a ϕ-coloring. Indeed, the latter equivalence can be obtained by setting Π, Γ such that $\Pi(v_i) = \Delta(x_i)$ and $\Gamma(e_j) = \Delta(y_j)$ for all $i \in [n], j \in [m]$. ◁

Since H_C has an edge coloring Γ which is not a ϕ-coloring, by Claim 12 we obtain a $y \in \{0,1\}^m$ which is outside $\mathsf{Range}(C)$. ◀

3.1 Warm-up: Algorithm for NC^0_2-Avoid

As mentioned in the introduction, [8] described a polynomial time algorithm for solving the range avoidance problem for the NC^0_2 circuit. In the following, we show an alternate proof of the same as an application of the framework described above.

▶ **Proposition 13.** *There is a polynomial time algorithm for* NC^0_2-Avoid *when* $m > n$.

Proof. Let $C : \{0,1\}^n \to \{0,1\}^m$ be the given NC^0_2 circuit. [8] observed that there are only four types of NC^0_2 functions possible: And, Or, Parity and constant functions. We can check the type of function in polynomial time. Suppose there is a constant function f in the circuit. Wlog, let this function be a constant zero function. Then, by setting the output of f to 1 and the other output bits arbitrarily, we get a string that is outside the range of the circuit.

Now we consider the other case when the circuit contains only $\{\wedge, \vee, \oplus\}$ gates, where inputs may be negated. We consider the graph $\mathcal{H}_C$ corresponding to the circuit C. Since $m > n$, the graph H_C corresponding to the circuit C indeed contains a cycle Q. Furthermore, we can find this cycle in polynomial time. Thus, it suffices to obtain an edge-coloring of Q which is not a ϕ-coloring. We consider the following cases based on the function types of the edges participating in the cycle Q:

Case 1: Suppose each edge in Q computes a Parity of two inputs which may be negated. Consider an edge f with endpoints x_i, x_j. Consider the path P starting at x_i and ending at x_j obtained by deleting f from Q. Let Γ color all the edges in $Q - f$ to R.
Let $x_i = b \in \{R, B\}$. Notice that setting the function to R fixes the color of every other vertex to either b or $\overline{b}$ depending on the function. In particular, it fixes $x_j = b' \in \{b, \overline{b}\}$. This fixes the value of the function computed at f, and hence the color of f. Thus, flipping the color of f would produce an edge-coloring that is not achievable. Therefore, the following coloring Γ is not a ϕ-coloring.

$$\Gamma(e) = \begin{cases} R & \text{if } e \in Q - f \\ B & \text{if } e = f \text{ and } b' = b \\ R & \text{if } e = f \text{ and } b' = \overline{b} \end{cases}$$

Case 2: Suppose there exists an edge e_1 in Q which computes And or or of two inputs which may be negated. Let the cycle Q be $e_1 e_2 \dots e_\ell e_1$. Consider the path $e_1 e_2 \dots e_{\ell-1}$ obtained by deleting the edge e_ℓ from Q. Consider the following edge-coloring $\Gamma : Q - e_\ell \to \{R, B\}$

$$\Gamma(e) = \begin{cases} B & \text{if } e \in Q - e_\ell,\ \phi(e) = \wedge \\ R & \text{if } e \in Q - e_\ell \text{ and } \phi(e) \in \{\vee, \oplus\} \end{cases}$$

Let $e_1 \cap e_\ell = \{x_i\}$ and $e_\ell \cap e_{\ell-1} = \{x_j\}$. We would like to show that by coloring the edges of $Q - e_\ell$ according to Γ, we end up fixing the colors of x_i, x_j thus fixing the color of e_ℓ or we obtain an inconsistency. In the former case, flipping the color of e_ℓ we obtain an edge-coloring which is not a ϕ-coloring.
It suffices to show that Γ fixes the color of e_ℓ or we obtain an inconsistent coloring for $Q - e_\ell$. We prove this by induction on the length of the path. For the base case, wlog let $\phi(e_1) =$And. According to our Γ, we color e_1 to B, which fixes the color of both vertices incident on e_1. By induction hypothesis, let Γ fix the colors of all vertices participating in the path $e_1 e_2 \dots e_{k-1}$ for $k \in [\ell]$. In the other case, when the coloring is inconsistent, we are already done. Thus, we would like to argue that Γ fixes the other endpoint y of e_k where $y \notin e_{k-1}$. Notice that, since one input feeding into e_k is already fixed Γ gives

a way to fix the other input y to $b \in \{R, B\}$ by setting the color of e_k appropriately or we get an inconsistency at this stage. Thus, inductively either we get an inconsistent coloring (in this case, color e_ℓ arbitrarily) or we end up fixing the colors of x_i, x_j which in turn fixes the color of the edge e_ℓ.
Hence, flipping the color of ℓ fixed by the above assignment produces an edge-coloring which is not a ϕ-coloring.

By the above case analysis, we have an edge-coloring of H which is not a ϕ-coloring. By theorem 1 we obtain a string outside the range of the circuit in polynomial time. ◀

3.2 Warm-up 2: Polynomial time algorithm for {AND, OR}-AVOID

As our next application, we show a polynomial time algorithm for solving the range avoidance problem when each output function computes an AND or OR function of k input bits for a fixed $k > 0$. Additionally, it satisfies the constraint that any two output functions share at most one input. Thus, the hypergraph H_C corresponding to C is a k-uniform linear hypergraph. The sub-hypergraphs we will use in this case are k-crown and C^*. As our next application, we show a polynomial time algorithm for solving the range avoidance problem when each output function computes an AND or OR function of k input bits for a fixed k. The sub-hypergraphs we will use in this case are k-crown and C^*. A *k-crown* is a k-uniform linear hypergraph consisting of k disjoint hyperedges $\{e_1, \dots e_k\}$ and an additional hyperedge e_0 intersecting each hyperedge $e_1, \dots e_k$ exactly once. C^* is a variant of the k-crown where the edges $e_1, \dots e_{k-2}$ intersect in a common vertex $v \notin e_0$. We will use the following extremal bound on these subhypergraphs.

▶ **Proposition 14** ([20]). *Let $\mathcal{H}$ be a k-uniform linear hypergraph with $m > \frac{k(k-2)(n-s)}{k-1}$ where s is the number of vertices with degree at least $(k-1)^2 + 2$. Then $\mathcal{H}$ contains a k-crown or C^*.*

▶ **Proposition 15.** *Let $C : \{0,1\}^n \to \{0,1\}^m$ be a multi-output circuit where $m > \frac{k(k-2)(n-s)}{k-1}$ and each output function computes AND_k or OR_k for fixed k. Additionally, any two output functions share at most one input. Then, there is a polynomial time algorithm for solving* AVOID *on C.*

Proof. We will show an edge coloring of the hypergraphs k-crown or C^* which is not a ϕ-coloring.

$$\Gamma(e) = \begin{cases} B \text{ if } e \in \{e_1, \dots e_k\}, \phi(e) = \text{AND} \\ R \text{ if } e \in \{e_1, \dots e_k\}, \phi(e) = \text{OR} \\ B \text{ if } e = e_0, \exists i \in [k] \text{ such that } \phi(e_i) = \text{OR} \\ R \text{ otherwise} \end{cases}$$

By Proposition 14 we have that if $m > \frac{k(k-1)(n-s)}{k-1}$ then $\mathcal{H}$ contains a C^* or k-crown. Furthermore, we can find these subhypergraphs in polynomial time. By the above argument, we can find an edge coloring of C^* and k-crown which is not a ϕ-coloring. By theorem 1 we can find a string outside the range of C in polynomial time. ◀

We remark that the above special case of AVOID ({AND, OR}-AVOID) can be solved by a direct algorithm similar to the algorithm for NC^0_2-AVOID due to [8]. Nevertheless, the above method is yet another demonstration of our framework for solving the AVOID problem.

3.3 Polynomial time Algorithm for One-Intersect-MAJ$_3$-Avoid

In this subsection, we show the first application of our formulation of the problem in terms of hypergraphs. We apply it to describe a deterministic polynomial time algorithm for One-Intersect-MAJ$_3$-Avoid.

For our purpose, we are interested in a special case of ϕ-coloring when $\mathcal{F} = \{\text{MAJ}\}$. We define this formally below:

▶ **Definition 16** (MAJ-coloring). *Let $H(V, E)$ be a k-uniform hypergraph. We say $\Gamma : E \to \{R, B\}$ is a* MAJ*-coloring if there exists a vertex coloring $\Pi : V \to \{R, B\}$ such that $\forall e \in E(H)$ we have $\Gamma(e) = B \iff \exists S \subseteq e, |S| \geq \left\lfloor \frac{|V|}{2} \right\rfloor + 1$ such that $\forall v \in S,\ \Pi(v) = B$.*

We have the following corollary from Theorem 1.

▶ **Corollary 17.** *Let $C : \{0,1\}^n \to \{0,1\}^m$ be a circuit where each output function computes a* MAJ *function. Let H_C be the corresponding hypergraph. Suppose H_C contains a subhypergraph H such that there is an edge-coloring of H which is not* MAJ*-coloring. Then there is a polynomial time algorithm to find a string outside the range of C.*

Thus it is sufficient to exhibit an explicit fixed size linear 3-uniform hypergraph H which satisfies the conditions of Corollary 17.

Hypergraph Structure: Wickets

A *wicket* is defined as a 3×3 grid with one edge removed. By Proposition 9, if $\mathcal{H}$ contains more than $o(n^2)$ edges, then it contains a wicket. We will show that there is an edge coloring of a wicket which is not a MAJ-coloring. We note that *Proposition* 9 requires the hypergraph to be linear and hence using wickets as subhypergraph would yield an algorithm for One-Intersect-MAJ$_3$-Avoid and not the more general case of MAJ$_3$-Avoid.

▶ **Lemma 18.** *Let $\mathcal{H}$ be a wicket. There exists an edge-coloring $\Gamma : E \to \{R, B\}$ such that it is not a* MAJ*-coloring. Furthermore, we can find this in polynomial time.*

Proof. Wlog let the edge e removed from the wicket be a column edge. Let $E = E_1 \cup E_2$ where E_1 be the set of three row edges and E_2 denote the set of column edges. We will show that the following $\Gamma : E \to \{R, B\}$ is not a MAJ-coloring: $\Gamma(e) = \begin{cases} R & \text{if } e \in E_1 \\ B & \text{if } e \in E_2 \end{cases}$.

Let U be the set of vertices upon which the edges of E_2 are incident. For the edges in E_1 to be colored R, there must be at least three vertices in U that should be colored R. Similarly, for the edges in E_2 to be B, there must be at least 4 vertices in U that should be colored B. Thus, totally there should be at least 7 distinct vertices in U. But $|U| = 6$, which is a contradiction. Hence, Γ is not a MAJ-coloring. ◀

The following is an easy corollary of Corollary 17 and Lemma 18.

▶ **Corollary 19.** *Let $C : \{0,1\}^n \to \{0,1\}^m$ be an instance of* One-Intersect-MAJ$_3$-Avoid *with $m = \Omega(n^2)$. Then we can find a $y \in \{0,1\}^m$ outside the range of C in polynomial time.*

3.4 Polynomial time Algorithm for One-Intersect-MAJ$_k$-Avoid

In this subsection, we show a second application of the framework proposed in Theorem 1. Let $C : \{0,1\}^n \to \{0,1\}^m$ be a multi-output circuit such that each output function computes MAJ of k input bits. Let $\mathcal{H}$ be a k-uniform hypergraph obtained from C as in the case of Theorem 1. Since every pair of output functions intersects in at most one input variable, we have that $\mathcal{H}$ is a linear k-uniform hypergraph.

Recall Proposition 11, which argued that a k-uniform hypergraph that has more than $\frac{n(n-1)}{k(k-1)}$ hyperedges must have a $k \times k$ grid contained in it. We will show that if $\mathcal{H}$ contains a $k \times k$- grid then we can solve range avoidance for the corresponding circuit in polynomial time and hence this will imply an algorithm for One-Intersect-MAJ$_k$-Avoid by using the same framework as Theorem 1.

▶ **Lemma 20.** *There exists an edge-coloring $\Gamma : E \to \{R, B\}$ of $k \times k$-grid which is not a* MAJ*-coloring. Furthermore, we can find Γ in polynomial time.*

Proof. Let $\mathcal{H}(V, E)$ be a $k \times k$ grid. Let $E = E_1 \cup E_2$ where E_1, E_2 are the sets of edges corresponding to the rows and columns of the grid, respectively. We define an edge-coloring

$$\Gamma(e) = \begin{cases} R & \text{if } e \in E_1 \\ B & \text{otherwise} \end{cases}$$

We will show that Γ is not a valid MAJ-coloring. We consider the following cases based on the parity of k:

Case 1: Suppose k is odd. For each $e \in E_1$ to be colored R, at least $\lfloor \frac{k}{2} \rfloor + 1$ of its vertices should be colored R. Since the edges in E_1 are pairwise disjoint, at least $k\left(\lfloor \frac{k}{2} \rfloor + 1\right)$ distinct vertices should be colored R. Similarly for edges in E_2, we have that at least $k\left(\lfloor \frac{k}{2} \rfloor + 1\right)$ vertices in V should be colored B. Hence, there should be at least $2k\left(\lfloor \frac{k}{2} \rfloor + 1\right) > k^2$ vertices in V, which is a contradiction.

Case 2: Suppose k is even. By the previous argument, for edges in E_2 to get color B at least $k\left(\frac{k}{2} + 1\right)$ vertices should be colored B. For edges in E_1 at least $k\left(\frac{k}{2}\right)$ vertices should be colored R. So totally, there are at least $k(k+1) > k^2$ vertices in V, which is again a contradiction.

In either case, we obtain a contradiction. Hence, Γ is not a MAJ-coloring. ◀

By proposition 11 and lemma 20, we have the following result:

▶ **Theorem 21.** *Let $C : \{0,1\}^n \to \{0,1\}^m$ be an instance of* One-Intersect-MAJ$_k$-Avoid *and $m > \frac{n(n-1)}{k(k-1)}$. There is a polynomial time algorithm to find a string outside the range of C.*

4 Polynomial time Algorithm for Monotone-NC$^0_3$-Avoid

In this section, we describe a deterministic polynomial time algorithm for Monotone-NC$^0_3$-Avoid for $m = \Omega(n^2)$. The algorithm proceeds in three steps. In the first step, we give a polynomial time reduction from Monotone-NC$^0_3$-Avoid to MAJ$_3$-Avoid. Next, we show a reduction from MAJ$_3$-Avoid to One-Intersect-MAJ$_3$-Avoid. Finally, we put all the pieces together and describe a polynomial time algorithm for solving MAJ$_3$-Avoid when $m = \Omega(n^2)$.

4.1 Reduction from Monotone-NC$^0_3$-Avoid to MAJ$_3$-Avoid

We show the following reduction:

▶ **Theorem 22.** *There is a polynomial time reduction from* Monotone-NC^0_3-Avoid *to* MAJ_3-Avoid.

Proof. Let $C : \{0,1\}^n \to \{0,1\}^m$ be a monotone circuit with $m > n$. We will obtain a circuit $C' : \{0,1\}^{n'} \to \{0,1\}^{m'}$ such that $n' \leq n$,$m' \leq m$ and $n' < m'$, where each output function computes majority of three input bits. Furthermore, we can find a $y = y_1 y_2 \dots y_m$ outside the $\mathsf{Range}(C)$ from $y' \notin \mathsf{Range}(C')$ in polynomial time by setting the remaining $m - m'$ many bits of y to arbitrary values. We remark that this proof technique is inspired by [8].

To begin with, note that there are only a few types of monotone functions that depend on 3 bits. For each of these cases we will show a reduction from $C : \{0,1\}^n \to \{0,1\}^m$ to a smaller circuit $C' : \{0,1\}^{n-1} \to \{0,1\}^{m-1}$ such that a string outside the range of C' gives a string which is outside the range of C. Let C_j denote the sub-circuit of C that computes the j-th bit of the output. Let $W_i := \{x \in \{0,1\}^3 \mid |x|_1 = i\}$ be the set of inputs with weight i, for $i \in \{0,1,2,3\}$. We describe the reduction for each type of function except when the function is the MAJ_3 function. We iteratively apply this sequence of reduction rules for each $j \in \{1, 2 \dots m\}$, we will end up with a circuit where each output bit is the MAJ_3 function in terms of input variables.

Case 1: $\boldsymbol{C_j^{-1}(0) = \emptyset}$**:** That is C_j computes a constant 1 function. Then setting $y_1 = 0$ and the remaining output bits arbitrarily gives a string outside the range of C.

Case 2: $\boldsymbol{C_j^{-1}(0) = W_0}$**:** Suppose C_j computes OR of three input variables x_1, x_2, x_3. Then setting $y_1 = 0$ and all the input variables x_1, x_2, x_3 to 0 we obtain a smaller circuit $C' : \{0,1\}^{n-3} \to \{0,1\}^{m-1}$ such that $y' \notin \mathit{Range}(C')$ then $y = 0y'$ is outside the range of C.

Case 3: $\boldsymbol{C_j^{-1}(0) = W_0 \cup \{\alpha\}}$ where $\alpha \in W_1$: Without loss of generality, we assume $\alpha = 001$. In this case, setting $y_1 = 0$ and the input variables x_1, x_2 to 0 we obtain a smaller circuit $C' : \{0,1\}^{n-2} \to \{0,1\}^{m-1}$. The same argument applies when $\alpha = 100$, and $\alpha = 010$.

Case 4: $\boldsymbol{C_j^{-1}(0) = W_0 \cup \{\alpha_1, \alpha_2\}}$ where $\alpha_1, \alpha_2 \in W_1$: Without loss of generality, let $\alpha_1 = 001$, and $\alpha_2 = 010$. By the property of monotone functions, we have that C_j evaluates to 1 on inputs $\{101, 110, 111\}$. Then setting $y_1 = 0$ and $x_1 = 0$, yields a circuit $C' : \{0,1\}^{n-1} \to \{0,1\}^{m-1}$. Same argument applies in the case when $\alpha_1 = 010$, $\alpha_2 = 100$ and also in the case when $\alpha_1 = 001, \alpha_2 = 100$.

Case 5: $\boldsymbol{C_j^{-1}(0) = W_0 \cup W_1 \cup \{\alpha\}}$ where $\alpha \in W_2$: Without loss of generality, assume $\alpha = 011$. Setting $y_1 = 1$ and $x_1 = 1$ we get a smaller circuit $C' : \{0,1\}^{n-1} \to \{0,1\}^{m-1}$. The other cases when $\alpha = 101$ and $\alpha = 110$ can be handled similarly.

Case 6: $\boldsymbol{C_j^{-1}(0) = W_0 \cup W_1 \cup \{\alpha_1, \alpha_2\}}$ where $\alpha_1, \alpha_2 \in W_2$: Without loss of generality, assume $\alpha_1 = 011, \alpha_2 = 110$. Setting $y_1 = 1$ and $x_1 = 1$ we get a smaller circuit $C' : \{0,1\}^{n-1} \to \{0,1\}^{m-1}$. The same argument applies when $W_2 \setminus \{\alpha_1, \alpha_2\} = 110$ or 011.

Case 7: $\boldsymbol{C_j^{-1}(0) = W_0 \cup W_1 \cup W_2}$**:** In this case, C_j computes AND of three input variables x_1, x_2, x_3. Then setting $y_1 = 1$ and all the input variables x_1, x_2, x_3 to 1 we obtain a smaller circuit $C' : \{0,1\}^{n-3} \to \{0,1\}^{m-1}$ such that $y' \notin \mathit{Range}(C')$ then $y = 1y'$ is outside the range of C.

Case 8: $\boldsymbol{C_j^{-1}(0) = W_0 \cup W_1 \cup W_2 \cup W_3}$**:** That is, C_j computes a constant zero function. Then setting $y_1 = 1$ and the remaining output bits arbitrarily gives a string outside the range of C.

It can be verified that the only function that is not covered in the above cases is when C_j computes Maj on 3 bits. Note that in each case we eliminate one output bit and at least one input bit. Thus, finally we are left with a circuit $C' : \{0,1\}^{n'} \to \{0,1\}^{m'}$ where $m' > n'$ and each output bit is computed by MAJ_3 function.

We note that we can check the type of function computed by a circuit by simply evaluating the function on all possible input values. Since the function depends on only three variables there are only 8 input possibilities to check. Hence, overall the reduction can be done in polynomial time. ◀

4.2 Polynomial time Algorithm for $\mathrm{MAJ_3}$-Avoid with Quadratic Stretch

We show that there is a deterministic polynomial time algorithm for $\mathrm{MAJ_3}$-Avoid. Recall that $\mathrm{MAJ_3}$-Avoid instance is a circuit $C : \{0,1\}^n \to \{0,1\}^m$ with constant depth, bounded fan-in and polynomial size, where each output function of the circuit is a majority of exactly three input variables. For each $i \in [m]$, let f_i denote the function corresponding to the i-th output bit, and let $I(f_i)$ denote the set of three variables that f_i depends on. We shall demonstrate the working of the algorithm in three steps.

For our purpose, we shall define sub-circuits of C called *clusters* (denoted by K) as follows: Let $\mathcal{O} = \{f_1, \dots f_m\}$ be the set of output functions of C. We define the relation $R : \mathcal{O} \times \mathcal{O}$ as follows: We say $(f_i, f_j) \in R$ if $\exists$ some input x that feeds into both f_i, f_j for $i, j \in [m]$. Let R' be the transitive closure of R. Observe that R' is an equivalence relation. We define a cluster to be an equivalence class of R'.

Step 1: Reduction to a single cluster

The following lemma shows that C must contain a cluster with more outputs than inputs and hence we can concentrate on such a cluster.

▶ **Lemma 23.** *Let $C : \{0,1\}^n \to \{0,1\}^m$ be a multi-output circuit with $m > n$, then there exists a cluster K such that $|K| > |I(K)|$ where $I(K) = \cup_{f \in K} I(f)$.*

Proof. Let $K_1, \dots K_t$ be the clusters such that $\cup_{i \in [t]} K_i$ covers the set of output functions of C. Observe that $\cup_{i \in [t]} I(K_i)$ covers the input set of C. By definition, $\forall i, j \in [t]$ such that $i \neq j$ we have that $K_i \cap K_j = \emptyset$ and $I(K_i) \cap I(K_j) = \emptyset$. Assume for the sake of contradiction that for each cluster K_i, we have $|K_i| \leq |I(K_i)|$. Then by the above observation, we have that $\left|\cup_{i \in [t]} K_i\right| \leq \left|\cup_{i \in [t]} I(K_i)\right|$ implying that $m < n$, which is a contradiction. ◀

Let K be a cluster guaranteed by proposition 23. Let C_K be the sub-circuit of C corresponding to the cluster K. Since, $|K| > |I(K)|$, there must be a string outside the range of C_K. Observe that if $y' \in \{0,1\}^{|K|} \notin \mathsf{Range}(C_K)$ then any $y \in \{0,1\}^m$ which agrees with y' is outside $\mathsf{Range}(C)$. Hence, it suffices to solve the problem on C_K.

It is easy to see that if there exist two output functions that intersect in three inputs, then we already have a string outside the range. Hence, we consider the cases when they intersect in fewer than three inputs. Motivated by this, we define the following problems: Two-Intersect-$\mathrm{MAJ_3}$-Avoid (One-Intersect-$\mathrm{MAJ_3}$-Avoid) is the following problem: Given $C : \{0,1\}^n \to \{0,1\}^m$ such that output function is $\mathrm{MAJ_3}$ function and any two functions share at most two (resp. one) inputs, the goal is to find $y \notin \mathsf{Range}(C)$.

Step 2 : A reduction to One-Intersect-$\mathrm{MAJ_3}$-Avoid

For the rest of the proof, $C = C_K$, $m = m'$ and $n = n'$. We show that there is a polynomial time reduction from Two-Intersect-$\mathrm{MAJ_3}$-Avoid problem to One-Intersect-$\mathrm{MAJ_3}$-Avoid problem. Finally, in step 3, we will show that there is a deterministic polynomial time algorithm for One-Intersect-$\mathrm{MAJ_3}$-Avoid problem.

▶ **Lemma 24.** *There is a polynomial time reduction from* Two-Intersect-MAJ$_3$-Avoid *to* One-Intersect-MAJ$_3$-Avoid.

Proof. Let $C : \{0,1\}^n \to \{0,1\}^m$ be a circuit such that each output bit is computed by a MAJ$_3$ function where $m > n$. By step 1, we know that the set of functions corresponding to C form some cluster K. Hence, we obtain C starting with an arbitrary function in K and iteratively adding a new function f to $\mathcal{O}$, such that $I(f) \cap I(\mathcal{O}) \neq \emptyset$. Our algorithm will follow this construction procedure, and eliminate functions that appear in that order by setting input variables.

We start with the following claim that helps us eliminate functions from the circuit by setting input variables. Let f_1, f_2 be two output functions of C such that $|I(f_1) \cap I(f_2)| = 2$. Initially $\mathcal{O} = \{f_1, f_2\}$, and we iteratively apply the following claim for each function $g \in K \setminus \mathcal{O}$ that is used to build the cluster by the above process. A variable in $I(K)$ is said to be *alive* if it is not set to a value in $\{0,1\}$ in the below process.

▷ Claim 25. $\forall g \in K \setminus \mathcal{O}$ such that $|I(g) \cap I(\mathcal{O})| = 2$, there is a setting of $g \in \{0,1\}$ and a consistent partial assignment to the input variables (or identification of variables), such that at most one variable in $\mathcal{O}$ remains alive.

Proof. We shall prove this by induction on the construction of the cluster. Initially, let $\mathcal{O} = \{f_1, f_2\}$, $I(\mathcal{O}) = \{x_1, x_2, x_3, x_4\}$. Without loss of generality, let $I(f_1) = \{x_1, x_2, x_3\}$ and $I(f_2) = \{x_2, x_3, x_4\}$. We observe that if $f_1 = 1$ and $f_2 = 0$, then $x_1 = 1, x_4 = 0, x_3 = \neg x_2$ and the number of variables in $I(\mathcal{O})$ is exactly 1.

By hypothesis, the number of variables in $I(K)$ which are alive after i steps is at most 1. Now consider a g such that $|I(g) \cap I(K)| = 2$, we would like to show that there is an assignment of values to g (which also sets some input variables) such that in the new cluster, $K \cup \{g\}$, the number of variables that are alive is still at most 1. The following cases arise depending on the two inputs g share with the cluster. Let $z_1, z_2 \in I(g) \cap I(K)$ and let $y = I(g) \setminus I(K), b \in \{0,1\}$.

Case 1: *Both z_1, z_2 are not alive:* There are two cases to consider. Suppose $z_1 = z_2 = b$. By setting $g = \overline{b}$, we can obtain a string that is outside the range of C. The other case is when $z_1 = b$, $z_2 = \overline{b}$. In this case, we set $g = b$ and $y = b$. Therefore, the number of variables in the new cluster is still at most 1.

Case 2: *Exactly one of z_1, z_2 is alive:* Again two cases arise. Consider the first case when $z_1 = b$, $z_2 = x$. Now, we set $g = x = y = \overline{b}$ eliminating all the variables. The other case is: $z_1 = b$, $z_2 = \neg x$. Similarly, we set $g = \overline{b}$, $x = b$, $y = \overline{b}$ which eliminates all the variables.

Case 3: *Both z_1, z_2 are alive:* If z_1, z_2 that are both alive, then it must be that $z_1 = x, z_2 = \neg x$. In this case, we set $g = y = 0$ so that the number of variables in the cluster is at most 1.

Hence, in all the cases there is an assignment value to g and consistent assignment for variables in $I(g)$ such that there is at most one variable which is alive in $I(\mathcal{O})$. ◁

Now, we consider the functions h such that $|I(h) \cap I(\mathcal{O})| = 1$. Let $w := I(h) \cap I(\mathcal{O})$. If w is assigned to $b \in \{0,1\}$, then we set h to $\overline{b}$ and the input variables in $I(h) \setminus \{w\}$ to $\overline{b}$. Consider the other case when w is alive. Wlog let $w = x$. We set the output of h and its other two inputs to $\overline{x}$. In this case, we have not yet assigned a fixed 0/1 value to h, which we shall fix in the next step. However, note that in either case, the number of alive variables in $\mathcal{O}$ is at most 1.

After at most $n-2$ iterations, we would have fixed each of the inputs to one of $0, 1, x$ or $\neg x$. This is because, in the iterative process, each function that we add to the set $\mathcal{O}$ covers at least one new variable. Now consider an $f \in K \setminus \mathcal{O}$, which is guaranteed since $m > n$. Observe that $I(f) \subseteq I(\mathcal{O})$. Let $I(f) = \{x_1, x_2, x_3\}$. The following two cases arise based on the number of alive variables.

Case 1: Suppose there is no variable in $\mathcal{O}$ which is alive. Then the value of f is already fixed by the inputs $I(\mathcal{O})$. Wlog let this be $b \in \{0,1\}$. Setting the output of f to $\overline{b}$ gives a string outside $\mathsf{Range}(C)$.

Case 2: Suppose there is exactly one variable that is alive. Wlog let $x_1 = x$.

Case 2(a): Consider the case when $x_2 = x_3 = b$ for $b \in \{0,1\}$. Notice that this already forces f to take value b. Thus, by setting f to $\overline{b}$ and the functions outputs of functions that were set to x to b we obtain a $y \in \{0,1\}^m$ which is outside $\mathsf{Range}(C)$.

Case 2(b): Suppose $x_2 = b, x_3 = \overline{b}$. Then setting the output of f to b, we fix the value of x to b. Thus, at this stage, there are no variables that are alive in $\mathcal{O}$. Since, $|\mathcal{O}| \leq n-1$ and $m > n$ there exists a $g \in K \setminus \mathcal{O}$ such that $I(g) \subseteq I(\mathcal{O})$. Since all inputs are fixed, we can now handle this using case 1.

Case 3: Suppose f has two variables that are alive. Let $x_1 = b$ where $b \in \{0,1\}$.

Case 3(a): Suppose $x_2 = x_3 = x$. By setting the output of f to $\overline{b}$, we fix the value of x_2, x_3 to $\overline{b}$. This eliminates all the variables from $\mathcal{O}$. Again, this reduces to case 1.

Case 3(b): Suppose $x_2 = x, x_3 = \overline{x}$. This fixes the value of f to b. Hence, setting the output of f to $\overline{b}$ gives a string outside the range.

Case 4: Suppose all the three variables of f are alive. Note that since $|\mathcal{O}| \leq n-1$ and $m > n$, there must be at least two more functions outside $\mathcal{O}$. If any of these functions satisfy the above cases, then we have already found a solution to the problem. Therefore, we consider the case when all three functions have all three variables that are alive. By PHP, there exist two functions at least two of whose inputs are set to $x(\overline{x})$. This fixes the output of these functions to $x(\overline{x})$. Setting one of the outputs to 1 and the other to 0 yields a string outside the range.

Thus, if there exist two functions in C that have at least two common inputs, then as described above, we already have a solution to the range avoidance problem. Otherwise, we have an instance of One-Intersect-MAJ$_3$-Avoid. Note that the above steps can be done in polynomial time. Hence, it suffices to show a polynomial time algorithm for this case. ◀

Having described all the ingredients of the proof, we are now ready to prove our first algorithm for MAJ$_3$-Avoid (which works only for quadratic stretch $m > cn^2$). We apply step 1 to obtain a sub-circuit C_K whose outputs form a cluster K. We note that there must be a cluster K with n' inputs and $m' > cn'^2$ outputs. Otherwise, if each cluster with n_i inputs has less than cn_i^2 outputs then we get $m = \sum cn_i^2 < cn^2$, which is a contradiction. Next, we observe that any two functions in C_K share at most two input variables since otherwise we can trivially obtain a string outside the range by assigning opposite values to the corresponding output bits. By step 2, we reduce our problem to an instance of One-Intersect-MAJ$_3$-Avoid in polynomial time. Since $m > cn^2$, Proposition 9 gives a polynomial time algorithm for solving One-Intersect-MAJ$_3$-Avoid.

This gives a polynomial time algorithm for solving Monotone-NC^0_3-Avoid with quadratic stretch, thus completing the proof of the following theorem from the introduction.

▶ **Theorem 5.** Monotone-NC^0_3-Avoid *when $m > cn^2$ for any constant c can be solved in deterministic polynomial time.*

5 A New Turan-type Bound and Application to Monotone-$\mathbf{NC}^0_3$-Avoid

In this section, we prove a new Turan-type theorem for 3-uniform linear hypergraphs. We show the application of the same to obtain an efficient algorithm for One-Intersect-MAJ$_3$-Avoid with $m > n$, thus proving Theorem 8.

5.1 Extremal Bounds for Loose $\mathcal{X}_{2\ell}$-Cycles

We first show an extremal bound for the loose $\mathcal{X}_{2\ell}$ cycle in a connected 3-uniform linear hypergraph.

▶ **Theorem 7.** *Any connected 3-uniform linear hypergraph with $m > n$ edges must contain a loose $\mathcal{X}_{2\ell}$ cycle.*

Proof. Let $\mathcal{H}$ be a connected 3-uniform linear hypergraph with $m > n$. We would like to show that there is a loose $\mathcal{X}_{2\ell}$ cycle in $\mathcal{H}$. Towards this, we construct *block graph* G corresponding to $\mathcal{H}$ as follows: let the set of m hyperedges in $\mathcal{H}$ be the vertices of G and add an edge (f, g) if the hyperedges f, g intersect at a vertex. Note that G is a simple graph since $\mathcal{H}$ is linear. Observe that G is connected. We have $|G| = m$ since the vertices in G correspond to the hyperedges in $\mathcal{H}$. Now we shall argue that finding a copy of loose $\mathcal{X}_{2\ell}$ cycle in $\mathcal{H}$ is equivalent to finding a subgraph G' in G where G' consists of an edge (f, f') and distinct odd walks from g to g' and h to h' without using edge (f, f'), where g, h and g', h' are neighbors of f, f' respectively in G. The equivalence follows from the observation that a walk of length ℓ corresponds to a walk of length $\ell - 1$ in G. Thus, our task is to show that $G' \subseteq G$.

For each vertex x_i in $\mathcal{H}$, there is a clique of size m_i in G, where $i \in [n], 1 \leq m_i \leq m$. Furthermore, each vertex in G participates in 3 cliques, since $\mathcal{H}$ is 3-uniform. The number of edges in $G = \sum_{i\in[n]} \binom{m_i}{2}$.

$$
\begin{aligned}
|E(G)| = \sum_{i\in[n]} \binom{m_i}{2} &= \sum_{i\in[n]} \frac{m_i^2}{2} - \sum_{i\in[n]} \frac{m_i}{2} \\
&= \sum_{i\in[n]} \frac{m_i^2}{2} - \frac{3m}{2} && \left(\because \sum_{i\in[n]} m_i = 3m\right) \\
&\geq \frac{(3m)^2}{2n} - \frac{3m}{2}
\end{aligned}
$$

The last inequality follows from the Cauchy-Schwartz inequality by taking u as all 1 vectors and v_i as m_i. Since $m > n$, we have $|E(G)| > n$. Hence, G contains a cycle. Let $e = (f, f')$ be an edge in this cycle. We would like to show that $G \setminus e$ contains $G' \setminus e$. Suffices to show that there is a walk of odd length from g to g'. The argument for odd walk from h to h' is symmetric. Together with the edges $e, (f, g), (f, h), (f', g'), (f', h')$ this gives the desired subgraph G'. Observe that $G \setminus e$ is still connected. Hence, there must be a path P from g to g'. If this path is of odd length then we have our desired path. Otherwise, we shall obtain a walk from g to g' of odd length. By handshaking lemma, we have that the average degree of $G \setminus e$ is $\geq \frac{2(|E|-1)}{m} \geq \frac{2\left(\frac{(3m)^2}{2m} - \frac{3m}{2} - 1\right)}{m} \geq 6 - \frac{2}{m}$. By averaging argument there exists a vertex f'' with minimum degree $6 - \frac{2}{m}$. Since $m > 2$, we have that the degree of f'' is at least 6. Since f'' participates in at most three cliques, by PHP we get that there is a clique of size at least three which contains f''. Hence, there is an odd cycle Q containing f''. Let P'

be a path from g to f'', whose existence is guaranteed as $G \setminus e$ is connected. Observe that $gP'f''Qf''P'gPg'$ is a walk of odd length from g to g'. The other walk containing h, h' can be obtained similarly. Hence, there is a copy of G'(loose $\mathcal{X}_{2\ell}$ cycle) in $G(\mathcal{H})$. This completes the proof. ◀

5.2 Polynomial time Algorithm for MONOTONE-NC^0_3-AVOID with Linear Stretch

In section 4.2, we saw that using wicket as our candidate for forbidden sub-hypergraph gives a polynomial time algorithm for MONOTONE-NC^0_3-AVOID with quadratic stretch. In order to improve this bound on the stretch requirement, we shall use different forbidden sub-hypergraphs which are loose $\mathcal{X}_{2\ell}$-cycles. It suffices to show that the structure has an edge-coloring which is not MAJ-coloring and that the extremal number for the structure is not too large.
First, we show that there exists an edge-coloring of $\mathcal{X}_{2\ell}$ which is not MAJ-coloring.

▶ **Lemma 26.** *Let $\mathcal{H}$ be $\mathcal{X}_{2\ell}$-cycle. Then there exists an edge-coloring of $\mathcal{H}$ which is not* MAJ*-coloring.*

Proof. Let $v_1e_1v_2 \dots v_{2\ell}e_{2\ell}v_1$ be the $\mathcal{X}_{2\ell}$ cycle with (e_i, e_j) as χ-structure where $i \equiv 0 \mod 2$, $j \equiv 1 \mod 2$. Consider the following edge-coloring-

$$\Gamma(e_k) = \begin{cases} R, & \text{if } i \equiv 0 \mod 2 \\ B, & \text{if } i \equiv 1 \mod 2 \end{cases}$$

Let $e_i \cap e_j = \{x\}$. First, consider the case when $\Pi(x) = R$. This forces the color of the other two endpoints of e_j to B. Iteratively, we get $\Pi(v_j) = B$ iff $j \equiv 1 \mod 2$. This implies $\Gamma(e_i) = B$ which is a contradiction. The other case when $\Pi(x) = B$ is similar. ◀

For our purpose, we shall work with loose $\mathcal{X}_{2\ell}$ cycles instead of $\mathcal{X}_{2\ell}$ cycles. In fact, the edge-coloring demonstrated in the proof of Lemma 26 is also an edge-coloring of loose $\mathcal{X}_{2\ell}$-cycle which is not a MAJ-coloring.

▶ **Corollary 27.** *Let $\mathcal{H}$ be a loose $\mathcal{X}_{2\ell}$-cycle. Then there exists an edge-coloring of $\mathcal{H}$ which is not a* MAJ*-coloring.*

Finally, combining all the results above, we shall now prove our first algorithm for MONOTONE-NC^0_3-AVOID with linear stretch.

▶ **Theorem 8** (Main Theorem)**.** *For $m > n$,* MONOTONE-NC^0_3-AVOID *can be solved in deterministic polynomial time.*

Proof. By Lemma 23 and Lemma 24, we first reduce the circuit to a sub-circuit corresponding to cluster, and then further to the case of ONE-INTERSECT-MAJ$_3$-AVOID. Hence, it suffices to solve ONE-INTERSECT-MAJ$_3$-AVOID. By theorem 7 we know that there exists a loose $\mathcal{X}_{2\ell}$ in the hypergraph corresponding to the given circuit. It suffices to show that we can find loose $\mathcal{X}_{2\ell}$ in $\mathcal{H}$ in polynomial time. By corollary 27 and theorem 17 we get a polynomial time algorithm to solve MAJ$_3$-AVOID. Indeed to find G' in the proof of Theorem 7, we need to find the paths P, Q', and a high degree vertex f'', which can be done in polynomial time. Hence, we can find the G' in polynomial time. ◀

Algorithm 1 Algorithm for solving MAJ$_3$-Avoid on input $C : \{0,1\}^n \to \{0,1\}^m$.

Step 1: Obtain a cluster K such that $|I(K)| < |K|$ ▷ by observation 23
Step 2: Reduction from Two-Intersect-MAJ$_3$-Avoid to One-Intersect-MAJ$_3$-Avoid
if $f_1, f_2 \in K$ such that $I(f_1) = \{x_1, x_2, x_3\}$ and $I(f_2) = \{x_2, x_3, x_4\}$ **then**
  Set $f_1 = 1, f_2 = 0$ and $x_1 = 1, x_2 = 0, x_4 = \neg x_3$, $\mathcal{O} = \{f_1, f_2\}$
  repeat
    Consider $g \in K \setminus \mathcal{O}$ such that $I(g) \neq \emptyset$
    if $|I(g) \cap I(\mathcal{O})| = 2$ **then**
      Let $I(g) = \{z_1, z_2, z_3\}$ and $z_3 = I(g) \setminus I(\mathcal{O})$. Let $b \in \{0,1\}$
      if Both z_1, z_2 are not alive: **then**
        If $z_1 = z_2 = b$ By setting $g = \overline{b}$, we obtain $y \notin \mathsf{Range}(C)$
        If $z_1 = b$, $z_2 = \overline{b}$ then set $g = b$ and $y = b$
      else if Exactly one of z_1, z_2 is alive: say $z_1 = b$, $z_2 = x$ **then**
        Setting $g = z_3 = \overline{b}$ eliminates all the variables
      else if Both z_1, z_2 are alive: that is $z_1 = x, z_2 = \neg x$ **then**
        We set $g = z_3 = 0$ so that the number of variables in the cluster is at most 1.
      end if
    else if $I(g) \cap I(\mathcal{O}) = \{z_1\}$ **then**
      If $z_1 = b \in \{0,1\}$, then Set g to $\overline{b}$ and $z_2 = z_3 = \overline{b}$
      If $z_1 = x$(alive) then set g to $\overline{x}$ and $z_2 = z_3 = \overline{x}$
      Update the cluster $\mathcal{O} = \mathcal{O} \cup \{g\}$ and $I(\mathcal{O}) = I(\mathcal{O}) \cup I(g)$
    end if
  until $\exists g \notin \mathcal{O}$ such that $I(g) \cap I(\mathcal{O}) \neq \emptyset$
end if
$\exists h_1, h_2, h_3 \notin \mathcal{O}$ such that $I(f) = \{w_1, w_2, w_3\} \subseteq I(\mathcal{O})$ since $|\mathcal{O}| \leq n-2$ and $|I(\mathcal{O})| = n$.
if $\nexists$ a variable which is alive **then**
  Setting the output of h_1 to $\overline{b}$ gives a string outside the range.
else if $\exists$ exactly one variable w_1 which is alive **then**
  if $w_2 = w_3 = b$ **then**
    Setting the output of h_1 to $\overline{b}$ gives a string outside the range.
  else if $w_2 = b, w_3 = \overline{b}$ **then**
    Set the output of h_1 to b, which fixes w_1 to b.
    We have $h_2 \notin \mathcal{O}$ and $I(h_2) \subseteq I(\mathcal{O})$. Same as the previous case.
  end if
else if h_1 has two variables that are alive and $w_3 = b$ **then**
  if $w_1 = w_2 = x$ **then**
    We set the output of h_1 to $\overline{b}$. This fixes all input variables (case 1).
  else if $w_1 = x, w_2 = \overline{x}$ **then**
    Setting the output of h_1 to $\overline{b}$ gives a string outside the range.
  end if
else if All three variables are alive **then**
  $\exists\ h_i, h_j$ whose inputs are set the same. Setting h_i to 1 and h_j to 0 gives $y \notin \mathsf{Range}(C)$.
end if
Step 3: Polynomial time algorithm for One-Intersect-MAJ$_3$-Avoid
Find a loose $\mathcal{X}_{2\ell}$ in H_C ▷ by Theorem 7
Output $y \notin \mathsf{Range}(C)$ ▷ by Corollary 17

6 Conclusion

We described the formulation of special cases of Avoid in terms of Turan-type problems in k-uniform hypergraphs and demonstrated some applications to solve monotone versions of Avoid under depth restrictions for the circuit. We exhibited several different fixed hypergraphs H - k-cage, weak Fano plane, 3×3 grid, $(3,3)$-butterfly, $(3,3)$-odd kite which can be also used to derive polynomial time algorithms for One-Intersect-MAJ$_3$-Avoid which in turn is used to solve Monotone-NC^0_3-Avoid when the stretch is quadratic. Finally, the improvement to linear stretch (Theorem 8) comes from using loose $\mathcal{X}_{2\ell}$ cycles as our hypergraph. We note that this is not a fixed-sized hypergraph; however, we can find it in polynomial time. Prior to this work, the best known polynomial time algorithm Monotone-NC^0_3-Avoid was via the algorithm for NC^0_3-Avoid which requires $m > \frac{n^2}{\log n}$ [6]. We extend our framework to solve One-Intersect-MAJ$_k$-Avoid for linear and quadratic stretch respectively. [6] give a polynomial time algorithm for solving NC^0_k-Avoid when $m > \frac{n^{k-1}}{\log n}$. it would be interesting to improve the stretch using the hypergraph framework even for monotone NC^0_6 circuits, which would in turn give an improvement for the case of NC^0_3-Avoid.

References

1 WG Brown, Paul Erdos, and VT Sós. Some extremal problems on r-graphs. In *New directions in the theory of graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich, 1971)*, pages 53–63. Academic Press New York, 1973.

2 Lijie Chen, Shuichi Hirahara, and Hanlin Ren. Symmetric Exponential Time Requires Near-Maximum Circuit Size. In *Proceedings of the 56th Annual ACM Symposium on Theory of Computing*, STOC 2024, pages 1990–1999, 2024. doi:10.1145/3618260.3649624.

3 Claton Collier-Cartaino, Nathan Graber, and Tao Jiang. Linear Turán Numbers of Linear Cycles and Cycle-Complete Ramsey Numbers. *Combinatorics, Probability and Computing*, 27(3):358–386, 2018. doi:10.1017/S0963548317000530.

4 Jakob Führer and Jozsef Solymosi. Caps and wickets. *Discrete Mathematics*, 348(3):114334, 2025. doi:10.1016/j.disc.2024.114334.

5 Zoltán Füredi and Miklós Ruszinkó. Uniform hypergraphs containing no grids. *Advances in Mathematics*, 240:302–324, 2013.

6 Karthik Gajulapalli, Alexander Golovnev, Satyajeet Nagargoje, and Sidhant Saraogi. Range Avoidance for Constant Depth Circuits: Hardness and Algorithms. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, (Proceedings of APPROX/RANDOM 2023)*, volume 275, pages 65:1–65:18, 2023. doi:10.4230/LIPICS.APPROX/RANDOM.2023.65.

7 W. T. Gowers and J. Long. The length of an s-increasing sequence of r-tuples. *Combinatorics, Probability and Computing*, 30(5):686–721, 2021. doi:10.1017/S0963548320000371.

8 Venkatesan Guruswami, Xin Lyu, and Xiuhan Wang. Range Avoidance for Low-Depth Circuits and Connections to Pseudorandomness. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (Proceedings of APPROX/RANDOM 2022)*, volume 245, pages 20:1–20:21, 2022. doi:10.4230/LIPIcs.APPROX/RANDOM.2022.20.

9 Ervin Gyori and Nathan Lemons. Hypergraphs with no cycle of a given length. *Combinatorics, Probability and Computing*, 21(1–2):193–201, 2012. doi:10.1017/S0963548311000691.

10 András Gyárfás and Gábor N. Sárközy. The linear turán number of small triple systems or why is the wicket interesting? *Discrete Mathematics*, 345(11):113025, 2022. doi:10.1016/j.disc.2022.113025.

11 Rahul Ilango, Jiatu Li, and R. Ryan Williams. Indistinguishability obfuscation, range avoidance, and bounded arithmetic. In *Proceedings of the 55th Annual ACM Symposium on Theory of Computing (STOC 2023)*, pages 1076–1089, 2023. doi:10.1145/3564246.3585187.

12 Peter Keevash. Hypergraph Turán Problems. *Surveys in Combinatorics*, June 2011.

13 Peter Keevash and Benny Sudakov. The Turán Number Of The Fano Plane. *Combinatorica*, 25, March 2004. doi:10.1007/s00493-005-0034-2.

14 Robert Kleinberg, Oliver Korten, Daniel Mitropolsky, and Christos Papadimitriou. Total Functions in the Polynomial Hierarchy. In *Proceedings of 12th Innovations in Theoretical Computer Science Conference (ITCS 2021)*, volume 185, pages 44:1–44:18, 2021. doi:10.4230/LIPIcs.ITCS.2021.44.

15 Oliver Korten. The Hardest Explicit Construction. In *2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 433–444, 2022. doi:10.1109/FOCS52979.2021.00051.

16 Neha Kuntewar and Jayalal Sarma. Range Avoidance in Boolean Circuits via Turan-type Bounds, 2025. doi:10.48550/arXiv.2503.17114.

17 Zeyong Li. Symmetric Exponential Time Requires Near-Maximum Circuit Size: Simplified, Truly Uniform. In *Proceedings of the 56th Annual ACM Symposium on Theory of Computing*, STOC 2024, pages 2000–2007. Association for Computing Machinery, 2024. doi:10.1145/3618260.3649615.

18 Hanlin Ren, Rahul Santhanam, and Zhikun Wang. On the Range Avoidance Problem for Circuits. In *63rd IEEE Annual Symposium on Foundations of Computer Science, FOCS 2022, Denver, CO, USA, October 31 - November 3, 2022*, pages 640–650. IEEE, 2022. doi:10.1109/FOCS54457.2022.00067.

19 Jozsef Solymosi. Wickets in 3-uniform hypergraphs. *Discrete Mathematics*, 347(6):114029, 2024. doi:10.1016/J.DISC.2024.114029.

20 Lin-Peng Zhang, Hajo Broersma, and Ligong Wang. A note on generalized crowns in linear r-graphs, 2024. arXiv:2401.12339.