

Approach of Agents with Restricted Fuel Tanks

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Abstract

Two mobile agents, modelled as points in the plane moving at speed 1, have to get at a distance at most 1 from each other. This task is known as approach or rendezvous in the plane. An adversary initially places both agents at distinct points, called their bases, at distance at most D , and wakes them up at possibly different times. Each of the agents has a fuel tank that allows them to traverse a trajectory of length D , and can be replenished at the base of the agent. The algorithm of each agent consists of a series of actions which are either moves at a chosen distance in a chosen direction or staying idle for a chosen period of time. For a given instance of the approach task, the execution time of an approach algorithm is the length of the period between the start of the later agent and the moment of approach.

Our goal is to design approach algorithms with optimal time complexity. We consider two independent *coherence assumptions*. One of them is *time coherence*, i.e., agents start simultaneously, and the other is *orientation coherence*: agents have compatible compasses, showing the same North direction. Our main result is establishing optimal time complexity of the approach problem with restricted fuel tanks. It turns out that this optimal complexity heavily depends on the above coherence assumptions. If both of them are satisfied then approach can be performed in time $O(D^2)$ and we show that this complexity is optimal. If any of the two coherence assumptions is missing then approach can be performed in time $O(D^2\sqrt{D})$ and we prove that this order of magnitude cannot be improved.

Our main technical contribution are lower bounds showing that, for each of the considered scenarios, our fairly natural approach algorithms are, in fact, optimal.

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1 Introduction

1.1 The background

Two mobile agents, modelled as points moving in the plane, have to get at a distance at most 1 from each other. This task is known as approach or rendezvous in the plane and has numerous applications. The final distance, normalized to 1, should be viewed as the distance at which agents can “see” each other, hence the goal of approach is mutual perception. In human interaction, people may want to meet in a large terrain, where meeting means seeing each other. In robotics applications, two mobile robots, independently deployed in a



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contaminated territory and collecting samples of the ground, may need to meet in order to exchange these sample and coordinate further actions. Robots have limited energy, either a fuel tank or an electric battery, that allows them to travel a certain maximum distance and then has to be replenished at the base.

1.2 The model and the problem

We consider two mobile agents modelled as points moving in the plane, that have to get at a distance at most 1 from each other. An adversary initially places both agents at distinct points, called their bases, at distance at most D , and wakes them up at possibly different times. Both agents execute the same deterministic algorithm. If they were identical then, in the case of simultaneous start and identical compasses, they would move along trajectories that are shifts of each other, and consequently, at any time, their distance would be the same as at the start, thus precluding approach, for any $D > 1$. In order to allow approach, this symmetry has to be broken. We do this in the way that is most common in the literature: agents have distinct labels that are integers from a set $\{1, \dots, L\}$, where L is some constant number. This may be viewed as the set of all identifiers used by the manufacturer of the robots. We make an assumption, standard in rendezvous/approach literature (cf. e.g., [4, 10]) that each agent only knows its own label that can be used as a parameter in a common deterministic algorithm they execute. Each agent is equipped with a clock and a compass showing the cardinal directions. Clocks of the agents tick at the same rate and the clock of each agent starts at its wake-up. Compasses may be inaccurate and arbitrarily distorted, i.e., for each agent, its North can point in any (fixed) direction. Each of the agents has a fuel tank that allows them to traverse a trajectory of length D , and can be replenished at the base of the agent.¹

The execution of an approach algorithm by an agent consists of a series of actions. An agent can either choose a direction α (according to its compass) and a distance x , in which case it travels distance x in direction α with speed 1, or it can stay put at the current point for a chosen time t . Whenever agents get at a distance at most 1, the algorithm is interrupted and the goal of approach is achieved. For a given instance of the approach task, the execution time of an approach algorithm is the length of the period between the start of the later agent and the moment of approach.

1.3 Our results

Our goal is to design approach algorithms with optimal time complexity. We consider two independent *coherence assumptions*. One of them is *time coherence*, i.e., agents start simultaneously, and the other is *orientation coherence*: agents have compatible compasses, showing the same North direction. The presence or absence of these two assumptions yields four possible scenarios. Our main result is establishing optimal time complexity of the approach problem for each of these scenarios. It turns out that this optimal complexity heavily depends on the above coherence assumptions. If both of them are satisfied then approach can be performed in time $O(D^2)$ and we show that this complexity is optimal. If any of the two coherence assumptions is missing then approach can be performed in time $O(D^2\sqrt{D})$ and we prove that this order of magnitude cannot be improved.

Our main technical contribution are lower bounds showing that, for each of the considered scenarios, our fairly natural approach algorithms are, in fact, optimal. In both our $\Omega(D^2\sqrt{D})$ lower bound proofs, we consider the most difficult instances of the approach problem, when

¹ The discussion of the size of the tank is postponed to Section 5.

bases of the agents are at a distance exactly D . For these instances, approach is only possible in short time periods in which both agents are close to their maximal range, i.e., at distance $D/2$ from their respective bases. We will prove that these time periods must be separated by time segments of length $\Omega(D)$. Due to lack of simultaneous start or of compass compatibility, any of these time periods may be necessary for approach, depending on the instance. In view of the above, the main difficulty is proving that the number of these time periods is $\Omega(D\sqrt{D})$, which will imply the lower bound $\Omega(D^2\sqrt{D})$ on the time of approach.

1.4 Related work

The task of rendezvous of mobile agents has been extensively studied, both when agents move in graphs, and when they move in the plane. For a survey of randomized rendezvous algorithms, see the classic book [1], and for a survey of deterministic rendezvous, see [16].

Deterministic rendezvous in networks modelled as graphs has been studied under two scenarios. In the synchronous scenario, agents move in synchronous rounds, in each round deciding either to stay put at the current node or to move to an adjacent neighbor. The goal is to get both agents at the same node in the same round. The time of rendezvous is the number of rounds used. A time-efficient synchronous rendezvous algorithm was designed in [17], although the problem of optimal synchronous rendezvous is still open. In the asynchronous scenario, an agent chooses the edge along which it wants to go but the adversary chooses the speed of the agent during the move. The goal is to meet at a node or inside an edge, and the cost is the number of traversed edges. A polynomial-cost algorithm for asynchronous rendezvous in arbitrary graphs was designed in [10].

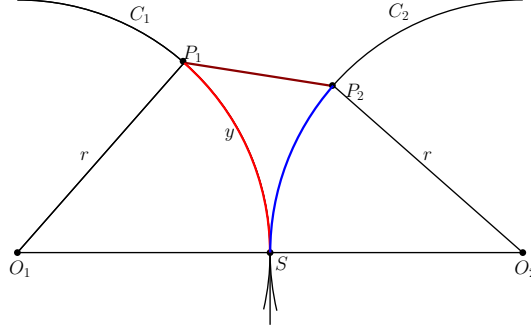
Rendezvous in the plane, often called approach, has also been studied both under the synchronous and asynchronous scenarios. In [8, 9], this task was investigated under various assumptions concerning attributes of agents, such as orientation, chirality and speed. An asynchronous approach algorithm in the plane working at cost polynomial in the initial distance and in the lengths of agents' labels was designed in [4]. The more general task of gathering many agents in the plane was studied in [6, 7, 12], under various assumptions concerning perception capabilities of the agents.

A special, well researched variation of rendezvous, both in graphs and in the plane, is treasure hunt. An inert treasure is hidden at a node of the graph or in a point in the plane, and a mobile agent starting at some other node or point in the plane has to find it. Thus, treasure hunt can be viewed as rendezvous in which one of the agents is permanently inert. Treasure hunt in the plane was considered, e.g., in [3, 15], see also surveys [13, 14]. Treasure hunt in graphs was investigated, e.g., in [2, 5, 11], under some constraints imposed on the mobile agent: in [2], the mobile agent had a restricted fuel tank that could be replenished at the base (as in the present paper), and in [11], the agent was tethered, i.e., attached to the base with a rope of fixed length. To the best of our knowledge, the general task of approach in the plane of agents with restricted fuel tanks has never been studied before.

2 Preliminaries

We will use the following geometric lemma which informally says that two points on tangent circles C_1 and C_2 of radius r located roughly symmetrically with respect to the tangent line of C_1 and C_2 are at a distance at most 1, provided that they are in arc-distance $O(\sqrt{r})$ from the common point of C_1 and C_2 . The proof of Lemma 1 is in the Appendix, Section A.

► **Lemma 1.** Let C_1 and C_2 be tangent circles of the same radius r , centered at O_1 and O_2 , respectively. Let S be the common point of C_1 and C_2 . For $i = 1, 2$, and for any point P on C_i , let $\alpha_i(P)$ be the length of the arc PS on C_i . Let $y = \sqrt{r}/4$. Let P_1 be a point on C_1 such that $\alpha_1(P_1) \leq y$, and let P_2 be a point on C_2 such that $\alpha_1(P_1) - 1/2 \leq \alpha_2(P_2) \leq \alpha_1(P_1) + 1/2$. Suppose that either P_1 on C_1 is counterclockwise from S and P_2 on C_2 is clockwise from S , or P_1 on C_1 is clockwise from S and P_2 on C_2 is counterclockwise from S . Then $|P_1P_2| < 1$. (See Figure 1.)



■ **Figure 1** Illustration for Lemma 1. The lemma says that, if the arc length of P_1S is at most $\sqrt{r}/4$, this length and the length of P_2S differ by at most $1/2$, then the distance between P_1 and P_2 is less than 1.

3 Algorithms

This section is devoted to our positive results. We design two approach algorithms: an algorithm with time complexity $O(D^2)$ for the strongest scenario where both coherence assumptions are satisfied, and an algorithm with time complexity $O(D^2\sqrt{D})$ working even in the weakest scenario where none of the coherence assumptions is satisfied.

For both algorithms, two variations are presented: one for agent Blue and the other for agent Red. This corresponds to a simplified version of our model, in which agents have labels 1 and 2, i.e., they know from the outset how symmetry between them is broken (they know which of them is Blue and which is Red). In Section 3.3, we show how these algorithms can be modified in order to conform with our model, where agents have distinct labels from the set $\{1, \dots, L\}$ and each agent only knows its label, i.e., it does not know if it is Blue or Red.

3.1 Time $O(D^2)$ for simultaneous start and common orientation

In this section, we design Algorithm **Faster** working in the strongest scenario where both coherence assumptions are satisfied. We prove that this algorithm has complexity $O(D^2)$.

Let O_1 be the base of agent Blue and O_2 the base of agent Red. Let C_1 be the circle of radius $D/2$ centered at O_1 and let C_2 be the circle of radius $D/2$ centered at O_2 . Let $n = 4\lceil\pi D\rceil$. Let $A_0, \dots, A_{n-1}$ (resp. $B_0, \dots, B_{n-1}$) be points on C_1 (resp. on C_2) in clockwise order, such that A_0 is North of O_1 , B_0 is North of O_2 , and all distances between points A_i and $A_{(i+1) \bmod n}$ and points B_i and $B_{(i+1) \bmod n}$ are equal. Each agent visits consecutive points on its circle but they start from antipodal points.

Algorithm **Faster** for agent Blue (Red, resp.): the algorithm works in n phases. In phase $i = 0, 1, \dots, n-1$, agent Blue (Red, resp.) goes from its base straight to A_i ($B_{(i+n/2) \bmod n}$, resp.) and comes back straight to its base.

The following theorem establishes the correctness and complexity of Algorithm **Faster**. Its proof is in the Appendix, Section B.

► **Theorem 2.** *Algorithm **Faster** guarantees approach in time $O(D^2)$.*

3.2 Time $O(D^2\sqrt{D})$ for arbitrary start delay and arbitrary orientations

In this section, we design Algorithm **Slower** working even in the weakest scenario where none of the coherence assumptions is satisfied. We prove that this algorithm has complexity $O(D^2\sqrt{D})$. Let O_1 be the base of agent Blue and O_2 the base of agent Red. Let C_1 be the circle of radius $D/2$ centered at O_1 and let C_2 be the circle of radius $D/2$ centered at O_2 . Let $n = \lceil 2\pi\sqrt{D} \rceil$ and let $m = \lceil 4\pi D \rceil$. Let $A_0, \dots, A_{n-1}$ (resp. $B_0, \dots, B_{m-1}$) be points on C_1 (resp. on C_2) in clockwise order, such that A_0 is North of O_1 according to agent's Blue compass, B_0 is North of O_2 according to agent's Red compass, and all distances between points A_i and $A_{(i+1) \bmod n}$ and points B_i and $B_{(i+1) \bmod m}$ are equal. Each agent visits consecutive points on its circle. However, agent Blue visits only $\Theta(\sqrt{D})$ points on C , waiting in each of them for time $\Theta(D^2)$, while agent Red visits $\Theta(D)$ points on C without any waiting.

Algorithm Slower for agent Blue: The algorithm works in phases. In phase $i \geq 0$, agent Blue goes from its base straight to $A_{i \bmod n}$, waits there for time $4\lceil 4\pi D^2 \rceil$, and comes back straight to its base.

Algorithm Slower for agent Red: The algorithm works in phases. In phase $i \geq 0$, agent Red goes from its base straight to $B_{i \bmod m}$ and comes back straight to its base.

The following result establishes the correctness and complexity of Algorithm **Slower**. Its proof is in the Appendix, Section C.

► **Theorem 3.** *Algorithm **Slower** guarantees approach in time $O(D^2\sqrt{D})$.*

3.3 Algorithms for agents with arbitrary labels

The algorithms presented so far assume that each agent knows in advance whether it is Blue or Red, and therefore it can execute the appropriate version of the algorithm. However, in our model we assume that the agents do not have prior knowledge of their roles. The only way to break symmetry is provided by distinct labels of agents from the set of natural numbers $\{1, \dots, L\}$, where the value of L is a constant not known to the agents. Each agent knows only its own label that can be used as a parameter in their common deterministic algorithm.

It turns out that our algorithms can be adjusted to the above scenario at the multiplicative cost in time complexity of the order of $O(\log L)$. Since we assume that L is a constant, this does not change the previously established complexities. The idea behind the generalizations is that the original algorithms presented earlier are executed by each agent several times, until the approach happens. The agent uses an appropriate binary encoding of its label in order to decide whether to act as Blue or Red in its i th execution. More precisely, the agent X with the binary encoding of its label equal to $l(X)$ executes the algorithm for Blue in the i th execution if the i th bit of $l(X)$ is equal to 0 and it executes the algorithm for Red otherwise. Using fairly standard encodings of size $O(\log L)$ and appropriate proof techniques one can prove that there will appear simultaneously a period in which one of the agents acts as Blue and the other one acts as Red in the first $O(\log L)$ executions of the original algorithm designed for agents Blue and Red. These facts will lead to the following result.

► **Theorem 4.** *There exist algorithms for agents with unique labels from the set $\{1, \dots, L\}$ which:*

- *guarantee approach in time $O(D^2 \log L)$ in the model with common orientation and simultaneous start (using the assumption that L is constant, this complexity is $O(D^2)$);*
- *guarantee approach in time $O(D^2 \sqrt{D} \log L)$ in the model without simultaneous start and without common orientation (using the assumption that L is constant, this complexity is $O(D^2)$).*

4 Lower bounds

This section is devoted to our negative results. We present three lower bounds. The first is the lower bound $\Omega(D^2)$ on the time approach that holds for the strongest scenario with simultaneous start and common orientation. This lower bound shows that our algorithm for this scenario has optimal time complexity. The two other lower bounds concern scenarios when one of the coherence assumptions is not satisfied. If either arbitrary delay between starting times is permitted or different orientations of agents are allowed, we prove that approach requires time $\Omega(D^2 \sqrt{D})$. This result, combined with the complexity $O(D^2 \sqrt{D})$ of our algorithm working even in the weakest scenario shows that $\Theta(D^2 \sqrt{D})$ is the optimal complexity of approach in all the scenarios except the strongest one.

4.1 Time $\Omega(D^2)$ for simultaneous start and common orientation

As a warm-up, in this section, we prove the lower bound $\Omega(D^2)$ on the time of approach that holds even for the strongest scenario with simultaneous start and common orientation. In fact we will show the following stronger lower bound implying our result: it turns out that this lower bound holds even for unbounded fuel tanks. Its proof is in the Appendix, Section D.

► **Theorem 5.** *The time of approach for two agents simultaneously starting at a distance at most D , with common orientation and unbounded tanks is $\Omega(D^2)$.*

Proof. Consider an arbitrary approach algorithm for two agents A and B . Let $a(t)$ (resp. $b(t)$) be a vector describing the position of agent A (resp. B) relative to its starting point in time t . Let $d(t) = a(t) - b(t)$. Let T be the approach time of the algorithm.

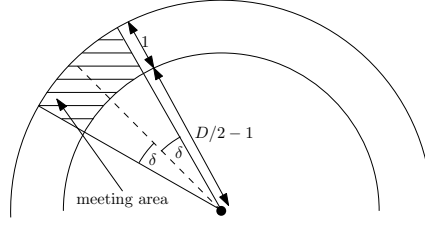
For $i, j \in [-\lfloor \frac{D}{4} \rfloor, \lfloor \frac{D}{4} \rfloor - 2]$ let $r_{i,j}$ be the rectangle with vertices in points $(2i, 2j), (2i + 2, 2j), (2i, 2j + 2), (2i + 2, 2j + 2)$.

We will consider only *even* rectangles, that is the rectangles $r_{i,j}$ where both i and j are even. We say that the algorithm *marks the rectangle* $r_{i,j}$ if there is a moment t of the execution of the algorithm such that the point $d(t)$ is inside $r_{i,j}$. As for any t_1, t_2 such that, $|t_1 - t_2| \leq 1$ the length of the vector $d(t_1) - d(t_2)$ is at most 2, the algorithm can mark at most $T + 1$ even rectangles.

Assume that the time of approach T is less than $\frac{D^2}{16}$. The algorithm can mark at most $\frac{D^2}{16} + 1$ even rectangles, and thus there exists an unmarked even rectangle $r_{i,j}$. By fixing the base of agent A at point $(0, 0)$ and the base of B at point $(2i + 1, 2j + 1)$ the adversary can guarantee that the approach does not occur. This contradiction concludes the proof. ◀

4.2 Geometric preliminaries for $\Omega(D^2 \sqrt{D})$ lower bounds

In this section we define the common notions and prove geometric properties useful in lower bounds from Sections 4.4 and 4.5. Some missing proofs will be provided in the full version of the paper.



■ **Figure 2** The meeting area of a walk: The dashed line is the walk bisector. The marked area is the meeting area, where $\delta = \arcsin(\frac{1}{D/2-1})$.

Consider an arbitrary approach algorithm for two agents A and B . Each agent's *range disc* is the disc with radius $D/2$ centered at the base of the agent. Each agent's *range circle* is the boundary of its range disc. We assume that the bases of the agents are at distance D , i.e., their range discs are tangent. Consider the trajectory of an agent between its wake-up and the approach. This trajectory can be partitioned into a sequence of sub-trajectories between consecutive visits of the base, and the final sub-trajectory finishing at the approach. These sub-trajectories are called *walks*. For each walk, we distinguish its *meeting part* between the first and last points of the walk at distance at least $D/2 - 1$ from the source. The bisector of the angle between these points is called the *walk bisector*. The angle between North and the walk bisector of agent A is called the *walk angle* of agent A . The angle between South and the walk bisector of agent B is called the *walk angle* of agent B . We define the angle $\delta = \arcsin(\frac{1}{D/2-1})$. The *meeting area of a walk* is the part of the plane bounded by two half-lines with origin at the base, forming angle δ with the walk bisector, and circles of radii $D/2$ and $D/2 - 1$ centered at the base (see Fig. 2).

► **Fact 1.** *If an agent goes back to its base at the end of a walk w , the intersection of w with the annulus between circles of radii $D/2 - 1$ and D centered at the base of this agent is included in the meeting area of the walk w .*

Proof. In order to get back to the base after leaving the disk Δ with the radius $D/2 - 1$ centered at the base, an agent can only move at distance at most 1 before getting back to the disk Δ . A possible part of the trajectory outside of Δ corresponds to an angle $\arcsin(\frac{1}{D/2-1})$. ◀

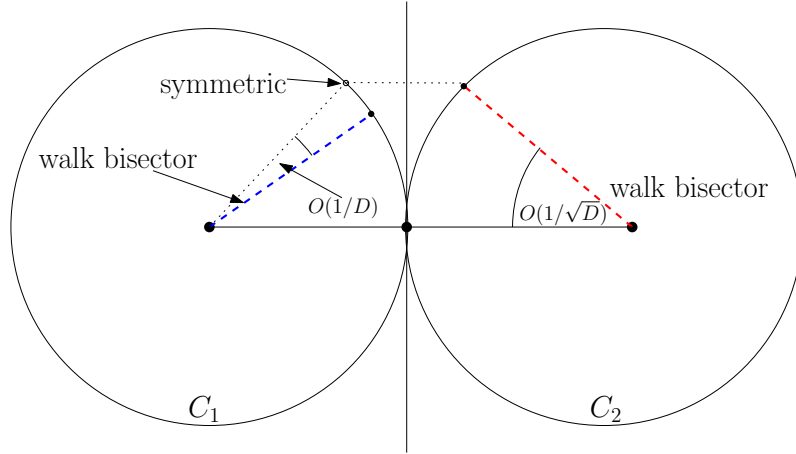
Note that approach can occur only when an agent is in the meeting area of its walk.

Consider a *geometric instance* I of the approach problem consisting of bases a of agent A and b of agent B . The angle between the segment ab and the half-line with direction North, originating at a is denoted by $\gamma(I)$. This is also the angle between the segment ab and the half-line with direction South, originating at b .

We define a *pair of walks* as a two-element set containing a walk of agent Blue and a walk of agent Red. We say that a pair of walks $\{w_1, w_2\}$ *supports* the geometric instance I if these walks are at distance at most 1.

Now, we provide some properties regarding “usefulness” of walks and pairs of walks with respect to the success of approach (cf. Fig. 3).

In the following lemma we focus on a single walk. We show that, a given walk w can belong to a pair of walks assuring approach for an instance I , only if the walk bisector of w is $O(1/\sqrt{D})$ -close to $\gamma(I)$. In other words, the angle between the segment connecting the bases and the walk bisector of w should be $O(1/\sqrt{D})$. Otherwise, the agent is at distance greater than 1 from any point reachable by the other agent during the walk w of the first agent. The proof of the lemma is in the Appendix, Section E.



■ **Figure 3** Illustration for Lemmas 6 and 7. The blue dashed segment is the walk bisector of agent Blue and the red dashed segment is the walk bisector of agent Red.

► **Lemma 6.** *If $|\alpha - \gamma(I)| > 3/\sqrt{D}$, for $D > 256$, then no pair of walks $\{w_1, w_2\}$ of agents A and B , such that α is the walk angle of one of the agents, supports the geometric instance I .*

Now, we analyze additional requirements which have to be satisfied for a pair of walks in order to make approach feasible, provided that walks from the pair are appropriately synchronized in time and both of them are close to the tangency point of their circles with radius $D/2$, as expressed in Lemma 6. In Lemma 7 we show that, apart from the requirements of Lemma 6, a pair of walks $\{w_1, w_2\}$ might lead to approach only if their walk bisectors are $O(1/D)$ -close. More precisely, if α and β are the angles of bisectors of these walks (clockwise from Blue and counterclockwise from Red) then approach might be possible only if $|\alpha - \beta| \in O(1/D)$. Let X_i be the common point of C_i and the bisector of the walk w_i , for $i \in \{1, 2\}$, where w_1 is the walk of Blue and w_2 is the walk of Red. Then, the lemma says that, in order to make approach feasible, X_1 and X_2 should be located almost symmetrically with respect to the line tangent to C_1 and C_2 . By moving one of these points on its circle by a distance $O(\frac{1}{D} \cdot 2\pi D) = O(1)$, the positions of those points would be ideally symmetric.

► **Lemma 7.** *Consider the set $\mathcal{I}$ of geometric instances supported by a set $\{w_A, w_B\}$ of walks. Let α be the walk angle of agent A , and β the walk angle of agent B . Then, for $D > 100$ and for any $I \in \mathcal{I}$, $\gamma(I) \in \left[\frac{\alpha + \beta}{2} - \frac{6}{D}, \frac{\alpha + \beta}{2} + \frac{6}{D} \right]$.*

Lemma 7 gives us the following corollary.

► **Corollary 8.** *Consider the set $\mathcal{I}$ of geometric instances supported by a set (w_A, w_B) of walks. Then all angles $\gamma(I)$, for $I \in \mathcal{I}$, belong to an interval of length at most $12/D$, for $D > 100$.*

4.3 High-level ideas of lower bounds for scenarios without simultaneous start or without common orientation

In this section we describe the ideas of the lower bounds $\Omega(D^2\sqrt{D})$ presented in Sect. 4.4 and 4.5.

4.3.1 Time-shifted instances

In order for a pair of walks $\{w_1, w_2\}$ to yield approach, both walks have to be sufficiently close geometrically (i.e., their trajectories should be close to each other). These requirements, combined with the assumption that the distance between the bases of agents is the largest still allowing approach, lead to Lemmas 6 and 7. Additionally, the walks w_1 and w_2 assuring approach should be somehow synchronized. This synchronization is necessary in order to assure that the agents get to the closest parts of the pair of walks at the same time. This property should be satisfied regardless of the delay of starting times of agents, chosen by the adversary.

Lemma 10 formalizes the above intuition and expresses some conclusions regarding time T needed for approach in the following way. First, we restrict attention to such instances I that the angle $\gamma(I)$ belongs to one of the angle-intervals from the set $\{g_i\}_{i=1}^z$ for $z = \Theta(\sqrt{D})$ such that:

- the length of each interval is $\Theta(1/\sqrt{D})$,
- the intervals g_i and $g_{1+i \bmod z}$ are separated by an interval of length $\Theta(1/\sqrt{D})$.

Then, by Lemma 6, for all instances I, I' such that $\gamma(I) \in g_i, \gamma(I') \in g_{i'}$ for $i \neq i'$, there is no pair of walks which simultaneously supports I and I' . Actually, Lemma 6 implies even a stronger property that no walk w_1 belongs to the sets $\{w_1, w_2\}, \{w_1, w_2'\}$ such that the former set supports an instance I and the latter supports I' such that $I \in g_i, I' \in g_{i'}, i \neq i'$.

Now, assume that one agent starts at time 0 and the start of the other agent is chosen uniformly at random in the period $[0, cT]$ for some constant c , where T is the time of approach of an analyzed algorithm. For this distribution, let p_i and q_i be random variables corresponding to the fractions of time serving the interval g_i by the agents Blue and Red, respectively. Let $r_i(t)$ be the expected fraction of time in which both agents simultaneously are in their walks which could support instances from g_i in some pairs, for time shift t .

In Lemma 11, we prove a lower bound on the time which agents need to spend in parallel in their walks designated for any interval g_i of length $\Omega(1/\sqrt{D})$, in order to achieve the approach for those time-shifted instances I for which the angle $\gamma(I)$ belongs to g_i . It turns out that this time has to be $\Omega(D\sqrt{D})$. An intuition behind this bound is that an interval of size $\Omega(1/\sqrt{D})$ requires $\Omega\left(\frac{1/\sqrt{D}}{1/D}\right) = \Omega(\sqrt{D})$ separate pairs of walks due to Lemma 7 and each change of a pair of walks takes $\Omega(D)$ time. Importantly, walks of both agents cannot help with approach for $\gamma(I) \in g_{i'}$ and $i' \neq i$, since g_i is separated from g_{i-1} and g_{i+1} by intervals of lengths $\Omega(1/\sqrt{D})$ as well – cf. Lemma 6.

In Lemmas 11 and 12 we take advantage of the fact that time shift is an appropriately chosen random variable and each walk can support instances from only one of the intervals g_i :

- In Lemma 11 we show that, for randomly chosen time-shift in $[0, 10T] \subset \mathbb{R}$ with uniform distribution, the expected value of time fraction when both agents perform walks supporting approach for the value of $\gamma(I)$ in g_i , for each i , is at least $1.1cp_iq_i$.
- Using purely algebraic arguments we show in Lemma 12 that the minimum of p_iq_i is $\Omega(1/D)$. This fact holds by substituting the maximum value of i which is $O(\sqrt{D})$ in the place of n in the lemma.

In Theorem 13 we combine the above observations (Lemmas 11 and 12) with Lemma 10, proving a lower bound on the time spent performing walks in the interval g_i , for each i and for each starting time of agent B .

4.3.2 Orientation-shifted instances

For the model without common orientation, we carefully choose the set $\mathcal{I}$ of orientation-shifted instances of size $\Theta(D\sqrt{D})$ such that no pair of walks can support more than one instance from $\mathcal{I}$. The set $\mathcal{I}$ consists of instances $I_{i,j}$ for natural i and j , such that the largest i is $O(\sqrt{D})$, the largest j is $O(D)$, $\gamma(I_{i,j})$ is i times an appropriate value of the order $\Theta(1/\sqrt{D})$ and $\phi(I_{i,j})$ is equal to j times an appropriate value of the order $\Theta(1/\sqrt{D})$. Thus,

- If $i_1 \neq i_2$ then the instances I_{i_1,j_1}, I_{i_2,j_2} cannot be supported by the same pair of walks due to the purely geometric argument regarding distance between tangency points of different geometric instances, see Lemma 6.
- If $i_1 = i_2$ and $j_1 \neq j_2$ then the instances I_{i_1,j_1}, I_{i_2,j_2} cannot be supported by the same pair of walks due to the fact that the differences between orientation shifts of those instances contradict the geometric argument regarding relative locations of walks bisectors of the walks from the set if both instances are supported by the considered pair of walks (see Lemma 7).

The above observations imply that $\Omega(D\sqrt{D})$ pairs of walks have to be scheduled in such a way that both agents performing walks of a pair have to be in the meeting parts of their walks at the same time. However, as an agent is at distance $\geq D/2$ from its base in a meeting part of its walk, there is time $\geq D$ between two consecutive pairs of walks supporting some instances from $\mathcal{I}$. This observation leads to the lower bound expressed in Theorem 15. However, some extra consideration is needed in order to tackle the case that an agent may move away by distance $> D/2$ from its base which prevents it from getting back to the base. This can happen in the final walk, at the end of which the agent does not return to the base. This special case is tackled in the final part of the proof of Theorem 15.

4.4 Time $\Omega(D^2\sqrt{D})$ for arbitrary start delay

In this section, we prove the lower bound $\Omega(D^2\sqrt{D})$ on the time of approach that holds if arbitrary start delay is permitted, even if agents have the same orientation. Some missing proofs of lemmas will be provided in the full version of the paper.

Consider an instance I of the approach problem consisting of bases a of agent A and b of agent B , and of a delay between the starting times of the agents. As I is a geometric instance with an additional delay parameter we will call it a *time-shifted instance*. We say that a pair of walks W of agents A and B supports the time-shifted instance I if W supports the geometric part of the instance I .

Let $z = 2\lfloor\sqrt{D}\rfloor$, and let $g_i = [(2i-1)\frac{2\pi}{z}, 2i\frac{2\pi}{z})$ for $i = 1, 2, \dots, z/2$.

The following corollary is implied directly by Lemma 6.

► **Corollary 9.** *Consider any walk w of one of the agents, and any walks w_1, w_2 of the other agent. Let I_1 and I_2 be time-shifted instances supported by pairs of walks $\{w, w_1\}$ and $\{w, w_2\}$, respectively. Then $\gamma(I_1)$ and $\gamma(I_2)$ cannot belong to different segments g_i .*

For any $i = 1, 2, \dots, z/2$, we define the set P_i of walks of agent A and the set Q_i of walks of agent B as follows.

We say that $w \in P_i$ (resp. $w \in Q_i$), if there exists a walk w' of agent B (resp. of agent A), such that the pair of walks $\{w, w'\}$ supports a time-shifted instance I with $\gamma(I) \in g_i$. Note that, due to Corollary 9, sets P_i , for different indices i , and sets Q_i , for different indices i , are pairwise disjoint.

Without loss of generality, we will assume that agent A starts first. Let 0 be the time when agent A starts, and $t \geq 0$ be the time when agent B starts. The local time of each agent is the time counted since its start. Assume that T is the approach time counted from the start of agent B , for a worst-case instance.

Denote by p_i the fraction of time spent by agent A in walks from set P_i in the period $[0, 11T]$. Denote by q_i the fraction of time spent by agent B in walks from set Q_i in the period $[0, T]$. (Recall that, we measure time for agent B according to its local time. In particular time 0 for agent B is equal to time t at which B starts the execution of the algorithm.) By definition, $\sum_i p_i \leq 1$ and $\sum_i q_i \leq 1$.

We define the indicator function $\mathbb{I}_A^i(s)$ as follows: $\mathbb{I}_A^i(s) = 1$, if at time s , agent A performs a walk from the set P_i and $\mathbb{I}_A^i(s) = 0$ otherwise. Analogously, $\mathbb{I}_B^i(s) = 1$ if at its local time s , agent B performs a walk from the set Q_i and $\mathbb{I}_B^i(s) = 0$ otherwise.

Note that $p_i = \frac{1}{11T} \int_0^{11T} \mathbb{I}_A^i(s) ds$ and $q_i = \frac{1}{T} \int_0^T \mathbb{I}_B^i(s) ds$. The *common period of walks* in P_i and Q_i is defined as the set of time points when agent A performs a walk from P_i while agent B performs a walk from Q_i . The *attempt period of walks* in P_i and Q_i is the subset of the common period of walks in P_i and Q_i consisting of time points such that both walks are in their meeting parts. This set of time points depends on the starting time of agent B . Moreover define $r_i(t) = \frac{1}{T} \int_0^T \mathbb{I}_A^i(t+s) \mathbb{I}_B^i(s) ds$. This is the fraction of the time segment $[t, t+T]$ occupied by the common period of walks in P_i and Q_i .

► **Lemma 10.** *Suppose that, for some $i = 1, 2, \dots, z/2$ and some starting time t of agent B , the inequality $r_i(t) \cdot T < \left(\frac{\pi\sqrt{D}}{12} - 1\right)(D-2)$ holds. Then there exists a time-shifted instance I such that $\gamma(I) \in g_i$ and t is the starting time of agent B , for which approach does not happen.*

► **Lemma 11.** *Assume that agent B starts in time taken uniformly at random from the range $[0, 10T]$. The expected value of r_i is not greater than $1.1p_iq_i$.*

Proof. The expected value of r_i is

$$\begin{aligned} \mathbb{E}[r_i] &= \frac{1}{10T} \int_0^{10T} r_i(t) dt = \frac{1}{10T} \int_0^{10T} \frac{1}{T} \int_0^T \mathbb{I}_A^i(t+s) \mathbb{I}_B^i(s) ds dt \\ &= \frac{1}{10T^2} \int_0^T \mathbb{I}_B^i(s) \int_0^{10T} \mathbb{I}_A^i(t+s) dt ds \\ &\leq \frac{1}{10T^2} \int_0^T \mathbb{I}_B^i(s) \int_0^{11T} \mathbb{I}_A^i(t) dt ds \\ &\leq \frac{11}{10} \frac{1}{11T} \int_0^{11T} \mathbb{I}_A^i(t) dt \cdot \frac{1}{T} \int_0^T \mathbb{I}_B^i(s) ds = 1.1p_iq_i \end{aligned}$$

To complete our considerations, we will also need the following algebraic lemma.

► **Lemma 12.** *Consider real numbers $p_i, q_i \in [0, 1]$, for $i = 1, \dots, n$, such that $\sum_{i=1}^n p_i \leq 1$ and $\sum_{i=1}^n q_i \leq 1$. Then $\min\{p_iq_i : i \leq n\} \leq \frac{1}{n^2}$.*

► **Theorem 13.** *The time of approach for two agents with arbitrary start delay, starting at a distance at most D , is $\Omega(D^2\sqrt{D})$.*

Proof. We apply Lemma 12 for $n = z/2 = \lfloor \sqrt{D} \rfloor$ and for p_i, q_i defined before in this section. By Lemma 12, there exists an i for which $p_iq_i \leq 1/n^2 \leq 4/D$. By Lemma 11, the expected value of r_i is at most $1.1p_iq_i$. Thus there exists an argument t for which $r_i(t) \leq 1.1p_iq_i$. Suppose that the approach occurs for an instance I for which $\gamma(I) \in g_i$ and the delay between starting times of the agents is t . By Lemma 10, we have $r_i(t) \cdot T \geq \left(\frac{\pi\sqrt{D}}{12} - 1\right)(D-2)$, where T is the length of time between the start of agent B and the approach. Hence

$$\left(\frac{\pi\sqrt{D}}{12} - 1\right)(D-2) \leq r_i(t)T \leq 1.1p_iq_iT \leq 5T/D.$$

Thus we get $T \geq \frac{D}{5} \left(\frac{\pi\sqrt{D}}{12} - 1\right)(D-2) \in \Omega(D^2\sqrt{D})$ which proves the theorem. ◀

4.5 Time $\Omega(D^2\sqrt{D})$ for arbitrary orientations

In this section, we prove the lower bound $\Omega(D^2\sqrt{D})$ on the time of approach that holds if orientations of agents may be different and arbitrary, even if they start simultaneously.

We will call the orientation of agent A the *global orientation*. Consider an instance I of the approach problem consisting of bases a of agent A and b of agent B , and of the orientation of agent B . As it is a geometric instance with an additional rotation parameter we will call it an *orientation-shifted instance*. Let $\phi(I)$ be the angle between the half-line indicating north according to the local orientation of agent B and the halfline indicating north in the global orientation. Thus, an orientation-shifted instance I is determined by the locations a and b of the bases of agents and the angle $\phi(I)$. Let $\{w_A, w_B\}$ be a pair of walks, where w_A, w_B are walks of agent A and agent B , respectively. Let $w_B(I)$ be the walk w_B of agent B rotated around the base of agent B by $\phi(I)$, i.e., the walk w_B expressed according to the global orientation in the orientation-shifted instance I . We say that the pair of walks $\{w_A, w_B\}$ *supports* the orientation-shifted instance I if $\{w_A, w_B(I)\}$ supports the part of the instance I determined by the locations of the bases.

Let $\mathcal{I}$ be a set of orientation-shifted instances $I_{i,j}$ for $i \in \{0, \dots, \lfloor 2\pi\frac{\sqrt{D}}{6} \rfloor - 1\}$ and $j \in \{0, \dots, \lfloor 2\pi\frac{D}{30} \rfloor - 1\}$ such that:

- the distance between the bases of agents A and B is exactly D .
- $\gamma(I_{i,j}) = \frac{6i}{\sqrt{D}}$,
- $\phi(I_{i,j}) = \frac{30j}{D}$.

► **Lemma 14.** *Let $\mathcal{S}$ be a collection of pairs of walks. If each instance $I \in \mathcal{I}$ is supported by a pair of walks from $\mathcal{S}$ then $|\mathcal{S}| \geq |\mathcal{I}|$.*

Proof. It is sufficient to prove that each pair of walks can support only one instance from the set $\mathcal{I}$. For the sake of contradiction assume that there exists a pair of walks $S \in \mathcal{S}$ supporting two different instances $I_{i_1, j_1}, I_{i_2, j_2} \in \mathcal{I}$ for $i_1, i_2 \in \{0, \dots, \lfloor 2\pi\frac{\sqrt{D}}{6} \rfloor - 1\}$ and $j_1, j_2 \in \{0, \dots, \lfloor 2\pi\frac{D}{30} \rfloor - 1\}$. Let us analyze two cases.

- $i_1 \neq i_2$. Let α be the walk bisector of the walk of agent A . By Lemma 6, we need both inequalities $|\alpha - \gamma(I_{i_1, j_1})| \leq 3/(2\sqrt{D})$ and $|\alpha - \gamma(I_{i_2, j_2})| \leq 3/(2\sqrt{D})$ in order to assure that S supports I_{i_1, j_1} and I_{i_2, j_2} . However, these inequalities imply that $|\gamma(I_{i_1, j_1}) - \gamma(I_{i_2, j_2})| \leq 3/\sqrt{D}$ and, by the definition of $\mathcal{I}$, $|\gamma(I_{i_1, j_1}) - \gamma(I_{i_2, j_2})| \geq 6/\sqrt{D}$ which gives a contradiction.
- $i_1 = i_2$ and $j_1 \neq j_2$.

Let α be the walk bisector of the walk of agent A and let $\beta(I_{i_k, j_k})$ be the walk bisector of the walk of agent B with respect to the global orientation for the instance I_{i_k, j_k} where $k \in \{1, 2\}$. By Lemma 7, we need that both $\gamma(I_{i_1, j_1}) \in \left[\frac{\alpha + \beta(I_{i_1, j_1})}{2} - \frac{6}{D}, \frac{\alpha + \beta(I_{i_1, j_1})}{2} + \frac{6}{D}\right]$ and $\gamma(I_{i_2, j_2}) \in \left[\frac{\alpha + \beta(I_{i_2, j_2})}{2} - \frac{6}{D}, \frac{\alpha + \beta(I_{i_2, j_2})}{2} + \frac{6}{D}\right]$ hold, in order to assure that S supports both I_{i_1, j_1} and I_{i_2, j_2} .

However, as $\gamma(I_{i_1, j_1}) = \gamma(I_{i_2, j_2})$, $\beta(I_{i_1, j_1}) = \beta(I_{i_2, j_2}) + \phi(I_{i_1, j_1}) - \phi(I_{i_2, j_2})$, and, given that $|\phi(I_{i_1, j_1}) - \phi(I_{i_2, j_2})| \geq \frac{30}{D}$, the segments

$$\left[\frac{\alpha + \beta(I_{i_1, j_1})}{2} - \frac{6}{D}, \frac{\alpha + \beta(I_{i_1, j_1})}{2} + \frac{6}{D}\right]$$

and

$$\begin{aligned} & \left[\frac{\alpha + \beta(I_{i_2, j_2})}{2} - \frac{6}{D}, \frac{\alpha + \beta(I_{i_2, j_2})}{2} + \frac{6}{D} \right] \\ &= \left[\frac{\alpha + \beta(I_{i_1, j_1})}{2} + \frac{\phi(I_{i_1, j_1}) - \phi(I_{i_2, j_2})}{2} - \frac{6}{D}, \right. \\ & \quad \left. \frac{\alpha + \beta(I_{i_1, j_1})}{2} + \frac{\phi(I_{i_1, j_1}) - \phi(I_{i_2, j_2})}{2} + \frac{6}{D} \right] \end{aligned}$$

are disjoint, which gives a contradiction. $\blacktriangleleft$

Using Lemma 14, we can prove the main theorem of this section.

► **Theorem 15.** *The time of approach for two agents with arbitrary orientations, starting at a distance at most D is $\Omega(D^2\sqrt{D})$, even if agents start simultaneously.*

Proof. The proof is by contradiction. Assume that there is an approach algorithm for two agents with arbitrary orientations, starting at a distance D , which completes the approach task in time at most $\frac{1}{32} \left(\frac{\sqrt{D}}{6} \cdot \frac{D}{30} - 1 \right) (D - 2) + D$ for all instances in $\mathcal{I}$.

First, consider only walks wholly contained inside the range circles of radius $D/2$ of the respective agents. We say that a pair of walks $\{w_1, w_2\}$ is *useful* if there exists a time t such that both w_1 and w_2 are in their meeting areas at time t .² The pair of walks $\{w_1, w_2\}$ cannot lead to approach in any instance, if it is not useful. (The reason is that, in such a case, at each time t the actual distance between the agents is larger than 1, regardless of the current instance.) A *usefulness witness* of a useful set $\{w_1, w_2\}$ is the smallest time t such that both walks are in their meeting areas at time t . Consider the sequence $t_1 < t_2 < \dots < t_p$, for some positive integer p , which consists of all usefulness witnesses of all pairs of walks (one walk for agent A , and one for agent B) of the algorithm. Then, by Lemma 14, at most p instances from $\mathcal{I}$ could potentially be supported by the algorithm. On the other hand, $t_l - t_{l-1} \geq D - 2$ for each $l \in [2, p]$ because at least one agent has to move back to its base from a point at distance $\geq D/2 - 1$, and then move away from the base at a distance $\geq D/2 - 1$, in the time period $[t_{l-1}, t_l]$. Therefore, p is at most $\frac{1}{32} \cdot \left(\frac{\sqrt{D}}{6} \cdot \frac{D}{30} - 1 \right)$ which is smaller than $\frac{1}{32}$ th of the size of $\mathcal{I}$ – a contradiction.

Thus, for at least $\frac{31}{32}|\mathcal{I}|$ instances in $\mathcal{I}$ the approach must occur when at least one agent leaves its range disc³. Denote the set of such instances by $\mathcal{I}'$.

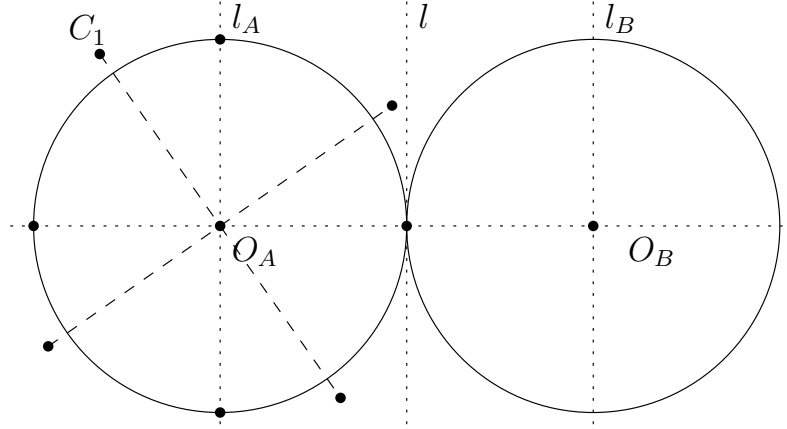
Recall that, by the definition of a walk, the agent gets back to its base at the end of each walk, except the final walk that ends with approach. As an agent is not able to get back to the base after leaving its range disc, the only walk of the agent after it leaves the range disc can be the final walk. Let's call the walk during which an agent leaves its range disc a *special walk*.

Hence the approach for all instances from the set $\mathcal{I}'$ (whose size is $\geq \frac{31}{32}|\mathcal{I}|$) must occur during the special walk of at least one agent. (However, many other walks of the other agent might occur during this special walk.)

Let $n = \lfloor 2\pi \frac{\sqrt{D}}{6} \rfloor$ and $m = \lfloor 2\pi \frac{D}{30} \rfloor$. For clarity of presentation we assume that both n and m are divisible by 4. One can easily verify that the proof holds in the general case as well, i.e., when n or m is not necessarily divisible by 4.

² Note that we work in the model with common start here, so both agents work according to the same time.

³ Recall that the range disc of an agent is the disk with radius $D/2$ centered at the base of the agent.



■ **Figure 4** The location of the point C_1 in the instance $I'_{a,b}$. The dots at the ends of the dashed segments illustrate all possible locations of C_1 for instances $I'_{a,b}$ for arbitrary $b \in [0, 3]$ and various values of $a \in [0, 3]$.

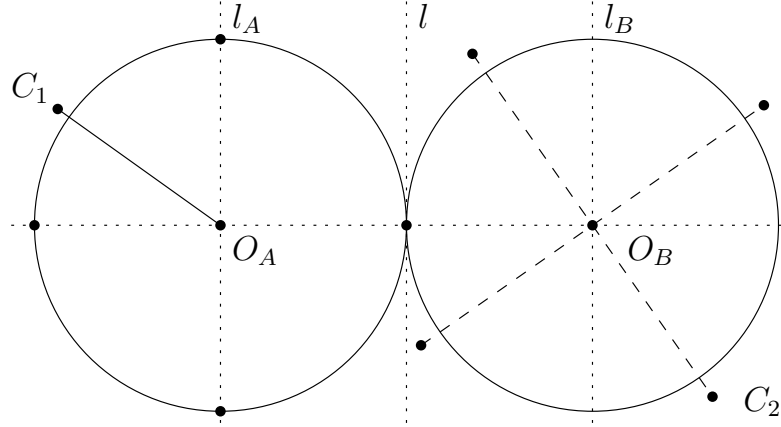
Now, let

$$\mathcal{I}_{i,j} = \{I'_{a,b} \mid I'_{a,b} = I_{i+a \cdot \frac{n}{4}, j+b \cdot \frac{m}{4}} \text{ for each } a, b \in [0, 3]\}$$

for $i \in [0, n/4 - 1]$ and $j \in [0, m/4 - 1]$. Since $|\mathcal{I}'| \geq \frac{31}{32}|\mathcal{I}|$ and $\mathcal{I}$ can be split into $|\mathcal{I}|/16$ subsets $\mathcal{I}_{i,j}$, there exists a subset $\mathcal{I}_{i,j}$ for some $i \in [0, n/4 - 1]$ $j \in [0, m/4 - 1]$ which is entirely included in $\mathcal{I}'$.

Thus, let $i \in [0, n/4 - 1]$ $j \in [0, m/4 - 1]$ be such that $\mathcal{I}_{i,j} \subset \mathcal{I}'$. Hence, none of the pairs of non-special walks of the agents are such that an approach occurs during executions of some of those pairs of walks for any of the 16 instances from $\mathcal{I}_{i,j}$.

We now show that the other agent must also leave its range disc in order to ensure approach for all instances from $\mathcal{I}_{i,j}$. The proof is by contradiction. Assume that it is not the case. Without loss of generality assume that A is the only agent which leaves its range disc. Let C_1 be a point in the only special walk of A , at distance $D/2 + \varepsilon_1$ from its base for some constant $\varepsilon_1 > 0$, such that A is at C_1 before approach happens for any of the instances from $\mathcal{I}_{i,j}$ (recall that approach cannot happen as long as both agents are inside their range discs, according to the definition of $\mathcal{I}_{i,j}$). Observe that, by choosing different values of $a \in [0, 3]$, we can change the location of C_1 with respect to the base O_B of B in the instances $I'_{a,j}$ for all $j \in [0, 3]$. Indeed, as we assume that the orientation of A corresponds to the global orientation, the value of $\gamma(I'_{a,b})$ determines relative location of C_1 and O_B as illustrated on Figure 4. The value of $\gamma(I'_{a,b})$ is determined by the value of a . Let $a \in [0, 3]$ be such that the base O_B of B is furthest from the point C_1 in $I'_{a,b'}$, for $b' \in [0, 3]$. Let l be the line tangent to the range circles of both agents, let l_A and l_B be the lines parallel to l , going through the bases O_A and O_B of the agents, respectively – see Figure 4. Then, in the instance $I'_{a,b}$ for $b \in [0, 3]$, the point C_1 is on the opposite side of l_A than the tangency point of the range circles of the agents. And therefore C_1 is further from the tangency point of the range circles of the agents than from the base of the agent A and so it is further from the base of B than $D + 1$. Indeed, as $|C_1 O_A| > D/2$, $|O_A O_B| = D$ and the angle $\angle C_1 O_A O_B$ is obtuse, the Pythagorean Theorem implies that $|C_1 O_B| > \sqrt{|C_1 O_A|^2 + |O_A O_B|^2} > \sqrt{(D/2)^2 + D^2} > D + 1$ for $D > 5$. As A has at this point only $D/2 - \varepsilon_1$ fuel left, the agent B must in this case travel further from its base than $D/2$ in order to ensure approach of the agents. But this fact contradicts our assumption that only one of the agents leaves its range disk during the execution of the algorithm.



■ **Figure 5** The locations of the points C_1 and C_2 in the instance $I'_{a,b}$. The dots at the ends of the dashed segments illustrate all possible locations of C_2 for instances $I_{a,b'}$, $b' \in [0, 3]$.

Let C_2 be any point in the special walk of B at distance $D/2 + \varepsilon_2 > 0$ from the base of agent B . Again, let b be such that the greatest distance between C_1 and C_2 amongst all the instances $I'_{a,b'}$ for $b' \in [0, 3]$ is obtained for $I'_{a,b}$. Then, the point C_2 is located on the opposite side of l_B than C_1 in $I'_{a,b}$ – see Figure 5. Inspecting the triangle $\Delta C_1 O_A C_2$ with the obtuse angle $\angle C_1 O_A C_2$, we obtain the following lower bound on the distance between C_1 and C_2 : $|C_1 C_2| > \sqrt{|C_1 O_A|^2 + |O_A C_2|^2} > \sqrt{(D/2)^2 + D^2} \geq D + 1$ for $D > 5$, by the Pythagorean Theorem. On the other hand, the amount of fuel left in the tank of A at the point C_1 and the amount of fuel left in the tank of B at the point C_2 are smaller than $D/2$ since the agents are outside of their range discs at these points. Thus, the approach cannot occur for the instance $I'_{a,b}$.

As our assumption led to an occurrence of an instance without approach, the algorithm working in time $\frac{1}{32} \cdot \left(\frac{\sqrt{D}}{6} \frac{D}{30} - 1 \right) (D - 2) + D$ cannot ensure the approach for all instances from $\mathcal{I}$, which concludes the proof. ◀

5 Discussion

In this section, we discuss two assumptions of our model. The first is that the size of the tank (i.e., the distance that an agent can travel without replenishing the tank) is D , and the second is that the time of approach is counted from the wake-up of the later agent.

Considering the first assumption, it is natural to ask if a smaller tank wouldn't be enough. First observe that an agent with tank of size x cannot go at distance larger than $x/2$ from its base, possibly apart from the last trip that does not end at the base, because then it would be unable to get back to the base in order to replenish its tank. This implies that the size of the tank must be strictly larger than $D - 1$. Indeed, consider two agents at distance exactly D . If the size x of the tank were strictly smaller than $D - 1$ then agents could never approach without going farther than $x/2$ from their bases, as their distance would be always larger than 1. It can also be shown that the adversary can place the bases of agents in such a way that approach could not happen during the final trip, when an agent does not have to get back to the base.

If the size of the tank were exactly $D - 1$ then the only way to approach without travelling farther than $(D - 1)/2$ from the base would be for both agents to go straight on the segment joining their bases, each at distance exactly $(D - 1)/2$. Clearly, for any hypothetical algorithm

of approach, the adversary can place the bases in order to avoid such a perfect guess of direction in any finite time. Similarly as before, the adversary can prevent approach during the final trip.

The remaining case is that of a tank of size $(D - 1) + \epsilon$, for some $0 < \epsilon < 1$. By slightly modifying our algorithms, it can be shown that, for each of our models, approach can be guaranteed with the same time complexity, whenever the tanks of the agents have size $(D - 1) + \epsilon$, for any constant $0 < \epsilon < 1$.

The second assumption is that the time of approach is counted from the wake-up of the later agent. An alternative way would be to count time from the wake-up of the earlier agent. However, for any $D > 2$, consider two agents with tanks of size D and bases at distance D . Before the wake-up of the later agent, the earlier agent cannot approach it without travelling farther than $D/2$ from its base. Since the adversary can make the delay between the wake-up times of the agents arbitrarily long, the time of approach counted from the wake-up of the earlier agent would be unbounded.

6 Conclusion

We considered the problem of approach of agents with restricted tanks and designed algorithms of optimal complexity, for any of the four scenarios yielded by the presence or absence of our two coherence assumptions. In this paper, we heavily relied on the synchronous character of the agents: they can stay put for a chosen amount of time and when they travel, they always move at speed 1. It is natural to ask how the problem of approach for agents with restricted tanks changes in the asynchronous setting, where agents can choose the direction and distance of their moves but the adversary decides the speeds of the agents. In this model, agents cannot choose to wait, as the adversary controls waiting times as well. The problem of asynchronous approach for agents with unrestricted tanks has been solved in [4]. What is the solution of it for agents with restricted tanks and how large tanks are needed?

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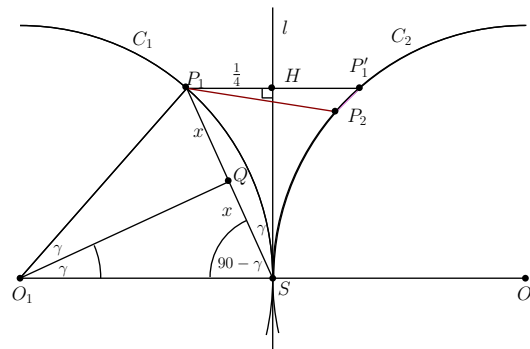
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Appendix

A Proof of Lemma 1

Proof. Without loss of generality, let P_1 be counterclockwise from S . Let l be the line



■ **Figure 6** Illustration for Lemma 1.

tangent to C_1 in point S and let O_1, O_2 be the centres of C_1 and C_2 , respectively. Let H be the common point of l and of the line parallel to the radius O_1S , passing through the point P_1 . Let Q be the common point of the bisector of the angle SO_1P_1 and of the segment

P_1S . As the triangle SO_1P_1 is an isosceles triangle, O_1QP_1 is a right triangle, and thus the triangles O_1QP_1 and SHP_1 are similar, because $|\angle QO_1P_1| = |\angle HSP_1|$ (cf. Fig. 6).

As $\alpha_1(P_1) \leq y = \frac{1}{4}\sqrt{r}$, the length x of the segment P_1Q satisfies $x = \frac{1}{2}|P_1S| \leq \alpha_1(P_1) \leq \frac{1}{4}\sqrt{r}$.

By similarity of the triangles O_1QP_1 and SHP_1 , we have $\frac{|O_1P_1|}{|P_1Q|} = \frac{|P_1S|}{|P_1H|}$ and thus

$$|P_1H| = \frac{|P_1Q| \cdot |P_1S|}{|O_1P_1|} = 2 \frac{x^2}{r} \leq 1/8. \quad (1)$$

Let P'_1 be the point axial symmetric to P_1 with respect to axis l . As l is also tangent to circle C_2 , and C_2 has the same radius as C_1 , the point P'_1 lies on C_2 .

In view of the assumption $\alpha_1(P_1) - 1/2 \leq \alpha_2(P_2) \leq \alpha_1(P_1) + 1/2$ we have

$$|P'_1P_2| \leq |\alpha_2(P'_1) - \alpha_2(P_2)| = |\alpha_1(P_1) - \alpha_2(P_2)| \leq \frac{1}{2}. \quad (2)$$

By the triangle inequality, (1), and (2) we have $|P_1P_2| \leq |P_1P'_1| + |P'_1P_2| = 2|P_1H| + |P'_1P_2| \leq \frac{1}{4} + \frac{1}{2} < 1$. $\blacktriangleleft$

B Proof of Theorem 2

Proof. Suppose that the distance between the base O_1 of agent Blue and the base O_2 of agent Red is $d \leq D$. Let C_1 and C_2 be circles of radius $D/2$ centered at O_1 and O_2 respectively, and let C'_1 and C'_2 be circles of radius $d/2$ centered at O_1 and O_2 respectively. Let S be the unique common point of circles C'_1 and C'_2 .

Consider points $A_0, \dots, A_{n-1}$ on circle C_1 , visited by agent Blue in its consecutive phases. For $i = 0, 1, \dots, n-1$, let A'_i be the intersection point of the segment O_1A_i with the circle C'_1 . Let j be the index for which point A'_j is closest to point S among all points A'_i . If there are two such closest points, j is chosen arbitrarily between their indices.

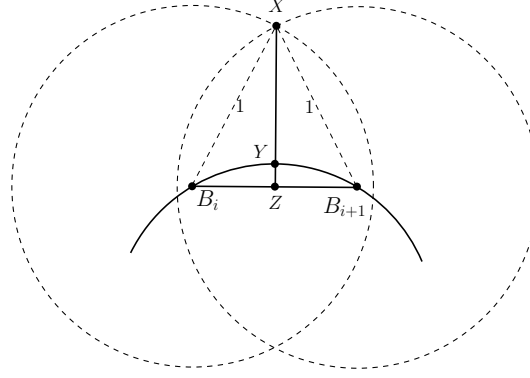
Consider the time point t in phase j when agent Blue is at point A'_j for the first time. At time t , agent Red is at some point B on circle C'_2 . We use Lemma 1 substituting $d/2$ for r , B for P_1 , and A'_j for P_2 . Denote by α the length of the arc SB on circle C'_2 . In view of the definition of n , we have $\alpha \leq \frac{1}{4}\sqrt{d/2}$ and the length of arc SA'_j on circle C'_1 is between $\alpha - 1/2$ and $\alpha + 1/2$. Thus both assumptions of Lemma 1 are satisfied. This implies $|A'_jB| < 1$. Hence approach occurs in phase j . Since $j < n = 4\lceil\pi D\rceil$ and the duration of each phase is D , phase j is completed in time $O(D^2)$ which proves the theorem. $\blacktriangleleft$

C Proof of Theorem 3

Proof. Suppose that the distance between the base O_1 of agent Blue and the base O_2 of agent Red is $d \leq D$. Consider two cases.

Case 1. $d = D$.

In this case, the circles C_1 and C_2 are tangent. Let S be the point of tangency. Let $j \in [0, n-1]$ be such an index that the point A_j is closest to the point S among all points A_i . If there are two such closest points, j is chosen arbitrarily from the indices of these two closest points. Let k be the index for which B_k is the closest point from the set $\{B_0, \dots, B_{m-1}\}$ to the point A_j . Consider the time point t when agent Blue visits the point A_j for the first time after both agents have been awoken. As agent Blue waits in this point for time $4\lceil 4\pi D^2 \rceil$, there is a time point $t' \in [t, t + 4\lceil 4\pi D^2 \rceil]$, when agent Red visits the point B_k . Now, we may use Lemma 1 substituting $d/2 = D/2$ for r , A_j for P_1 , and B_k for P_2 . The definition of n



■ **Figure 7** Illustration for Claim 16.

implies that the length α of arc SA_j on the circle C_1 is at most $\frac{1}{4}\sqrt{D/2}$ and the length of arc SB_k on the circle C_2 is not smaller than $\alpha - 1/2$ and not larger than $\alpha + 1/2$. Thus both assumptions of Lemma 1 are satisfied. This implies the inequality $|A_j B_k| < 1$. Hence approach occurs at time t' . Since t is the first time agent Blue visits point A_j when both agents are awake, the approach occurs in time at most $4\lceil 4\pi D^2 \rceil (\lceil 2\pi\sqrt{D} \rceil + 2) = O(D^2\sqrt{D})$.

Case 2. $d < D$.

Let the *approach area* of an agent in some time period be the area equal to all points at distance ≤ 1 from all points visited by the agent during the given time period. The following claim provides significant property of the approach area of agent Red during its execution of the algorithm. Then, in Claim 17, we will show that the property stated in Claim 16 guarantees that A_j belongs to the approach area of agent Red for some $j \in [0, n-1]$.

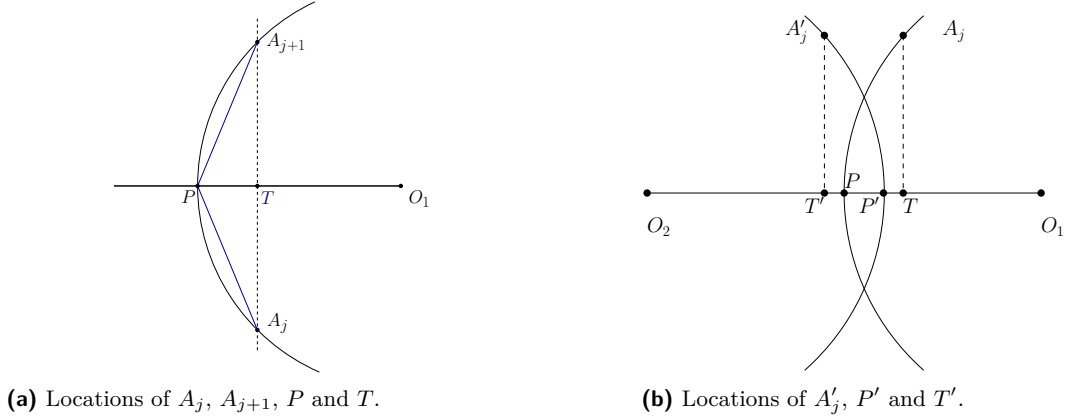
▷ **Claim 16.** Each point within distance of $\frac{1}{2}D + \frac{\sqrt{15}-1}{4}$ from the base of agent Red belongs to its approach area.

In order to prove the claim, notice that, as the length of the circle C_2 is $2\pi \cdot \frac{1}{2}D = \pi D$, the distance between B_i and $B_{(i+1) \bmod m}$ is smaller than one, and therefore the set of points at distance at most 1 from the set of points $\{B_0, \dots, B_{m-1}\}$ covers the whole circle C_2 . As agent Red moves via a straight line from its base to each of the points B_i , its approach area covers C_2 . Now we will show that agent Red also approaches the additional annulus of width at least $\frac{\sqrt{15}-1}{4}$ outside of C_2 .

Fix two consecutive points $B_i, B_{(i+1) \bmod m}$ on C_2 . (For brevity, we will write B_{i+1} instead of $B_{(i+1) \bmod m}$ in the rest of the proof.) Consider the circles of radius one with the centres in B_i and B_{i+1} , respectively (see Fig. 7). The union of the corresponding discs is the area approachable from both B_i and B_{i+1} . Let X be a common point of these two discs at the largest distance from O_2 . The point X is one of the points furthest from O_2 at distance 1 from B_i . Our goal is to bound the distance from X to O_2 .

Consider the segment going through the point X and perpendicular to $B_i B_{i+1}$. Let Y, Z be, respectively, the common point of this segment with the circle C_2 and with the segment $B_i B_{i+1}$ (see Fig. 7).

As X is a common point of circles of the same radius, the point Z divides the segment of $B_i B_{i+1}$ into two segments of equal lengths. Similarly, the point Y divides the arc $B_i B_{i+1}$ in two arcs of equal lengths. As there are $m = \lceil 4\pi D \rceil$ points B_j , the length of $B_i B_{i+1}$ is at most $1/4$ and, as $B_i X$ is a radius of length 1, we have $|XZ| = \sqrt{1 - |B_i Z|^2} \geq \sqrt{1 - \frac{1}{16}} = \frac{\sqrt{15}}{4}$.

(a) Locations of A_j , A_{j+1} , P and T .(b) Locations of A'_j , P' and T' .

■ **Figure 8** Illustration of notations from the proof of Claim 17.

Moreover, the arc B_iY has length $|\widehat{B_iY}| \leq \frac{|B_iB_{i+1}|}{2} \leq \frac{1}{2} \cdot \frac{2\pi \cdot \frac{1}{2}D}{|4\pi D|} < \frac{1}{4}$. As the segment B_iY is a hypotenuse of the triangle B_iZY we have $|YZ| \leq |B_iY| \leq \frac{1}{4}$ and $|XY| = |XZ| - |YZ| \geq \frac{\sqrt{15}}{4} - 1/4 = \frac{\sqrt{15}-1}{4}$.

Thus the point X is at distance at least $\frac{1}{2}D + \frac{\sqrt{15}-1}{4}$ from O_1 , which implies that all points at this distance are approachable by agent Red. This proves the claim. $\diamond$

Now we will show that there is at least one point A_i , which agent Red can approach.

▷ **Claim 17.** There is at least one point A_i , for $i \in [0, n-1]$, with distance at most $\frac{1}{2}D + \frac{\sqrt{15}-1}{4}$ from O_2 .

In order to prove the claim notice that, if there is a point A_i , for $i \in [0, n-1]$, located inside the disc limited by circle C_2 , then the claim holds. So, assume that there is no such point A_i .

Let $j \in [0, n-1]$ be an index for which A_j is the closest point to O_2 among all the points from $\{A_0, \dots, A_{n-1}\}$. If there are two such indices, let j be such that $|O_2A_j| = |O_2A_{j+1}|$ and O_2A_j is the smallest distance from O_2 to all A_i s. Let P be the common point of the circle C_1 and the line O_1O_2 such that P is located on the segment O_1O_2 (cf. Fig. 8a). Moreover, let T be the projection of the point A_j on the line O_1O_2 . We want to bound the distance from P to A_j from above. Observe that, the distance between A_j and P is maximized in the instance in which there are two points A_j and A_{j+1} closest to O_2 symmetric with respect to the line O_1O_2 , as presented in Fig. 8a. Therefore, in order to bound the length A_jP from above, we will restrict our consideration to this specific case.

Let l be the line perpendicular to O_1O_2 going through A_j and A_{j+1} . Let T be the common point of the line O_1O_2 and the line l (see Fig. 8a).

As A_j and A_{j+1} are two consecutive points from the cyclic sequence $A_0, \dots, A_{n-1}$, the length of the arc A_jA_{j+1} is at most $\sqrt{D}$. Thus the angle $\angle A_jPA_{j+1}$ is at least $\pi \cdot \frac{2\pi D - \sqrt{D}}{2\pi D} = \pi - \frac{1}{2\sqrt{D}}$.

As the line PT is perpendicular to the segment A_jA_{j+1} , we have $\alpha = \angle A_jPT = \frac{1}{2}\angle A_jPA_{j+1} = \frac{\pi}{2} - \frac{1}{4\sqrt{D}}$. One can verify that, for all $x \geq 1$, $\tan(\frac{\pi}{2} - \frac{1}{4x}) > 3x$. Thus

$$|PT| = \frac{|A_jT|}{\tan \alpha} < \frac{\frac{\sqrt{D}}{2}}{3\sqrt{D}} = \frac{1}{6}. \quad (3)$$

Let P' be a common point of the line O_1O_2 and the circle C_2 . Moreover, let T' be the point on O_1O_2 such that $|P'T'| = |PT|$ and $|O_2T'| \leq D$. Let A'_jT' be the segment symmetric to A_jT with respect to the bisector of the segment O_1O_2 . As the circles C_1 and C_2 have the same radius, the point A'_j lies on circle C_2 (see Fig. 8b).

The length of $A_jA'_j$ satisfies the inequality (see Fig. 8b) $|A_jA'_j| < |PT| + |P'T'| < \frac{1}{3}$, where the last inequality follows from (3). Thus by the triangle inequality, $|O_2A_j| \leq |O_2A'_j| + |A'_jA_j| < D + \frac{1}{3} < D + \frac{\sqrt{15}-1}{4}$. This completes the proof of the Claim. $\diamond$

Let A_j be a point whose existence is guaranteed by Claim 17. Let t be the time point when agent Blue visits point A_j for the first time after both agents have been awoken. As agent Blue waits in this point for time $4\lceil 4\pi D^2 \rceil$, agent Red visits all of his points B_i during this time and, by Claims 16 and 17, the approach is successful. Thus the time to complete approach is at most $4\lceil 4\pi D^2 \rceil (\lceil 2\pi\sqrt{D} \rceil + 2)$. This implies that algorithm **Slower** guarantees approach in time $O(D^2\sqrt{D})$. $\blacktriangleleft$

D Proof of Theorem 5

Proof. Consider an arbitrary approach algorithm for two agents A and B . Let $a(t)$ (resp. $b(t)$) be a vector describing the position of agent A (resp. B) relative to its starting point in time t . Let $d(t) = a(t) - b(t)$. Let T be the approach time of the algorithm.

For $i, j \in [-\lfloor \frac{D}{4} \rfloor, \lfloor \frac{D}{4} \rfloor - 2]$ let $r_{i,j}$ be the rectangle with vertices in points $(2i, 2j), (2i + 2, 2j), (2i, 2j + 2), (2i + 2, 2j + 2)$.

We will consider only *even* rectangles, that is the rectangles $r_{i,j}$ where both i and j are even. We say that the algorithm *marks the rectangle* $r_{i,j}$ if there is a moment t of the execution of the algorithm such that the point $d(t)$ is inside $r_{i,j}$. As for any t_1, t_2 such that, $|t_1 - t_2| \leq 1$ the length of the vector $d(t_1) - d(t_2)$ is at most 2, the algorithm can mark at most $T + 1$ even rectangles.

Assume that the time of approach T is less than $\frac{D^2}{16}$. The algorithm can mark at most $\frac{D^2}{16} + 1$ even rectangles, and thus there exists an unmarked even rectangle $r_{i,j}$. By fixing the base of agent A at point $(0, 0)$ and the base of B at point $(2i + 1, 2j + 1)$ the adversary can guarantee that the approach does not occur. This contradiction concludes the proof. $\blacktriangleleft$

E Proof of Lemma 6

Proof. Let S be the tangent point of range circles of the agents. Let l be the line tangent to the range circle C of agent A at point S . Let P be the point of the meeting area of the walk w_A , closest to line l (see Fig. 9). Without loss of generality, suppose that α is the walk angle of agent A . Denote by σ the angle SaP . Thus, $\sigma = |\alpha - \gamma| - \delta$.

Let H be the intersection of l with the line containing P , parallel to the segment aS . Let Q be the midpoint of segment PS . Due to similarity of triangles PHS and PQa we have

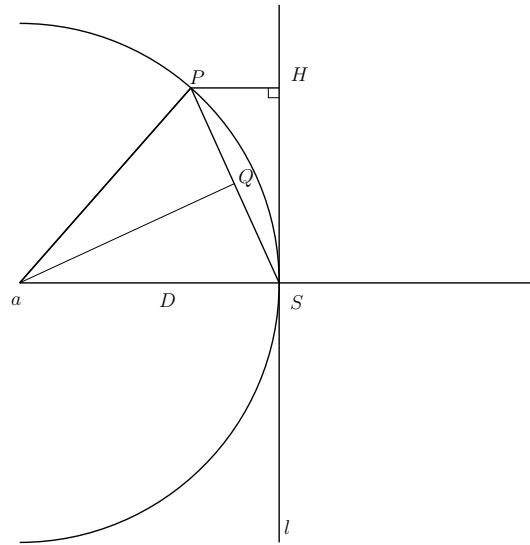
$$\frac{|PH|}{|PQ|} = \frac{|PS|}{|Pa|} = \frac{2|PQ|}{D/2}$$

and thus $|PH|D = 4|PQ|^2$. Hence $|PH| > 1$ if and only if $|PQ| > \sqrt{D}/2$. Suppose that $D > 256$.

If $\sigma < 2$, then

$$|PQ| = \frac{D}{2} \sin\left(\frac{\sigma}{2}\right) \geq \frac{D}{2} \cdot 0.8 \frac{\sigma}{2} > 0.4D \left(\frac{|\alpha - \gamma|}{2} - \frac{\delta}{2} \right) \geq 0.4D \left(\frac{1.5}{\sqrt{D}} - 4/D \right) > \frac{\sqrt{D}}{2}.$$

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■ **Figure 9** Illustration for Lemma 6.

If $\sigma \geq 2$, then

$$|PQ| = \frac{D}{2} \sin\left(\frac{\sigma}{2}\right) \geq \frac{D}{2} \sin 1 > \frac{\sqrt{D}}{2}.$$

Hence $|PH| > 1$. This implies that the range circle of agent B must be at distance larger than 1 from the meeting area of w_A , which proves the lemma. ◀