

# The Bend Number of Cocomparability Graphs

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## Abstract

We introduce a new complexity measure for cocomparability graphs of posets or in other words, intersection graphs of piecewise linear functions, the bend number. We prove that cocomparability graphs of bounded bend number are not too plentiful and give two hierarchies of classes of cocomparability graphs, depending on whether the piecewise linear functions are restricted to slopes of  $\pm 1$  (diagonal case) or not (general case). These hierarchies give a gradation between permutation graphs and cocomparability graphs.

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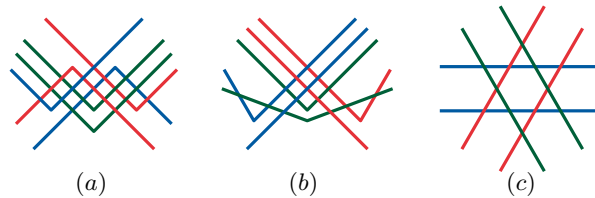


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## 1 Introduction

The *cocomparability graph* of a poset  $(P, \preceq)$  is a graph whose vertices are the elements of  $P$ , with two vertices connected by an edge if and only if they are incomparable in  $(P, \preceq)$ . In 1983, Golumbic, Rotem and Urrutia [16] showed that cocomparability graphs can be also equivalently characterised as intersection graphs of  $x$ -monotone curves that project orthogonally to the interval  $[0, 1]$  on the  $x$ -axis; more explicitly,  $G = (V, E)$  is a co-comparability graph if and only if, for an interval  $I$ , we can associate to each vertex  $v$  a continuous function  $f_v: I \rightarrow \mathbb{R}$  so that two vertices  $v, w$  are adjacent if and only if  $f_v(x) = f_w(x)$  for some  $x \in I$ . We then call such a collection of functions  $(f_v; v \in V)$  a *cocomparability representation* of  $G$ .

In practice, it is often convenient, and sometimes necessary, to consider representations that admit a simple combinatorial description, for instance, to restrict the functions  $f_v$  to be piecewise linear with not too many bends.



■ **Figure 1** Cocomparability representations of the octahedral graph  $K_{2,2,2}$  in the diagonal setting (a) and the general setting (b), (c).

In this paper, we initiate a systematic study of the classes of graphs admitting a cocomparability representation by piecewise linear curves with at most  $k$  bends per curve. In fact, we separately consider two natural settings: the *diagonal setting*, where the curves are made of alternating segments of slopes  $+1$  and  $-1$ , and the *general setting* where arbitrary slopes are allowed (see Figure 1). Within each of the two settings, we distinguish *bounded* representations, in which the common domain  $I$  is a bounded interval  $[a, b]$ , *grounded* representations, in which  $I$  is of the form  $[a, \infty)$ , and *unbounded* representations, in which  $I$  is the whole  $\mathbb{R}$ . Moreover, in the diagonal setting, we may further distinguish representations in which all the curves have precisely  $k$  bends (so-called *strict representations*), from those whose curves have at most  $k$  bends (the *relaxed representations*). And finally, in the strict diagonal setting, we may impose the restriction that the leftmost segments of all the curves have the same slope, say  $+1$  (so-called *synchronous representation*), or we may allow the leftmost slopes to differ (*asynchronous representation*).

The aim of this paper is to clarify the relative expressive power of the various forms of  $k$ -bend cocomparability representations. In particular, we prove that in both the diagonal and the general setting, the class of (diagonal/general)  $k$ -bend cocomparability graphs is a strict subclass of the class of (diagonal/general)  $(k + 1)$ -bend cocomparability graphs.

## Previous results

Intersection graphs of (not necessarily  $x$ -monotone) piecewise linear curves with bounded bend number have been extensively studied. Asinowski, Cohen, Golumbic, Limouzy, Lipshteyn and Stern [3, 4] have introduced the graph class  $B_k$ -VPG, which are the intersection graphs of piecewise-linear curves whose segments are horizontal and vertical and which have at most  $k$  bends. They showed that  $B_1$ -VPG is a strict superset of  $B_0$ -VPG, and conjectured

that  $B_k$ -VPG is a strict superset of  $B_{k-1}$ -VPG for each  $k$ . The conjecture has been subsequently proven by Chaplick, Jelínek, Kratochvíl and Vyskočil [8]. Up to a 45-degree rotation, every diagonal cocomparability representation with  $k$  bends per curve is also a  $B_k$ -VPG-representation. However, the arguments used by [8] to distinguish  $B_k$ -VPG from  $B_{k-1}$ -VPG do not yield cocomparability graphs, so they are not helpful for our purposes.

A notable subclass of  $B_1$ -VPG are the intersection graphs of L-shapes, where an L-shape is a union of a horizontal and a vertical segment in which the bottom endpoint of the vertical segment coincides with the left endpoint of the horizontal one. This class of graphs are known as *L-graphs*. It is known that all planar graphs are L-graphs [17], while every L-graph is also an intersection graph of straight line segments [21]. Furthermore, complements of planar graphs are known to belong to  $B_{17}$ -VPG [14].

Cocomparability graphs have been considered in the context of bend-bounded representations as well. In particular, various authors studied the relationship between the necessary number of bends in various representations and the dimension of the poset represented by the cocomparability graph. Already in the seminal paper by Golumbic, Rotem and Urrutia [16], it has been pointed out that a cocomparability graph of poset dimension  $d \geq 2$  can be represented via piecewise linear curves with at most  $d - 2$  bends and arbitrary slopes, as in our general setting. Later, Cohen, Golumbic, Trotter and Wang [9] showed that cocomparability graphs of posets of dimension  $d \geq 2$  are in  $B_{d-1}$ -VPG; however, their argument does not yield  $x$ -monotone curves (even after rotation), and therefore not a bend-bounded cocomparability representation of the type we consider in this paper. More refined results on the relationship of poset dimension and  $B_k$ -VPG membership were obtained by Chakraborty, Das, Mukherjee and Sahoo [6], again with no attempt to obtain cocomparability representations.

To our knowledge, the first work to explicitly consider bend-bounded diagonal cocomparability representations is by Nesterenko [22]. His main motivation was the class of grounded-unit-L graphs, which are intersection graphs of L-shapes whose top endpoints belong to a common horizontal line and all the horizontal segments of the L-shapes have length 1. This subclass of cocomparability graphs was previously considered by Chakraborty, Gajjar and Rusu [7]. In our terminology, Nesterenko showed grounded-unit-L graphs are a strict superclass of the graphs with asynchronous unbounded 1-bend cocomparability representation, while being incomparable to those with bounded 1-bend cocomparability representation, and being a strict subclass of those with a synchronous 2-bend representation.

Nesterenko [22] also pointed out that cocomparability graphs of posets with dimension  $d$  have a synchronous representation with  $d - 1$  bends, slightly strengthening the result from [9]; on the other hand, he pointed out that cocomparability graphs with 1-bend representations may represent posets of arbitrary dimension. An open problem of Nesterenko [22] was to clarify the inclusions between the various graph classes defined by bend-bounded diagonal cocomparability representations. This problem serves as a starting point for our paper.

## 2 Notation

Throughout this paper, we work with  $x$ -monotone piecewise linear curves with a finite number of bends. For  $k \geq 1$ , a  $k$ -bend curve is a concatenation of a sequence  $s_1, s_2, \dots, s_{k+1}$  of nonvertical segments and nonvertical rays, where the rightmost point of  $s_i$  coincides with the leftmost point of  $s_{i+1}$ . In particular,  $s_2, \dots, s_k$  are necessarily segments, while each of  $s_1$  and  $s_{k+1}$  is either a segment or a ray. A 0-bend curve is a nonvertical line. We call  $s_i$  the  $i$ -th segment of the curve (even when it is in fact a ray), and the common endpoint of  $s_i$  and  $s_{i+1}$  is the  $i$ -th bend of the curve. We always assume that  $s_i$  and  $s_{i+1}$  have different slopes,

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so the decomposition of a curve into segments is unique. A  $k$ -bend curve is necessarily the graph of a continuous function  $f: I \rightarrow \mathbb{R}$  defined on an interval  $I$ . Where there is no risk of confusion, we identify the function  $f$  with its graph, and refer to  $k$ -bend curves as ‘functions’.

A *diagonal curve* is a  $k$ -bend curve whose every segment has a slope equal to  $+1$  or  $-1$ . The segments with slope  $+1$  are the *increasing segments*, while the others are *decreasing*. If a segment  $s_i$  is increasing, then  $s_{i+1}$  (if it exists) is decreasing and their common point is then a *peak* of the curve. Bends that are not peaks are the *valleys* of the curve. A  $k$ -bend *representation* of a graph  $G = (V, E)$  associates to each vertex  $v \in V$  a  $k'$ -bend curve  $f_v$  for some  $k' \leq k$ , so that all the curves  $\{f_v; v \in V\}$  have the same domain  $I$ , and in such a way, that two vertices  $v, w$  are adjacent in  $G$  if and only if the two curves  $f_v$  and  $f_w$  intersect. A  $k$ -bend representation is *pure* if any two of its curves share at most finitely many points, no two curves share an endpoint, no three curves share a common point, and no bend of a curve belongs to another curve. We will henceforth assume that all considered representations are pure.

Even in the diagonal setting, by choosing independently various combinations of bounded/unbounded, strict/non-strict, synchronous/non-synchronous restrictions, we might in theory get twelve representation models. The following proposition shows, however, that many of these models have the same expressive power.

### ► Proposition 1.

1. Any unbounded representation  $\Gamma$  can be transformed into a grounded representation  $\Gamma'$  of the same graph, and any grounded representation  $\Gamma'$  can be transformed into a bounded representation  $\Gamma''$  of the same graph, in such a way that the number of bends and the slopes of the segments of each curve are preserved.
2. If a graph  $G$  has a grounded or bounded  $k$ -bend representation  $\Gamma$ , then it also has a strict  $k$ -bend representation  $\Gamma'$  on the same domain. Moreover, if  $\Gamma$  is diagonal, then so is  $\Gamma'$ .
3. If a graph  $G$  has a synchronous diagonal  $k$ -bend representation  $\Gamma$ , then it also has a strict synchronous diagonal  $k$ -bend representation on the same domain.
4. If a graph  $G$  has a strict synchronous diagonal  $k$ -bend representation which is bounded or grounded, then it also has an unbounded strict synchronous diagonal  $k$ -bend representation.

**Proof.** (1) For an unbounded representation  $\Gamma$ , choose a constant  $a$  small enough so that any bend of  $\Gamma$  and any intersection of two curves from  $\Gamma$  appears to the right of the line  $x = a$ . Then obtain  $\Gamma'$  by restricting the functions in  $\Gamma$  to the interval  $[a, +\infty)$ . In a similar way, choosing  $b$  large enough and restricting to the domain  $[a, b]$ , we obtain  $\Gamma''$ .

(2) We can add any number of extra bends and segments in a small neighborhood of the leftmost point of a curve without affecting the intersection graph.

(3) If  $\Gamma$  is bounded, we can add more bends near the right endpoint of any curve with fewer than  $k$  bends, until we get a strict representation. If  $\Gamma$  is grounded or unbounded, we proceed as follows: choose a  $b \in \mathbb{R}$  large enough so that all the crossing points and bends of  $\Gamma$  are to the right of the line  $x = b$ . Then add to each curve of  $\Gamma$  a suitable number of bends in a small neighborhood of the point where the curve crosses the line  $x = b$ , so that all curves have exactly  $k$  bends. Since the leftmost segments of all the curves in  $\Gamma$  are parallel, the same is true for rightmost segments after the above modification. In particular, the modification does not create any new intersections between curves.

(4) We can turn  $\Gamma$  into an unbounded representation by simply extending the leftmost and the rightmost segment of each curve into infinity. Since  $\Gamma$  is strict and synchronous, this cannot create any new intersections between curves. ◀

We are now ready to formally define the graph classes of interest. Let us begin with the diagonal case. For each  $k \in \mathbb{N}_0$ , by Proposition 1 we should consider five classes:

- $\mathfrak{D}_k^\wedge$  is the class of graphs that have a synchronous diagonal  $k$ -bend representation. By Parts 3 and 4 of Proposition 1, any such representation can be transformed to a representation that is additionally strict and unbounded. Thus, a  $\mathfrak{D}_k^\wedge$ -representation of a graph  $G$  is a synchronous strict unbounded diagonal  $k$ -bend representation of  $G$ .
- $\mathfrak{D}_k^\bar{\phantom{D}}$  is the class of graphs that have a strict unbounded diagonal  $k$ -bend representation. Such a representation is then called a  $\mathfrak{D}_k^\bar{\phantom{D}}$ -representation.
- $\mathfrak{D}_k^\leq$  is the class of graphs that have an unbounded diagonal  $k$ -bend representation, which we call a  $\mathfrak{D}_k^\leq$ -representation.
- $\mathfrak{D}_k^\dagger$  is the class of graphs that have a grounded diagonal  $k$ -bend representation; by Part 2 of Proposition 1, we may additionally assume that the representation is strict. A  $\mathfrak{D}_k^\dagger$ -representation is thus a strict grounded diagonal  $k$ -bend representation.
- $\mathfrak{D}_k^H$  is the class of graphs that have a bounded diagonal  $k$ -bend representation, and by Part 2 of Proposition 1, we may again assume that the representation is strict. A  $\mathfrak{D}_k^H$ -representation is then a strict bounded diagonal  $k$ -bend representation.

For general (as opposed to diagonal)  $k$ -bend representations, the distinction between synchronous and non-synchronous representations does not apply, and the distinction between strict and non-strict representations is mostly irrelevant, since one can always add new bends by replacing a single bend by two or more bends close to each other. Thus, we only need to consider three types of general representations, for each  $k \in \mathbb{N}_0$ .

- $\mathfrak{G}_k^\bar{\phantom{G}}$  is the class of graphs that have an unbounded  $k$ -bend representation.
- $\mathfrak{G}_k^\dagger$  is the class of graphs that have a grounded  $k$ -bend representation.
- $\mathfrak{G}_k^H$  is the class of graphs that have a bounded  $k$ -bend representation.

## Our results

Our goal is to characterise the inclusions among the graph classes introduced above. For the diagonal case, we succeeded in obtaining a complete characterisation of all the inclusions.

### ► Theorem 2 (Main result – diagonal case).

- For any odd  $k \geq 1$ , we have  $\mathfrak{D}_{k-1}^H \subsetneq \mathfrak{D}_k^\wedge \subsetneq \mathfrak{D}_k^\bar{\phantom{D}} = \mathfrak{D}_k^\leq = \mathfrak{D}_k^\dagger \subsetneq \mathfrak{D}_k^H$ .
- For any even  $k \geq 2$ , we have  $\mathfrak{D}_{k-1}^H \subsetneq \mathfrak{D}_k^\wedge \subsetneq \mathfrak{D}_k^\bar{\phantom{D}} \subsetneq \mathfrak{D}_k^\leq \subsetneq \mathfrak{D}_k^\dagger \subsetneq \mathfrak{D}_k^H$ .

The proof of this result appears in Section 3. We remark that for the case  $k = 0$ , one easily observes the inclusions  $\mathfrak{D}_0^\wedge \subsetneq \mathfrak{D}_0^\bar{\phantom{D}} = \mathfrak{D}_0^\leq \subsetneq \mathfrak{D}_0^\dagger \subsetneq \mathfrak{D}_0^H$ .

For the general case, we are able to show that increasing the number of bends strictly increases the expressive power of the model considered.

### ► Theorem 3 (Main result – general case). For any $k \geq 0$ , $\mathfrak{G}_k^\bar{\phantom{G}} \subseteq \mathfrak{G}_k^\dagger = \mathfrak{G}_k^H \subsetneq \mathfrak{G}_{k+1}^\bar{\phantom{G}}$ .

We prove this in Section 4. We do not know whether the inclusion  $\mathfrak{G}_k^\bar{\phantom{G}} \subseteq \mathfrak{G}_k^H$  is strict (except for several small values of  $k$ ), but we conjecture that it is.

Before we turn to the proofs of our results, we introduce several graph constructions that are used throughout the paper. For two disjoint graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$ , the *complete join* (or just *join*) of  $G_1$  and  $G_2$  is the graph obtained from the disjoint union of  $G_1$  and  $G_2$  by adding all the edges between  $V_1$  and  $V_2$ . For a graph  $G = (V, E)$ , a set of vertices  $A \subseteq V$  and an integer  $t \geq 1$ , we say that a set  $B \subseteq V \setminus A$  is a *t-extension* of  $A$  if the following holds:

- every vertex from  $B$  is adjacent to exactly  $t$  vertices from  $A$ ,

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- for each set  $S \subseteq A$  containing  $t$  vertices, there is a unique vertex  $x_S \in B$  which is adjacent to the vertices in  $S$  (and therefore to no vertex in  $A \setminus S$ ), and
- any two vertices in  $B$  are adjacent.

For a pair of graphs  $G$  and  $H$ , we let  $G + H$  denote the disjoint union of  $G$  and  $H$ .

### 3 Diagonal setting

In this section, we focus on the relationships between the classes  $\mathfrak{D}_k^\nearrow$ ,  $\mathfrak{D}_k^\equiv$ ,  $\mathfrak{D}_k^\leq$ ,  $\mathfrak{D}_k^\uparrow$ , and  $\mathfrak{D}_k^H$ . We begin by noticing that these classes form a hierarchy.

► **Theorem 4.** For any  $k \geq 0$ ,  $\mathfrak{D}_k^\nearrow \subseteq \mathfrak{D}_k^\equiv \subseteq \mathfrak{D}_k^\leq \subseteq \mathfrak{D}_k^\uparrow \subseteq \mathfrak{D}_k^H \subseteq \mathfrak{D}_{k+1}^\nearrow$ .

**Proof.** Any  $\mathfrak{D}_k^\nearrow$ -representation is also a  $\mathfrak{D}_k^\equiv$ -representation, and any  $\mathfrak{D}_k^\equiv$ -representation is also a  $\mathfrak{D}_k^\leq$ -representation, showing that  $\mathfrak{D}_k^\nearrow \subseteq \mathfrak{D}_k^\equiv \subseteq \mathfrak{D}_k^\leq$ . The inclusions  $\mathfrak{D}_k^\leq \subseteq \mathfrak{D}_k^\uparrow \subseteq \mathfrak{D}_k^H$  follow from Part 1 of Proposition 1. It remains to prove that  $\mathfrak{D}_k^H \subseteq \mathfrak{D}_{k+1}^\nearrow$ .

If we are given a  $\mathfrak{D}_k^H$ -representation  $\Gamma$  of  $G$  with domain  $[a, b]$  for some  $a, b \in \mathbb{R}$ , we define a  $\mathfrak{D}_{k+1}^\nearrow$ -representation  $\Gamma'$  as follows. Let  $f$  be a function in  $\Gamma$ . If the leftmost segment of  $f$  has slope  $-1$ , we add to  $f$  a new leftmost segment with slope  $+1$  with domain  $(-\infty, a]$ , and we extend the rightmost segment of  $f$  (which has slope  $(-1)^{k+1}$ ) to  $+\infty$ . On the other hand, if the leftmost segment of  $f$  has slope  $+1$ , we extend it to  $-\infty$  and add an additional rightmost segment to  $f$  with slope  $(-1)^{k+1}$  and domain  $[b, +\infty)$ . This new representation is clearly a  $\mathfrak{D}_{k+1}^\nearrow$ -representation since we added one new bend to each function, and it is still a representation of  $G$  since we did not introduce any new crossings. This shows that  $\mathfrak{D}_k^H \subseteq \mathfrak{D}_{k+1}^\nearrow$ , finishing the proof. ◀

The remainder of this section is dedicated to determining which of these inclusions are strict. To do so systematically, we differentiate between odd  $k$  and even  $k$ .

#### 3.1 Odd number of bends

► **Theorem 5.** For  $k$  odd,  $\mathfrak{D}_k^\equiv = \mathfrak{D}_k^\leq = \mathfrak{D}_k^\uparrow$ .

**Proof.** By Theorem 4, we have  $\mathfrak{D}_k^\equiv \subseteq \mathfrak{D}_k^\leq \subseteq \mathfrak{D}_k^\uparrow$ . Now, let  $G$  be a graph in  $\mathfrak{D}_k^\uparrow$  and let  $\Gamma$  be a  $\mathfrak{D}_k^\uparrow$ -representation of  $G$ . Recall that we can assume that  $\Gamma$  is strict. We construct a  $\mathfrak{D}_k^\equiv$ -representation of  $G$  from  $\Gamma$  by extending the leftmost segment of each function to  $-\infty$ . This new representation  $\Gamma'$  is clearly a  $\mathfrak{D}_k^\equiv$ -representation, but we need to show that it is a representation of  $G$ . Clearly, if two functions cross in  $\Gamma$ , they still cross in  $\Gamma'$ . Now, let  $f, g$  be two functions that do not cross in  $\Gamma$ , say  $f \leq g$ . We need to show that  $f, g$  still do not cross in  $\Gamma'$ . This is clearly true if the leftmost segment of  $f$  has greater or equal slope than  $g$ , so we may assume this slope is  $-1$  while the leftmost segment of  $g$  has slope  $+1$ . Since  $k$  is odd and the representation is strict, this means that the rightmost segments of  $f$  and  $g$  have slopes  $+1$  and  $-1$  respectively, a contradiction to  $f \leq g$ . ◀

Our next two results deal with constructions that fail to be representable within a restricted class of curves. They both depend on a geometric counting argument which we now introduce. For a set of points  $P$  and an unbounded  $k$ -bend curve  $f$ , we say that  $f$  *cuts off* a subset  $A \subseteq P$  from  $P$ , if the points from  $A$  are all strictly below the curve  $f$ , while the points of  $P \setminus A$  are strictly above it.

► **Lemma 6.** For every  $m \in \mathbb{N}$  and  $t \in \mathbb{N}_0$  there is a constant  $c = c(m, t)$  such that every nonempty finite set of points  $P$  in the plane has at most  $c|P|^t$  subsets of size at most  $m$  that can be cut off from  $P$  by an unbounded diagonal curve with at most  $t$  peaks.

**Proof.** Fix  $m$  and fix a point set  $P$  of size  $N \geq 1$ . We say that a set  $A \subseteq P$  is *small* if  $|A| \leq m$ .

Let us proceed by induction on  $t$ . For  $t = 0$ , the curves under consideration are lines of slopes  $+1$  or  $-1$  and 1-bend diagonal curves with a single valley. There are at most  $m + 1$  small subsets of  $P$  that can be cut off by a line of slope  $+1$ , and likewise at most  $m + 1$  cut off by lines of slope  $-1$ . If  $A$  is a small set cut off by a diagonal 1-bend curve with a valley at a point  $X$ , then  $A$  is the union of two sets  $A^+$  and  $A^-$  where  $A^+$  is cut off by the line of slope  $+1$  passing through  $X$  and  $A^-$  by a line of slope  $-1$  through  $X$ . As there are at most  $(m + 1)$  choices for  $A^+$  and for  $A^-$ , there are at most  $(m + 1)^2$  such sets  $A$ .

Suppose now that  $t = 1$  and consider the curves that have one peak  $Y$  and no valleys. Any small set cut off by a curve of this form can be identified in two steps:

1. Choose a line  $\ell$  of slope  $+1$ .
2. Choose a point  $Y \in \ell$  so that the 1-bend diagonal curve with a single peak in  $Y$  cuts off a small subset of  $P$ .

In the first step, we can only make  $N + 1$  combinatorially distinct choices, since we only need to know which points of  $P$  are below  $\ell$ . In the second step, there are at most  $m + 1$  distinct small sets that can be cut off once  $\ell$  is fixed. Thus, at most  $(m + 1)(N + 1)$  small sets can be cut off by curves with a single peak and no valleys.

For a curve  $f$  with a single peak  $Y$  and up to two valleys  $X_1, X_2$  with  $X_1$  left of  $X_2$ , the set of points cut off by  $f$  is the union of the points cut off by the decreasing line through  $X_1$ , the points cut off by the 1-bend curve with peak at  $Y$ , and the points cut off by the increasing line through  $X_2$ . The number of small sets obtainable in this way is at most  $(m + 1)^3(N + 1)$ .

For  $t \geq 2$ , consider a curve  $f$  with  $t$  peaks, and let  $X$  be the valley lying between the first and second peak of  $f$ . Consider the unbounded diagonal curve  $f_1$  whose bends are exactly the bends of  $f$  strictly to the left of  $X$ , and the unbounded diagonal curve  $f_2$  whose bends are the bends of  $f$  strictly to the right of  $X$ ; in particular,  $f_1$  and  $f_2$  both pass through  $X$ ,  $f_1$  has 1 peak and  $f_2$  has  $t - 1$  peaks. The set of points cut off by  $f$  is the union of the set cut off by  $f_1$  and the set cut off by  $f_2$ . By induction, there are at most  $c(m, 1)c(m, t - 1)N^t$  distinct small sets cut off by a curve with  $t$  peaks. ◀

In our applications of Lemma 6, we will assume that  $m$  and  $t$  are constants, while  $|P|$  will grow to  $+\infty$ . We then write the bound  $c(m, t)|P|^t$  from the lemma simply as  $\mathcal{O}_{m,t}(|P|^t)$ .

► **Theorem 7.** *For every  $k$  odd, there is a graph  $G \in \mathfrak{D}_k^\wedge \setminus \mathfrak{D}_{k-1}^H$ .*

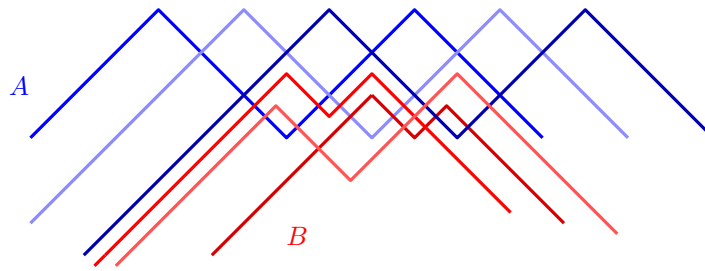
**Proof.** We have  $\mathfrak{D}_0^H \subsetneq \mathfrak{D}_1^\wedge$ , since  $\mathfrak{D}_0^H$  only contains bipartite permutation graphs, while  $\mathfrak{D}_1^\wedge$  is known to be the class of all permutation graphs [22]. Now assume  $k \geq 3$ , and let  $t = \frac{k+1}{2}$ .

The separation is witnessed by the graph  $G(n)$  consisting of a clique  $A$  of size  $n$  with a  $t$ -extension  $B$ . We easily observe that for any  $n$ ,  $G(n)$  has a  $\mathfrak{D}_k^\wedge$ -representation, see Figure 2.

We will show that for  $n$  large enough,  $G(n)$  has no  $\mathfrak{D}_{k-1}^H$ -representation. For contradiction, consider a  $\mathfrak{D}_{k-1}^H$ -representation  $\Gamma$  of  $G(n)$ . For any  $a \in A$  and  $b \in B$  nonadjacent vertices, the curve  $f_a$  is either completely above or completely below  $f_b$ . Without loss of generality, suppose that  $f_a$  is above  $f_b$ , otherwise we turn  $\Gamma$  upside down. It follows that any other vertex  $b' \in B$  nonadjacent to  $a$  has its representation below  $f_a$ , because  $f_b$  and  $f_{b'}$  must cross. We conclude that for any pair of nonadjacent vertices  $a \in A$  and  $b \in B$ , the curve  $f_a$  is above  $f_b$ .

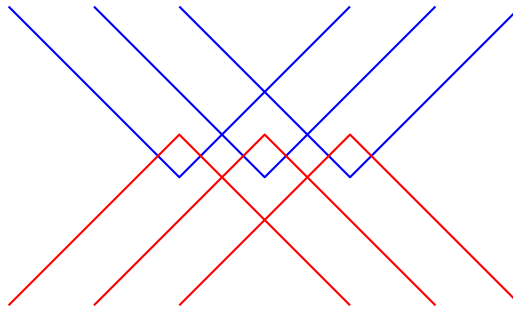
Let  $P$  be the set of all the endpoints and bends of all the curves of  $\Gamma$  representing vertices in  $A$ . In particular,  $|P| = n(k + 2)$ , since we may assume that  $\Gamma$  is strict. Every curve representing a vertex from  $B$  crosses exactly  $t$  curves representing vertices in  $A$ , and hence it

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■ **Figure 2** A  $\mathfrak{D}_3^>$ -representation of the graph  $G(3)$  from the proof of Theorem 7. For readability, colors of the sets  $A, B$  vary slightly.

cuts off at most  $t(k + 2)$  points of  $P$ . Moreover, two curves representing distinct vertices from  $B$  must cut off distinct subsets of  $P$ . Thus, at least  $\binom{n}{t}$  distinct subsets of  $P$  of size at most  $(k + 2)t$  are cut off in this way. However, the curves of  $\Gamma$  have at most  $t - 1$  peaks, and therefore can cut off at most  $\mathcal{O}_{k,t}(n^{t-1})$  distinct subsets of  $P$  of size at most  $t(k + 2)$  by Lemma 6. For  $n$  large enough, this yields a contradiction. ◀



■ **Figure 3** A  $\mathfrak{D}_1^-$ -representation of the graph  $\overline{C}_6$ , which is not a permutation graph and therefore has no  $\mathfrak{D}_1^>$ -representation.

► **Theorem 8.** For every  $k$  odd, there is a graph  $G \in \mathfrak{D}_k^- \setminus \mathfrak{D}_k^>$ .

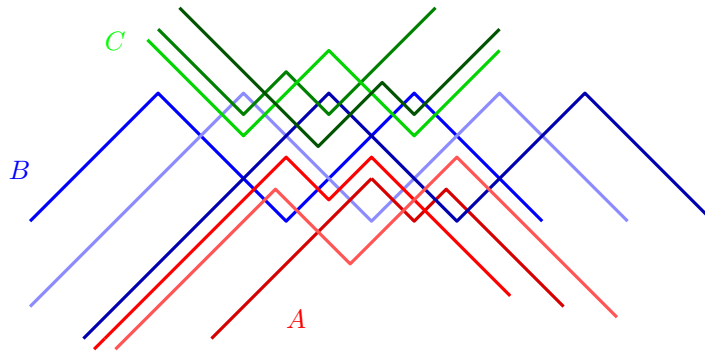
**Proof.** For  $k = 1$  the theorem holds, since  $\mathfrak{D}_1^>$  is the class of permutation graphs, while  $\mathfrak{D}_1^-$  contains the graph  $\overline{C}_6$  (the complement of the 6-cycle) which is not a permutation graph (see Figure 3). Suppose that  $k \geq 3$  and let  $t = \frac{k+1}{2}$ .

Let  $G(n)$  be the graph consisting of a clique  $B$  of size  $n$  and two disjoint  $t$ -extensions  $A$  and  $C$  of  $B$ . There are no edges between  $A$  and  $C$ . We easily observe that  $G$  has a  $\mathfrak{D}_k^-$ -representation, see Figure 4.

We will show that for  $n$  large enough,  $G(n)$  has no  $\mathfrak{D}_k^>$ -representation.

Suppose for contradiction that  $G(n)$  has a  $\mathfrak{D}_k^>$ -representation  $\Gamma$ . In  $\Gamma$ , for any choice of pairwise nonadjacent vertices  $a \in A, b \in B$  and  $c \in C$ , the curve  $f_b$  must be between  $f_a$  and  $f_c$ . Suppose without loss of generality that  $f_b$  is below  $f_a$  and above  $f_c$ . Since all vertices of  $A$  are adjacent to  $a$  and all vertices of  $C$  are adjacent to  $c$ , it follows that for any choice of  $a' \in A, b' \in B$  and  $c' \in C$  pairwise nonadjacent,  $f_{b'}$  is below  $f_{a'}$  and above  $f_{c'}$ .

Each curve in  $\Gamma$  has  $t$  peaks and  $t - 1$  valleys. Let us turn  $\Gamma$  upside down, and then transform it into a bounded representation by limiting its domain to a bounded interval covering all intersection points and bends. Call the modified representation  $\Gamma'$ . We now repeat the argument of the proof of Theorem 7: let  $P$  be the set of bends and endpoints of



■ **Figure 4** A  $\mathfrak{D}_3^-$ -representation of the graph  $G(3)$  from the proof of Theorem 8. For readability, the colors of the sets  $A, B, C$  vary slightly.

the curves representing  $B$  in  $\Gamma'$ . Each curve representing a vertex  $a \in A$  in  $\Gamma'$  has at most  $t - 1$  peaks, and cuts off from  $P$  a subset of size at most  $t(k + 2)$ . By Lemma 6, there are at most  $\mathcal{O}_k(n^{t-1})$  such subsets of  $P$ , whereas  $A$  has  $\binom{n}{t}$  vertices, and each must correspond to a different set being cut off. For  $n$  large enough, this yields a contradiction. ◀

We complete the case of  $k$  odd by a result for odd and even numbers of bends.

► **Theorem 9.** *For any  $k$  and for any graph  $G \in \mathfrak{D}_k^- \setminus \mathfrak{D}_k^\uparrow$  we have  $G + G \in \mathfrak{D}_k^H \setminus \mathfrak{D}_k^+$ .*

**Proof.** Fix  $G \in \mathfrak{D}_k^- \setminus \mathfrak{D}_k^\uparrow$ . It follows that  $G$  is also in  $\mathfrak{D}_k^H$ , and we easily see that  $\mathfrak{D}_k^H$  is closed under disjoint unions, hence  $G + G \in \mathfrak{D}_k^H$ . For contradiction, assume that  $G + G$  has a  $\mathfrak{D}_k^+$ -representation  $\Gamma$ . If every component of  $G$  is in  $\mathfrak{D}_k^\uparrow$ , then  $G$  itself is in  $\mathfrak{D}_k^\uparrow$ , a contradiction. Hence,  $G$  has a connected component  $C$  not belonging to  $\mathfrak{D}_k^\uparrow$ .

Let  $C_1$  and  $C_2$  be two components isomorphic to  $C$  inside  $G + G$ . In  $\Gamma$ , the representations of  $C_1$  and  $C_2$  must include curves with increasing unbounded segments as well as those with decreasing unbounded segments, otherwise  $C$  would have a  $\mathfrak{D}_k^\uparrow$ -representation. Since each copy of  $C$  is connected, the left endpoints of its curves must appear consecutively on the grounding line of  $\Gamma$ . It follows that in  $\Gamma$ , a curve from the representation of  $C_1$  crosses a curve from the representation of  $C_2$ , which is impossible. ◀

### 3.2 Even number of bends

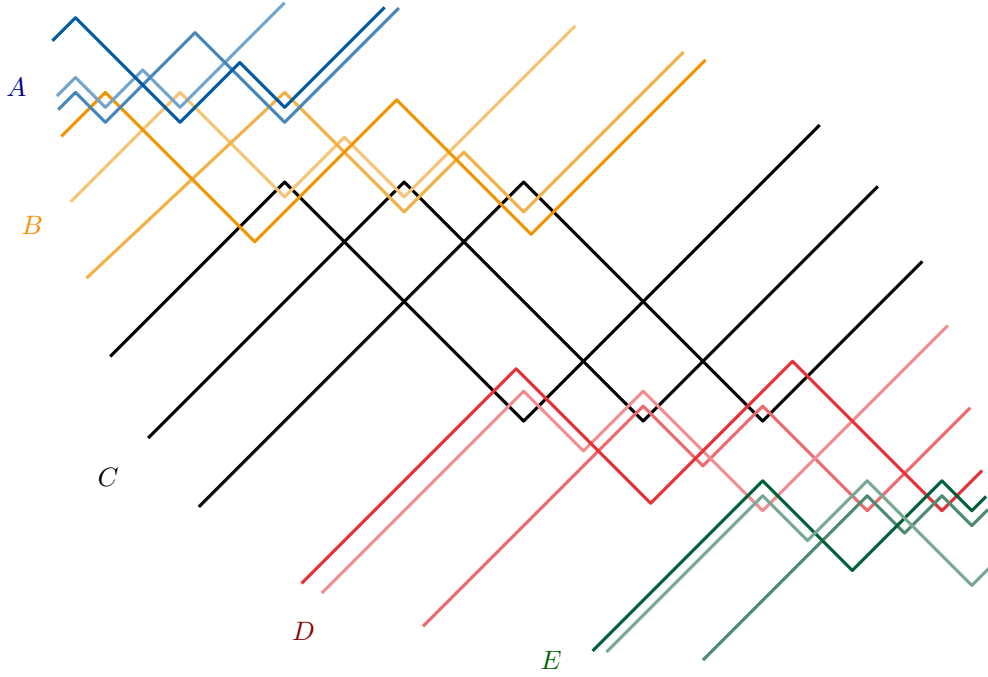
Let  $k = 2t$  be the number of bends. We will start by defining a graph  $H_t(n)$  which, as we will prove, is in  $\mathfrak{D}_k^\uparrow \setminus \mathfrak{D}_{k-1}^H$  if  $n$  is large enough. This construction will further be used as a building block in the subsequent constructions of graphs in  $\mathfrak{D}_k^- \setminus \mathfrak{D}_k^\uparrow$ ,  $\mathfrak{D}_k^\leq \setminus \mathfrak{D}_k^-$ ,  $\mathfrak{D}_k^+ \setminus \mathfrak{D}_k^\leq$  and  $\mathfrak{D}_k^H \setminus \mathfrak{D}_k^+$ .

We define  $H_t(n)$  to be the graph consisting of a base clique  $C = [n]$  and four cliques  $A, B, D$  and  $E$ , where  $B$  and  $D$  are disjoint  $t$ -extensions of  $C$ , while  $A$  is a  $t$ -extension of  $B$  and  $E$  is a  $t$ -extension of  $D$ . There are no other edges in  $H_t(n)$ . We call  $C$  the *central clique* of  $H_t(n)$ .

It is straightforward to show that  $H_t(n)$  belongs to  $\mathfrak{D}_{2t}^\uparrow$ . We leave the details of the argument for the full version of our paper and refer to Figure 5.

The main part of this section is an analysis of the possible  $k$ -bend representations of  $H_t(n)$ . The aim is to prove that if for every  $n$ , the graph  $H_t(n)$  can be represented within a given class  $\mathfrak{D}$ , then it also has such a representation satisfying certain additional restrictions. This analysis will serve us for two purposes: firstly, it will allow us to show that  $H_t(n)$

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■ **Figure 5**  $\mathfrak{D}_4^>$ -Representation of  $H_2(3)$ . For readability, colors of the sets  $A, B, D, E$  vary slightly.

has no  $\mathfrak{D}_{2t-1}^H$ -representation, and secondly, it will impose constraints on the possible  $\mathfrak{D}_{2t}^>$ -representations, which will turn out useful when we use  $H_t(n)$  as a building block for further constructions. To avoid duplicating our efforts, and highlight the generality of our arguments, we will now fix a number of bends  $b \geq 1$  and a class  $\mathfrak{D} \in \{\mathfrak{D}_b^>, \mathfrak{D}_b^=, \mathfrak{D}_b^<, \mathfrak{D}_b^H\}$ , and study the properties of  $\mathfrak{D}$ -representations of  $H_t(n)$  (we omit  $\mathfrak{D}_b^<$ , since it is convenient for us to work with  $\mathfrak{D}$ -representations that are strict).

Since we only want to investigate  $H_t(n)$  for large enough  $n$ , our proof is making extensive use of pigeonhole and Ramsey-type arguments. More precisely, in order to obtain a  $\mathfrak{D}$ -representation of  $H_t(n)$  with certain additional desirable properties, we typically start with a  $\mathfrak{D}$ -representation  $\Gamma$  of a larger graph  $H_t(N)$  with central clique  $C$ . We then identify within  $C$  a subset  $C'$  of size  $n$ , and we delete from  $\Gamma$  all the vertices of  $C \setminus C'$ , as well as all the vertices in  $B$  and  $D$  adjacent to at least one deleted vertex in  $C$ , and all the vertices in  $A$  and  $E$  adjacent to at least one deleted vertex from  $B$  or  $D$ . The remaining vertices will form a representation  $\Gamma'$  of  $H_t(n)$  with central clique  $C'$ . We will say that  $\Gamma'$  is the *restriction* of  $\Gamma$  generated by the set  $C'$ .

The above process will be used iteratively to impose increasingly stronger restrictions on the  $\mathfrak{D}$ -representations of  $H_t(n)$ . To avoid repetitive language, we will use the phrase “we can enforce property  $P$ ” as a shorthand for the following claim: “For every  $n \in \mathbb{N}$  there is an  $N \in \mathbb{N}$  such that every  $\mathfrak{D}$ -representation of  $H_t(N)$  can be restricted to a representation of  $H_t(n)$  that satisfies property  $P$ .”

Let us assume that for every  $n$ , the graph  $H_t(n)$  has a  $\mathfrak{D}$ -representation. We let  $f_v$  denote the curve representing the vertex  $v$  in the  $\mathfrak{D}$ -representation of  $H_t(n)$  being considered.

The first property we consider does not even need to be enforced by restrictions.

▷ **Property 1.** We may assume without loss of generality that for every  $a \in A$ ,  $c \in C$  and  $e \in E$  we have  $f_a > f_c > f_e$ .

Proof. Since  $A, C$  and  $E$  are disjoint cliques, all functions in one of them lie above or below all functions of another. If  $C$  is not in the middle, then  $A$  is in the middle without loss of generality. However, this is impossible, because the functions of  $D$  are supposed to intersect functions of  $C$  and  $E$  while not intersecting any from  $A$ . Thus,  $C$  is in the middle. If  $A$  were below  $C$  and  $E$  above it, we could simply relabel the vertices, since  $H_t(n)$  admits an automorphism that exchanges  $A$  with  $E$  and  $B$  with  $D$ .  $\triangleleft$

$\triangleright$  **Property 2.** We may enforce that all the curves representing vertices in  $C$  have parallel leftmost segments.

Proof. Pick an arbitrary  $\mathfrak{D}$ -representation  $\Gamma$  of  $H_t(2n - 1)$ , with central clique  $C$ . Clearly  $C$  contains a subset of  $n$  curves whose leftmost segments are parallel. By restricting to this subset, we enforce the property.

If  $C$  does not contain  $n$  upwards-starting curves, then it contains a set  $C''$  of  $n$  downwards starting curves. We restrict  $\Gamma$  by this set  $C''$ , and turn the resulting representation upside down. To ensure that Property 1 is preserved, we again relabel vertices according to the automorphism that maps  $A$  to  $E$  and  $B$  to  $D$ .  $\triangleleft$

We may in fact assume without loss of generality that in the representations satisfying Property 2 the leftmost segments in the curves representing  $C$  are all increasing. This is automatic when  $\mathfrak{D} = \mathfrak{D}_b^+$ , and for  $\mathfrak{D} \in \{\mathfrak{D}_b^-, \mathfrak{D}_b^+, \mathfrak{D}_b^H\}$ , this can be achieved by turning the representation upside down and relabeling vertices to ensure Property 1 also holds. From now on, we will restrict ourselves to representations where the leftmost segments of the curves representing the central clique are all increasing.

Note that all curves in  $C$  have exactly  $b$  bends. Since they start upwards, they have exactly  $\lceil \frac{b}{2} \rceil$  peaks and  $\lfloor \frac{b}{2} \rfloor$  valleys. We will say that a point  $(x, y) \in \mathbb{R}$  lies above another point  $(x', y')$ , if  $y - y' > |x - x'|$ . A set of points is *independent*, if no point of the set lies above any other. Note that since all functions in the diagonal representation are Lipschitz-continuous with Lipschitz constant 1, if a point on the curve  $f$  lies above a point on the curve  $g$ , then  $f \geq g$  on the interval in between those two points.

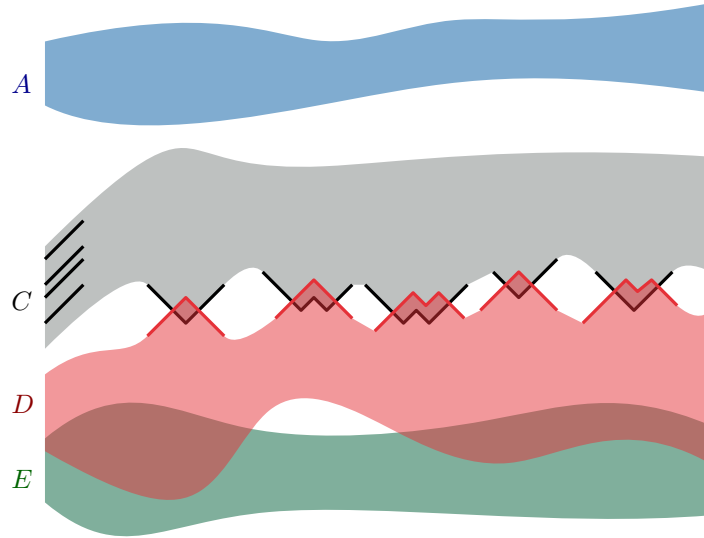
From now on, we will scale our  $\mathfrak{D}$ -representations in such a way, that the  $x$ -coordinate every bend and every intersection point is in the interval  $(0, 1)$ . Furthermore, the domain of a bounded representation is taken to be  $[0, 1]$  and the domain of a grounded representation is  $[0, +\infty)$ .

$\triangleright$  **Property 3.** We may enforce that for any  $c \in C$  and  $d \in D$  we have  $f_c(0) > f_d(0)$  and  $f_c(1) > f_d(1)$ .

Proof. Start with a representation of  $H_t(n + 2t)$  satisfying the previously enforced properties. Choose  $d \in D$  so that  $f_d(0)$  is maximised. Note that if there is a  $c \in C$  such that  $f_c(0) < f_d(0)$ , then  $f_d$  crosses  $f_c$ . This is because  $d$  is adjacent to a vertex  $e \in E$ , and  $f_e < f_c$  by Property 1. Since  $d$  has  $t$  neighbors in  $C$ , there can be at most  $t$  vertices  $c \in C$  for which  $f_c(0) < f_d(0)$ . By deleting these at most  $t$  vertices, we enforce that for every  $c \in C$  and  $d \in D$ ,  $f_c(0)$  is greater than  $f_d(0)$ . By a similar argument, we also enforce that  $f_c(1) > f_d(1)$ , after deleting another set of at most  $t$  vertices.  $\triangleleft$

Note that Property 3 implies that for any  $c \in C$  and  $d \in D$ ,  $f_c$  and  $f_d$  intersect an even number of times. Moreover, if  $P$  and  $Q$  are two consecutive intersections of  $f_c$  and  $f_d$  so that  $f_d$  stays above  $f_c$  between  $P$  and  $Q$ , then  $f_d$  has at least one peak and  $f_c$  at least one valley between  $P$  and  $Q$ , and the leftmost such peak of  $f_d$  lies above the leftmost such valley of  $f_c$ .

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■ **Figure 6** Visualization of properties 1-3.

From this point on, we will denote some of the valleys of the functions representing  $C$  as *active*. To determine which valleys are active, we first consider all of them as active, that is  $\lfloor \frac{b}{2} \rfloor$  per function, then do  $\lfloor \frac{b}{2} \rfloor$  steps: In step  $i$ , we consider all functions with exactly  $\lfloor \frac{b}{2} \rfloor - i + 1$  active valleys. As long as an active valley  $v_1$  of such a function  $f_1$  representing  $c_1 \in C$  lies above an active valley of such a function  $f_2$  representing  $c_2 \in C$ , we delete  $c_2$  from  $C$  and deactivate  $v_1$ . This procedure deletes at most half of the elements of  $C$  in each step, so at least  $\frac{N}{2^{\lfloor \frac{b}{2} \rfloor}} \geq n$  elements remain. The procedure enforces the following two properties.

▷ **Property 4.** If a peak of a function  $f_d$  lies above a valley  $v$  of a function  $f_c$ , then  $v$  is active.

*Proof.* Assume for a contradiction that some function  $f_d$  obeys  $f_d(x) > f_c(x)$ , where  $(x, y)$  is a valley of  $f_c$  that is not active. Since all valleys were active before the procedure, there was a certain step in which  $(x, y)$  was deactivated. In this step another function  $f_{c'}$  was deleted, whose valley  $(x', y')$  obeys  $y - y' > |x - x'|$ . Since  $f_d$  is Lipschitz continuous with Lipschitz constant 1, we get

$$f_d(x') \geq f_d(x) - |f_d(x) - f_d(x')| > f_c(x) - 1 \cdot |x - x'| = y - |x - x'| > y' = f_{c'}(x').$$

Since  $f_d(0) < f_{c'}(0)$ , by continuity  $f_d$  crosses  $f_{c'}$  and should have therefore been removed when  $c'$  and all of its neighbors were removed, a contradiction. ◁

▷ **Property 5.** For every  $i \in [\lfloor \frac{b}{2} \rfloor]$ , the active valleys of all functions representing  $C$  with exactly  $i$  active valleys are independent.

*Proof.* This is true after step  $\lfloor \frac{b}{2} \rfloor - i + 1$  because it would not have ended otherwise. It stays true as the functions with  $i$  active valleys are not changed in any of the further steps. ◁

▷ **Property 6.** We may enforce that all functions representing  $C$  have  $m$  active valleys, for some  $m \in \{0, 1, \dots, \lfloor \frac{b}{2} \rfloor\}$ .

*Proof.* Let  $0 \leq m \leq \lfloor \frac{b}{2} \rfloor$  be a number that maximizes the number of functions of  $C$  with exactly  $m$  active valleys. At least  $\frac{N}{\lfloor \frac{b}{2} \rfloor + 1} \geq n$  elements of  $C$  are represented by such functions by the pigeon-hole principle. Then delete all other elements of  $C$ . ◁

From Properties 5 and 6 we can now deduce that all active valleys of functions representing  $C$  are independent. By Property 4, only active valleys can be used for intersection with functions representing  $D$ .

Since the active valleys are independent, they form a natural left-to-right sequence. It is convenient to view such a sequence as a word over the alphabet whose symbols are the vertices in the central clique  $C$ . We first introduce some related terminology.

We will say that a word  $w$  is  $m$ -regular over an alphabet  $X$ , if it contains exactly  $m$  occurrences of each symbol from  $X$ , and no other symbols. We say that a word  $w$  is an  $m$ -immediate repetition over an alphabet  $X$  if it has the form  $w = x_1^m x_2^m \cdots x_n^m$ , where  $x^m$  denotes an  $m$ -fold repetition of the symbol  $x \in X$ , and the  $n$ -tuple  $(x_1, \dots, x_n)$  is a permutation of the alphabet  $X$ .

We will now introduce the *pattern* of the active valleys, which is a word  $w$  over the alphabet  $C$  whose  $i$ -th symbol  $w_i$  is the vertex whose curve contains the  $i$ -th active valley from the left. Note that  $w$  is  $m$ -regular over  $C$ , by Property 6.

► **Lemma 10.** *For any  $m, n \in \mathbb{N}$ , there is an  $N \in \mathbb{N}$  such that for every  $m$ -regular word  $w$  over the alphabet  $[N]$ , there is a set  $C \subseteq [N]$  of size  $n$  such that the restriction of  $w$  to the symbols from  $C$  yields a word  $w'$  of one of the following two forms:*

1. *there is an integer  $j \in [m-1]$  such that  $w'$  is a concatenation of two words  $w'_1, w'_2$  where  $w'_1$  is  $j$ -regular over  $C$  (and hence  $w'_2$  is  $(m-j)$ -regular over  $C$ ), or*
2. *the word  $w'$  is an  $m$ -immediate repetition over the alphabet  $C$ .*

**Proof.** Set  $N = mn^2$ , and let  $w$  be an  $m$ -regular word over  $[N]$ . To each symbol  $x \in [N]$  we associate the interval  $I_x = [x^i, x^j]$  where  $x^i, x^j$  are the indices of the first and last occurrence of  $x$  in  $w$ .

We then construct an interval graph  $G(w)$  as the graph on the vertex set  $[N]$  where distinct vertices  $x, y \in [N]$  are adjacent if and only if the intervals  $I_x$  and  $I_y$  intersect. This graph is an interval graph and therefore perfect. In particular, it satisfies  $\alpha(G(w)) \cdot \omega(G(w)) \geq N$ , where  $\alpha(G(w))$  and  $\omega(G(w))$  denote the size of the largest independent set and the largest clique of  $G(w)$ , respectively.

We thus have  $\alpha(G(w)) \geq n$  or  $\omega(G(w)) \geq mn$ . If the former is true, we let  $C$  be an independent set of size  $n$  in  $G(w)$ , and we see that Case 2 of the lemma occurs.

On the other hand, if  $\omega(G(w)) \geq mn$ , we first fix a clique  $Q$  of size  $mn$  in  $G(w)$ . The intervals representing the vertices of  $Q$  have nonempty intersection by Helly's theorem, and since all these intervals have distinct endpoints by construction, their common intersection is a non-degenerate interval  $J$ . Choose a  $p \in J$  such that  $p+1$  is also in  $J$ . Let us call the prefix  $w_1 w_2 \cdots w_p$  the *left part* of  $w$ , and the suffix  $w_{p+1} \cdots w_N$  the *right part* of  $w$ . Each symbol  $x \in Q$  thus appears both in the left part and in the right part.

For  $j \in [m-1]$ , let  $Q_j$  be the set of those symbols  $x \in Q$  that have exactly  $j$  occurrences in the left part. Choose the value of  $j$  for which the size of  $Q_j$  is maximised, and set  $C = Q_j$ . We then have  $|C| \geq |Q|/(m-1) \geq n$ , and the restriction of  $w$  to the symbols of  $C$  satisfies Case 1 of the lemma. ◀

▷ **Property 7.** We can enforce that the pattern  $w$  is a concatenation of  $\ell \geq 1$  words  $w = I_1 \cdots I_\ell$  where every  $I_j$  is an  $m_j$ -immediate repetition of the alphabet  $C$ , and  $\sum_{j=1}^{\ell} m_j = m$ . We then call  $I_j$  the  $j$ -th interval of  $w$ .

**Proof.** Apply Lemma 10 iteratively to the intervals in the first case, until the second case is encountered (which is definitely the case if  $m = 1$ ). Note that removing letters from the alphabet of the pattern is equivalent to deleting elements of  $C$ . ◀

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We may assume that the order of valleys in the first interval  $I_1$  is  $(1, \dots, n)$  by relabeling the vertices in  $C$  if necessary.

▷ **Property 8.** We can enforce that in each of the intervals  $I_j$ , the  $n$  symbols  $1, 2, \dots, n$  appear either in the increasing order or in the decreasing order.

*Proof.* For this step choose  $N = n^{2^{\ell-1}}$ . For each interval consider the sequence of letters of the alphabet, ordered by their occurrences in the interval. In each interval choose one representative of each letter to have a permutation of  $(1, \dots, n)$  as subsequence. We consider just those permutations. By the Erdős-Szekeres Lemma [11], each sequence of  $n^2$  numbers contains a subsequence of  $n$  numbers that is either monotonically increasing or monotonically decreasing. With the assumption that the sequence for  $I_1$  is  $(1, \dots, N)$ , it is possible to find a subset of  $\sqrt{N}$  letters such the permutation for  $I_2$  is increasing or decreasing. By removing all letters not in the subset and repeating the process for each of the intervals  $I_3, \dots, I_\ell$ , we end up with at least  $N^{\frac{1}{2^{\ell-1}}} = n$  letters such that each interval contains the subsequence  $(1, \dots, n)$  or  $(n, \dots, 1)$  after relabeling. ◁

For a set  $I = \{c_1 < c_2 < \dots < c_t\}$  of  $t$  vertices from  $C$ , we let  $d_I \in D$  denote the vertex in  $D$  adjacent to the vertices in  $I$ . For such  $I$  we further define the *crossing type* of  $I$  as the  $t$ -tuple  $(i(1), i(2), \dots, i(t)) \in [\ell]^t$  such that for every  $j \in [t]$ , the curve of  $d_I$  has a peak above a valley of the curve of  $c_j$  which occurs in the interval  $I_{i(j)}$ . The crossing type need not be determined uniquely, in which case we assign one fixed crossing type to each  $I \in \binom{C}{t}$  arbitrarily.

▷ **Property 9.** We can enforce that all  $t$ -element subsets of  $C$  have the same crossing type.

*Proof.* The crossing types define a coloring of the set  $\binom{C}{t}$ , and we apply Ramsey's Theorem to this coloring. ◁

► **Lemma 11.** *Let  $\Gamma$  be a  $\mathcal{D}$ -representation of  $H_t(n)$  satisfying the previously enforced properties. Let  $I$  be a  $t$ -element subset of  $C$  that does not contain any two consecutive integers, and contains neither 1 nor  $n$ . Let  $p$  be a peak of the curve representing  $d_I$ . Then there is at most one vertex  $c \in I$  whose curve has a valley below  $p$ .*

*Proof.* Since all valleys are independent, every peak lies above a consecutive set of valleys. Say a peak  $p$  of  $d_I$  lies above valleys of vertices  $i$  and  $j > i + 1$  from  $C$ . If these valleys are in the same interval, Property 8 makes sure that a valley of the function of  $i + 1$  lies in between those two valleys, a contradiction since  $i + 1 \notin I$ . If the valleys of  $i$  and  $j$  are in different intervals, then the end of one interval and the beginning of the other are contained in  $I$ , which is impossible, since  $I$  contains neither 1 nor  $n$ . ◀

► **Theorem 12.** *For every even  $k = 2t$  and every  $n$  large enough,  $H_t(n) \in \mathcal{D}_k^\wedge \setminus \mathcal{D}_{k-1}^H$ .*

*Proof.* We already noted that  $H_t(n)$  has a  $\mathcal{D}_k^\wedge$ -representation for any  $n$ . For contradiction, assume that  $H_t(n)$  also has a  $\mathcal{D}_{k-1}^H$ -representation for  $n$  arbitrarily large.

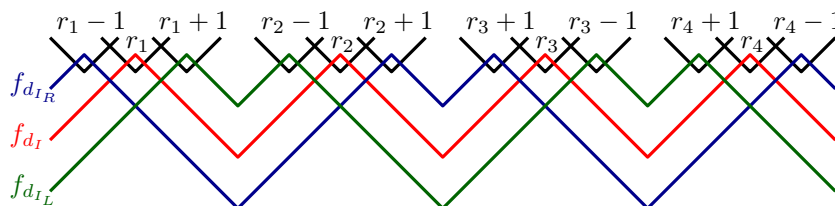
It follows that for arbitrarily large  $n$ , the graph  $H_t(n)$  has a  $\mathcal{D}_{k-1}^H$ -representation with all the properties listed above. We say that a peak  $p$  of a function representing  $d \in D$  *collects* an element  $c \in C$  if it lies above one of the valleys of  $f_c$ . Consider the  $t$ -subset  $I = \{i_1, \dots, i_t\} \subseteq C, 0 \leq i_1 - 4 \leq i_2 - 8 \leq \dots \leq i_t - 4t$ . Since  $I$  does not contain two consecutive integers, nor both  $n$  and 1, every peak of  $f_{d_I}$  collects a different element of  $I$

by Lemma 11. Let  $(r_1, \dots, r_t) \in C^t$  be the order of the elements of  $I$  given by the order of  $x$ -coordinates of the peaks of  $f_{d_I}$  that collect them. By Property 8 for each  $i \in [t]$ , we can set

$$\sigma_i := \begin{cases} 1, & \text{if } I_i \text{ has the same order as } I_1 \\ -1, & \text{if } I_i \text{ has the reversed order from } I_1 \end{cases}$$

$$I_L := \{r_i - \sigma_{c_{r_i}}(-1)^i \mid i \in [t]\}$$

$$I_R := \{r_i + \sigma_{c_{r_i}}(-1)^i \mid i \in [t]\}$$



■ **Figure 7**  $I$ ,  $I_L$  and  $I_R$  depicted by the corresponding collecting functions. The figure depicts a situation when the intervals containing  $r_1$  and  $r_2$  are increasing, while the intervals containing  $r_3$  and  $r_4$  are decreasing.

We will proceed to show that  $\forall x \in [0, 1] : f_{d_I}(x) > \min_{i \in \{L, R\}} f_{d_{I_i}}(x)$  which is a contradiction since  $E$  contains an element that is adjacent to  $d_I$  but neither  $d_{I_L}$  nor  $d_{I_R}$ .

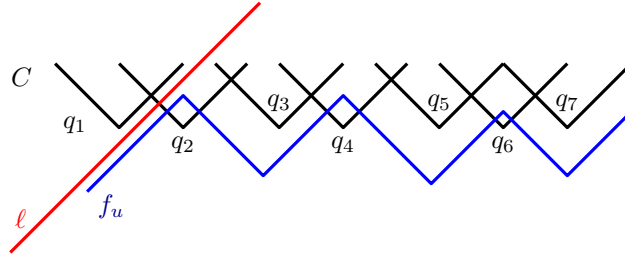
To see this, first note that neither of  $I_L$  and  $I_R$  contain a pair of consecutive numbers, nor do they contain  $n$  and further their elements  $r_i \pm 1$  are in the same order as the  $r_i$ , since  $|r_i - r_j| \geq 4 \forall i \neq j \in [t]$ . Therefore  $r_i \pm 1$  is in the same interval as  $r_i$  by Property 9. The peak of  $f_{d_I}$  collecting  $r_i$  is therefore always in between the peaks of  $f_{d_{I_L}}$  and  $f_{d_{I_R}}$  collecting  $r_i \pm \sigma_{c_{r_i}}$ . Therefore  $f_{d_{I_L}}$  is below  $f_{d_I}$  between the peaks collecting  $r_i$  and  $r_{i+1}$  if  $i$  is even, because in this case the valley of  $r_i - \sigma_{c_{r_i}}(-1)^i$  is to the left of the valley of  $r_i$  and the valley of  $r_{i+1} - \sigma_{c_{r_{i+1}}}(-1)^{i+1}$  is to the right of the valley of  $r_{i+1}$ . The argumentation is the same for  $f_{d_{I_R}}$  if  $i$  is odd. Finally the first segment of  $f_{d_I}$  is above the function  $f_{d_{I_L}}$  because the valley where  $r_1 + \sigma_{c_{r_1}}$  is collected is the right neighbor of the valley of  $r_1$ . Therefore the function of  $d_I$  is lower than either of the functions representing  $d_{I_R}$  and  $d_{I_L}$  at any point, which completes the contradiction. ◀

For our next theorem, we exploit the properties of the  $\mathfrak{D}$ -representations of  $H_t(n)$  in a different way.

► **Lemma 13.** *Let  $k = 2t$  be even. For any  $\mathfrak{D}_k^\wedge$ -representation  $\Gamma$  of  $H_t(n)$  with  $n$  large enough, there is a line  $\ell$  of slope 1 such that  $\Gamma$  contains a function fully above  $\ell$  as well as a function fully below  $\ell$ .*

**Proof.** Recall that our proof of Theorem 12 has in fact shown that in a representation of  $H_t(n)$ , there will be a vertex  $v_I \in D$  and two other vertices  $v_{I_L}$  and  $v_{I_R}$  with the property that each of the  $t$  peaks of the curve of  $v_I$  is needed to collect a distinct valley of a curve from  $C$ , while each of the  $t - 1$  valleys of  $v_I$  between these  $t$  peaks is above a valley of  $v_{I_L}$  or  $v_{I_R}$ . In this situation, we say that  $v_I$  is *protected* by  $v_{I_L}$  and  $v_{I_R}$ . In a  $\mathfrak{D}_{k-1}^H$ -representation, such a protected vertex yielded a contradiction, since it could not have been crossed by any of its neighbors in  $E$ . In a  $\mathfrak{D}_k^\wedge$ -representation, this merely means that any  $E$ -neighbor of a protected vertex must collect its rightmost valley.

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■ **Figure 8** Construction of  $\ell$  in Lemma 13.

Suppose that we have chosen  $n$  large enough to be able to construct  $2t + 1$  protected vertices  $w_1, w_2, \dots, w_{2t+1}$  in  $D$ , and let  $q_i$  be the rightmost valley in the representation of  $w_i$ . Note that the points  $q_1, \dots, q_{2t+1}$  must be independent, since for each of them, there is a neighbor in  $E$  that must collect it while avoiding all the others. Suppose that  $q_1, \dots, q_{2t+1}$  were numbered left-to-right (see Figure 8). Consider now a vertex  $u$  of  $E$  adjacent to  $w_2, w_4, \dots, w_{2t}$ . Each peak in the representation of  $u$  must collect a different valley among  $q_2, \dots, q_{2t}$ , since a single peak collecting more than one of these points would also collect an odd-numbered valley between them.

Let  $\ell$  now be a line of slope  $+1$  chosen directly above the leftmost segment of  $u$ , so that the valley  $q_1$  is above it. It follows that  $f_u$  is below  $\ell$ , while the entire representation of  $w_1$  is above it, since  $q_1$  is the rightmost bend of the representation of  $w_1$ . ◀

► **Lemma 14.** *Let  $k \geq 2$  be even. Then a graph  $G$  belongs to  $\mathcal{D}_k^-$  if and only if  $G$  is the complete join of two graphs  $G_1$  and  $G_2$  from  $\mathcal{D}_k^+$ .*

**Proof.** Let  $G \in \mathcal{D}_k^-$ . Then the functions where the leftmost segment goes upward induce a graph  $G_1 \in \mathcal{D}_k^+$  and the functions where the leftmost segment goes downward induce a graph  $G_2 \in \mathcal{D}_k^+$  (where the representation is obtained by mirroring at a horizontal line). Note that each function for  $G_1$  crosses each function for  $G_2$  since  $k$  is even. Conversely, given  $G_1, G_2 \in \mathcal{D}_k^+$  we obtain the representation of the corresponding graph  $G$  by combining the representation of  $G_1$  and the mirrored representation of  $G_2$  (after adding bends such that each function has  $k$  bends). ◀

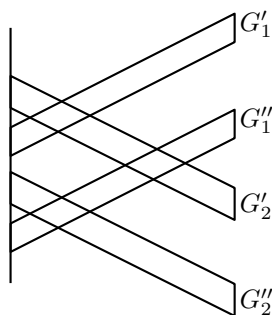
► **Theorem 15.** *For  $k$  even, there is a graph  $G \in \mathcal{D}_k^- \setminus \mathcal{D}_k^+$ .*

**Proof.** We will show that the conclusion of the theorem holds for the graph  $G$  which is the join of two copies of  $H_t(n)$ , with  $t = \frac{k}{2}$ . Since we already know that  $H_t(n)$  is in  $\mathcal{D}_k^+$  by Theorem 12, Lemma 14 shows that  $G$  is in  $\mathcal{D}_k^-$ . We now prove that  $G$  is not in  $\mathcal{D}_k^+$ .

Suppose that we are given a  $\mathcal{D}_k^+$ -representation  $\Phi$  of  $G$ . Let  $G_1$  and  $G_2$  denote the two copies of  $H_t(n)$  whose join is  $G$ . By Lemma 13, for each  $G_i$  there is a line  $\ell_i$  of slope  $+1$  such that in  $\Phi$  there are curves representing vertices of  $G_i$  both fully above and fully below  $\ell_i$ . Suppose without loss of generality that  $\ell_1$  is above  $\ell_2$ . Then  $G_1$  has a vertex  $v_1$  represented by a curve  $f_1$  above  $\ell_1$ , and  $G_2$  has a vertex  $v_2$  represented by a curve  $f_2$  below  $\ell_2$ . In particular,  $f_1$  and  $f_2$  are disjoint, contradicting the fact that  $v_1$  and  $v_2$  are adjacent in  $G$ . ◀

► **Theorem 16.** *For  $k \geq 2$  even, if  $G$  is an arbitrary graph in  $\mathcal{D}_k^- \setminus \mathcal{D}_k^+$ , then the following holds.*

1.  $G + K_1 \in \mathcal{D}_k^{\leq} \setminus \mathcal{D}_k^-$ .
2.  $G + G \in \mathcal{D}_k^{\text{H}} \setminus \mathcal{D}_k^+$ .



■ **Figure 9** A  $\mathfrak{D}_k^+$ -representation of  $H$  from the third part of Theorem 16. Representations of  $G'_1, G'_2, G''_1, G''_2$  can be placed in corresponding infinite parallelograms.

3. Let us decompose  $G$  into  $G_1$  and  $G_2$  as in Lemma 14, then take two copies  $G'_1, G''_1$  of  $G_1$  and two copies  $G'_2$  and  $G''_2$  of  $G_2$ , and construct a graph  $H$  by adding all edges between  $G'_1$  and  $G'_2$ , between  $G'_2$  and  $G''_1$ , and between  $G''_1$  and  $G''_2$ . Then  $H \in \mathfrak{D}_k^+ \setminus \mathfrak{D}_k^<$ .

**Proof.** Fix a graph  $G \in \mathfrak{D}_k^- \setminus \mathfrak{D}_k^>$ . Since  $G \notin \mathfrak{D}_k^>$ , any  $\mathfrak{D}_k^-$ -representation  $\Gamma$  of  $G$  contains at least one function  $f_\Gamma$  whose leftmost segment has slope  $+1$  and at least one function  $g_\Gamma$ , whose leftmost segment has slope  $-1$ . Since  $k$  is even, any function starting with an increasing segment must cross any function starting with a decreasing one, in particular  $G$  is connected.

To prove the first part of the theorem, assume that  $G + K_1$  can be represented in  $\mathfrak{D}_k^-$  and fix one such representation  $\Phi$ . Let  $\Gamma' \subset \Phi$  be the set of functions in  $\Phi$  corresponding to  $G$  and let  $h$  be the function in  $\Phi$  representing the single vertex in  $K_1$ . Then, without loss of generality, the leftmost segment of  $h$  has slope  $-1$  and therefore  $h$  crosses  $f_{\Gamma'}$ , contradicting the assumption that  $\Phi$  is a representation of  $G + K_1$ . On the other hand, we can obtain a  $\mathfrak{D}_k^<$ -representation of  $G + K_1$  by placing a function consisting of two pieces with slopes  $-1$  and  $+1$  very far above  $\Gamma'$ .

The second part of the theorem is already covered by Theorem 9.

To prove the last part of the theorem, we assume that  $H$  can be represented in  $\mathfrak{D}_k^<$  with a representation  $\Phi$ . Then, let  $f \in \Phi$  be a function representing a vertex in  $G'_1$ , we can assume that the leftmost segment of  $f$  has slope  $+1$ . Further, by our choice of  $G$  and  $H$ , there is at least one other function  $g \in \Phi$ , representing a vertex of either  $G''_1$  or  $G''_2$  which has leftmost segment with slope  $-1$ . Since  $k$  is even,  $f$  and  $g$  cross each other. But this is impossible since vertices in  $G'_1$  are not adjacent to any vertices in  $G''_1$  nor to any vertices in  $G''_2$ , so  $\Phi$  cannot be a representation of  $H$ . On the other hand, in Figure 9 we explain how to obtain a  $\mathfrak{D}_k^+$ -representation of  $H$ . ◀

## 4 General setting

This section is concerned with the relationship between the classes  $\mathfrak{G}_k^-$ ,  $\mathfrak{G}_k^+$  and  $\mathfrak{G}_k^H$ . The general setting allows for arbitrary slopes, which makes more graphs representable. In fact, you need about half as many bends for similar graphs. However it also turns the investigation quite difficult. Surprisingly, the bounded and grounded representation model do not produce different graph classes though.

► **Theorem 17.** For any  $k \in \mathbb{N}_0$ ,  $\mathfrak{G}_k^+ = \mathfrak{G}_k^H$ .

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**Proof.** Since any number of functions defined on the positive reals can be restricted to a large enough interval without missing any crossings or bends,  $\mathfrak{G}_k^+ \subseteq \mathfrak{G}_k^H$ .

For the other inclusion, let  $G \in \mathfrak{G}_k^H$ . We use a projective transformation that transforms a  $\mathfrak{G}_k^H$  representation of  $G$  to a  $\mathfrak{G}_k^+$  representation of  $G$  as follows. Suppose the  $\mathfrak{G}_k^H$ -representation is in the *source plane* embedded in  $\mathbb{R}^3$  as the plane  $\{(x, y, 0)\}$  with the lines  $x = 0$  and  $x = 1$  bounding the representation with domain  $(0, 1]$ . We fix the *perspective point*  $P = (0, 0, 1)$  above the source plane, and fix the target plane  $\{(-1, y, z)\}$ . We now map each point from the source plane to the target plane by projecting it via a line passing through  $P$ . Explicitly, this maps a point  $(x, y, 0)$  to a point  $(-1, -y/x, 1 + 1/x)$ . From the construction, one can verify that this preserves collinearity and changes  $x$ -monotone curves in the strip  $0 < x \leq 1$  into  $z$ -monotone ones in the half-plane in the target plane with  $2 \leq z < +\infty$ . Similarly, each  $x$ -monotone line-segment with an open end in line  $x = 0$  is transformed into a  $z$ -monotone ray with arbitrarily large  $z$ -values. On the other hand, observe that each  $x$ -monotone line-segment that does not end in line  $x = 0$  is transformed to another  $z$ -monotone line-segment. Thus, we indeed obtain a  $\mathfrak{G}_k^+$  representation of the same graph.  $\blacktriangleleft$

As in the diagonal setting, we can observe that the remaining classes form a hierarchy.

► **Theorem 18.** *For any  $k \in \mathbb{N}_0$ , we have  $\mathfrak{G}_k^- \subseteq \mathfrak{G}_k^H \subseteq \mathfrak{G}_{k+1}^-$ .*

**Proof.** The first inclusion follows from Proposition 1. For the other inclusion, let  $G \in \mathfrak{G}_k^H$ . By Theorem 17 we can fix some representation of  $G$  in  $\mathfrak{G}_k^+$ . We now extend all the functions of  $G$  to  $\mathbb{R}$  by another constant segment on the negative reals in a continuous way. This is a representation of  $G \in \mathfrak{G}_{k+1}^-$ , since we added at most one bend and no functions intersect on the negative reals.  $\blacktriangleleft$

For very small values of  $k$ , we can see that there are strict inclusions between  $\mathfrak{G}_k^-$  and  $\mathfrak{G}_k^H$ . In particular, it is easy to see that the class  $\mathfrak{G}_0^-$  is just the class of multipartite graphs and is therefore strictly included in  $\mathfrak{G}_0^H$ , which is exactly the class of permutation graphs [22]. We cannot exactly characterize the graphs in  $\mathfrak{G}_1^-$ , but we can show that for every  $n \in \mathbb{N}$ , the cocomparability graph of the *standard example*  $S_n$  [24] admits a  $\mathfrak{G}_1^-$ -representation (see Figure 3), and we can therefore conclude that  $\mathfrak{G}_0^H \subsetneq \mathfrak{G}_1^-$ . We remark that we believe that it is possible to prove the strict inclusions  $\mathfrak{G}_1^- \subsetneq \mathfrak{G}_1^H \subsetneq \mathfrak{G}_2^- \subsetneq \mathfrak{G}_2^H$ , but the argument is tedious and the graphs witnessing the strictness of the inclusions do not generalize to higher values of  $k$ . Therefore we do not include these results. Our next aim is to prove that  $\mathfrak{G}_k^H$  is a strict subclass of  $\mathfrak{G}_{k+1}^-$  for all  $k \geq 2$ .

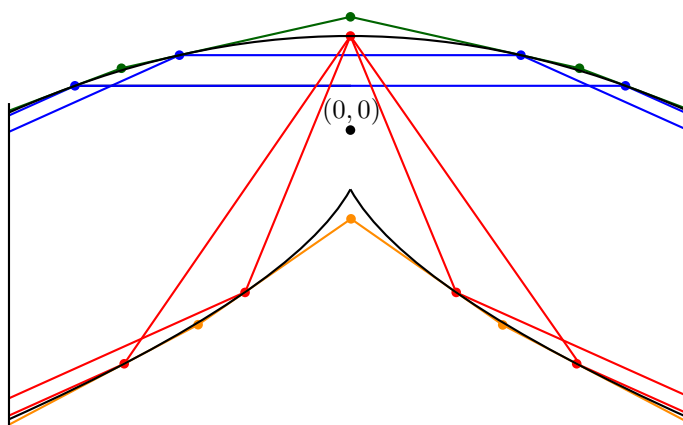
The graph  $G_t \equiv G_t(n)$  consists of three cliques  $A, B \cup C, D$ , where  $B$  and  $C$  have size  $n$  while  $A$  and  $D$  are  $t$ -extension of  $B$  and  $C$ , respectively. There are no edges in  $G_t$  other than those implied by the previous rules.

► **Theorem 19.** *For any  $t \geq 4$  and  $n$  large enough, the graph  $G_t(n)$  is in  $\mathfrak{G}_{t-1}^- \setminus \mathfrak{G}_{t-2}^H$ .*

**Proof.** The proof that  $G_t(n) \in \mathfrak{G}_{t-1}^-$  can be found in the full version of our paper, see Figure 10 for illustration.

Toward a contradiction, assume that  $G_t \in \mathfrak{G}_{t-2}^H$  and fix a representation. Let  $a \in A, d \in D$ . Since the corresponding functions  $f_a$  and  $f_d$  do not intersect, without loss of generality, we may assume that  $f_a \geq f_d$ . Since  $A$  and  $D$  are nonadjacent,  $f_a \geq f_d$  holds not only for these specific functions, but for all  $a \in A, d \in D$ .

We will delete elements of  $B$  and all elements of  $A$  adjacent to it and an equivalent amount of elements from  $C$  and  $D$  in the same manner similarly to Theorem 12. The goal is for every  $n \in \mathbb{N}$  to find  $N \in \mathbb{N}$  in order to arrive at a representation of a  $G_t(n)$  subgraph of



■ **Figure 10** Representation of  $G_4$  with only two functions in  $B$  (blue) and  $C$  (red) that are touched twice in both (non-middle) bends by one function each in  $A$  (green) and  $D$  (orange). In a proper representation, different functions of  $B$  or  $C$  would be touched by the functions of  $A$  and  $D$  in the positive and negative reals.

$G_t(N)$  that fulfills some property, then prove that such a representation cannot exist. In order to state the property, we need to define the so-called first level of an arrangement of functions, which is the graph of the second highest function at every point, formally  $L_1(f_1, \dots, f_m)(x) = \max \{y \in \mathbb{R} : |\{i \in [m] : f_i(x) \geq y\}| = 2\}$

▷ **Property 10.** No function in  $A$  crosses the first level of the functions in  $B$ .

*Proof.* Agarwal, Aronov, Chan, and Sharir [2] prove that the first level of a set of  $s$  segments has complexity  $\mathcal{O}(s \alpha(s))$ , where  $\alpha$  is the inverse Ackermann function and complexity means the number of intersection points plus discontinuities. Since  $B$  contains only continuous functions, there are no discontinuities. The number of segments of some subset of  $k$  functions of  $B$  is  $(t-1)k$ . Therefore, the number of intersection points and bends along the first level is in  $\mathcal{O}(tk \alpha(tk))$  and so is the number of segments between them.

We define a graph  $H$  with vertex set  $B$  where we add an edge between  $b_i$  and  $b_j$  if there exists  $x \in [0, 1]$ , such that  $f_{b_i}(x)$  and  $f_{b_j}(x)$  are the two largest values among  $\{f_b(x) \mid b \in B'\}$ . In this case one of these functions contributes a segment to the first level and all  $x$ -values of this segment qualify as witnesses to the existence of the edge  $b_i b_j$ . Thus, the number of edges of  $H$  is  $\mathcal{O}(tn \alpha(tn))$  so its average degree is in  $\mathcal{O}(t \alpha(tn))$ . By Turán's Theorem  $H$  contains an independent set of size  $\Omega\left(\frac{n}{t \alpha(tn)}\right)$ . Restricting the set  $B$  to this independent set, the property follows, since if any function in  $A$  crosses the first level, it drops below two or more functions of  $B$ , which consequently have an edge in  $H$ , a contradiction. ◁

We are particularly interested in the upper envelope or maximum of all functions  $f_b$  representing  $b \in B$ . A *chunk* of the upper envelope is a connected subset of it that is contained in the graph of a single function, i.e. a connected subset of  $\{(x, f_b(x)) \in \mathbb{R}^2 \mid f_b(x) = \max_{b' \in B} f_{b'}(x)\}$ . A *chunk set* is a set of  $t$  chunks. A chunk set is *crossable*, if there is a function  $f_a$  representing some vertex  $a \in A$  that intersects all of its chunks. A set of functions  $\{f_{b_1}, \dots, f_{b_t}\}$  is *crossable* if there exists a crossable chunk set  $\{c_1, \dots, c_t\}$ , such that for all  $i \in [t]$ ,  $c_i$  is part of  $f_{b_i}$ . By the definition of  $A$  and Property 1, every set of  $t$  functions is crossable, so for contradiction, we are going to show that this is not true. We will even be able to show that the number of crossable chunk sets is less than  $\binom{n}{t}$ .

## 10:20 The Bend Number of Cocomparability Graphs

To do so, we group the chunk sets using a parameter  $\ell = \lfloor \sqrt{c} \rfloor$  where  $c$  is the number of chunks. It is a well-known fact that the upper envelope of  $s$  segments has complexity at most  $\mathcal{O}(s \alpha(s))$ ; see for instance Sharir and Agarwal [23], chapter 2. So the total number of chunks is  $c \in \mathcal{O}(nt \alpha(nt))$ . We can therefore identify the chunks with numbers in  $[c]$  and the chunk sets with sets in  $\binom{[c]}{t}$ . An *early* chunk set includes a number  $< \ell$ . A *short* chunk set includes two numbers of difference  $< \ell$ . A *group* is a set of disjoint chunk sets of the form  $\{\{i_1 - j, \dots, i_{t-1} - j, c - k\ell - j\} \mid 0 \leq j < \ell\}$  for some  $k \in \mathbb{N}_0$ . Note that the *maximum set*  $\{i_1, \dots, i_{t-1}, c - k\ell\}$  has maximum  $c - k\ell$  and is neither short nor early. Further note that each chunk set is short, early or in a group. Two distinct chunk sets contained in the same group are not only disjoint but alternating, that is, if the smallest number of them is in one of the sets, then the second smallest is in the other and so on. We will use this fact and the following observation in the subsequent claim:

► **Observation 20.** *Two piecewise linear functions with  $t - 2$  bends cross  $\leq 2t - 3$  times.*

Proof. They define at most  $2t - 3$  intervals between the  $2t - 4$  bends (and on either end), where both functions are (affine) linear, thus only cross once. ◀

▷ **Claim 21.** *At most  $4t^2$  of the chunk sets in a group are crossable.*

Proof. Consider the subset  $S = \{s_1, \dots, s_k\}$  of crossable chunk sets of a group and fix  $i, j \in [k]$  as well as functions  $f_1, \dots, f_k$  that witness the crossability of  $s_1, \dots, s_k$ . If the sets of chunks intersected by  $f_i$  and  $f_j$  are disjoint, observe that between their intersections with any two chunks  $c_i \in s_i, c_j \in s_j$ ,  $f_i$  and  $f_j$  have to cross by the intermediate value theorem. Since the chunks alternate, there are at least  $2t - 1$  such crossings, a contradiction to the above observation. Therefore a chunk in  $s_i$  is crossed by  $f_j$  or vice-versa. We construct a tournament on  $S$ , such that  $(s_i, s_j)$  is an arc in the former case and  $(s_j, s_i)$  is an arc otherwise. However we can also deduce from the observation that any function in  $A$  crosses at most  $t \cdot (2t - 3) \leq 2t^2 - 1$  chunks, because a function in  $A$  only crosses  $t$  functions of  $B$ , each of which is crossed at most  $2t - 3$  times. For the tournament on  $S$ , this means that the indegree of any vertex is at most  $2t^2 - 1$ , so the number of edges is  $\binom{k}{2} \leq (2t^2 - 1)k \Rightarrow k \leq 4t^2$ . ◀

Using the above claim, we can now prove that the number of crossable chunk sets is too small. Recall that each chunk set is early, short or in a group. For the early chunk sets, we have  $\leq \ell$  possibilities to choose the smallest number and  $\leq \binom{c}{t-1}$  possibilities to choose the rest. For the short chunk sets, we have  $\leq \ell c$  possibilities to place two numbers of distance  $< \ell$  and  $\leq \binom{c}{t-2}$  possibilities to choose all other numbers. Observe that there are  $\leq \binom{c}{t}/\ell$  groups by the restriction of the maximum to some  $c - \ell k$ . It follows that the number of early chunk sets  $N_E$  is at most  $\ell \binom{c}{t-1} \leq \ell c^{t-1}$  and the number of short chunk sets  $N_S$  is at most  $\ell c \binom{c}{t-2} \leq \ell c^{t-1}$ . For the total number of crossable chunk sets  $N_c$  we get the upper bound

$$N_c \leq N_E + N_S + \frac{4t^2}{\ell} \binom{c}{t} \leq 2\ell c^{t-1} + \frac{4t^2 c^t}{\ell} \in \Theta(t^2 c^{t-1/2}) \subseteq \mathcal{O}(t^2 (nt \alpha(nt))^{t-1/2}) \subseteq o_t(n^t). \blacktriangleleft$$

## 5 Concluding remarks and open problems

Our investigation of the bend number and the related graph classes leads to several open questions. We highlight the ones that we consider the most interesting.

We first present some combinatorial problems. We introduced the graph classes  $\mathfrak{G}_k^-$  and  $\mathfrak{G}_k^H$ , and while we can prove that  $\mathfrak{G}_k^- \subseteq \mathfrak{G}_k^H$ , we do not know whether these inclusions are strict:

► **Conjecture 22.** For any  $k \in \mathbb{N}_0$ ,  $\mathfrak{G}_k^-$  is a proper subset of  $\mathfrak{G}_k^H$ .

For  $k = 0$ , the conjecture holds, since  $\mathfrak{G}_0^-$  is the class of complete multipartite graphs, while  $\mathfrak{G}_0^H$  is the class of permutation graphs. For  $k = 1$  and  $k = 2$  we can also verify that the conjecture holds: in fact, using the graphs  $G_t(n)$  used in the proof of Theorem 19, we are able to verify that  $G_2(n) + G_2(n)$  is in  $\mathfrak{G}_1^H \setminus \mathfrak{G}_1^-$ , while  $G_3(n) + G_3(n)$  is in  $\mathfrak{G}_2^H \setminus \mathfrak{G}_2^-$ . Theorem 19 shows this does not straightforwardly generalize to higher  $k$ . Verifying Conjecture 22 would complete our understanding of the hierarchy induced by the classes  $\mathfrak{G}_k^-$  and  $\mathfrak{G}_k^H$ .

Another natural question is to determine the maximum number of bends needed to represent every poset of size  $n$ . In the diagonal setting, by rotating the representation by 45 degrees, we can view it as a VPG-representation with at most  $(k + 2)n$  bends and endpoints. Since rows and columns without these points are unnecessary the whole arrangement of curves can be embedded to an integer grid of size  $\mathcal{O}(kn) \times \mathcal{O}(kn)$ , thus encoded using  $\mathcal{O}(kn \log(kn))$  bits. Hence there are no more than  $2^{\mathcal{O}(kn \log(kn))}$  cocomparability graphs in  $\mathfrak{D}_k^H$ . Since there are  $2^{\Omega(n^2)}$   $n$ -vertex cocomparability graphs, some of them must require  $\Omega\left(\frac{n}{\log n}\right)$  bends per curve in a diagonal representation. We do not know if this bound is tight, and we are not aware of a similar lower bound for  $\mathfrak{G}_k^H$ -representations. However enumeration of intersection graphs has been investigated for various classes of intersection graphs [1, 10, 12, 15, 18, 19, 26, 27], and in particular for some natural subclasses of cocomparability graphs, such as permutation graphs and interval graphs [1, 12, 18, 26, 27]. Since our classes naturally refine the hierarchy of subclasses of cocomparability graphs, determining their size could be an interesting problem.

Natural questions also arise on the complexity-theoretic side, mainly determining if any of our classes can be recognized in polynomial time may be interesting. It can be shown that the recognition problem for “diagonal” classes  $\mathfrak{D}_k^>, \mathfrak{D}_k^=, \mathfrak{D}_k^<, \mathfrak{D}_k^+, \mathfrak{D}_k^H$  is in NP. However, for the classes  $\mathfrak{G}_k^-$  and  $\mathfrak{G}_k^H$ , even membership in NP is not clear. Note that both  $\mathfrak{D}_1^>$  and  $\mathfrak{G}_0^H$  are equal to the class of permutation graphs, which are known to be recognisable in polynomial time. We expect that recognition will turn out to be NP-hard for classes with higher bend-number, but it may be interesting to pin down the boundary of tractable recognition.

Lastly, there are several well-studied computational problems that remain NP-hard when restricted to cocomparability graphs but can be solved efficiently for interval or permutation graphs [5, 13, 20, 25]. It would be interesting to determine where exactly these problems become NP-hard, and our classes give a natural setting to investigate this.

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