Asynchronous Latency and Fast Atomic Snapshot

João Paulo Bezerra
LTCI, Télécom Paris, Institut Polytechnique de Paris, France

Luciano Freitas
Nomadic Labs, Paris, France

Petr Kuznetsov
LTCI, Télécom Paris, Institut Polytechnique de Paris, France

Matthieu Rambaud
LTCI, Télécom Paris, Institut Polytechnique de Paris, France

Abstract

This paper introduces a novel, *fast* atomic-snapshot protocol for asynchronous message-passing systems. In the process of defining what "fast" means exactly, we spot a few interesting issues that arise when conventional time metrics are applied to long-lived asynchronous algorithms. We reveal some gaps in latency claims made in earlier work on snapshot algorithms, which hamper their comparative time-complexity analysis. We then come up with a new unifying time-complexity metric that captures the latency of an operation in an asynchronous, long-lived implementation. This allows us to formally grasp latency improvements of our atomic-snapshot algorithm with respect to the state-of-the-art protocols: optimal latency in fault-free runs without contention, short constant latency in fault-free runs with contention, the worst-case latency proportional to the number of active concurrent failures, and constant *amortized* latency.

2012 ACM Subject Classification Theory of computation → Distributed algorithms

Keywords and phrases Asynchronous systems, time complexity, atomic snapshot, crash faults

Digital Object Identifier 10.4230/LIPIcs.DISC.2025.15

Related Version *Extended Version*: `https://arxiv.org/abs/2408.02562` [11]

Funding *João Paulo Bezerra*: Mazars, TrustShare Innovation Chair.
Petr Kuznetsov: CHIST-ERA-22-SPiDDS-05 (REDONDA project) and Agence Nationale de la Recherche (ANR-23-CHR4-0009).

1 Introduction

The *distributed snapshot* abstraction [14, 26] allows us to determine a consistent view of the global system state. Originally proposed in the asynchronous fault-free message-passing context, it was later cast to shared-memory models [3] as a vector of shared variables, exporting an *update* operation that writes to one of them and a *snapshot* operation that returns the current vector state. Atomic snapshot can be implemented from conventional read-write registers in a *wait-free* manner, i.e., tolerating unpredictable delays or failures of any number of processes. By applying the reduction from shared memory to message-passing [6], one can get an asynchronous distributed atomic-snapshot implementation that tolerates up to a minority of faulty processes. The *atomic-snapshot object* (ASO) is, in a strong sense, equivalent to *lattice agreement* (LA) [8, 17][1]: one can implement the other with no time overhead. A long line of results improve time and space complexities of ASO and LA algorithms in shared-memory [5, 4, 7] and message-passing [17, 21, 19, 16, 18] models.

[1] Lattice agreement can be seen as a weak version of consensus, where decided values form totally ordered *joins* of proposed values in a *join semi-lattice*.

Table 1 Comparative time complexity of atomic-snapshot algorithms in asynchronous message-passing models. The table shows results for Single-Writer Multi-Reader (**SWMR**) implementations.

	Fault-free w/o contention	Fault-free w/ contention	Worst-case	Amortized constant
Faleiro et al. [17]	2	16	$O(k)$	yes
Imbs et al. [21]	2	$O(n)$	$O(n)$	no
Garg et al. [18]	≥ 6	≥ 8	$O(k)$	yes
Garg et al. [18] + Zheng et al. [27]	$O(\log n)$	$O(\log n)$	$O(\log n)$	no
Delporte et al. [16]	2	$O(n)$	$O(n)$	no
This paper	2	8	$O(k)$	yes

In this paper, we focus on the *latency* of operations in message-passing ASO implementations. We propose an LA (and, thus, ASO) algorithm that is *faster* than (or matches) state-of-the-art solutions in all execution scenarios: with or without failures and with or without contention. The comparative analysis of our algorithm with respect to the existing work appeared to be challenging: as we show, earlier work considered diverging metrics and execution scenarios, and sometimes used over-simplified reasoning. We observed that conventional metrics [13, 6, 2] are not always suitable for *long-lived* asynchronous algorithms. Besides, prior latency analyses of ASO and LA algorithms [17, 21, 19, 16, 18] used different ways to measure time, which complicated the comparison. We therefore propose a unifying time-complexity analysis of prior asynchronous ASO and LA algorithms with respect to a new metric, which we take as a contribution on its own.

Lamport [25] proposed to measure time in asynchronous systems as the length of the longest chain of *causally related* messages, the metric used to to determine the best-case latency of consensus [25] and Crusader Agreement [1]. However, as we show in this paper, the metric may produce counter-intuitive results for protocols involving all-to-all communication. For instance, in the failure-free case, n-process reliable broadcast [12] exhibits a causal chain of n hops, even though, intuitively one expects it to terminate in one.

Building upon the classical approach by Canetti and Rabin [13], Abraham et al. [2] recently proposed an elegant metric to grasp the good-case latency of broadcast protocols. We observe, however, that the metric does not really apply to executions of *long-lived* abstractions, which may contain *holes* – periods of inactivity when no protocol messages are in transit. Moreover, we get diverging results when applying [2] and [13] to *operation latency*, i.e., the time between invocation and response events of a given operation.

We therefore extend the round-based approach to long-lived abstractions (such as ASO and LA) and establish a framework to measure the time between arbitrary events, subsequently showing that the results align with those from earlier classical metrics [6, 13].

To summarize, our main contribution is a novel LA (and, thus, ASO) protocol that is generally *faster* than prior solutions, i.e., it exhibits shorter latency of its operations in various scenarios. In our complexity analysis, we compared our protocol to the original long-lived LA algorithm by Faleiro et al. [17][2], the first direct message-passing ASO implementation by Delporte et al. [16], the ASO algorithm based on the *set-constraint broadcast* by Imbs et al. [21], and the ASO algorithms by Garg et al. based on generic construction of ASO from *one-shot* LA with constant latency in fault-free runs [18] or $\log n$ worst-case latency by Zheng et al. [27] (where n is the number of processes).

[2] We consider the ASO protocol built atop the lattice agreement protocol proposed in [17].

As shown in Table 1, in a fault-free run, the latency of an operation of our protocol is the optimal two rounds if there is no contention and eight rounds in the presence of contention (four rounds if we ignore the "buffering" period when a value is submitted but not yet proposed), regardless of the number of contending operations. Moreover, the worst-case latency of our algorithm is proportional to the number of *active failures* k, i.e., the number of faulty processes whose messages are received within the operation's interval, therefore the *amortized* latency (averaged over a large number of operations in a long-lived execution) converges to the fault-free constant.

Our protocol can be seen as a novel combination of techniques employed separately in prior work. These include the use of *generalized* (long-lived) lattice agreement as a basis for ASO [23], the *helping* mechanism where all the learned lattice values are shared [23], relaying of messages to all replicas instead of quorum-based rounds [21, 18, 22, 15], and buffering proposed values until previous proposals get committed [17]. Similar to earlier proposals [17], our algorithm involves $O(n^2)$ (all-to-all) communication, which is compensated by its constant (amortized) latency. An interesting open question is whether one can reduce the communication cost in good runs, while maintaining constant amortized latency.

The paper is organized as follows. In Section 2, we present our model assumptions, and in Section 3, we state the problem of atomic snapshot and relate it to generalized lattice agreement. In Section 4, we present our protocol and analyze its correctness. In Section 5, we discuss several gaps in the complexity analyses of earlier work. In Section 6, we present a comparative analysis of time metrics. Certain proofs and a detailed discussion of time complexity of earlier protocols are delegated to the appendix.

2 System Model

Processes and Channels. We consider a system of n *processes* (or *nodes*). Processes communicate by exchanging messages $m = (s, r, data)$ with a sender s, a receiver r, and a message content *data*.

A process is an automaton modeled as a tuple $(\mathcal{I}, \mathcal{O}, \mathcal{Q}, q_0, \pi)$, where $\mathcal{I}$ is a set of inputs (messages and application calls) it can receive, $\mathcal{O}$ is a set of outputs (messages and application responses), $\mathcal{Q}$ is a (potentially infinite) set of possible internal states, $q_0 \in \mathcal{Q}$ is an initial state and $\pi : 2^{\mathcal{I}} \times \mathcal{Q} \to 2^{\mathcal{O}} \times \mathcal{Q}$ is a transition function mapping a set of inputs and a state to a set of outputs and a new state. Each process i is assigned an algorithm A_i which defines $(\mathcal{I}, \mathcal{O}, \mathcal{Q}, q_0, \pi)$, a *distributed algorithm* is an array $[A_1, ..., A_n]$.

Events and Configurations. Application calls and responses are tuples $(i, aReq)$ and $(i, aRep)$ with a process identifier, a request, and a reply respectively.

An *event* e is a tuple (R, P, S) where R is a set of received messages and/or application calls, P is the set of nodes producing the event and S is a set of messages sent and/or application responses. We denote $\mathsf{receive}(e)$ as the set of messages received in the event, conversely, $\mathsf{send}(e)$ is the set of messages sent. A *message hop* is a pair (e, e') in which e' receives at least one message that was sent in e.

Messages in transit are stored in the *message buffer*.[3] A *configuration* C is an $(n+1)$-array $[M, s_1, ..., s_n]$ with the buffer's state $M = C[0]$ and the local state $s_i = C[i]$ of each node i $(i = 1, \ldots, n)$. Let C_0 denote the *initial configuration* in which every s_i is an initial state and the buffer M is empty.

[3] We assume that every message in the message buffer is unique.

Executions. An *execution* (or *run*) is an alternating sequence $C_0e_1C_1e_2...$ of configurations and events, where for each $j > 0$ and $i = 1, \ldots, n$:

1. $\mathsf{receive}(e_j) \subseteq C_{j-1}[0]$;
2. $e_j.S$ consists of messages and application outputs that the nodes in $e_j.P$ produce, given their algorithms, their states in C_{j-1} and their inputs in $e_j.R$; the nodes in $e_j.P$ carry their states from C_{j-1} to C_j, accordingly;
3. for the nodes $i \notin e_j.P$, $C_{j-1}[i] = C_j[i]$.

Each triple $C_{j-1}e_jC_j$ is called a *step*. In this paper, we consider algorithms defined by deterministic automata, and we assume a default initial state. Thus, we sometimes skip configurations and simply write $e_1e_2 \ldots$.

In an *infinite* execution, a process is *correct* if it takes part in infinitely many steps, and *faulty* otherwise. We only consider infinite executions in which $f < n/2$, where f is the number of faulty processes and n is the total number of processes. Moreover, in an infinite execution, messages exchanged among correct processes are eventually received, i.e., if there is an event e from a correct process sending a message m to another correct process, then there is e' succeeding e such that $m \in \mathsf{receive}(e')$.

We also assume that the communication channels neither alter nor create messages. Finally, we assume that the channels are *FIFO*: messages from a given source to a given destination arrive in the order they were sent. A FIFO channel can be implemented by attaching sequence numbers to messages, without extra communication or time overhead.

3 Lattice Agreement and Atomic Snapshot

3.1 Lattice Agreement

A *join semi-lattice* is defined as a tuple $(\mathcal{L}, \sqsubseteq)$, where $\sqsubseteq$ is a partial order on a set $\mathcal{L}$, such that for any pair of values u and v in $\mathcal{L}$, there exists a unique *least upper bound* $u \sqcup v \in \mathcal{L}$ ($\sqcup$ is called the *join* operator). Also, u and v in $\mathcal{L}$ are said to be *comparable* if $u \sqsubseteq v \vee v \sqsubseteq u$.

The (generalized) *Lattice Agreement* abstraction LA [17] defined over $(\mathcal{L}, \sqsubseteq)$ can be accessed by every node with operation $\mathsf{Propose}(v)$, $v \in \mathcal{L}$ (we say that the node *proposes* v) which triggers the reply event $\mathsf{Learn}(w)$ (we say that the node *learns* w). Each node may invoke Propose any number of times but does so *sequentially*, that is, it initiates a new operation only after the previous one has returned.[4] The abstraction must satisfy:

▶ **Definition 1** (Lattice Agreement (LA))**.**

- **Validity.** *Any value learned by a node is the join of some set of proposed values that includes its last proposal.*
- **Stability.** *The values learned by any node increase monotonically, with respect to* $\sqsubseteq$.
- **Consistency.** *All values learned are comparable, with respect to* $\sqsubseteq$.
- **Liveness.** *If a correct node proposes* v*, it eventually learns a value* w.

[4] Following [23], without loss of generality, we slightly modified the conventional LA interface [8, 17] by introducing the explicit Propose operation that combine proposing and learning the values, the properties of the abstraction are adjusted accordingly.

3.2 Atomic Snapshot Object (ASO)

An atomic snapshot object (ASO) stores a vector of values $R = [r_1, ..., r_m]$ and exports two operations: update(i, v) and snapshot(). The update(i, v) operation writes the value v in $R[i]$ and returns OK, and snapshot() returns the entire vector R. An ASO implementation guarantees that every operation invoked by a correct process eventually completes. It also ensures that each of its operations appears to take effect in a single instance of time within its interval, i.e., it is *linearizable* [20].

We say that an ASO is *single writer* **SW** (resp. *multi writer* **MW**) if for each of its registers $R[i]$, only a single process can call update(i, v) (resp. every process can call update(i, v)). In this paper, we focus mostly on **SWMR** atomic snapshot objects. In Table 1 we give results only for **SWMR**. A **MWMR** ASO can be devised from **SWMR** by adding an additional "read" phase when updating values (see Section 3.3 for more details).

Next, we show that ASO can be implemented on top of LA with no additional overhead.

3.3 From LA to ASO

To implement a **SWMR** ASO on top of LA, we build upon the transformation algorithm introduced in [23]. We consider a partially ordered set $\mathcal{L}^*$ of $(m+n)$-vectors (recall that m is the size of the ASO vector and n is the number of nodes), defined as follows.

A vector position $\ell \in 1, \ldots, m$ is defined as a tuple $(w, v) \in R_\ell$, where v is an element of a value set V equipped with a total order $\leq^V$, and $w \in \mathbb{N}$ is the number of write operations on position ℓ. A total order on R_ℓ is defined in the natural way: for any two tuples $(w_1, v_1) \leq^{R_\ell} (w_2, v_2) \equiv (w_1 < w_2) \vee (w_1 = w_2 \wedge v_1 \leq^V v_2)$. For each process $i = 1, \ldots, n$, the vector position $m + i$ stores the number of snapshot operations executed by i.

The lattice $\mathcal{L}^*$ of $(m+n)$-position vectors is then the composition $R_1 \times \ldots \times R_m \times \mathbb{N}^n$. The partial order $\sqsubseteq^*$ on $\mathcal{L}^*$ is then naturally defined as the compositions of $<^{R_1} \times \ldots \times <^{R_m} \times \leq^n$. The composed join operator $\sqcup^*$ is the composition of max operators, one for each position in the $(m+n)$-position vectors. The construction implies a join semi-lattice [23].

In Algorithm 1, we show how to implement an **SWMR** atomic snapshot on top of LA defined over the semi-lattice $(\mathcal{L}^*, \sqsubseteq^*, \sqcup^*)$. For simplicity, we assume that $m = n$, i.e., the size of the array is the total number of nodes, and that each node i has a dedicated register i where it can write. Elements of $\mathcal{L}^*$ are then $2n$-vectors.

When a node i calls update(i,v), it increments its local *writing* sequence number w and proposes a $2n$-vector with (w, v) in position i and initial values in all other positions to the LA object. The vector learned from this proposal is ignored. When the node i calls snapshot(), it increments its local *reading* sequence number r proposes a $2n$-vector with r in position $n + i$ and initial values in all other positions to the LA object. The values in the first n positions of the returned vector is then returned as the snapshot outcome.

Algorithm 1 can be extended to implement a **MWMR** ASO: to update a position j in the array, a node first takes a snapshot to get the current state, gets up-to-date sequence number in position j and proposes its value with a higher sequence number. With this modification, the update operation takes two LA operations instead of one. We refer the reader to [23] for further details.

▶ **Theorem 2.** *Algorithm 1 implements ASO.*

Proof. See Appendix A. ◀

Algorithm 1 LA → **SWMR** ASO conversion.

```
1: Distributed objects:
2:     LA instance on (ℒ*, ⊑*, ⊔*)

3: upon startup
4:     w ← 0
5:     r ← 0

6: operation update(i, v)
7:     w ← w + 1
8:     V ← 2n-vector with (w, v) in position i and initial values in all other positions
9:     LA.Propose(V)

10: operation snapshot()
11:     r ← r + 1
12:     V ← 2n-vector with r in position n + i and initial values in all other positions
13:     return (LA.Propose(V))[1..n]
```

4 LA Protocol

In Algorithm 2, we describe our protocol for solving LA. To guarantee amortized constant complexity, the protocol relies on two basic mechanisms, employed separately in earlier work [17, 23]. First, when a node receives a request (e.g., a value from the application), it first adds the request to a buffer (*MPool*) and then relays it before starting a proposal. This ensures that "idle" nodes also help in committing the request. Second, the node relays every learned value so that nodes that are "stuck" can adopt values from other nodes.

4.1 Overview

The protocol is based on helping: every node tries to commit every proposed value it is aware of. As long as the node has *active* proposals that are not yet committed, it buffers newly arriving proposals in the local variable *MPool*. Intuitively, in the worst case, an LA.Propose operation has to wait until *one* of the concurrently invoked LA.Propose operations complete. Once this happens, the currently buffered value is put in the local dictionary *Pending* and shared with the other nodes (lines 31 and 32) via a **PROPOSE** message. In turn, the other nodes relay the message to each other (line 38). The dictionary maps a value to the number of times it is "supported" by the nodes (using **PROPOSE** messages). Once a value v in the dictionary assembles a quorum of $n-f$ of $\langle \mathbf{PROPOSE}, v \rangle$ messages, i.e., $Pending[v] \geq n-f$ (line 39), the value is added to the *Validated* variable. Once every value currently stored in *Pending* is in *Validated* (line 41), the operation completes with *Validated* as the learned value. As the final element of the helping mechanism, each process broadcasts every value it learns (lines 45 and 51), ensuring that processes that might otherwise remain "stuck" can complete their current proposal.

In summary, the algorithm relies on four main ideas: 1) buffering incoming requests when already proposing, 2) sharing every received proposal so all processes are quickly aware of active ones, 3) initiating a new proposal only after all currently seen proposals have been validated, and 4) broadcasting learned values to help other processes make progress.

Message Complexity. The protocol is comprised of three all-to-all communication phases: processes send and relay requests at lines 23 and 27, proposals at lines 32 and 38, and accepted values at lines 45 and 51. The total number of messages is therefore $O(n^2)$. However, a value in a **PROPOSE** message can include up to n distinct requests, and a value in a **ACCEPT** message may have arbitrary size. Therefore, in the extended version [11], we present a refined protocol description in which processes exchange $O(n^2)$ messages *per individual request*. This efficiency is achieved by relaying only the differences between current and previously received proposals and the learned values in phases 2 and 3, thus eliminating redundant messages with the same requests.

Algorithm 2 Long-Lived LA: code for node x.

14: **upon** Startup
15: $MPool, Proposing, Validated, Learned \leftarrow \bot$
16: $Pending \leftarrow \emptyset$

17: **operation** Propose(v)
18: SendRequest(v)
19: wait until $v \sqsubseteq Learned$
20: **return** $Learned$

21: **operation** SendRequest(v)
22: $MPool \leftarrow MPool \sqcup v$
23: **send** $\langle$**REQUEST**, $v\rangle$ **to** every other node

24: **upon** Receive $\langle$**REQUEST**, $v\rangle$ **from** a node
25: **if** $v \not\sqsubseteq MPool \sqcup Proposing \sqcup Learned$ **then**
26: $MPool \leftarrow MPool \sqcup v$
27: **send** $\langle$**REQUEST**, $v\rangle$ **to** every other node

28: **upon event** $(MPool \neq \bot) \wedge (Proposing = \bot)$
29: $Proposing \leftarrow MPool$
30: $MPool \leftarrow \bot$
31: $Pending[Proposing] \leftarrow 1$
32: **send** $\langle$**PROPOSE**, $Proposing\rangle$ **to** every other node

33: **upon** Receive $\langle$**PROPOSE**, $v\rangle$ **from** a node
34: **if** $v \in Pending$.keys() **then**
35: $Pending[v] + +$
36: **else**
37: $Pending[v] \leftarrow 1$
38: **send** $\langle$**PROPOSE**, $v\rangle$ **to** every node

39: **upon** exists v s.t. $Pending[v] = n - f$
40: $Validated \leftarrow Validated \sqcup v$

41: **upon event** $\bigsqcup Pending$.keys() $\sqsubseteq Validated$
42: **if** $Learned \sqsubset Validated$ **then**
43: $Learned \leftarrow Validated$
44: $Proposing \leftarrow \bot$
45: **send** $\langle$**ACCEPT**, $Learned\rangle$ **to** every node

46: **upon** Receive $\langle$**ACCEPT**, $w\rangle$ **from** a node
47: **if** $(Proposing \sqcup Learned \sqsubseteq w)$ **then**
48: $Validated \leftarrow Validated \sqcup w$
49: $Learned \leftarrow w$
50: $Proposing \leftarrow \bot$
51: **send** $\langle$**ACCEPT**, $Learned\rangle$ **to** every node

4.2 Correctness

Validity and Stability are immediate. We now proceed with Consistency and Liveness.

▶ **Lemma 3.** *If nodes i and j learn, resp., values w_i and w_j, then w_i and w_j are comparable.*

Proof. Suppose that $(w_i \not\sqsubseteq w_j) \wedge (w_j \not\sqsubseteq w_i)$. Then there must exist $v_i \sqsubseteq w_i$ and $v_j \sqsubseteq w_j$ such that $v_i \not\sqsubseteq w_j$ and $v_j \not\sqsubseteq w_i$.

Let Q_i (resp. Q_j) be the quorum i used to include v_i *Validated* at line 39. Since $Q_i \cap Q_j \neq \emptyset$, there is a common node x that sent $\langle \mathbf{PROPOSE}, v_i \rangle$ to i and $\langle \mathbf{PROPOSE}, v_j \rangle$ to j, but since channels are FIFO, either i received v_j or j received v_i from x before learning a value, therefore adding the value to *Pending*. Suppose it was i that received v_j before v_i, from the condition of line 41, i could not have learned w_i if $v_j \not\sqsubseteq$ *Validated*. ◀

▶ **Lemma 4.** *If a correct node x sets Proposing $= v$, x eventually learns a value with v.*

Proof. A node x sends a **PROPOSE** message to every other node whenever it adds a new value to *Pending* (line 38). If x is correct, it will receive at least $n - f$ **PROPOSE** messages for every value in *Pending*, adding the value to *Validated*. Therefore, the condition in line 41 is never satisfied from some point on only if x keeps adding a new value to *Pending* before all the current ones are validated.

Since each node proposes only one value at a time (until it learns a value, lines 28, 44, 50), for x to indefinitely add new values to *Pending*, there must be at least one other node that keeps learning values and proposing new ones. Without loss of generality, let y be one such node. Since faulty nodes eventually crash and stop taking steps, y must be correct. Every time y learns a new value w it sends $\langle \mathbf{ACCEPT}, w \rangle$ to x, and because channels are FIFO, x receives the **ACCEPT** message before the new value proposed by y. Eventually (because x sent its proposal to y), one of the received values w contains $x's$ *Proposing* and the condition on line 46 is satisfied, x then learns w. ◀

▶ **Lemma 5.** *If a correct node calls Propose(v), it eventually sets Proposing $= v'$, $v \sqsubseteq v'$.*

Proof. Let a correct node x call Propose(v), x then includes v in *MPool* (line 22). If x is not currently proposing, that is, the current value of *Proposing* is $\perp$, then it meets the condition in line 28 and immediately sets *Proposing* $=$ *MPool*. Otherwise, by Lemma 4, it eventually learns a value and sets *Proposing* $=\perp$ in lines 44 and 50, thus meeting the condition in line 28 and setting *Proposing* $=$ *MPool*. ◀

Lemmas 3, 4 and 5 imply:

▶ **Theorem 6.** *Algorithm 2 implements Generalized Lattice Agreement.*

▶ **Corollary 7.** *Algorithms 1 and 2 implement Atomic Snapshot.*

4.3 Time Metric

We now define the latency metric we are going to use in evaluating time complexity. Our metric is inspired by the metric proposed by Abraham et al. [2] (which in turn rephrases the original metric by Canetti and Rabin [13]). The distinguishing feature of our approach is that it also applies to *long-lived* executions and executions with *holes* (illustrated in Figure 1).[5]

Algorithm 3 describes the iterative method that assigns rounds to events in an execution. We give an informal description of the metric below.

[5] In Section 6, we show that the three metrics are equivalent in "hole-free" executions.

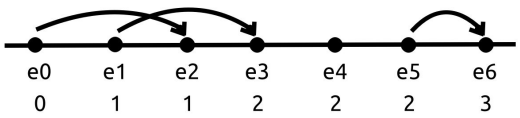


Figure 1 Example of round assignment using **IRA**. Arrows represent message transmissions and the number below an event corresponds to its round. A "hole" in communication appears betwen events e_3 and e_5.

▶ **Definition 8** (**I**terative **R**ound **A**ssignment - Informal)**.** *Algorithm 3 assigns round* 0 *to the initial event, and defines the end of round i as the last event that receives a message sent in round $i-1$. In addition, if there are no more messages to be received (or in transit), the event inherits the round number of its immediate predecessor.*

Algorithm 3 Iterative Round Assignment (**IRA**).

52: $e_0^* := e_0$
53: e_0 is assigned round 0
54: $r := 0$
55: **for** i=1... **do**
56: **if** e_i does not receive a message **then**
57: e_i is assigned round r
58: **else**
59: Let e_j be the oldest event from which e_i receives a message
60: Let r' be the round assigned to e_j ($r' \leq r$)
61: Let e' be the most recent event among $e_{r'}^*$ and e_j
62: All events after e' and up to e_i receive round $r'+1$
63: $e_{r'+1}^* := e_i$
64: $r = r'+1$

▶ **Definition 9** (**IRA** - Arbitrary Events)**.** *To measure the* latency between two events e_i and e_j*, we assign rounds according to Algorithm 3,* starting from e_i*, with* all events *up to and including e_i receiving round* 0*. The latency between e_i and e_j is then given by the round assigned to e_j.*

We say that an application request (or simply request, when there is no ambiguity) *completes* once the receiving node learns a value which includes the request. For a specific node i, we are interested in measuring the latency between the event e_C in which i receives a value v from the application software, and an event e_R, in which i learns a value w with v.

4.4 Time Complexity of Algorithm 2

We define latency as the number of rounds spanning between the moment a correct process receives an application call and the moment it returns from the operation. In evaluating the latency of our protocol, we consider two types of executions: (1) the *fault-free* case, when all processes are correct, and (2) the *worst-case*, when only a majority of processes are correct.

A snapshot operation *op precedes* another operation *op*′ if the response event of *op* happens before the call event for *op*′. Two operations are said to be *concurrent* if none precedes the other. For ASO protocols, we analyze latency in fault-free runs of an operation *op* in two distinct scenarios: (a) without contention, i.e., when no other operation overlaps in time with *op*, and (b) with contention, i.e., when there might be an arbitrary number of concurrent operations.

Garg et al. [19] use the notion of *amortized time complexity*, i.e., the average operation latency taken over a large number of operations in an execution. In some protocols, including ours, the latency of an operation is only affected by the number of faulty processes whose messages are received during the operation's interval (we call these processes *active-faulty*). Intuitively, faulty processes take a finite amount of steps, so in these protocols a failure can only affect a finite number of operations. In this paper, we also distinguish ASO protocols with *constant* time complexity. In Appendix B, we establish the optimality of our protocol under no-contention and prove the following results:

▶ **Theorem 10.** *An operation completes in at most* 2 *rounds in fault-free runs w/o contention.*

Consider an execution of our algorithm, and let F ($|F| \leq f$) be its set of faulty processes.

▶ **Theorem 11.** *An operation op takes at most* 8 *rounds to complete if, during its interval, no correct node receives a message from a faulty one.*

We say that there are k *active faulty nodes during an operation op* if, in between the call and return events for *op*, a message is received from a total of k distinct faulty nodes.

▶ **Theorem 12.** *An operation op takes $O(k)$ rounds to complete, where k is the number of active faulty nodes during op.*

▶ **Corollary 13.** *Algorithms 1 and 2 together have an amortized time complexity of* 8 *rounds.*

5 Measuring Latency of ASO Protocols

We conclude the paper with an overview of time complexity of earlier LA and ASO protocols [17, 16, 21, 18, 19]. We highlight certain gaps in their latency analyses and discuss the ways to fix them. Formalities and proofs can be found in the full version [11].

The First Message-Passing LA Protocol. Faleiro et al. [17] came up with the first LA implementation for asynchronous message-passing systems. They use the metric of [6] to measure latency and conclude that it takes $O(n)$ rounds to output from a lattice agreement operation in their protocol.

We show the somewhat surprising result that this protocol has constant latency of 16 rounds in fault-free runs. The upper bound holds as long as no message from faulty processes is received during the interval of the operation, implying that their LA protocol has constant amortized time complexity. We conjecture that the protocol has $O(k)$ worst-case latency, where k is the number of actual failures in the execution.

The First Direct ASO Implementation. Delporte et al. [16] is the first paper to directly implement ASO in message passing systems, instead of using an atomic register implementation [6] and the shared-memory snapshot construction [3].

In fault-free runs without contention, the latency of their protocol is only 2 rounds. In fault-free runs with contention, we support the claim of a bound of $O(n)$ rounds from [19].

ASO with SCD-Broadcast. Imbs et al. [21] introduce the abstraction of *Set Constrained Delivery Broadcast* (SCD − Broadcast), and show that it allows for implementing LA and ASO with no complexity overhead. In their complexity analysis, they assume bounded message delays and show that the latency of their ASO algorithm *in faulty-free and contention-free runs* is 2 rounds. We show that an operation of their resulting ASO algorithm can take $\Omega(n)$ rounds in fault-free runs *with contention*. We conjecture that this bound is tight, and so the time complexity of their ASO protocol is $\Theta(n)$.

A Generic ASO Algorithm. Garg et al. [18, 19] give a generic construction for atomic snapshot which uses any one-shot LA protocol as a building block (with constant latency overhead). The protocol thus inherits the asymptotic complexity of the underlying LA algorithm. They also provide a protocol for one-shot LA with 2 rounds latency in fault-free runs (using [13]'s metric). Their protocol requires 2 rounds of communication plus two lattice agreement invocations in the good case w/o contention and three lattice invocations with contention, making it at least 6 and 8 message delays, respectively.

For the worst-case latency analysis, they assume an additional requirement over communication channels: if a process executes $\mathsf{send}(m)$, sending m to a *correct* process, then m is eventually received (even if the sender is faulty). Using this assumption, they show a worst-case latency of $O(\sqrt{f})$ for their LA protocol.

In this paper, we assume a weaker channel that only guarantees delivery of messages among correct processes. We show that under this model, the LA protocol of [18] has an execution that takes $\Omega(f)$ rounds. We conjecture the upper bound of their protocol to be $O(f)$, and also that when using the stronger assumption, both our (Section 4) and [21]'s protocol have $O(\sqrt{f})$ worst-case latency.

The generic ASO construction may also be combined with the one-shot LA protocol presented in [27], which has worst-case latency of $O(\log f)$, providing an object whose update and snapshot operations take $O(\log f)$ in both fault-free fault-prone executions. For the sake of completeness, we also provide the time complexity analysis for the one-shot LA protocols from [17] and [21] in the full version [11].

6 Comparative Analysis of Time Measurement Metrics

In this section, we recall metrics used in the literature [6, 13, 2, 24] for measuring time in asynchronous systems. We exhibit executions where the metrics by Attiya et al. [6] and Canetti and Rabin [13] yield arbitrary results due to the presence of *holes* – "periods of silence" during which no messages are in transit – which are common in long-lived protocols. We show that in a subset of executions without holes, which we refer to as *covered executions*, these metrics align with the one proposed by Abraham et al. [2]. This is not surprising, as these metrics were designed for *distributed tasks*, which assume finite hole-free executions. We also recall Lamport's longest causal chain metric [24] and show that it is not suitable for comparing the ASO protocols we consider here.

Next, we show that the metric from [2] diverges from [6] and [13] when naïvely applied to measure time between arbitrary events. We then show that, after employing our refined method from Section 4.3 (Definition 9), they match when measuring rounds between arbitrary events in covered executions. Finally, we show that both our metric and that of [2] yield equivalent results in cases where [2] is applicable.

Altogether, we establish that our metric generalizes [2] and aligns with classical metrics [6, 13] when applied to distributed tasks. A summary of the comparative analysis is presented in Table 2. Proofs are provided in the extended version [11].

6.1 Definitions

Timed Executions. We assume a global clock, not accessible to the nodes. A *timed event* $\bar{e}$ is a pair (t, e) in which t is a non-negative real number, we also say that $\bar{e}$ is a *time assignment* of e. A *timed execution* is an alternating sequence $C_0\bar{e}_1C_1\ldots$ where $\bar{e}_1 = (t_1, e_1), \bar{e}_2 = (t_2, e_2), \ldots$, where events $e_1, e_2, \ldots$ are equipped with monotonically increasing times $t_1, t_2, \ldots$:

Table 2 Comparison between asynchronous time metrics. Metrics that are *timed* make use of time assignments to determine the number of rounds between events. We compare each metric against **CR**, evaluating the number of rounds resulting from applying them over entire (covered) executions and between arbitrary events. Blue stands for "good" features and red – for "bad" ones. The equivalence of NTR to CR holds as long as one uses Definition 9.

	Timed	Equivalent to **CR** (Covered Executions)	Equivalent to **CR** (Arbitrary Events)	Admits Holes
CR [13]	Yes	-	-	No
Round [6, 9]	Yes	Yes	Yes	No
NTR [2]	No	Yes	Yes	No
LCC [25]	No	No	No	Yes
IRA	No	Yes	Yes	Yes

1. $t_m > t_l$ whenever $m > l$;
2. $t_l \to \infty$ as $l \to \infty$.[6]

A *time assignment* of E is a timed execution $\overline{E}$ in which every event e_i in E is matched with a timed event (t_i, e_i) in $\overline{E}$ and the sequences of configurations in E and $\overline{E}$ are the same. Notice that an execution allows for infinitely many time assignments.

Let m be a message sent in $\overline{e}_l$ and received in $\overline{e}_m$, the *delay* of m is then defined as $t_m - t_l$. For a finite timed execution $\overline{E} = C_0\overline{e}_1 ... \overline{e}_l C_l$, we define $t_{start}(\overline{E}) = t_1$, $t_{end}(\overline{E}) = t_l$ (we use t_{start} and t_{end} when there is no ambiguity) and $duration(\overline{E}) = t_{end} - t_{start}$.

In the subsequent discussion, given an execution E, let $\mathcal{T}(E)$ denote the set of all timed executions $\overline{E}$ based on E.

Time Metrics. It is conventional to measure the execution time by the number of communication *rounds*, typically calculated using the "longest message delay." These metrics can be applied to both *executions* and *timed executions*. The first metric we consider is defined in Definition 14 [6]. When applied to timed executions, this metric assumes a known upper bound on message delays, which can be normalized to one time unit without loss of generality. To apply this metric to an *execution*, we consider the maximum duration of all possible timed executions that adhere to the upper-bound communication constraint.

▶ **Definition 14** (**Round** metric). *Given a timed execution $\overline{E}$, in which the maximum message delay is bounded by one unit of time, $\overline{E}$* takes *$duration(\overline{E})$* rounds.

By extension, an execution E takes $\sup_{\overline{E} \in \mathcal{T}(E)} duration(\overline{E})$ rounds.

In the metric proposed by Attiya and Welch [9, 10], the time assignments are scaled so that the maximum message delay is always 1, thus, the metric produces the same results for executions as Definition 14. A more general metric introduced by Canetti and Rabin [13] captures the time complexity of any finite execution. Let $\overline{E}$ be a timed execution, and let $\delta_{\overline{E}}$ be the maximum message delay in it. Then $\overline{E}$ *takes $duration(\overline{E})/\delta_{\overline{E}}$* **CR** *rounds.*

▶ **Definition 15** (**CR** metric). *A finite execution E takes $\sup_{\overline{E} \in \mathcal{T}(E)} duration(\overline{E})/\delta_{\overline{E}}$ rounds, where $\delta_{\overline{E}}$ is the maximum message delay of each corresponding timed execution.*

[6] We require this property to avoid the case where a never-terminating execution has a finite time duration.

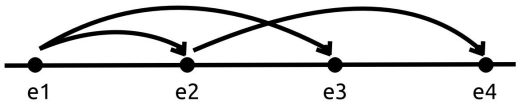


Figure 2 Example of an execution with 2 rounds in the Round, CR and NTR metrics.

▶ **Example 16.** Figure 2 shows an execution with four events, where we assign a delay of δ to the message exchanges (e_1, e_3) and (e_2, e_4), and a delay of $\delta - \epsilon$ $(\epsilon > 0)$ to (e_1, e_2). By making ϵ arbitrarily small, the number of rounds in this execution converges to 2 in the **CR** metric. The same result is obtained in the Round metric by setting $\delta = 1$.

Recently, Abraham et al. [2] proposed an approach that can be applied to executions without relying on time assignments. We call this metric **n**on-**t**imed **r**ounds (**NTR**):

▶ **Definition 17** (**NTR** metric). *Given an execution E, each event in E is assigned a round number as follows:*

- *The first event e_0 is assigned round 0. We also write $e_0^* = e_0$;*
- *For any $r \geq 1$, let e_r^* be the last event where a message of round $r-1$ is delivered. All events after e_{r-1}^* until (and including) event e_r^* are in round r.*

The number of rounds in E is the round assigned to its last event.

▶ **Example 18.** Coming back to Figure 2, if we assign a round to each event based on Definition 17 then e_1 gets round 0, e_2 and e_3 get round 1 and e_4 is assigned round 2. The execution has therefore 2 rounds according to **NTR**.

Lamport [25] proposed a metric for latency based on the causal chain of messages. The *Longest Causal Chain* (**LCC**) was used to show best-case latency of protocols such as consensus [25] and Crusader Agreement [1].

▶ **Definition 19** (**L**ongest **C**ausal **C**hain). *Let e be an event in E and M the set of messages received by e, then e is assigned round $k+1$, where k is the maximum round of an event originating a message in M. If $M = \emptyset$, then $k = 0$. The number of rounds in an execution becomes the highest round assigned to one of its events.*

This metric, however, diverges from **CR** and **NTR**.

▶ **Example 20** (Reliable Broadcast). In the *reliable broadcast* primitive [12], a dedicated source *broadcasts* a message and, if the source is correct, then all correct nodes should deliver the message. Furthermore, if a correct process delivers a message, then every correct process eventually delivers it. The following protocol satisfies this property:

- When the source invokes *broadcast(m)*, it delivers m and sends it to everyone;
- When a process receives m for the first time, it delivers m and sends it to everyone.

In Figure 3, we depict an execution of this protocol with four processes: p_1, p_2, p_3 and p_4. Here, p_1 is the source and broadcasts m, the message is received by p_2 which then sends m to everyone. Process p_3 receives m from p_2 before receiving it from p_1, and finally, p_4 receives m from p_1 in the last event. This execution has 2 **LCC** rounds, while having 1 round according to **CR** and **NTR**.

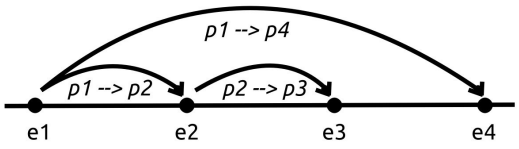
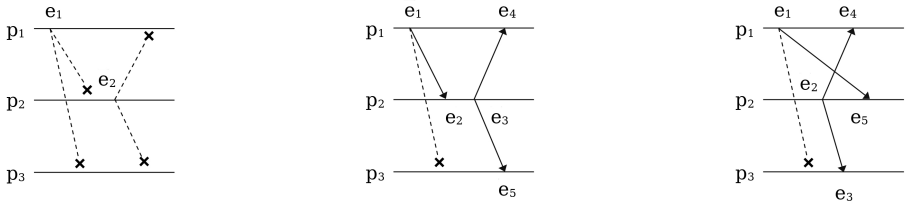


Figure 3 Example of a reliable broadcast protocol execution.



(a) Execution with undefined $\delta_{\overline{E}}$. **(b)** Execution where the number of rounds is unbounded according to Round and **CR**. **(c)** Covered execution.

Figure 4 Examples of non-covered and covered executions.

Example 20 shows that the **LCC** metric diverges from the others in cases where a fast exchange of messages happens in the interval of one slow message. This is the case for several ASO protocols in the literature (including ours) which heavily rely on relaying values to speed up the validation phase, making the metric unsuitable for our use case. On the other hand, **CR** and **NTR** provide equivalent results in *covered* executions, described next.[7]

6.2 Covered Executions and Holes

Consider an execution $E = C_0 e_1 C_1 ... e_l C_l$ illustrated in Figure 4a where no process receives a message from another process, i.e., events may add messages to the buffer but no event removes a message from it. $\delta_{\overline{E}}$ is not defined in any time assignment $\overline{E}$.

Now consider an execution $E' = C_0 e_1 C_1 ... e_l C_l ... e_m C_m$ in which:

- A message m is sent in e_1 and received in e_l;
- A message m' is sent in e_{l+1} and received in e_m;
- No message from $e_1 ... e_l$ is received in $e_{l+1} ... e_m$.

In this example, illustrated in Figure 4b with 5 events, $\delta_{\overline{E'}}$ exists for any time assignment of E', but we can still assign an arbitrary time difference to e_l and e_{l+1} without affecting $\delta_{\overline{E'}}$, which results in the number of **CR** rounds to be unbounded.

The two executions in the examples above have events whose time difference is unrelated to message delays. By consequence, the duration of these executions can grow irrespective of any bound imposed by message exchanges. Similarly, in Figure 4b, since there is no message being received in $e_3 e_4 e_5$ from $e_1 e_2$, there is no round assignment defined when using **NTR** to e_3, e_4 and e_4.

We then restrict the analysis of these metrics to executions that are *covered*. Formally:

[7] The Round and **CR** metrics also provide equivalent results in covered executions [11].

Figure 5 An execution in which there are 2 **CR** rounds between e_2 and e_4. However, the difference of the rounds assigned to e_2 and e_4 using **NTR** is 1.

▶ **Definition 21** (Covered Execution). *A* hole *in an execution is a pair* (e_l, e_{l+1}) *in which no event in* e_{l+1}... *receives a message from* ...e_l*, in other words, there are no message hops among the two sequence of events. An execution is* covered *iff it has no holes.*

Abraham et al. [2] introduce **NTR** as an equivalent to **CR**, however, no formal proof is provided. The next result corroborates this claim in covered executions. Later in Example 24, we show that using **NTR** naively to measure time between events may *not* match **CR**.

▶ **Theorem 22.** *A finite covered execution* E *has* k ***CR*** *rounds iff it has* $\lceil k \rceil$ ***NTR*** *rounds.*

6.3 Time Between Arbitrary Events

In long-lived executions (such as those of atomic snapshot algorithms) we are interested in measuring time between two events, for instance, between an application call and response. Definition 15 can easily be adapted to measure the number of rounds between two events as follows:

▶ **Definition 23** (Generalized **CR** metric). *Let* E *be an execution, let* $\mathcal{T}(E)$ *denote the set of all timed executions* $\overline{E}$ *based on* E*, and* $\delta_{\overline{E}}$ *- the maximum message delay in* $\overline{E}$*. Let* e_i *and* e_j $(j > i)$ *be events in* E*, and* t_i *and* t_j *time assignments in* $\overline{E}$ *for them respectively. Then we say that in between* e_i *and* e_j *there are:* $\sup_{\overline{E} \in \mathcal{T}(E)} (t_j - t_i)/\delta_{\overline{E}}$ **CR** rounds.

An appealing way of defining time between two events e_i and e_j using a non-timed metric is to assign rounds according to **NTR**, and then take the difference of rounds assigned to e_i and e_j. As illustrated in Example 24, this definition can diverge from generalized **CR**.

▶ **Example 24.** Consider the execution shown in Figure 5. We can assign times to e_1, e_3 and e_4 such that the two message hops have delay of δ. Now consider the number of rounds between e_2 and e_4, since we can assign a time for e_2 that is arbitrarily close to e_1's assignment, there are 2 **CR** rounds between e_2 and e_4. However, the round assignments using **NTR** to e_2 and e_4 are 1 and 2 respectively, so simply taking the difference between them leads to a value that diverges from **CR**.

We then give the following definition, using the approach described in Section 4.3:

▶ **Definition 25** (Generalized **NTR**). *Given an execution* E*, let* e_i *and* e_j $(j > i)$ *be events in* E*. The number of rounds between* e_i *and* e_j *is given by the round assigned to* e_j *according to the following:*

- *All events up to (and including)* e_i *are assigned round* 0*. We also write* $e_0^* = e_i$*;*
- *For any* $r \geq 1$*, let* e_r^* *be the last event where a message of round* $r - 1$ *is delivered. All events after* e_{r-1}^* *until (and including) event* e_r^* *are in round* r*.*

▶ **Theorem 26.** *Let* E *be a covered execution and* e_i *and* e_j $(j > i)$ *be events of* E*. There are* k *rounds in between* e_i *and* e_j *according to* ***CR*** *(Definition 23) iff there are* $\lceil k \rceil$ *rounds in between them according to* ***NTR*** *(Definition 25).*

6.4 Relating IRA to NTR

▶ **Theorem 27.** *Let E be a finite covered execution and suppose that all events of E are assigned rounds according to **IRA** after all iterations of the algorithm. It holds that:*

1. *Round 0 is composed only of e_0 (the initial event).*
2. *The final event of round $i+1$ is the last event to receive a message from round i.*

▶ **Corollary 28.** ***IRA** and **NTR** assign the same rounds to events in covered executions.*

References

1 Ittai Abraham, Naama Ben-David, Gilad Stern, and Sravya Yandamuri. On the round complexity of asynchronous crusader agreement. *Cryptology ePrint Archive*, 2023.

2 Ittai Abraham, Kartik Nayak, Ling Ren, and Zhuolun Xiang. Good-case latency of byzantine broadcast: a complete categorization. *CoRR*, abs/2102.07240, 2021. arXiv:2102.07240.

3 Yehuda Afek, Hagit Attiya, Danny Dolev, Eli Gafni, Michael Merritt, and Nir Shavit. Atomic snapshots of shared memory. *J. ACM*, 40(4):873–890, 1993. doi:10.1145/153724.153741.

4 James Aspnes, Hagit Attiya, Keren Censor-Hillel, and Faith Ellen. Limited-use atomic snapshots with polylogarithmic step complexity. *J. ACM*, 62(1):3:1–3:22, 2015. doi:10.1145/2732263.

5 James Aspnes and Keren Censor-Hillel. Atomic snapshots in o(log3 n) steps using randomized helping. In Yehuda Afek, editor, *Distributed Computing - 27th International Symposium, DISC 2013, Jerusalem, Israel, October 14-18, 2013. Proceedings*, volume 8205 of *Lecture Notes in Computer Science*, pages 254–268. Springer, 2013. doi:10.1007/978-3-642-41527-2_18.

6 Hagit Attiya, Amotz Bar-Noy, and Danny Dolev. Sharing memory robustly in message-passing systems. *J. ACM*, 42(1):124–142, January 1995. doi:10.1145/200836.200869.

7 Hagit Attiya, Faith Ellen, and Panagiota Fatourou. The complexity of updating snapshot objects. *J. Parallel Distributed Comput.*, 71(12):1570–1577, 2011. doi:10.1016/J.JPDC.2011.08.002.

8 Hagit Attiya, Maurice Herlihy, and Ophir Rachman. Atomic snapshots using lattice agreement. *Distributed Comput.*, 8(3):121–132, 1995. doi:10.1007/BF02242714.

9 Hagit Attiya and Jennifer Welch. *Distributed computing: fundamentals, simulations, and advanced topics*, volume 19. John Wiley & Sons, 2004.

10 Hagit Attiya and Jennifer L Welch. Multi-valued connected consensus: A new perspective on crusader agreement and adopt-commit. In *27th International Conference on Principles of Distributed Systems*, 2024.

11 João Paulo Bezerra, Luciano Freitas, and Petr Kuznetsov. Asynchronous latency and fast atomic snapshot. *arXiv preprint arXiv:2408.02562*, 2024.

12 Christian Cachin, Rachid Guerraoui, and Luís Rodrigues. *Introduction to reliable and secure distributed programming.* Springer Science & Business Media, 2011.

13 Ran Canetti and Tal Rabin. Fast asynchronous byzantine agreement with optimal resilience. In *Proceedings of the twenty-fifth annual ACM symposium on Theory of computing*, pages 42–51, 1993. doi:10.1145/167088.167105.

14 K. Mani Chandy and Leslie Lamport. Distributed snapshots: Determining global states of distributed systems. *ACM Trans. Comput. Syst.*, 3(1):63–75, 1985. doi:10.1145/214451.214456.

15 George Danezis, Lefteris Kokoris-Kogias, Alberto Sonnino, and Alexander Spiegelman. Narwhal and tusk: a dag-based mempool and efficient BFT consensus. In *EuroSys*, pages 34–50. ACM, 2022. doi:10.1145/3492321.3519594.

16 Carole Delporte-Gallet, Hugues Fauconnier, Sergio Rajsbaum, and Michel Raynal. Implementing snapshot objects on top of crash-prone asynchronous message-passing systems. *IEEE Transactions on Parallel and Distributed Systems*, 29(9):2033–2045, 2018. doi:10.1109/TPDS.2018.2809551.

17 Jose M. Faleiro, Sriram Rajamani, Kaushik Rajan, G. Ramalingam, and Kapil Vaswani. Generalized lattice agreement. In *Proceedings of the 2012 ACM Symposium on Principles of Distributed Computing*, PODC '12, pages 125–134, New York, NY, USA, 2012. Association for Computing Machinery. doi:10.1145/2332432.2332458.

18 Vijay Garg, Saptaparni Kumar, Lewis Tseng, and Xiong Zheng. Amortized constant round atomic snapshot in message-passing systems. *arXiv preprint arXiv:2008.11837*, 2020. arXiv:2008.11837.

19 Vijay K. Garg, Saptaparni Kumar, Lewis Tseng, and Xiong Zheng. Fault-tolerant snapshot objects in message passing systems. In *2022 IEEE International Parallel and Distributed Processing Symposium (IPDPS)*, pages 1129–1139, 2022. doi:10.1109/IPDPS53621.2022.00113.

20 Maurice Herlihy and Jeannette M. Wing. Linearizability: A correctness condition for concurrent objects. *ACM Trans. Program. Lang. Syst.*, 12(3):463–492, 1990. doi:10.1145/78969.78972.

21 Damien Imbs, Achour Mostéfaoui, Matthieu Perrin, and Michel Raynal. Set-constrained delivery broadcast: Definition, abstraction power, and computability limits. In *Proceedings of the 19th International Conference on Distributed Computing and Networking*, pages 1–10, 2018. doi:10.1145/3154273.3154296.

22 Idit Keidar, Eleftherios Kokoris-Kogias, Oded Naor, and Alexander Spiegelman. All you need is DAG. In *PODC*, pages 165–175. ACM, 2021. doi:10.1145/3465084.3467905.

23 Petr Kuznetsov, Thibault Rieutord, and Sara Tucci-Piergiovanni. Reconfigurable Lattice Agreement and Applications. In Pascal Felber, Roy Friedman, Seth Gilbert, and Avery Miller, editors, *23rd International Conference on Principles of Distributed Systems (OPODIS 2019)*, volume 153 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 31:1–31:17, Dagstuhl, Germany, 2020. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.OPODIS.2019.31.

24 Leslie Lamport. Time, clocks, and the ordering of events in a distributed system. *Communications*, 1978.

25 Leslie Lamport. Lower bounds for asynchronous consensus. *Distributed Computing*, 19:104–125, 2006. doi:10.1007/S00446-006-0155-X.

26 Friedemann Mattern. Efficient algorithms for distributed snapshots and global virtual time approximation. *J. Parallel Distributed Comput.*, 18(4):423–434, 1993. doi:10.1006/JPDC.1993.1075.

27 Xiong Zheng, Vijay K. Garg, and John Kaippallimalil. Linearizable Replicated State Machines With Lattice Agreement. In Pascal Felber, Roy Friedman, Seth Gilbert, and Avery Miller, editors, *23rd International Conference on Principles of Distributed Systems (OPODIS 2019)*, volume 153 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 29:1–29:16, Dagstuhl, Germany, 2020. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.OPODIS.2019.29.

A Linearizability of Algorithm 1

The *history* of an execution E is the subsequence of E consisting of invocations and responses of ASO operations (update and snapshot). A history is *sequential* if each of its invocations is followed by a matching response. An execution is linearizable if, to each of its operation (update or snapshot, except, possibly, for incomplete ones), we can assign an indivisible point within its interval (called a *linearization point*), so that the operations put in the order of its linearizaton points constitute a *legal* sequential history of ASO (called a *linearization*), i.e., every snapshot operation returns a vector where every position contains the last value written to it (using an update operation), or the initial value if there are no such prior updates. Equivalently, a linearizable execution E with history H should have a linearization S, a legal sequential history that (1) no node can locally distinguish a completion of H and

S and (2) S respect the *real-time order* of H, i.e., if operation op completes before operation op' in H, then op' cannot precede op in S.

We show that every execution of Algorithm 1 is linearizable. Consider an execution of Algorithm 1, let H be its history. Every operation (snapshot or update) is associated with a unique sequence number and performs a Propose operation on the LA object. If there is an LA.Propose operation that returns (w, v) in position i, by Validity of LA, there is an operation update(i, v) executed by node i with sequence number w that started before the LA.Propose completed and invoked a LA.. In this case, we say that the update operation is *successful*. Notice that by Validity of LA, the update must have invoked LA.Propose with a vector containing (w, v) in position i.

Now we order complete snapshot operations and complete successful update operations in the order of the values returned by their LA.Propose operations (by Consistency of LA, these values are totally ordered. As each of these LA.Propose returns a value containing its unique sequence number (Stability of LA) , this order respects the real-time order of H. A successful update operation performed by node i with (w, v) in position i that has no complete LA.Propose is placed right before the first snapshot whose LA.Propose returns this value. By construction, the resulting sequential history is legal and locally indistinguishable from a completion of H.

Finally, Liveness implies that every operation invoked by a correct process eventually completes.

B Time Complexity of Algorithm 2

We establish the optimality of our protocol under no-contention. A protocol implementing LA *tolerates* k faults if it satisfies all the properties of Definition 1 in every execution with at most k faulty processes.

▶ **Theorem 29.** *Let $\mathcal{P}$ be a distributed protocol that implements LA and tolerates at least one faulty process. Then, there exists a fault-free run of $\mathcal{P}$ in which an LA operation requires at least two rounds of communication to complete without contention.*

Proof. Consider an operation op initiated by node x, with call event e_C and response event e_R. Suppose op completes in at most one round in fault-free, contention-free executions.

We first show that there exists an execution $E = e_1, \ldots, e_C, \ldots, e_R$ such that:

- x is the only process to take a step in e_R,
- no message sent by x in $e_C, \ldots, e_R$ is received by any other process before e_R.

If multiple processes perform steps in the same event e, we can conceptually "split" e into a sequence of events $e^1, e^2, \ldots$, where each process takes the step in its own dedicated event. Since their steps are independent, these split events are indistinguishable from the original e from each process's perspective. This reasoning also applies to e_R.

Now, assume for the sake of contradiction that in every fault-free, contention-free execution containing both e_C and e_R, there exists some process $y \neq x$ that receives a message m–sent by x in the interval $e_C, \ldots, e_R$–before e_R occurs.

Let e_M denote the event where y receives m. We define rounds from e_C's perspective:

- all events up to e_C are in round 0,
- round 1 ends at the last event e_L that receives a message originating in round 0.

If m is sent after e_C, then we can construct E so that all messages from round 0 are received before m. This ensures that e_M occurs after e_L, meaning e_M is in round 2. Since e_R

occurs after e_M, it too is assigned round 2–contradicting our assumption that *op* completes in one round.

If instead m is sent in e_C, we can again construct the execution so that all round 0 messages are received before or at the same time as m, making $e_M = e_L$. Since e_R occurs after e_M, it is again assigned to round 2–a contradiction.

These contradictions hold regardless of whether *op* is concurrent with any other operation. Hence, such an execution E must exist. Now consider an extension E' of E where all messages sent by x after (and including) e_C are indefinitely delayed, while messages from other nodes are not.

Suppose a node z invokes a new operation op' after e_R, making op' non-concurrent with *op*. Since protocol $\mathcal{P}$ tolerates at least one faulty process, and x appears to have crashed in E', node z must eventually complete op' without any process receiving any messages from x.

Let v and w be the value proposed and the value learned by x in *op*, and let v' and w' be the corresponding values for z in op'. By Validity, we know $v \sqsubseteq w$, and by Consistency, we know $w \sqsubseteq w'$, hence $v \sqsubseteq w'$.

However, since no process receives a message from x since e_C, no one could have known about v, contradicting the requirement that w' must contain v.

Finally, after op' completes, we can allow all delayed messages from x to be received, making all processes correct in the final execution E'. This completes the proof. ◀

▶ **Theorem 30.** *An operation completes in at most* 2 *rounds in fault-free runs w/o contention.*

Proof. Consider a contention-free request with call event e_C and return event e_R invoked by a node i. There are no call events for other nodes between e_C and e_R, but some messages from previous proposals may still be in transit.

Suppose v is the value to be proposed for the application call. If i is not proposing (has $Proposing = \perp$) when it receives v, then it directly sends $\langle \mathbf{PROPOSE}, v \rangle$ to everyone. Let e_P be the last event in which a process receives $\langle \mathbf{PROPOSE}, v \rangle$ from i, then every process also sends $\langle \mathbf{PROPOSE}, v \rangle$ by at most e_P. Now take e_F as the final event in which a process receives $\langle \mathbf{PROPOSE}, v \rangle$ in the execution, and e_S as the corresponding sending event. It must be that e_S happens between e_C and (potentially including) e_P. Also, because the channels are FIFO, every previous proposal must have been validated before e_F, and i will learn a value containing v by at most e_F. Let e_C be assigned round 0, then e_P happens at most in round 1. As a consequence, e_S is assigned either 0 or 1, thus e_F can be assigned at most round 2. Then, by the end of round 2, i already has v validated.

Now suppose that i is proposing when it receives v, so it still has a value v' in *Pending* that is not validated, w.l.o.g. assume that v' is the only one. This value must be from a call that already finished, and the corresponding node sent $\langle \mathbf{ACCEPT}, w \rangle$ containing v' before e_C. Consider two pairs of events: (e_A, e'_A) and (e_C, e'_C). In the first pair, e_A is the event where $\langle \mathbf{ACCEPT}, w \rangle$ was first sent, and e'_A is the *last* event in which $\langle \mathbf{ACCEPT}, w \rangle$ is received from e_A. In the second, e_C is the usual application call event and e'_C is the *last* event in which $\langle \mathbf{REQUEST}, v \rangle$ is received from i. There are two cases to consider: 1) e'_A happens before e'_C and 2) e'_C happens before e'_A.

If it is the first case, then at the moment e'_C happens, every node was already able to propose v (since there was no other value to be learned). Take the last event e_L in which a $\langle \mathbf{PROPOSE}, v \rangle$ (or a value containing v) is received, and e_S as the corresponding sending event, it follows that i validates v by at most e_L and can learn a value containing it. Let e_C be assigned round 0, e'_C and e_S can be assigned at most round 1, and since e_L receives a message from e_S, it can be assigned at most round 2. If it is the second case, then all

nodes received $\langle \mathbf{REQUEST}, v \rangle$ and put v in *MPool* before e'_A. Every node proposes v by at most e'_A (since they can adopt w and stop any current proposal). Let e_L be the last event in which a process receives a proposal for v and e_S it's corresponding sending event, similarly to the above cases, e_S happens between e_C and e'_A. Now, let e_A and e_C be assigned round 0. e_S can be assigned at most round 1 (e_S happens before or at e'_A) and e_L at most 2, which concludes the proof. ◀

▶ **Lemma 31.** *Consider an event in which a correct node sends $\langle \boldsymbol{PROPOSE}, v \rangle$ and the first event in which a correct node learns a value including v. If no correct node receives a message from a faulty one between these two events, then there are at most 3 rounds between them.*

Proof. A message sent by a correct node is received by every correct node in the execution, and since correct nodes do not receive messages from faulty ones in the interval we are analyzing, we can consider only events originated from correct nodes. Therefore, we only refer to correct nodes in the following.

Let x be the node sending $\langle \mathbf{PROPOSE}, v \rangle$, e_P be the corresponding event and e'_P the last event a node receives $\langle \mathbf{PROPOSE}, v \rangle$ from x. Because x also sends $\langle \mathbf{REQUEST}, v \rangle$, by e'_P every node received the request and must be proposing. Any value learned after e'_P contains v since all nodes have v in *Pending*.

Now, at the configuration just after applying e'_P, let V be the set in which $w \in V$ satisfies: there exists a (correct) node where w is in *Pending* but is not yet validated. Consider a value $w \in V$ that is the last whose $\langle \mathbf{PROPOSE}, w \rangle$ is received by any node, where e'_L is the event in which $\langle \mathbf{PROPOSE}, w \rangle$ is last received and e_L the corresponding sending event. It follows that some node learns a value containing v by at most e'_L.

Next, take the first event e_F in which a node sent $\langle \mathbf{PROPOSE}, w \rangle$, and e'_F the event in which the last $\langle \mathbf{PROPOSE}, w \rangle$ from e_F is received. Note that e_L happens at most at e'_F and e_F at most at e'_P. Let e_P be assigned round 0, then e'_P (and thus e_F) can be assigned at most round 1, e'_F (and thus e_L) at most 2 and lastly, e'_L can be assigned at most round 3. Therefore, there are at most 3 rounds between a propose and the first learn event for v. ◀

▶ **Theorem 32.** *An operation op takes at most 8 rounds to complete if, during its interval, no correct node receives a message from a faulty one.*

Proof. Let v be the value received from the application call for *op*, e be the event in which node i proposes v (or a value containing v) and e' the event in which a value including v is learned for the first time. From Lemma 31, there are at most 3 rounds between e and e'. Since the node that learns v sends $\langle \mathbf{ACCEPT}, v \rangle$ to everyone, i receives and adopts it in one extra round. We conclude that in at most 4 rounds every correct node can learn v.

If i is already proposing a value when it receives a call for v, it sends $\langle \mathbf{REQUEST}, v \rangle$ to everyone and put it in *MPool*, so it is proposed next. Let e_P be the event in which i initiated its previous proposal to v, and consider the worst case where the application call e_C with v happens just after e_P. From e_P to the event in which i learns its previous proposal e'_P (and thus starts proposing v), there are at most 4 rounds, and from e'_P to the learning event of v there are also at most 4 rounds. Therefore, the operation completes in at most 8 rounds. ◀

▶ **Theorem 33.** *An operation op takes $O(k)$ rounds to complete, where k is the number of active faulty nodes during op.*

Proof. We show that an operation *op* in Algorithm 2 takes $O(k)$ rounds to complete, where k is the number of active faulty nodes during *op*.

Messages from and to faulty nodes may not arrive, however, a message sent by (and to) a faulty node at round r is received at most by round $r+1$. Moreover, since channels are FIFO, when a node i receives a message from another node j, i must also have received all previous messages j sent to i, irrespective of them being correct or faulty.

If a correct node receives $\langle \textbf{PROPOSE}, v' \rangle$ (even from a faulty node) in round r, every correct node will have v' added to *Pending* by the end of round $r+1$, and will have v' validated by the end of round $r+2$. Also, faulty nodes wait for its current proposal to finish before starting a new one, in which case they send an **ACCEPT** message for the last learned value before sending the new proposal.

We say that a node *introduces a new value* w during the operation if it is the first node to send a $\langle \textbf{PROPOSE}, w \rangle$ for w in the interval of the operation. A node can introduce a new value coming from an internal source, i.e., the value was buffered and proposed when the node had already finished its previous proposal, or from an external source, i.e., after receiving a proposal originated from another node before the operation started.

Let v be the value received from the application call for *op* and e_C (as well as all previous events) be assigned round 0. If there are no active faulty nodes, a correct node learns a value containing v by at most round 7 (by Lemma 31, here, we include the time v can remain buffered). Also by the end of round 5, every correct node has sent a **PROPOSE** message for v and has v validated by the end of round 6 (including buffering time, a correct node proposes v in round 4 at the latest). By that point, all correct nodes are waiting for their proposals to complete and, therefore, cannot introduce a value from an internal source. In order to delay a correct node from leaning a value containing v by round 7, every correct node should receive a new value in a **PROPOSE** message before, which is added to *Pending* but is not validated. Using a simple inductive argument, $2k+1$ new proposals originated from faulty nodes are necessary to delay a correct node from learning a value from round 7 to round $7+2k$.

Suppose that there is an execution where it takes $8+2k+1$ rounds for node i to complete an operation. But there are only k active faulty nodes, which means that at least $k+1$ extra proposals were introduced by active faulty nodes.

Let f_0 be an active faulty node that introduced more than one of the $2k+1$ values that delayed the operation (assuming w.l.o.g. that there are *exactly* $2k+1$ new proposals). Let w and w' be the first and the second values introduced by f_0 respectively. If w' was received by f_0 from an internal source, f_0 should have finished its previous proposal (and learned a value containing w) before proposing w'. But because w was one of the values that delayed the operation, and since channels are FIFO, f_0 needs to add v to *Pending* before validating w (at least a majority of correct nodes sent a **PROPOSE** for v before sending a **PROPOSE** for w). f_0 then learns a value containing v and sends **ACCEPT** with that value to everyone. The **ACCEPT** message is received by correct processes before $\langle \textbf{PROPOSE}, w' \rangle$, and they would be able to adopt it.

So f_0 must have received $\langle \textbf{PROPOSE}, w' \rangle$ from an external source at most by round 1, which means it issued proposals for w' that can be received by at most round 2. We can also conclude that at least $k+1$ values were introduced by active faulty nodes from external sources. Now let w_{k+1} be the $(k+1)$th such value used to delay correct nodes from learning v. The earliest round w_{k+1} can delay is $7+k$, which means that by round $7+k$ all correct nodes already sent a propose for w_{k+1}, but by the end of round $5+k$ no correct node has done it (otherwise w_{k+1} would have been validated in round $7+k$ by every correct process). Take the first active faulty node f_1 from which a correct node received $\langle \textbf{PROPOSE}, w_{k+1} \rangle$. Since the earliest this message is received is in round $6+k$, the earliest it could be sent is

in round $5 + k$, so f_1 first received $\langle \mathbf{PROPOSE}, w_{k+1} \rangle$ from another distinct active faulty node, f_2, which sent it in round $4 + k$ the earliest. But w_{k+1} was introduced from an external source and it needs to be received by a faulty node at round 1. Following the chain above, for the node f_{k+6} to receive it in round 1, there would be necessary a chain of $k + 6$ active nodes, although there are only k.

Therefore, an operation takes less than $8 + 2k + 1$ rounds to complete. ◀