

Natural Calamities Demand More Rescuers: Exploring Connectivity Time Dynamic Graphs

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Abstract

We study the exploration problem by mobile agents in Connectivity Time dynamic graphs. The Connectivity Time model was introduced by Michail et al. [JPDC 2014] and is arguably one of the weakest dynamic graph connectivity models. We prove that exploration is impossible in such graphs using $\frac{(n-1)(n-2)}{2}$ mobile agents starting from an arbitrary initial configuration, even when agents have full knowledge of system parameters, global communication, full visibility, and infinite memory. We then present an exploration algorithm that uses $\frac{(n-1)(n-2)}{2} + 1$ agents equipped with global communication, 1-hop visibility and $O(\log n)$ memory.

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1 Introduction

The exploration of graphs by mobile agents is a well-studied problem in distributed computing and has foundational importance in theoretical computer science. Originating from early work by Shannon [39], the objective is for mobile agents to collectively visit every node in a given network. Depending on the requirements, the task may involve visiting each node at least once (exploration with termination) or repeatedly over time (perpetual exploration). This problem is not only of theoretical interest but also has practical implications for systems involving autonomous agents, such as robots, software agents, or vehicles, where exploration helps in fault detection, information dissemination, or data collection across the network.

The graph exploration problem has been studied under a wide range of assumptions. These include whether the nodes are uniquely labelled or anonymous, whether agents have distinct identities or are indistinguishable, and the mode of communication or interaction among agents, such as using whiteboards, tokens, face-to-face meetings, or vision-based mechanisms. Variations also arise based on the degree of synchrony among agents (asynchronous, semi-synchronous, or fully-synchronous), the extent of their knowledge about the network, and the amount of memory available to them (refer to [2, 6, 7, 11, 13, 14, 21, 22, 37], for a comprehensive overview, refer to [9]). Despite the diversity in models, most of the prior research is on static graphs, meaning the graph structure remains fixed throughout the exploration. While this assumption works well for traditional networks, where changes typically result from failures but it falls short in capturing the behaviour of today's highly dynamic networks.

The dynamic nature of modern networks presents significant challenges in addressing various algorithmic problems in mobile computing and related domains, as the underlying network topology evolves over time. From the perspective of mobile agents, this means



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that agents must carry out their tasks in an environment that evolves over time steps. A foundational model capturing such dynamic behaviour was introduced by Kuhn et al. [32]. In their framework, they defined a stability property known as T -Interval Connectivity (for $T \geq 1$), which requires that in every sequence of T consecutive rounds, there exists a stable, connected spanning subgraph, although additional edges may appear or disappear in each round. Later, Michail et al. [33] proposed a more relaxed and natural notion of connectivity in dynamic networks, particularly suitable for networks that may be disconnected at individual time steps. They introduced the concept of Connectivity Time, defined as follows:

Let V be a fixed set of nodes and $S = \{(u, v), |, u, v \in V\}$ be the set of possible edges. Let $\mathcal{P}(S)$ denote the power set of S . A synchronous dynamic network is modeled as a dynamic graph $G = (V, E)$, where $E : \mathbb{N} \rightarrow \mathcal{P}(S)$ maps each round number $r \in \mathbb{N} \cup \{0\}$ to the set of edges present at that round. The static graph at round r is denoted by $\mathcal{G}_r = (V, E(r))$.

Definition. [33] *The Connectivity Time of a dynamic graph $G = (V, E)$ is the minimum integer $T \in \mathbb{N}$ such that $\forall r \in \mathbb{N} \cup \{0\}$, the union graph $G_{r,T} := (V, \bigcup_{i=r}^{r+T-1} E(i))$ is connected.*

This model generalizes T -Interval Connectivity, but unlike T -Interval Connectivity, it allows temporary disconnections. Thus, Connectivity Time is strictly weaker than T -Interval Connectivity. In the next section, we discuss the model and problem definition.

2 Model and Problem Definition

Dynamic graph model. We consider a dynamic network modeled as a sequence of undirected graphs $G = (V, E)$, where the node set V remains fixed over time and satisfies $|V| = n$. Define $S = \{(u, v) | u, v \in V\}$ as the set of all possible edges, and let $\mathcal{P}(S)$ denote its power set. The function $E : \mathbb{N} \rightarrow \mathcal{P}(S)$ maps each round number $r \in \mathbb{N} \cup \{0\}$ to the set of edges $E(r)$ present at that round, yielding the snapshot graph $\mathcal{G}_r = (V, E(r))$. The dynamic graph G is thus given as a sequence $\langle \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \dots \rangle$. We assume the presence of a dynamic adversary that may insert or delete any edge at the beginning of each round. The degree of a node v at round r is denoted by $\deg_r(v)$. The diameter of $\mathcal{G}_r$ is denoted by D_r .

Each snapshot graph $\mathcal{G}_r$ is unweighted, undirected, and anonymous. Moreover, the graph is port-labelled: for any node $v \in \mathcal{G}_r$, the incident edges are assigned distinct local port numbers in the range $[0, \deg_r(v) - 1]$. For an edge (u, v) , the port numbers at u and v are independently assigned and unrelated. Port labellings can differ across rounds; i.e., port numbers at a node in $\mathcal{G}_r$ may not match those in $\mathcal{G}_{r'}$ for $r \neq r'$. Nodes do not have any storage capability. A node is referred to as **hole** in round r if no agent is present at that node, and as **multinode** if two or more agents occupy it at round r . In this work, the graph $G (= \langle \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \dots \rangle)$ maintains the Connectivity Time property for some T , i.e., for every $r \geq 0$, the graph $G_{r,T} := (V, \bigcup_{i=r}^{r+T-1} E(i))$ is connected.

Agent model. We consider ℓ mobile agents that are initially placed arbitrarily on the nodes of G . Each agent has a unique identifier from the range $[1, n^c]$, where c is some constant, and knows only its own ID. Agents are equipped with $O(\log n)$ memory and execute the algorithm under a fully synchronous scheduler, i.e., in each round t , every agent executes a Communicate-Compute-Move (CCM) cycle:

- **Communicate:** Agents communicate as per the communication model.
- **Compute:** Based on its local view and any received information, the agent performs computation, including deciding whether and where to move.
- **Move:** The agent moves through a chosen port or stays idle.

The time complexity is measured by the number of synchronous rounds. We refer to $\mathcal{G}_r$ together with the agents' positions as the *configuration*. With a slight abuse of notation, we may denote the configuration at round r by $\mathcal{G}_r$.

Visibility model. We adopt a standard visibility framework where agents have l -hop visibility. In the l -hop model [1, 34], at the beginning of round r , an agent can see the subgraph induced by nodes within distance l from its current location in $\mathcal{G}_r$, including the presence or absence of agents in that neighbourhood. When $l = D_r$, this provides full visibility at round r .

Communication model. In this work, we consider the global communication model [8, 20, 30, 31, 36]. The global communication allows agents to exchange messages with any agent located in the same connected component of $\mathcal{G}_r$, utilizing the graph's links. The global communication between two different connected components of $\mathcal{G}_r$ is not possible, as there is no edge between two different connected components.

Problem definition. A node v is visited by round r if at least one agent is at node v at round t , where $t \in [0, r]$. An algorithm achieves *exploration* if every node is visited at least once. And, an algorithm achieves *perpetual exploration* if every node is visited infinitely often.

3 Related Work

Exploration of dynamic graphs has been widely studied in centralized settings, where agents have full knowledge of the network's evolution. Notably, optimal exploration schedules have been analyzed under 1-Interval Connectivity [18] and extended in subsequent works [15–17]. Specific topologies such as rings and cactuses have also been explored under T -Interval and 1-Interval Connectivity, respectively [26, 27].

Distributed exploration, with limited agent knowledge, has received less attention. Probabilistic methods like random walks were introduced in early foundation work [3], while deterministic approaches focus on periodic graphs and carrier models [18, 19, 27, 28]. Perpetual exploration and exploration with termination have been studied in 1-Interval Connected rings using 2 or 3 agents under Fsync and Ssync models [4, 5, 12]. Other results include exploration with $O(n)$ agents in toroidal networks [24], and single-agent strategies with partial foresight [25]. A significant advancement in this area is the work by Gotoh et al. [23], which investigates the fundamental limits of exploration in time-varying graphs under Fsync and Ssync schedulers. In their model, the network is derived from a fixed footprint graph from which edges are deleted dynamically. Agents are able to detect missing edges indirectly when their attempted movements fail. Additionally, port numbers at nodes remain fixed throughout the computation, inherited from the footprint. In contrast, our model assumes the absence of a global footprint. More importantly, port numbers are not fixed over time, as they depend on the local degree of $\mathcal{G}_r$. Consequently, agent movements are always successful, and agents may be unable to detect the change of topology. These characteristics make our model weaker than the one studied in [23], as it places fewer constraints on the dynamic behaviour of the network and provides the agents with less information. In this work, we study perpetual exploration in the Connectivity Time dynamic graphs.

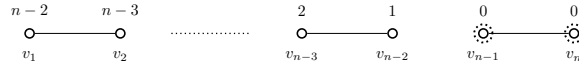
Recently, Saxena et al. [38] studied exploration under various connectivity models, including Connectivity Time, and showed that exploration is impossible with at most n agents (under model assumptions consistent with ours). However, their result did not

yield tight bounds. In this work, we strengthen their impossibility result by proving that exploration is not solvable even with up to $(n-2)(n-1)/2$ agents, and complement this with a matching algorithmic upper bound. Our contributions are presented in the next section.

4 Our Contribution

In this work, we present the following two results:

1. Exploration is impossible in the Connectivity Time dynamic graphs by $\frac{(n-2)(n-1)}{2}$ agents starting from an arbitrary initial configuration, even if agents have infinite memory, full visibility, global communication, and knowledge of all parameters (refer Theorem 4).¹
2. We present a perpetual exploration algorithm for the Connectivity Time dynamic graphs using $\frac{(n-2)(n-1)}{2} + 1$ agents starting from arbitrary initial configuration, where each agent has 1-hop visibility, global communication, and $O(\log n)$ memory (refer Theorem 17).



■ **Figure 1** Initial configuration $\mathcal{C}_0$ of $\frac{(n-2)(n-1)}{2}$ agents.

5 Impossibility Result

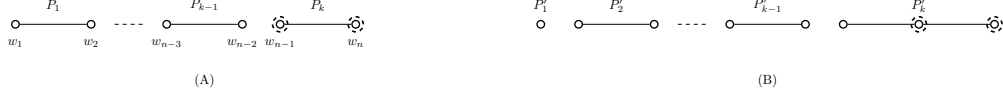
In this section, we show that exploration is impossible to solve using $\frac{(n-2)(n-1)}{2}$ agents. Before proceeding with the construction of $G = \langle \mathcal{G}_0, \mathcal{G}_1, \dots \rangle$, we first outline the high-level idea behind the impossibility result.

High-level idea. While there remains a non-empty set of unexplored nodes, the goal is to transfer agents from explored and occupied nodes to those unexplored ones. As it can be difficult to differentiate between an unexplored node with an explored but unoccupied node, the algorithm may require to transfer agents from explored and occupied nodes to explored but currently unoccupied nodes as well. If an algorithm succeeds in keeping all explored nodes occupied, then eventually, as the adversary must pick an edge across the cut, some agent will move to an unexplored node and hence the node becomes visited. However, the adversary is powerful: if b agents move from a node with x agents to a node with $y < x$ agents according to some deterministic algorithm, making the new counts $y' = y + b$ and $x' = x - b$, the adversary can flip the roles of these two nodes in the next graph instance, effectively undoing progress as the algorithm performs the reverse operation. The proofs formalize this idea: with fewer than $(n-1)(n-2)/2 + 1$ agents, the adversary can always choose such a flip whereas with that many agents it cannot always do that.

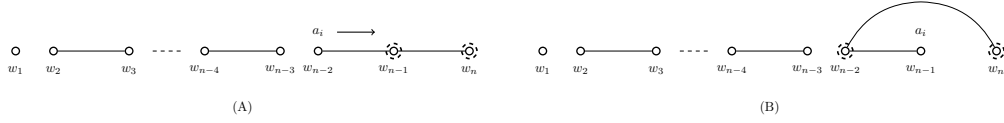
Dynamic graph G . Let $n = 2k$ for some $k \in \mathbb{N}$, $n \geq 4$ and $T \geq 2$. We give an initial configuration with $\frac{(n-2)(n-1)}{2}$ agents such that the exploration is impossible to solve. Let $v_1, v_2, \dots, v_n$ be nodes. At the beginning of round 0, consider k many one length paths as follows: $P_1 (= v_1 \sim v_2)$, $P_2 (= v_3 \sim v_4)$, $\dots$, $P_{k-1} (= v_{n-3} \sim v_{n-2})$ and $P_k (= v_{n-1} \sim v_n)$. Let $\alpha_r(w)$ represent the number of agents located at a node w at the beginning of round r , and

¹ If exploration is impossible then so is perpetual exploration and if perpetual exploration is possible then so is exploration.

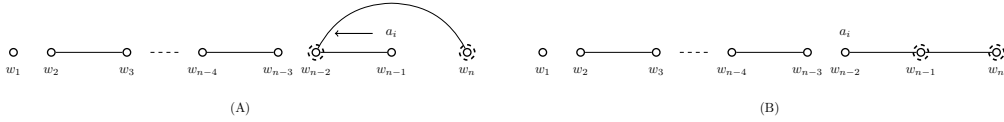
let $\beta_r(w)$ represent the number of agents located at the node w at the end of round r .² Let $\alpha_0(v_i) = n - i - 1$ for $i \in [1, n - 2]$, and $\alpha_0(v_{n-1}) = \alpha_0(v_n) = 0$ (i.e., nodes v_{n-1} and v_n are holes). Let's denote this configuration by $\mathcal{C}_0$ (refer to Fig. 1). The total number of agents is $\sum_{i=1}^n \alpha_0(v_i) = \sum_{i=1}^{n-2} i = \frac{(n-2)(n-1)}{2}$. In round $r \geq 0$, the adversary maintains $\mathcal{G}_r$ as follows:



■ **Figure 2** (A) Graph $\mathcal{G}_{iT-2}$, (B) Graph $\mathcal{G}_{iT-1}$.



■ **Figure 3** (A) An agent moves from node w_{n-2} to node w_{n-1} at round $r - 1$ in $\mathcal{G}_{r-1}$, (B) $\mathcal{G}_r$ with respect to $\mathcal{G}_{r-1}$.



■ **Figure 4** (A) An agent moves from node w_{n-1} to node w_{n-2} at round $r - 1$ in $\mathcal{G}_{r-1}$, (B) $\mathcal{G}_r$ with respect to $\mathcal{G}_{r-1}$.

(a) $r \in [0, T - 2]$: It maintains $\mathcal{C}_0$ in these rounds.

(b) $r \in [iT - 1, (i + 1)T - 2]$, where $i \geq 1$ & $i \pmod{2} \neq 0$:

At round $iT - 2$, there are k many one length paths, say $P_1 (= w_1 \sim w_2)$, $P_2 (= w_3 \sim w_4)$, $\dots$, $P_{k-1} (= w_{n-3} \sim w_{n-2})$, $P_k (= w_{n-1} \sim w_n)$ (we separately show that nodes w_{n-1} and w_n are holes at round $iT - 2$, with $w_n = v_n$). Note that at the end of round $T - 2$, $w_i = v_i$, for every i . We can see $\mathcal{G}_{iT-2}$ in Fig. 2(A). Based on the movement of agents at round $iT - 2$, the adversary forms k paths, say $P'_1, P'_2, \dots, P'_k$, at the beginning of round $iT - 1$ as follows.

If $\beta_{iT-2}(w_1) \geq \beta_{iT-2}(w_2)$, then $P'_1 = w_1$. Else, $P'_1 = w_2$. For $j \in [2, k - 1]$, P'_j is defined as follows.

$$P'_j = \begin{cases} w_{2j-3} \sim w_{2j-1}, & \text{if } \beta_{iT-2}(w_{2j-3}) < \beta_{iT-2}(w_{2j-2}) \text{ and } \beta_{iT-2}(w_{2j-1}) \geq \beta_{iT-2}(w_{2j}) \\ w_{2j-2} \sim w_{2j-1}, & \text{if } \beta_{iT-2}(w_{2j-3}) \geq \beta_{iT-2}(w_{2j-2}) \text{ and } \beta_{iT-2}(w_{2j-1}) \geq \beta_{iT-2}(w_{2j}) \\ w_{2j-3} \sim w_{2j}, & \text{if } \beta_{iT-2}(w_{2j-3}) < \beta_{iT-2}(w_{2j-2}) \text{ and } \beta_{iT-2}(w_{2j-1}) < \beta_{iT-2}(w_{2j}) \\ w_{2j-2} \sim w_{2j}, & \text{if } \beta_{iT-2}(w_{2j-3}) \geq \beta_{iT-2}(w_{2j-2}) \text{ and } \beta_{iT-2}(w_{2j-1}) < \beta_{iT-2}(w_{2j}) \end{cases}$$

If $\beta_{iT-2}(w_{n-3}) \geq \beta_{iT-2}(w_{n-2})$, then $P'_k = w_{n-2} \sim w_{n-1} \sim w_n$. Else, $P'_k = w_{n-3} \sim w_{n-1} \sim w_n$. We can see $\mathcal{G}_{iT-1}$ in Fig. 2(B).

Without loss of generality, let $P'_1 (= w_1)$, $P'_2 (= w_2 \sim w_3)$, $\dots$, $P'_{k-1} (= w_{n-4} \sim w_{n-3})$, $P'_k (= w_{n-2} \sim w_{n-1} \sim w_n)$. If $\alpha_{iT-1}(w_{n-2}) = 0$, the adversary maintains the graph $\mathcal{G}_r$ as

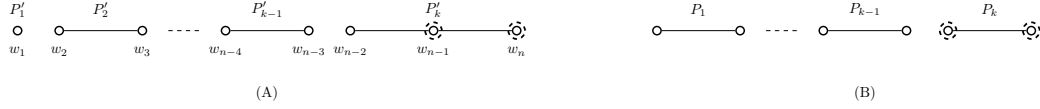
² The parameter $\beta_r(w)$ is the number of agents at node w after completing CCM cycle at round r .

$\mathcal{G}_{iT-1}$ for every $r \in [iT, (i+1)T-2]$. If $\alpha_{iT-1}(w_{n-2}) > 0$, the adversary maintains the graph $\mathcal{G}_r$ for every $r \in [iT, (i+1)T-2]$ as follows.

If an agent from node w_{n-2} moves to node w_{n-1} at round $r-1$, then adversary at the beginning of round r maintains the following path: $P'_1 = w_1$, $P'_2 = w_2 \sim w_3, \dots$, $P'_{k-1} = w_{n-4} \sim w_{n-3}$, and $P'_k = w_{n-1} \sim w_{n-2} \sim w_n$ (refer Fig. 3(A) at round $r-1$, and refer Fig. 3(B) at round r). Otherwise, it maintains the graph $\mathcal{G}_r$ as $\mathcal{G}_{r-1}$.

If an agent from node w_{n-1} moves to node w_{n-2} at round $r-1$, then adversary at the beginning of round r maintains the following path: $P'_1 = w_1$, $P'_2 = w_2 \sim w_3, \dots$, $P'_{k-1} = w_{n-4} \sim w_{n-3}$, and $P'_k = w_{n-2} \sim w_{n-1} \sim w_n$ (refer Fig. 4(A) at round $r-1$, and refer Fig. 4(B) at round r). Otherwise, it maintains the graph $\mathcal{G}_r$ as $\mathcal{G}_{r-1}$.

If $i = 1$, then it is not difficult to observe that the number of agents at node $\alpha_{T-1}(w_{n-2}) \leq 1$ as $\alpha_{T-2}(w_{n-3}) + \alpha_{T-2}(w_{n-2}) = 3$. We show later for every i , $\alpha_{iT-1}(w_{n-2}) \leq 1$. Thus, the way we change the graph, the agent always stays between node w_{n-2} and w_{n-1} and can not access node w_n in round r . We denote this configuration by $\mathcal{C}_{1-2-3}$.



■ **Figure 5** (A) Graph $\mathcal{G}_{iT-2}$, (B) Graph $\mathcal{G}_{iT-1}$.

(c) $r \in [iT-1, (i+1)T-2]$, where $i \geq 1$ & $i \pmod{2} = 0$:

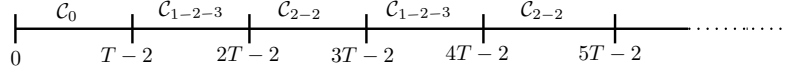
At the end of round $iT-2$, there are k many paths, say $P'_1 (= w_1)$, $P'_2 (= w_2 \sim w_3)$, $\dots$, $P'_{k-1} (= w_{n-4} \sim w_{n-3})$, $P'_k = w_{n-2} \sim w_{n-1} \sim w_n$ (we separately show that nodes w_{n-1} and w_n are holes at the beginning of round $iT-2$, with $w_n = v_n$, and at most one agent occupy node w_{n-2}). We can see $\mathcal{G}_{iT-2}$ in Fig. 5(A). At round $iT-1$, the adversary forms k many one length paths, say $P_1, P_2, \dots, P_k$. Based on the movement of agents at round $iT-2$, the adversary forms k paths, say $P_1, P_2, \dots, P_k$, at the beginning of round $iT-1$ as follows.

If $\beta_{iT-2}(w_2) \geq \beta_{iT-2}(w_3)$, then $P_1 = w_1 \sim w_2$. Else, $P_1 = w_1 \sim w_3$. For $j \in [2, k-2]$, P_j is defined as follows.

$$P_j = \begin{cases} w_{2j-2} \sim w_{2j}, & \text{if } \beta_{iT-2}(w_{2j-2}) < \beta_{iT-2}(w_{2j-1}) \text{ and } \beta_{iT-2}(w_{2j}) \geq \beta_{iT-2}(w_{2j+1}) \\ w_{2j-1} \sim w_{2j}, & \text{if } \beta_{iT-2}(w_{2j-2}) \geq \beta_{iT-2}(w_{2j-1}) \text{ and } \beta_{iT-2}(w_{2j}) \geq \beta_{iT-2}(w_{2j+1}) \\ w_{2j-2} \sim w_{2j+1}, & \text{if } \beta_{iT-2}(w_{2j-2}) < \beta_{iT-2}(w_{2j-1}) \text{ and } \beta_{iT-2}(w_{2j}) < \beta_{iT-2}(w_{2j+1}) \\ w_{2j-1} \sim w_{2j+1}, & \text{if } \beta_{iT-2}(w_{2j-2}) \geq \beta_{iT-2}(w_{2j-1}) \text{ and } \beta_{iT-2}(w_{2j}) < \beta_{iT-2}(w_{2j+1}) \end{cases}$$

At the end of round $iT-2$, there is at most one agent in path P'_k , and this agent is either at node w_{n-2} or w_{n-1} (we show this separately). If there is no agent in path P'_k , then P_{k-1} and P_k are defined as follows. If $\beta_{iT-2}(w_{n-4}) \geq \beta_{iT-2}(w_{n-3})$, then $P_{k-1} = w_{n-3} \sim w_{n-2}$ and $P_k = w_{n-1} \sim w_n$. Else, $P_{k-1} = w_{n-4} \sim w_{n-2}$ and $P_k = w_{n-1} \sim w_n$. If there is one agent in path P'_k , then P_{k-1} and P_k are defined as follows. Based on agent's position at node w_{n-2} or w_{n-1} at the end of round $iT-2$, the cases are as follows.

- If an agent is at node w_{n-2} at the end of round $iT-2$, the path P_{k-1} and P_k are defined as follows. If $\beta_{iT-2}(w_{n-4}) \geq \beta_{iT-2}(w_{n-3})$, then $P_{k-1} = w_{n-3} \sim w_{n-2}$ and $P_k = w_{n-1} \sim w_n$. Else, $P_{k-1} = w_{n-4} \sim w_{n-2}$ and $P_k = w_{n-1} \sim w_n$.
- If an agent is at node w_{n-1} at the end of round $iT-2$, the path P_{k-1} and P_k are defined as follows. If $\beta_{iT-2}(w_{n-4}) \geq \beta_{iT-2}(w_{n-3})$, then $P_{k-1} = w_{n-3} \sim w_{n-1}$ and $P_k = w_{n-2} \sim w_n$. Else, $P_{k-1} = w_{n-4} \sim w_{n-1}$ and $P_k = w_{n-2} \sim w_n$.



■ **Figure 6** It shows how the agent configurations evolve periodically over the dynamic graph, with specific configurations like C_0 , C_{1-2-3} , and C_{2-2} reappearing at regular intervals.

We can see $\mathcal{G}_{iT-1}$ in Fig. 5(B). It maintains $\mathcal{G}_r$ as $\mathcal{G}_{iT-1}$ for every $r \in [iT, (i+1)T-2]$. Later, we show that there is no agent in P_k , and one of the nodes in P_k is v_n . We denote this configuration by C_{2-2} .

Fig. 6 shows how the configuration evolves over time.

► **Lemma 1.** *Dynamic graph G maintains the Connectivity Time property. (Proof in Appendix, see Section A.1)*

We define two notations as follows.

- **(N1)** $r \in [iT-1, (i+1)T-2]$, where $i \geq 1$ & $i \pmod{2} = 0$ (or $r \in [0, T-2]$): In this case, either configuration C_0 or C_{2-2} is true. Therefore, at round r , there are k one length paths $P_1, P_2, \dots, P_k$. Let $P_j (= w_{2j-1} \sim w_{2j})$, for every $j \in [1, k]$. If $\alpha_r(w_{2j-1}) \geq \alpha_r(w_{2j})$, then we denote w_{2j-1} as w_{2j-1}^r and w_{2j} as w_{2j}^r . Otherwise, we denote w_{2j-1} as w_{2j}^r and w_{2j} as w_{2j-1}^r .
- **(N2)** $r \in [iT-1, (i+1)T-2]$, where $i \geq 1$ & $i \pmod{2} \neq 0$: In this case, configuration C_{1-2-3} is true. Therefore, at round r , there are k paths $P'_1, P'_2, \dots, P'_k$. Let $P'_j (= w_{2j-2} \sim w_{2j-1})$, for every $j \in [2, k-1]$, $P'_1 (= w_1)$ and $P'_k (= w_{n-2} \sim w_{n-1} \sim w_n)$. If $\alpha_r(w_{2j-2}) \geq \alpha_r(w_{2j-1})$, then we denote w_{2j-2} as w_{2j-2}^r and w_{2j-1} as w_{2j-1}^r . Otherwise, we denote w_{2j-2} as w_{2j-1}^r and w_{2j-1} as w_{2j-2}^r . We consider w_1 as w_1^r , w_{n-2} as w_{n-2}^r , w_{n-1} as w_{n-1}^r , and w_n as w_n^r .

► **Lemma 2.** *The following inequality holds $\forall l \in [1, n-2]$ and for every round $r \geq 0$.*

$$\sum_{i=1}^l \alpha_r(w_i^r) \geq \sum_{i=1}^l (n-i-1)$$

Proof. Proving our lemma is equivalent to showing that the following two inequalities hold for every $j \in [1, k-1]$ at round r :

$$(A) \quad \sum_{i=1}^{2j} \alpha_r(w_i^r) \geq \sum_{i=1}^{2j} (n-i-1), \quad (B) \quad \sum_{i=1}^{2j-1} \alpha_r(w_i^r) \geq \sum_{i=1}^{2j-1} (n-i-1).$$

We use mathematical induction to prove inequalities (A) and (B) for every $r \geq 0$. Both inequalities are true for $r = 0$ as C_0 includes k paths: $P_1 (= v_1 \sim v_2)$, $P_2 (= v_3 \sim v_4)$, ..., $P_{k-1} (= v_{n-3} \sim v_{n-2})$, and an additional path $P_k (= v_{n-1} \sim v_n)$. We define $\alpha_0(v_i) = n-i-1$ for $i \in [1, n-2]$, with $\alpha_0(v_{n-1}) = \alpha_0(v_n) = 0$. Therefore, $w_i^0 = v_i$ for every $i \leq n-2$.

Assuming the statement is true for round $r \geq 0$, we aim to show that it also holds for round $r+1$. There are five possible cases to consider: **Case 1:** $r, r+1 \in [0, T-2]$, **Case 2:** $r, r+1 \in [iT-1, (i+1)T-2]$, where i is an odd number, **Case 3:** $r, r+1 \in [iT-1, (i+1)T-2]$, where i is an even number, **Case 4:** $r = iT-2$, where i is odd, and **Case 5:** $r = iT-2$, where i is even. Due to the length of the proof, we present only the proofs of Case 1 and Case 4 here. The proofs of Case 2 and Case 3 follow similar ideas to Case 1, and Case 5 builds upon the approach used in Case 4. The proof of Case 2, 3 and 5 is in Appendix, see Section A.2.

Case 1. In this case, agents are in configuration C_0 at round r and $r+1$.

Proof of (A). Due to the induction hypothesis, the following inequality holds:

$$\sum_{i=1}^{2j} \alpha_r(w_i^r) \geq \sum_{i=1}^{2j} (n - i - 1) \quad (1)$$

Since the dynamic graph does not change between rounds 0 and $T - 2$, no matter how agents move, for every $j \in [1, k - 1]$, we have the following:

$$\alpha_r(w_{2j-1}^r) + \alpha_r(w_{2j}^r) = \alpha_{r+1}(w_{2j-1}^{r+1}) + \alpha_{r+1}(w_{2j}^{r+1}) \quad (2)$$

Therefore, the following inequality holds using Eq. (1) and Eq. (2):

$$\sum_{i=1}^{2j} \alpha_{r+1}(w_i^{r+1}) = \sum_{i=1}^{2j} \alpha_r(w_i^r) \geq \sum_{i=1}^{2j} (n - i - 1) \quad (3)$$

Proof of (B). The inequality $\alpha_r(w_1^r) \geq n - 2$ holds due to the induction hypothesis, and the inequality $\alpha_r(w_1^r) + \alpha_r(w_2^r) \geq (n - 2) + (n - 3)$ is valid due to proof of (A). Therefore, regardless of how the agents move in round r , we have $\alpha_{r+1}(w_1^{r+1}) \geq n - 2$ as $\alpha_r(w_1^r) + \alpha_r(w_2^r) = \alpha_{r+1}(w_1^{r+1}) + \alpha_{r+1}(w_2^{r+1})$ and $\alpha_{r+1}(w_1^{r+1}) \geq \alpha_{r+1}(w_2^{r+1})$. This shows (B) holds for $j = 1$. For $j \in [2, k - 1]$, we use a contrapositive argument. Suppose for some smallest value $j \geq 2$, the inequality (B) does not hold for round $r + 1$. Then the following inequality must be true:

$$\sum_{i=1}^{2j-1} \alpha_{r+1}(w_i^{r+1}) < \sum_{i=1}^{2j-1} (n - i - 1) \quad (4)$$

Due to Eq. (3), we get:

$$\sum_{i=1}^{2j-2} \alpha_{r+1}(w_i^{r+1}) \geq \sum_{i=1}^{2j-2} (n - i - 1) \quad (5)$$

Therefore, using Eq. (4) and Eq. (5), the inequality $\alpha_{r+1}(w_{2j-1}^{r+1}) < n - (2j - 1) - 1 = n - 2j$ holds. Rewrite Eq. (3) as:

$$\alpha_{r+1}(w_{2j}^{r+1}) \geq \sum_{i=1}^{2j} (n - i - 1) - \sum_{i=1}^{2j-1} \alpha_{r+1}(w_i^{r+1}) \quad (6)$$

From Eq. (4) and Eq. (6), the inequality $\alpha_{r+1}(w_{2j}^{r+1}) > n - 2j$ holds. Thus, due to our assumption (i.e., Eq. 4), we have $\alpha_{r+1}(w_{2j-1}^{r+1}) < n - 2j$ and $\alpha_{r+1}(w_{2j}^{r+1}) > n - 2j$. Therefore, we have $\alpha_{r+1}(w_{2j-1}^{r+1}) < \alpha_{r+1}(w_{2j}^{r+1})$. This leads to a contradiction because, due to (N1), the inequality $\alpha_{r+1}(w_{2j-1}^{r+1}) \geq \alpha_{r+1}(w_{2j}^{r+1})$ holds. This shows that our initial assumption is incorrect. Therefore, inequality (B) holds for round $r + 1$.

Case 4. In this case, at round $r = iT - 2$, where i is an odd number, the configuration is either $\mathcal{C}_0$ or $\mathcal{C}_{2-2}$, and at round $r + 1$, it changes to $\mathcal{C}_{1-2-3}$

Proof of (A). Due to the induction hypothesis, the following inequality is true:

$$\sum_{i=1}^{2j} \alpha_r(w_i^r) \geq \sum_{i=1}^{2j} (n - i - 1) \quad (7)$$

The adversary constructs the dynamic graph in round $r + 1$ based on the agents' positions at the end of round r . Let $p = \max\{\beta_r(w_{2j-1}^r), \beta_r(w_{2j}^r)\}$. Therefore, $\alpha_{r+1}(w_{2j-1}^r) = p$ and the value of $\alpha_{r+1}(w_{2j}^r)$ is as follows (recall configuration $\mathcal{C}_{1-2-3}$):

$$\alpha_{r+1}(w_{2j}^{r+1}) = \max\left\{\alpha_r(w_{2j-1}^r) + \alpha_r(w_{2j}^r) - p, \max\{\beta_r(w_{2j+1}^r), \beta_r(w_{2j+2}^r)\}\right\} \quad (8)$$

The adversary forms the dynamic graph at round $iT - 1$, ensuring the following equality:

$$\sum_{i=1}^{2j-1} \alpha_{r+1}(w_i^{r+1}) = \sum_{i=1}^{2j-2} \alpha_r(w_i^r) + p \quad (9)$$

Using Eq. (9), the following equality holds.

$$\sum_{i=1}^{2j} \alpha_{r+1}(w_i^{r+1}) = \sum_{i=1}^{2j-1} \alpha_{r+1}(w_i^{r+1}) + \alpha_{r+1}(w_{2j}^{r+1}) \geq \sum_{i=1}^{2j-2} \alpha_r(w_i^r) + p + \alpha_{r+1}(w_{2j}^{r+1}) \quad (10)$$

Due to Eq. (8), we have $\alpha_{r+1}(w_{2j}^{r+1}) \geq \alpha_r(w_{2j-1}^r) + \alpha_r(w_{2j}^r) - p$. Thus, using Eq. (10):

$$\sum_{i=1}^{2j} \alpha_{r+1}(w_i^{r+1}) \geq \sum_{i=1}^{2j} \alpha_r(w_i^r) \quad (11)$$

Using Eq. (7) and Eq. (11), the inequality (A) holds for round $r + 1$.

Proof of (B). For $j = 1$, the inequality $\alpha_r(w_1^r) \geq n - 2$ holds, and due to the proof of (A), the inequality $\alpha_r(w_1^r) + \alpha_r(w_2^r) \geq n - 2 + n - 3$ holds. Therefore, $\alpha_{r+1}(w_1^{r+1}) \geq n - 2$ holds regardless of how agents move at round r due to the following reason. If $\alpha_{r+1}(w_1^{r+1}) < n - 2$, then $\beta_r(w_1^r) < n - 2$ and $\beta_r(w_2^r) < n - 2$ holds as per the dynamic graph construction at round $iT - 1$, $\alpha_{r+1}(w_1^{r+1}) = \max\{\beta_r(w_1^r), \beta_r(w_2^r)\}$. Therefore, $\beta_r(w_1^r) \leq n - 3$ and $\beta_r(w_2^r) \leq n - 3$, and hence the inequality $\beta_r(w_1^r) + \beta_r(w_2^r) \leq 2(n - 3)$ holds. Since $\beta_r(w_1^r) + \beta_r(w_2^r) = \alpha_r(w_1^r) + \alpha_r(w_2^r)$, the inequality $\alpha_r(w_1^r) + \alpha_r(w_2^r) \leq 2(n - 3)$. This leads to a contradiction as $\alpha_r(w_1^r) + \alpha_r(w_2^r) \geq n - 2 + n - 3$. Therefore, our assumption is wrong, and it implies $\alpha_{r+1}(w_1^{r+1}) \geq n - 2$. Now we prove (B) when $j \geq 2$. Due to (A), the following inequalities hold for $j \geq 2$.

$$\sum_{i=1}^{2j-2} \alpha_r(w_i^r) \geq \sum_{i=1}^{2j-2} (n - i - 1) \quad \text{and} \quad \sum_{i=1}^{2j} \alpha_r(w_i^r) \geq \sum_{i=1}^{2j} (n - i - 1) \quad (12)$$

In Eq. (9), the lower bound of p is $\left\lceil \frac{\alpha_r(w_{2j-1}^r) + \alpha_r(w_{2j}^r)}{2} \right\rceil$. Using Eq. (9), we have:

$$\sum_{i=1}^{2j-1} \alpha_{r+1}(w_i^{r+1}) \geq \sum_{i=1}^{2j-2} \alpha_r(w_i^r) + \left\lceil \frac{\alpha_r(w_{2j-1}^r) + \alpha_r(w_{2j}^r)}{2} \right\rceil \quad (13)$$

Using Eq. (12), we have (by taking the sum of inequalities of Eq. (12)):

$$\begin{aligned} 2 \left(\sum_{i=1}^{2j-2} \alpha_r(w_i^r) \right) + \alpha_r(w_{2j-1}^r) + \alpha_r(w_{2j}^r) &\geq 2 \left(\sum_{i=1}^{2j-2} (n - i - 1) \right) + (n - 2j) + (n - 2j - 1) \\ \Rightarrow \sum_{i=1}^{2j-2} \alpha_r(w_i^r) + \frac{\alpha_r(w_{2j-1}^r) + \alpha_r(w_{2j}^r)}{2} &\geq \sum_{i=1}^{2j-2} (n - i - 1) + n - 2j - \frac{1}{2} \end{aligned} \quad (14)$$

Since $\lceil x \rceil \geq x$ for every $x \in \mathbb{R}$, the following holds:

$$\sum_{i=1}^{2j-2} \alpha_r(w_i^r) + \left\lceil \frac{\alpha_r(w_{2j-1}^r) + \alpha_r(w_{2j}^r)}{2} \right\rceil \geq \sum_{i=1}^{2j-2} (n-i-1) + n-2j = \sum_{i=1}^{2j-1} (n-i-1) \quad (15)$$

Due to Eq. (13) and (15), inequality (B) holds for round $r+1$. This completes the proof. ◀

► **Corollary 3.** *At round $iT-1$, $\alpha_{iT-1}(w_{n-2}^{iT-1}) \leq 1$, where i is an odd number.*

Proof. Due to Lemma 2, the following two inequalities hold at round $iT-1$.

$$\sum_{i=1}^{2n-3} w_i^{iT-1} \geq \sum_{i=1}^{2n-3} (n-i-1) \quad \text{and} \quad \sum_{i=1}^{2n-2} w_i^{iT-1} \geq \sum_{i=1}^{2n-2} (n-i-1) \quad (16)$$

Therefore, due to Eq. (16), $\alpha_{iT-1}(w_{n-2}^{iT-1}) \leq 1$. This completes the proof. ◀

► **Theorem 4.** *If the initial configuration contains at least two holes, then a group of $(n-2)(n-1)/2$ agents cannot solve the exploration problem in dynamic graphs that maintain the Connectivity Time property. This impossibility holds even if agents have infinite memory, full visibility, global communication, and know the parameters k , $n(\geq 4)$, $T(\geq 2)$.*

Proof. Our dynamic graph construction satisfies the Connectivity Time property as established in Lemma 1. To prove that exploration is impossible, it suffices to show that node v_n remains a hole at every round $r \geq 0$. If $r \in [0, T-2]$, node v_n is not accessible to the agents because node v_n is in path P_k , where v_{n-1} is a hole.

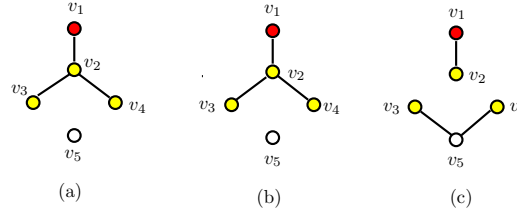
As per Corollary 3, $\alpha_{T-1}(w_{n-2}^{T-1}) \leq 1$. Therefore, at the beginning of round $T-1$, there can be at most one agent at node w_{n-2} . If there is no agent, then node v_n is not accessible to the agents because node v_n is in path P'_k , where w_{n-2} , w_{n-1} are holes.

If there is an agent at node w_{n-2} at the beginning of round $T-1$, it has the option to move to node w_{n-1} during round $r \in [T-1, 2T-3]$. In this scenario, the adversary constructs the following graph at round $r+1$ according to our dynamic graph construction method: the adversary maintains the paths $P'_1, P'_2, \dots, P'_{k-1}$ unchanged and modifies P'_k from $w_{n-2} \sim w_{n-1} \sim v_n$ to $w_{n-1} \sim w_{n-2} \sim v_n$. In this manner, at round $r+1$, the node v_n is two hops away from the agent's position. The adversary follows the same procedure; if the agents move in later rounds from node w_{n-1} to w_{n-2} , this ensures that during rounds $r \in [T-1, 2T-2]$, the agents consistently remain at nodes w_{n-2} and w_{n-1} . Consequently, the agent can not reach node v_n in any of these rounds.

At the beginning of round $2T-1$, as per our dynamic graph construction, if there was an agent in path P'_k at round $2T-2$ (there is at most one agent due to Corollary 3), it removes such a node from P'_k at the beginning of round $2T-1$. Therefore, as per dynamic graph construction at the beginning of round $2T-1$, the node v_n is in path $P_k = w_{n-1} \sim w_n (= v_n)$, where node w_{n-1} is a hole. The adversary maintains the same configuration at every round $r \in [2T-1, 3T-2]$. Therefore, the agent can not reach node v_n in any of these rounds. This idea can be extended for $r \geq 3T-1$ using Corollary 3. It is important to note that this is independent of the agents' power. Therefore, the proof remains valid even if the agents have full visibility, global communication, and know all parameters. This completes the proof. ◀

6 Connectivity Time Dynamic Graph Exploration

The Connectivity Time dynamic graph has high dynamicity, and the high dynamicity may prevent even basic coordination if agents are significantly restricted in their ability to perceive their surroundings or communicate with one another. Assume agents have 1-hop visibility.



■ **Figure 7** (a) Graph $\mathcal{G}_r$, where $r \pmod 3 = 0$, (b) Graph $\mathcal{G}_r$, where $r \pmod 3 = 1$, (c) Graph $\mathcal{G}_r$, where $r \pmod 3 = 2$. This figure is an example of the Connectivity Time for $T = 3$.

Consider two paths: $P = w_1 \sim w_2 \sim w_3 \sim w_4 \sim w_5$ and $P' = w_5 \sim w_2 \sim w_3 \sim w_4 \sim w_1$. The adversary can create either P or P' , such that only at w_3 , the 1-hop view remains unchanged. The movement of agent(s) at w_3 remains the same for both P and P' irrespective of the hole position. This can cause exploration to fail. Now, assume agents have global communication but no 1-hop visibility. Then, local views at each node remain the same in the above example, making it difficult again. To address these challenges, we assume that agents are equipped with 1-hop visibility and global communication capabilities in our algorithm. Even stronger assumptions are considered in the study of 1-Interval Connected dynamic networks (e.g., in [10], [35], authors use full visibility), and are instrumental in enabling tractable algorithmic solutions under highly dynamic conditions. Since 1-Interval Connectivity is a stronger assumption than the Connectivity Time, the Connectivity Time model imposes greater difficulty. It is important to emphasize that these assumptions are not fundamental to the definition of the problem but are adopted solely to overcome the adversarial nature of the Connectivity Time model. Moreover, we complement our algorithmic results with lower bounds and impossibility results (refer to Section 5) that hold even when agents possess *stronger capabilities*, including full knowledge of all system parameters, full visibility, and global communication. This highlights the inherent difficulty of perpetual exploration in such settings.

High-level idea. Suppose that at round r , agents know the map of $\mathcal{G}_r$. Let w be a multinode and v a hole in $\mathcal{G}_r$, connected via a shortest path $v_1 \sim v_2 \sim \dots \sim v_p$, with $v_1 = w$, $v_p = v$, and all intermediate nodes $v_2, \dots, v_{p-1}$ occupied by some agent. Let a_i denote an agent at v_i for $1 \leq i \leq p-1$. In round r , each a_i moves to v_{i+1} . The multinode v_1 remains non-empty, and each v_i ($2 \leq i \leq p-1$) receives an agent from v_{i-1} and sends one to v_{i+1} , preserving non-hole status. Finally, the hole v_p is filled. This strategy is known as *pipeline strategy*, which pushes an agent to the nearest hole without creating new holes and has been used in prior works (e.g., [29]).

Challenges. A key challenge arises from the agents' inability to reconstruct the dynamic graph due to limited knowledge of the adversary's behavior. However, with *1-hop visibility* and *global communication*, agents can share local views to collaboratively form a partial network snapshot. While this may not capture the full graph, it often suffices for executing the pipeline procedure.

The core difficulty lies in ensuring every node is visited at least once. Even if agents can fill holes, some nodes may remain unvisited. Under the Connectivity Time model, two nodes may never belong to the same connected component in any single round. For example, let $V = \{v_1, v_2, v_3, v_4, v_5\}$ with $T = 3$, where v_1 is a multinode, and nodes v_2, v_3, v_4 are not holes; node v_5 is initially a hole. Define $\mathcal{G}_r = (V, E(r))$ as follows: if $r \pmod 3 = 0$ or 1 ,

let $E(r) = \{(v_1, v_2), (v_2, v_3), (v_2, v_4)\}$ (see Fig. 7(a), (b)), and if $r \bmod 3 = 2$, let $E(r) = \{(v_1, v_2), (v_5, v_3), (v_5, v_4)\}$ (see Fig. 7(c)). Although $\mathcal{G}_{r,3} = (V, \bigcup_{i=r}^{r+2} E(i))$ is connected for all r , no direct path exists between v_1 and v_5 in any single round. Thus, v_1 cannot pipeline to v_5 , despite being a multinode. This illustrates how the adversary can isolate nodes round by round, impeding exploration. To address this, we use an *enhanced pipeline*. If at round r , agents detect that there exists at least one hole and at least one multinode in their connected component of $\mathcal{G}_r$, then the standard pipeline fills it. If not, then each node in the component has at least one agent. Let $\text{Count}(v)$ be the number of agents at node v at round r . If two nodes v and w in the same component satisfy $\text{Count}(w) \geq \text{Count}(v) + 2$, agents initiate a redistribution along a shortest path $v_1 \sim v_2 \sim \dots \sim v_p$ from w to v (with $v_1 = w, v_p = v$), transferring one agent via pipelining to balance the load of agents. Since the graph can be highly sparse and disconnected, enhanced pipeline gradually builds a configuration of agents on the nodes such that pipeline becomes feasible along every connected path in each round.

In the following sections, we use two parameters: $\text{Count}(v)$, which denotes the number of agents at node v , and $a_i.\text{ID}$, which denotes the ID of agent a_i .

6.1 Map Construction

We begin by presenting an algorithm that allows agents to construct a partial map of the network, specifically, the subgraph induced by the nodes currently occupied by agents. A similar strategy was proposed in [29]; we adapt and modify it to suit our setting and terminology. This serves as a subroutine in our exploration algorithm.

At round r , since $\mathcal{G}_r$ may be disconnected, it consists of p connected components, denoted $G_1, G_2, \dots, G_p$, where each $G_i = (V_i, E_i)$ with $V_i \subseteq V$ and $E_i \subseteq E(r)$. For each connected component G_i , we further divide it into several subgraphs considering nodes that contains agent(s). In this context, we define:

► **Definition 5 (Connected component of G_i without holes).** *The component G_i can be partitioned into subgraphs $G_i^1, G_i^2, \dots, G_i^k$, where each $G_i^j = (V_i^j, E_i^j)$ satisfies the following:*

- *Every node $v \in V_i^j$ have at least one agent, for all $j \in [1, k]$.*
- *The sets of nodes and edges are pairwise disjoint, i.e., $V_i^j \cap V_i^l = \emptyset$ and $E_i^j \cap E_i^l = \emptyset$ for all $j \neq l$, where $j, l \in [1, k]$.*
- *There exists no edge $e = (u, v) \in E_i$ such that $u \in V_i^j$ and $v \in V_i^l$, where $j \neq l$, i.e., the subgraphs are disconnected from each other within G_i .*

We refer to the collection of subgraphs G_i^j as the connected component of G_i without holes, and denote by $AC(G_i)$.

In other words, $AC(G_i)$ denotes the collection of connected components obtained by removing all holes and their associated edges from G_i . Recall that agents use global communication, but communication is limited within each connected component. That is, agents located in G_i cannot communicate with agents in G_j for $i \neq j$, as communication happens through the graph links. An agent a_j in component G_i performs the following steps at round r to compute $AC(G_i)$. The algorithm is divided into two phases as described below.

Phase 1. (1-hop view collection) Let a_j be the agent with the minimum ID located at a node $v \in G_i$. The agent a_j performs the following for each port $p \in \{0, 1, \dots, \deg_r(v) - 1\}$:

- Let u be the neighbor of v reachable via port p . Set $ID_v = a_j.\text{ID}$.
- If at least one agent is present at u : define $C_v^p = (\text{Count}(v), ID_v, p, ID_u)$, where ID_u is the minimum ID among the agents at node u .

- If no agent is present at u (i.e., it is a hole): define $C_v^p = (\text{Count}(v), ID_v, p, \perp)$, where $\perp$ denotes a hole via port p .

Let $C_v = \{C_v^0, C_v^1, \dots, C_v^d\}$ denote the 1-hop view at node v , where $d = \deg_r(v) - 1$. The agent a_j broadcasts C_v to all agents in G_i using global communication.

Phase 2. (Graph Reconstruction) After receiving all 1-hop views from agents in G_i , agent a_j proceeds as follows:

- Define $V' = \{ID_v \mid C_v\}$, where each unique ID represents a node.
- Construct the edge set E' as follows: for each pair of tuples $(\text{Count}(v), ID_v, p, ID_u)$ and $(\text{Count}(u), ID_u, q, ID_v)$, where $ID_u \neq \perp$ and $ID_v \neq \perp$, add an undirected edge (ID_v, ID_u) with port labels $\pi(ID_v, ID_u) = p$ and $\pi(ID_u, ID_v) = q$, where $\pi(v_1, v_2)$ denotes the outgoing port from node v_1 to v_2 .
- For each tuple $(\text{Count}(v), ID_v, p, \perp)$, mark port p at node ID_v as leading to a hole.³

We denote this algorithm as $\text{MAP}()$. The correctness of $\text{MAP}()$ is as follows. We show that at round r , if agent a_i is in connected component G_i of $\mathcal{G}_r$, then it constructs $AC(G_i)$ by using $\text{MAP}()$. Let G'_i be the map constructed by agent a_i using $\text{MAP}()$.

► **Theorem 6.** *Let agent a_i be located at some node in the connected component G_i . Then, using the subroutine $\text{MAP}()$, agent a_i constructs $AC(G_i)$. (Proof in Appendix, see Section A.3)*

From Theorem 6 and Definition 5, we derive three key observations.

- **Observation 7.** *Using $\text{MAP}()$, the agents in G_i construct $AC(G_i)$, including information of the number of agents at each node v and the ports from node v (if any) that lead to a hole.*
- **Observation 8.** *Since all agents within the same connected component exchange information via global communication, the output of $\text{MAP}()$ is identical for every agent in that component.*
- **Observation 9.** *If there is no hole in G_i , then $AC(G_i)$ is the same as G_i , as per Def. 5.*

6.2 Perpetual Exploration Algorithm

In this section, we present the algorithm $\text{EXP_ALGO}()$, which solves perpetual exploration when agents are equipped with 1-hop visibility and global communication. The following is a detailed description of the algorithm for agent a_i at node v during round r : if node v is a multinode and agent a_i is not the minimum ID agent, then it stays at node v . If agent a_i is the minimum ID agent at node v , it follows the following steps at round r .

- Agent a_i broadcasts C_v .
- After receiving all C_u s, agent a_i use $\text{MAP}()$ algorithm. Let's call this constructed graph G' . Using Theorem 6, the graph G' is nothing but a collection of partitioned subgraphs $G_j^1, G_j^2, \dots, G_j^k$ of G_j , where G_j is the connected component for which a_i is a part. The following four cases may arise in the constructed map G' , and in each case, agent a_i makes a corresponding decision.

Case 1 (there is no multinode and no information of a port towards a hole):

Agent a_i stays at node v .

³ This step does not introduce any new holes or edges leading to holes during Phase 2; it only marks the port(s) at node v that lead to a hole.

Case 2 (there is no multinode, but there is information of a port leading to a hole):

Assume there are k nodes, each with an agent that has a port leading to a hole. Let $b_1, b_2, \dots, b_k$ be agents at node $u_1, u_2, \dots, u_k$, respectively in G' , where one of the ports of u_j leads to a hole for every $j \in [1, k]$. If $a_i.ID = \min\{b_j.ID : j \in [1, k]\}$, then it moves to the node which leads to the hole via the minimum port. Otherwise, it stays at its position.

Case 3 (both a multinode and information of a port towards a hole are present):

Suppose there are k multinodes in G' . Let $b_1, b_2, \dots, b_k$ be agents at node $u_1, u_2, \dots, u_k$, respectively in G' where each node u_i is a multinode in G' . Without loss of generality, let $b_1.ID = \min\{b_j.ID : j \in [1, k]\}$. Without loss of generality, let b_1 be in G_j^1 . One of the nodes in G_j^1 leads to a hole as there is a hole in G_j , and using Definition 5. Assume there are k' nodes in G_j^1 , each with an agent that has a port leading to a hole. Let $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_{k'}$ be agents at node $\bar{u}_1, \bar{u}_2, \dots, \bar{u}_{k'}$, respectively in G_j^1 such that one of the ports lead to a hole from node u_j , for every $j \in [1, k]$. Without loss of generality, let $\bar{a}_1.ID = \min\{\bar{a}_j.ID : j \in [1, k']\}$.

Since, agent a_i is aware of G' , it consider a shortest path P between u_1 and $\bar{u}_1$. If there are multiple shortest paths between u_1 and $\bar{u}_1$, then it selects the one that is lexicographically shortest among all other shortest paths. Let $P = w_1(= u_1) \sim w_2 \sim \dots \sim w_y(= \bar{u}_1)$. If agent a_i is at some node w_j , for $1 \leq j < y$, then it moves to node w_{j+1} . Else if agent a_i is at node w_y , then it moves to the node via the minimum available port, which leads to a hole. Otherwise, it stays at node v .

Case 4 (there is a multinode but there is no information of a port towards a hole):

Assume there are k nodes in G' . Due to Observation 8, the graph G' is the graph G_1 . Define: $i' = \max\{\text{Count}(x) : x \in V_1\}$, and $j' = \min\{\text{Count}(x) : x \in V_1\}$, where V_1 is the set of nodes in G_1 . Let $v_1, v_2, \dots, v_p$ be the nodes in G_1 such that $\text{Count}(v_k) = i'$ for each $k \in [1, p]$, and let $\bar{a}_k$ denote the minimum ID agent at node v_k . Similarly, let $w_1, w_2, \dots, w_q$ be the nodes in G_1 such that $\text{Count}(w_{k'}) = j'$ for each $k' \in [1, q]$, and let $\bar{b}_{k'}$ denote the minimum ID agent at node $w_{k'}$. Without loss of generality, let $\bar{a}_1$ be the agent among $\{\bar{a}_k : k \in [1, p]\}$ with the minimum ID, and $\bar{b}_1$ be the agent among $\{\bar{b}_{k'} : k' \in [1, q]\}$ with the minimum ID. If $\text{Count}(v_1) < \text{Count}(w_1) + 2$, agent a_i stays at node v . Otherwise (i.e., $\text{Count}(v_1) \geq \text{Count}(w_1) + 2$), agent a_i find a shortest path P between v_1 and w_1 . If there are multiple shortest paths between v_1 and w_1 , then it selects the one that is lexicographically shortest. Let $P = z_1(= v_1) \sim z_2 \dots \sim z_y(= w_1)$. If agent a_i is at some node z_j , for $1 \leq j < y$, then it moves to node z_{j+1} . Else, it stays at v .

In the next section, we show the correctness of our algorithm.

6.3 Correctness and Analysis of the Algorithm

Before proving correctness, we introduce the following notation. Let n be the number of nodes, and define $l = \frac{(n-2)(n-1)}{2} + 1$. For each $i \in [0, l]$, let $S_i := \{v \in V : \text{Count}(v) = i\}$. Let L denote the largest integer such that $S_L \neq \emptyset$ at round 0.

► **Lemma 10.** *Let $\mathcal{G}_r$ be the configuration at round r . If there exists at least one hole and at least one multinode in the same connected component of $\mathcal{G}_r$, then the total number of holes in $\mathcal{G}_r$ decreases by at least one at the end of round r . (Proof in Appendix, see Section A.4)*

► **Lemma 11.** *Let $\mathcal{G}_r$ be the configuration at round r , and suppose that there exists a connected component G_1 of $\mathcal{G}_r$ such that G_1 contains at least one multinode but no hole. Define $i' = \max\{\text{Count}(x) : x \in V_1\}$ and $j' = \min\{\text{Count}(x) : x \in V_1\}$, where V_1 is the*

set of nodes in G_1 . If $i' \geq j' + 2$, then at the end of round r , one of the nodes $v \in V(G_1)$ with $\text{Count}(v) = i'$ reduces its Count by 1, and one of the nodes $w \in V(G_1)$ with $\text{Count}(w) = j'$ increases its Count by 1. (*Proof in Appendix, see Section A.5*)

► **Remark 12.** Based on Lemma 11, we observe that one agent reaches a node $w_1 \in S_{j'}$ at round r . As a result, $\text{Count}(w_1)$ becomes $j' + 1$ at the beginning of round $r + 1$, implying that $w_1 \in S_{j'+1}$. Whenever at round r , a node becomes a part of S_p , i.e., at round $r - 1$, it was not a part of S_p , we call this event *join*. Whenever at round r , a node does not remain a part of S_p , i.e., at round $r - 1$, it was a part of S_p but it is not part of S_p at round r , we call this event *leave*.

We have an observation based on EXP_ALGO(), which is as follows.

► **Observation 13.** A node v with $\text{Count}(v) \geq 1$ at round r can become a hole by the end of round r only if $\text{Count}(v) = 1$ and v belongs to a connected component of $\mathcal{G}_r$ that contains a hole but no multinode.

We now show that perpetual exploration is achieved using l agents.

► **Lemma 14.** If $|S_0| = 1$ at some round r , then node $v \in S_0$ is visited by some agent within the next T rounds. (*Proof in Appendix, see Section A.6*)

► **Lemma 15.** If $|S_0| \geq 2$, then $\exists i, j (\geq i) \in [1, l]$ such that $S_i \neq \emptyset$, $S_j \neq \emptyset$, $j \geq i + 2$ and $S_k = \emptyset$ for all $i < k < j$. (*Proof in Appendix, see Section A.7*)

► **Lemma 16.** If initially $|S_0| \geq 2$, then within $O(n^4 \cdot T)$ round $|S_0| \leq 1$.

Proof. At round 0, there are two possible cases: Case 1: $|S_1| = 0$, or Case 2: $|S_1| \geq 1$.

Case 1. To maintain the Connectivity Time, some node from S_0 must be in the same connected component as a node from S_i , $i \geq 2$, within the first T rounds. By Lemma 10, this causes $|S_0|$ to decrease by at least one in the next T rounds. If $|S_1| = 0$ throughout $[0, n \cdot T]$, then S_0 eventually becomes empty. Otherwise, if $|S_1| \geq 1$ at some round $r \in [0, n \cdot T]$, we are in Case 2.

Case 2. Suppose that at round $r \geq 0$, $|S_1| \geq 1$. To show that $|S_0| \leq 1$, we proceed by contrapositive argument. Assume that $|S_0| \geq 2$ throughout the interval $[r, r + (n^4 + 1) \cdot T]$. Then, by Lemma 15, there exist indices i and j with $j \geq i + 2$ such that $S_i \neq \emptyset$, $S_j \neq \emptyset$, and $S_k = \emptyset$ for all $k \in [i + 1, j - 1]$.

According to Remark 12, nodes may join or leave the set S_p at each time step. A key question is: how many times can nodes join the set S_p ? By Lemma 11, a node can join S_p at round $t \geq r$ only if at least one node from S_q (with $q \geq p + 1$) and one node from S_{p-1} belong to the same connected component G_1 of $\mathcal{G}_t$, and no node from $S_{p'}$ for $p' < p - 1$ is in G_1 . We have l many agents. Then in the worst case, nodes can join the set S_p at most l times, as each time one agent can move from S_q to S_{p-1} . If at most l times nodes joins set S_p , the size of S_q decreases due to Lemma 11, eventually making $S_q = \emptyset$ for every $q > p$.

As there are L such sets with $L \leq l$, the total number of join events between rounds r and $r + (n^4 + 1) \cdot T$ can not be more than $l^2 \leq n^4$. Now divide the interval $[r, r + (n^4 + 1) \cdot T]$ into $n^4 + 1$ consecutive sub-intervals of length T : each sub-interval is of the form $[r + kT, r + (k + 1)T - 1]$ for $k = 0, 1, \dots, n^4 + 1$. If at least one node leaves some set S_j during each sub-interval, then there are $n^4 + 1$ such leave events in total. Since each leave corresponds to a join, this implies there are also $n^4 + 1$ join events. But this contradicts our earlier conclusion that there can be at most n^4 join events. By the pigeonhole principle, this contradiction implies that our assumption is false. Therefore, within $O(n^4 \cdot T)$ rounds, we must have $|S_0| \leq 1$.

Since Case 2 is reached within $O(n \cdot T)$ rounds from any initial configuration, $|S_0| \leq 1$ holds within $O(n^4 \cdot T)$ rounds overall. This completes the proof. ◀

► **Theorem 17.** *EXP_ALGO() solves perpetual exploration in Connectivity Time dynamic graphs using $\frac{(n-2)(n-1)}{2} + 1$ synchronous agents equipped with 1-hop visibility, global communication, and $O(\log n)$ bits of memory. The agents have no knowledge of n, T, l .*

Proof. If $|S_0| \geq 2$ initially, then by Lemma 16, $|S_0| \leq 1$ within $O(n^4 \cdot T)$ rounds. Suppose that at some round $r \geq 0$, $|S_0| \leq 1$. If $S_0 = \emptyset$, then all nodes are occupied, and by Observation 13, no node becomes a hole in future rounds. Thus, perpetual exploration is achieved.

If $S_0 = \{v\}$, then by Lemma 14, node v is visited within the next T rounds. Thus, all nodes are visited by some agent within $O(n^4 \cdot T)$ rounds, and the invariant $|S_0| \leq 1$ is preserved thereafter. This process repeats indefinitely, ensuring perpetual exploration.

Each agent requires $O(\log n)$ bits to distinguish itself. Since the computation is round-local (i.e., agents do not retain information from previous rounds), $O(\log n)$ bits of memory are sufficient. This completes the proof. ◀

► **Remark 18.** Lemma 16 shows that at least $n - 1$ nodes are occupied in the first $O(n^4 \cdot T)$ rounds. Theorem 17 ensures that from this point onward, at most one hole exists and, by Lemma 14, it is revisited within each T rounds. Therefore, this also implies that each node of the network is visited by some agent in the first $O(n^4 \cdot T)$ rounds.

7 Conclusion and Future Work

In this work, we have shown that exploration is impossible in Connectivity Time dynamic graphs by $\frac{(n-2)(n-1)}{2}$ agents starting from an arbitrary initial configuration, even if agents have infinite memory, full visibility, global communication, and knowledge of all parameters. We then presented an algorithm that solves perpetual exploration using one extra agent with only 1-hop visibility, global communication, and $O(\log n)$ memory. One can study whether this extra agent helps to reduce the assumptions of 1-hop visibility and/or global communication which we require for exploration.

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A

 Appendix

A.1 Proof of Lemma 1

Proof. For $r \geq 0$, let $\mathcal{G}_r, \mathcal{G}_{r+1}, \dots, \mathcal{G}_{r+T-1}$ be consecutive T sequence of graphs, where $\mathcal{G}_i = (V, E(i))$ for $i \in [r, r+T-1]$. Suppose the above dynamic graph G does not satisfy the Connectivity Time property for some round r , i.e., $G_{r,T} := (V, \cup_{r+T-1}^{r+1} E(i))$ is not connected. It is important to note that there exists a round r' between r and $r+T-1$ such that $r' = iT - 1$, for some $i \in \mathbb{N}$.

If i is **odd**, then in each round $t \in [r, iT - 2]$, there are k one length paths in $\mathcal{G}_t$: $P_1(= w_1 \sim w_2), P_2(= w_3 \sim w_4), \dots, P_{k-1}(= w_{n-3} \sim w_{n-2})$ and $P_k(= w_{n-1} \sim w_n)$. As per the dynamic graph construction at round $iT - 1$, the adversary changes paths as follows: $P'_1(= w'_1), P'_2(= w'_2 \sim w'_3), \dots, P'_{k-1}(= w'_{n-4} \sim w'_{n-3})$ and $P'_k(= w'_{n-2} \sim w'_{n-1} \sim w'_n)$, where $w'_{2j-1} \in \{w_{2j-1}, w_{2j}\}$ and $w'_{2j} = \{w_{2j-1}, w_{2j}\} \setminus \{w'_{2j-1}\}$, for every $j \in [1, k]$. Taking the union of the edges from $\mathcal{G}_i$ for $r \leq i \leq r+T-1$ creates a path of length n .

Similarly, if i is **even**, then in each round $t \in [r, iT - 2]$, there are k paths in $\mathcal{G}_t$: $P'_1(= w_1), P'_2(= w_2 \sim w_3), \dots, P'_{k-1}(= w_{n-4} \sim w_{n-3})$, and $P'_k(= w_{n-2} \sim w_{n-1} \sim w_n)$. By using a similar argument, we can show that the way we modify the construction at round $iT - 1$ results in the union of edges from $\mathcal{G}_i$ for $r \leq i \leq r+T-1$, which forms a path of length n .

This shows our assumption is wrong. Therefore, this dynamic setting satisfies the Connectivity Time property. ◀

A.2 Proof of Lemma 2 (Remaining Cases)

Proof. The proof for Cases 2, 3, and 5 is as follows.

Case 2. In this case, round $r, r+1 \in [iT-1, (i+1)T-2]$, where i is an odd number, and agents are in configuration $\mathcal{C}_{1-2-3}$. The proof of Eq. (A) and Eq. (B) for round $r+1$ is as follows.

Proof of (B). Due to the induction hypothesis, the following inequality holds for $j \geq 1$:

$$\sum_{i=1}^{2j-1} \alpha_r(w_i^r) \geq \sum_{i=1}^{2j-1} (n-i-1) \quad (1)$$

Since the dynamic graph does not change for every round $r \in [iT-1, (i+1)T-2]$, no matter how agents move, for every $j \in [1, k-2]$, we have the following

$$\alpha_r(w_{2j}^r) + \alpha_r(w_{2j+1}^r) = \alpha_{r+1}(w_{2j}^{r+1}) + \alpha_{r+1}(w_{2j+1}^{r+1}) \quad (2)$$

Therefore, inequality (B) holds for round $r+1$ using Eq. (1) and Eq. (2).

Proof of (A). We use a contrapositive argument. Suppose for some smallest value $j \geq 1$, the inequality does not hold. Then the following inequality must be true:

$$\sum_{i=1}^{2j} \alpha_{r+1}(w_i^{r+1}) < \sum_{i=1}^{2j} (n-i-1) \quad (3)$$

Due to proof of (B), we have:

$$\sum_{i=1}^{2j-1} \alpha_{r+1}(w_i^{r+1}) \geq \sum_{i=1}^{2j-1} (n-i-1) \quad (4)$$

Therefore, using Eq. (3) and Eq. (4), the inequality $\alpha_{r+1}(w_{2j}^{r+1}) < n-2j-1 \implies \alpha_{r+1}(w_{2j-1}^{r+1}) \leq n-2j-2$ holds. Due to proof of (B), we have:

$$\sum_{i=1}^{2j+1} \alpha_{r+1}(w_i^{r+1}) \geq \sum_{i=1}^{2j+1} (n-i-1) \quad (5)$$

We now rewrite Eq. (5) as:

$$\alpha_{r+1}(w_{2j+1}^{r+1}) \geq \sum_{i=1}^{2j+1} (n-i-1) - \sum_{i=1}^{2j} \alpha_{r+1}(w_i^{r+1}) \quad (6)$$

From Eq. (3) and Eq. (6), the inequality $\alpha_{r+1}(w_{2j+1}^{r+1}) > n-2j-2$ holds. Therefore, due to our assumption (i.e., Eq. (3)), we have $\alpha_{r+1}(w_{2j-1}^{r+1}) \leq n-2j-2$ and $\alpha_{r+1}(w_{2j+1}^{r+1}) > n-2j-2$. This leads to a contradiction because, due to (N2), the inequality $\alpha_{r+1}(w_{2j}^{r+1}) \geq \alpha_{r+1}(w_{2j+1}^{r+1})$ holds. This shows that our initial assumption is incorrect. Therefore, inequality (A) holds for round $r+1$.

Case 3. In this case, round $r, r+1 \in [iT-1, (i+1)T-2]$, where i is an even number. The proof is similar to Case 1.

Case 5. In this scenario, let $r = iT-2$, where i is an even integer. At round r , the configuration is $\mathcal{C}_{1-2-3}$. As per (N2), there are k paths: $P'_1(= w_1^r)$, $P'_2(= w_2^r \sim w_3^r)$, $\dots$, $P'_{k-1}(= w_{n-4}^r \sim w_{n-3}^r)$, and $P'_k(= w_{n-2}^r \sim w_{n-1}^r \sim w_n^r)$. At round $r+1$ as per (N1), there are k paths: $P_1(= w_1^{r+1} \sim w_2^{r+1})$, $P_2(= w_3^{r+1} \sim w_4^{r+1})$, $\dots$, $P_{k-1}(= w_{n-3}^{r+1} \sim w_{n-2}^{r+1})$, and $P_k(= w_{n-1}^{r+1} \sim w_n^{r+1})$. It holds that $\alpha_r(w_{2j-1}^{r+1}) \leq \alpha_r(w_{2j}^{r+1})$ for every $j \in [1, k-1]$.

Proof of (B). Due to the induction hypothesis, the following inequality is true:

$$\sum_{i=1}^{2j-1} \alpha_r(w_i^r) \geq \sum_{i=1}^{2j-1} (n - i - 1) \quad (7)$$

Since $\alpha_{r+1}(w_1^r) = \max\{\alpha_r(w_1^r), \max\{\beta_r(w_2^r), \beta_r(w_3^r)\}\}$, $\alpha_{r+1}(w_1^r) \geq \alpha_r(w_1^r) \geq n - 2$. Therefore, for $j = 1$, the inequality (A) holds at round $r + 1$. For $j \geq 2$, the inequality (A) holds at round $r + 1$ for the following reason. Let $p = \max\{\beta_r(w_{2j-2}^r), \beta_r(w_{2j-1}^r)\}$ for $j \geq 2$. Therefore, the value of $\alpha_{r+1}(w_{2j-1}^r)$ is as follows.

$$\alpha_{r+1}(w_{2j-1}^{r+1}) = \max\left\{\alpha_r(w_{2j-2}^r) + \alpha_r(w_{2j-1}^r) - p, \max\{\beta_r(w_{2j}^r), \beta_r(w_{2j+1}^r)\}\right\} \quad (8)$$

The adversary forms the dynamic graph at round $iT - 1$, ensuring the following equality is true.

$$\sum_{i=1}^{2j-2} \alpha_{r+1}(w_i^{r+1}) = \sum_{i=1}^{2j-3} \alpha_r(w_i^r) + p \quad (9)$$

Using Eq. (9), the following equality holds.

$$\sum_{i=1}^{2j-1} \alpha_{r+1}(w_i^{r+1}) = \sum_{i=1}^{2j-2} \alpha_{r+1}(w_i^{r+1}) + \alpha_{r+1}(w_{2j-1}^{r+1}) \geq \sum_{i=1}^{2j-3} \alpha_r(w_i^r) + p + \alpha_{r+1}(w_{2j-1}^{r+1}) \quad (10)$$

Due to Eq. (8), $\alpha_{r+1}(w_{2j-1}^{r+1}) \geq \alpha_r(w_{2j-2}^r) + \alpha_r(w_{2j-1}^r) - p$. Therefore, we have the following from Eq. (10).

$$\begin{aligned} \sum_{i=1}^{2j-1} \alpha_{r+1}(w_i^{r+1}) &\geq \sum_{i=1}^{2j-3} \alpha_r(w_i^r) + p + \alpha_{r+1}(w_{2j-1}^{r+1}) \geq \sum_{i=1}^{2j-3} \alpha_r(w_i^r) + p + \alpha_r(w_{2j-2}^r) + \alpha_r(w_{2j-1}^r) - p \\ &\implies \sum_{i=1}^{2j-1} \alpha_{r+1}(w_i^{r+1}) \geq \sum_{i=1}^{2j-1} \alpha_r(w_i^r) \end{aligned} \quad (11)$$

Using Eq. (7) and Eq. (11), the inequality (B) holds at round $r + 1$.

Proof of (A). Due to the proof of (B), the following inequalities are true for any $j \geq 1$.

$$\sum_{i=1}^{2j-1} \alpha_r(w_i^r) \geq \sum_{i=1}^{2j-1} (n - i - 1) \quad (12)$$

$$\sum_{i=1}^{2j+1} \alpha_r(w_i^r) \geq \sum_{i=1}^{2j+1} (n - i - 1) \quad (13)$$

Due to Eq. (9), the following inequality is true for $j \geq 1$.

$$\sum_{i=1}^{2j} \alpha_{r+1}(w_i^{r+1}) = \sum_{i=1}^{2j-1} \alpha_r(w_i^r) + p \quad (14)$$

The lower bound of p is $\left\lceil \frac{\alpha_r(w_{2j}^r) + \alpha_r(w_{2j+1}^r)}{2} \right\rceil$. Therefore, we get the following inequality from Eq. (14).

$$\sum_{i=1}^{2j} \alpha_{r+1}(w_i^{r+1}) \geq \sum_{i=1}^{2j-1} \alpha_r(w_i^r) + \left\lceil \frac{\alpha_r(w_{2j}^r) + \alpha_r(w_{2j+1}^r)}{2} \right\rceil \quad (15)$$

Using Eq. (12) and Eq. (13), we get the following inequality (by taking the sum of Eq. (12) and Eq. (13)):

$$\begin{aligned}
2 \left(\sum_{i=1}^{2j-1} \alpha_r(w_i^r) \right) + \alpha_r(w_{2j}^r) + \alpha_r(w_{2j+1}^r) &\geq 2 \left(\sum_{i=1}^{2j-1} (n-i-1) \right) + (n-2j-2) + (n-2j-1) \\
\Rightarrow \sum_{i=1}^{2j-1} \alpha_r(w_i^r) + \frac{\alpha_r(w_{2j}^r) + \alpha_r(w_{2j+1}^r)}{2} &\geq \sum_{i=1}^{2j-1} (n-i-1) + \frac{2n-4j-3}{2} \\
\Rightarrow \sum_{i=1}^{2j-1} \alpha_r(w_i^r) + \frac{\alpha_r(w_{2j}^r) + \alpha_r(w_{2j+1}^r)}{2} &\geq \sum_{i=1}^{2j-1} (n-i-1) + n-2j-1 - \frac{1}{2} \quad (16)
\end{aligned}$$

We know that the following inequality is true.

$$\sum_{i=1}^{2j-1} \alpha_r(w_i^r) + \left\lceil \frac{\alpha_r(w_{2j}^r) + \alpha_r(w_{2j+1}^r)}{2} \right\rceil \geq \sum_{i=1}^{2j-1} \alpha_r(w_i^r) + \frac{\alpha_r(w_{2j}^r) + \alpha_r(w_{2j+1}^r)}{2} \quad (17)$$

Due to Eq. (16) and Eq. (17), the following holds.

$$\sum_{i=1}^{2j-1} \alpha_r(w_i^r) + \left\lceil \frac{\alpha_r(w_{2j}^r) + \alpha_r(w_{2j+1}^r)}{2} \right\rceil \geq \sum_{i=1}^{2j-1} (n-i-1) + n-2j-1 = \sum_{i=1}^{2j} (n-i-1) \quad (18)$$

Due to Eq. (15) and Eq. (18), the following holds.

$$\sum_{i=1}^{2j} \alpha_{r+1}(w_i^{r+1}) \geq \sum_{i=1}^{2j} (n-i-1)$$

This completes the proof. $\blacktriangleleft$

A.3 Proof of Theorem 6

Proof. Let edge (u, v) be in graph G_i^j , for some j , which is a subgraph of G_i , and $\pi(u, v) = p$, $\pi(v, u) = q$. Since $(u, v) \in E_i^j$, there is at least one agent at each node u and v using Definition 5. Let agent a be the minimum ID agent at node u , and agent b the minimum ID agent at node v . Since node u and v are in the same connected component, agent a gets information of C_v , and agent b gets information of C_u . Since q is an outgoing port of node v , agent a gets information about C_v^q . And since p is an outgoing port of node u , agent b gets information about C_u^p , $C_v^q = (Count(v), b.ID, q, a.ID)$ and $C_u^p = (Count(u), a.ID, p, b.ID)$. As per MAP(), agent a add edge $(a.ID, b.ID)$, where $a.ID$ and $b.ID$ are two nodes in G_i' , and $\pi(a.ID, b.ID) = p$, and $\pi(b.ID, a.ID) = q$. Therefore, $G_i' = AC(G_i)$. This completes the proof. $\blacktriangleleft$

A.4 Proof of Lemma 10

Proof. Let $G_1, G_2, \dots, G_k$ be the connected components of $\mathcal{G}_r$. Without loss of generality, suppose G_1 contains at least one hole and at least one multinode. By Theorem 6 and Observation 8, all agents in G_1 possess a common map of the anonymous copy $AC(G_1)$, denoted by $G' = \{G_1^1, G_1^2, \dots, G_1^m\}$. Suppose there are k' multinodes in G' , located at nodes $u_1, u_2, \dots, u_{k'}$, with the minimum ID agents $b_1, b_2, \dots, b_{k'}$ occupying them. Without loss of generality, let b_1 be the agent with the minimum ID among them, i.e., $b_1.ID = \min\{b_i.ID : i \in [1, k']\}$, and assume b_1 resides in G_1^1 .

Since G_1 contains at least one hole, there exists at least one node in G_1^1 from which a port leads to a hole. Let there be k'' such nodes, denoted $\bar{u}_1, \bar{u}_2, \dots, \bar{u}_{k''}$, with corresponding minimum-ID agents $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_{k''}$ occupying them. Let $\bar{a}_1$ be the agent with the minimum ID among them, i.e., $\bar{a}_1.ID = \min\{\bar{a}_j.ID : j \in [1, k'']\}$. Since all agents in G_1 share the same map G' , they identify the same pair of nodes: u_1 (a multinode) and $\bar{u}_1$ (adjacent to a hole). Let $P = (v_1 = u_1 \sim v_2 \sim \dots \sim v_y = \bar{u}_1)$ denote the lexicographically shortest path from u_1 to $\bar{u}_1$ in G' . This path is unique due to the deterministic selection criteria based on IDs and shared knowledge of G' .

Each agent on this path identifies its current position and acts accordingly: if an agent is at node v_i for $i < y$, it moves to v_{i+1} . The agent at $v_y = \bar{u}_1$ selects the minimum available port that leads to a hole and moves through it. Thus, a hole gets filled without creating a new hole elsewhere. Therefore, the total number of holes in $\mathcal{G}_r$ decreases by at least one at the end of round r . This completes the proof. ◀

A.5 Proof of Lemma 11

Proof. Let $G_1, G_2, \dots, G_k$ be the connected components of $\mathcal{G}_r$. Without loss of generality, let G_1 be the connected component that contains at least one multinode but no hole. Since G_1 contains a multinode but no hole, by Theorem 6 and Observation 9, every agent in G_1 constructs the map of G_1 using the procedure $\text{MAP}()$. Let $v_1, v_2, \dots, v_p$ be the nodes in G_1 such that $\text{Count}(v_k) = i'$ for each $k \in [1, p]$, and let a_k denote the minimum ID agent at node v_k . Similarly, let $w_1, w_2, \dots, w_q$ be the nodes such that $\text{Count}(w_{k'}) = j'$ for each $k' \in [1, q]$, and let $b_{k'}$ denote the minimum ID agent at node $w_{k'}$. Without loss of generality, let a_1 be the agent among $\{a_k : k \in [1, p]\}$ with the minimum ID, and b_1 be the agent among $\{b_{k'} : k' \in [1, q]\}$ with the minimum ID. Since all agents share the same reconstructed map G_1 , they all identify the same pair of nodes v_1 and w_1 as the nodes with maximum and minimum Count values, respectively. Let $P = (v_1 = u_1 \sim u_2 \sim \dots \sim u_y = w_1)$ be the lexicographically shortest path from v_1 to w_1 in G_1 , which is uniquely determined by the map and agent ID choices. Each agent on this path identifies its position and moves accordingly: if an agent is at node u_i for $i < y$, it moves to u_{i+1} . As a result, the value of $\text{Count}(v_1)$ decreases by 1, and the value of $\text{Count}(w_1)$ increases by 1. This completes the proof. ◀

A.6 Proof of Lemma 14

Proof. To maintain the Connectivity Time property, node v must be connected to some node w at round t , where $t \in [r, r + T]$. Let G_1 be the connected component of $\mathcal{G}_t$ at round t such that the node v is part of the graph G_1 . If G_1 has at least one multinode, then one agent moves to node v as agents in G_1 execute $\text{EXP_ALGO}()$ due to Lemma 10.

Otherwise, all nodes in G_1 except node v contain exactly one agent. At round t , let $w_1, w_2, \dots, w_p$ be neighbours of node v in G_1 , and agent a_i be at node w_i . As per $\text{EXP_ALGO}()$, the minimum ID agent ID among a_i s moves to node v . This completes the proof. ◀

A.7 Proof of Lemma 15

Proof. Suppose the lemma does not hold. It implies $|S_i| \geq 1$ for every $i \in [1, L]$, where L is the largest index satisfying $S_L \neq \emptyset$. The value L is $\leq n - 2$. Assume $L > n - 2$. Without loss of generality, let $L = n - 1$. Since $|S_i| \geq 1$ for every $i \in [1, L]$, $\sum_{i=1}^L |S_i| \geq n - 1$. Therefore, the total number of nodes is $|S_0| + \sum_{i=1}^L |S_i| \geq 2 + n - 1 = n + 1$ as $|S_0| \geq 2$ and $\sum_{i=1}^L |S_i| \geq n - 1$. This leads to the contradiction as n many nodes are present. Therefore, $L \leq n - 2$.

Since $|S_i| \geq 1$ for every $i \in [1, L]$, $L \leq \cup_{i=1}^L |S_i| \leq n-2$. Therefore, $L \leq n-2$. Define $X = \sum_{i=1}^L i \cdot |S_i|$. The value X denotes the total number of agents when $\mathcal{C}$ is false. The maximum value of X occurs when $|S_i| = 1$ for $1 \leq i \leq L-1$, and $|S_L| = n - |S_0| - (L-1) \leq n - L - 1$ as $|S_0| \geq 2$. Thus, $X \leq 1 + 2 + \dots + (L-1) + L \cdot (n - L - 1) = L \cdot (L-1)/2 + L \cdot (n - L - 1)$. After simplifying, we get that $X \leq L \cdot (2n - L - 3)/2 \leq (n-2)(n-1)/2$. This leads to the contradiction as the number of agents in the system is $\frac{(n-2)(n-1)}{2} + 1$. Thus, our initial assumption must be incorrect. This completes the proof. $\blacktriangleleft$