

Flipping Odd Matchings in Geometric and Combinatorial Settings

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Abstract

We study the problem of reconfiguring *odd matchings*, that is, matchings that cover all but a single vertex. Our reconfiguration operation is a so-called *flip* where the unmatched vertex of the first matching gets matched, while consequently another vertex becomes unmatched. We consider two distinct settings: the *geometric setting*, in which the vertices are points embedded in the plane and all occurring odd matchings are crossing-free, and a *combinatorial setting*, in which we consider odd matchings in general graphs.

For the latter setting, we provide a complete polynomial time checkable characterization of graphs in which any two odd matchings can be reconfigured into each another. This complements the previously known result that the flip graph is always connected in the geometric setting [2]. In the combinatorial setting, we prove that the diameter of the flip graph, if connected, is linear in the number of vertices. Furthermore, we establish that deciding whether there exists a flip sequence of length k transforming one given matching into another is NP-complete in both the combinatorial and the geometric settings. To prove the latter, we introduce a framework that allows us to transform partial order types into general position with only polynomial overhead. Finally, we demonstrate that when parameterized by the flip distance k , the problem is fixed-parameter tractable (FPT) in the geometric setting when restricted to convex point sets.

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1 Introduction

In many areas of discrete mathematics and theoretical computer science, one is interested not only in finding individual solutions to combinatorial problems, but also in understanding how these solutions relate to each other – specifically, whether it is possible to gradually transform one solution into another through a sequence of small, valid changes (so-called *flips*), while maintaining feasibility at each step. This framework is known as *reconfiguration*. It provides insight into the structure of the solution space and has applications in various areas. For example in optimization [21], as every optimization problem can be reformulated as a reconfiguration problem by introducing an appropriate threshold on the objective functions [18], as well as combinatorial enumeration and robustness [4, 22]. Moreover, many classical results in combinatorics naturally fall within this framework, e.g., the 5-color theorem [13] or Vizing’s theorem [3, 27]. We also refer to the following surveys [23, 26]. A prominent example is the reconfiguration of triangulations of a convex n -gon; since each binary tree corresponds to the geometric dual of such a triangulation, this problem is equivalent to performing rotations on binary trees [17, 25].

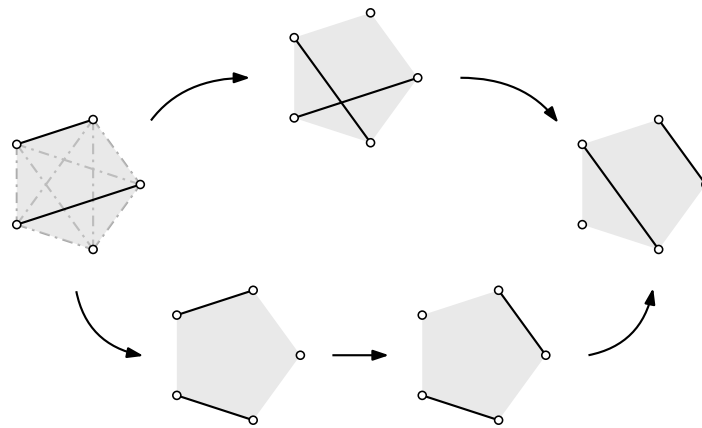
This leads to structural and algorithmic questions: (1) Can any structure of a certain class be transformed into any other using the given flip operation? (2) What is the worst-case number of flips required to perform such a transformation? (3) Given two specific structures, how can we compute the minimum number of flips needed to reconfigure one into the other? These questions can also be rephrased in terms of the flip graph, where vertices represent the structures and edges correspond to flips: (1) Is the flip graph *connected*? (2) What is the *diameter* of the flip graph? (3) What is the computational *complexity* of finding shortest paths between vertices in the flip graph?

A central example is the reconfiguration of *matchings* in graphs, particularly *perfect matchings*, which are sets of pairwise non-adjacent edges covering every vertex. In this setting, a small local change corresponds to removing a set of edges and inserting another along an alternating cycle, while maintaining a valid matching throughout.

We study the reconfiguration of *odd matchings*, which are matchings that cover all but one vertex in a graph. This variant arises naturally in graphs without perfect matchings due to parity constraints, or in point sets of odd cardinality. In such cases, we refer to the graph or point set as being of odd *order*, where the order is the number of vertices or points. Each flip adds an edge between the unmatched vertex (which we also refer to as *isolated*) and another vertex v and removes the matching edge vw that was previously incident to v . This way, w becomes the new isolated vertex. We consider two distinct settings: In the *geometric setting*, the vertices are points in the plane, and matchings consist of straight lines between points that do not share endpoints and that do not cross. Each such *geometric flip* yields a (plane) odd matching. In the *combinatorial setting*, we consider odd matchings in an input graph G , and a *combinatorial flip* consists of removing and adding an edge of G such that the result is again an odd matching. For an illustration of both settings, see Figure 1.

Problem description

We study the problem FLIPPING ODD MATCHINGS, which concerns reconfiguring one odd matching into another through a sequence of valid flip operations. We examine the problem in both the geometric and the combinatorial setting with their corresponding flip operations. The associated decision problem asks whether, given two odd matchings M and M' and an integer $k \in \mathbb{N}$, there exists a sequence of at most k flip operations that transforms M into M' within the respective setting.



■ **Figure 1** This figure illustrates the two distinct settings and their corresponding flip types. The odd matching on the left must be reconfigured into the odd matching on the right. The clockwise flip sequence represents a combinatorial reconfiguration, while the counterclockwise sequence is geometric. Dotted-dashed lines in the initial matching indicate additional edges present in the graph.

Our contribution

In this paper, we present a number of novel results on reconfiguring odd matchings in geometric and combinatorial settings. In particular, we provide a complete characterization of graphs in which each two odd matchings can be flipped into each other. Moreover, if the corresponding (combinatorial) flip graph is connected, its diameter is linear in the number of vertices of the input graph. Furthermore, we show that the decision variant of **FLIPPING ODD MATCHINGS** is **NP**-complete in both considered settings, but fixed-parameter tractable in the case of convex point sets when parameterized in the flip distance.

A detailed summary of our results is given in Table 1.

■ **Table 1** Current status of reconfiguring odd matchings; new results are highlighted in light blue.

	Setting	Connectedness	Diameter	Complexity
Geometric	Convex	Yes [2]	at most $3n/2 - \mathcal{O}(1)$, Corollary 16 at least $n - 2$ [2]	FPT, Theorem 13
	General position	Yes [2]	$\Omega(n)$ and $\mathcal{O}(n^2)$ [2]	NP-complete, Theorem 18
Combinatorial	Rectangular grids	Yes, Remark 5	$\Theta(n)$, Theorem 10	NP-complete, Theorem 23
	Planar graphs	Characterization, Theorem 3	if connected: $\Theta(n)$, Theorem 10	NP-complete, Theorem 23
	General graphs	Characterization, Theorem 3	if connected: $\Theta(n)$, Theorem 10	NP-complete, Theorem 23

Due to space constraints, full technical details and proofs for statements marked with (★) are given in the full version of our paper [1].

Related work

In the geometric setting, it was recently shown in [2] that for a set of n points in general position in the plane, the flip graph of odd matchings is connected, with diameter bounded between $\Omega(n)$ and $\mathcal{O}(n^2)$. In contrast, for perfect matchings in the geometric setting, the connectedness of the flip graph remains an open question when flips are restricted to alternating cycles of sublinear length. However, if no such restriction is imposed, the flip graph is known to be connected [16]. Recent work shows that finding shortest flip sequences is NP-hard for point sets in general position, when the flip size is restricted to 4-cycles [6].

In the combinatorial setting, the connectivity of the flip graph for perfect matchings has been studied in [7]. The authors showed that the problem is PSPACE-complete for several graph classes, including split graphs and bipartite graphs of bounded bandwidth with maximum degree five. In contrast, they proved that it can be solved in polynomial time for strongly orderable graphs, outerplanar graphs, and cographs. In these positive cases, they also provided flip sequences of linear length.

Odd matchings have also been studied in the combinatorial setting, particularly in the special case of the sliding block puzzle known as *Gourds* [11, 19]. Here, the underlying graph is a triangular grid graph, where matching edges are assigned distinct colors, and in each flip, the added edge inherits the color of the removed edge. The connectedness of the flip graph is fully characterized when the underlying graph is a solid triangular grid graph [11]. For triangular grid graphs with holes, sufficient conditions for connectedness are known, but a complete characterization remains open [19].

We refer to [8] for a survey of additional results and open questions on the reconfiguration of planar graphs in both the combinatorial and geometric settings. Reconfiguration in general is also closely linked to *Gray codes* [22], which aim to list all feasible configurations in an order such that each configuration differs from the previous one only by a single flip. Such an ordering corresponds to identifying a Hamiltonian cycle in the flip graph, providing a compact and systematic traversal of the solution space [14, 15, 17, 24].

2 Preliminaries and basic observations

Geometric and combinatorial settings. We consider odd matchings in two different settings. In the *geometric setting*, let S be a set of n points embedded in the plane. A line segment between two points of S is an *edge*. A *matching* on S is a set of edges whose endpoints are pairwise distinct, said to be *plane* if no two edges cross. An *odd matching* M on S is a plane matching on S where exactly one point in S is not an endpoint of any edge of M . A *geometric flip* in an odd matching removes one edge and inserts another one so that the result remains an plane matching. In essence, an odd matching on a point set S is simply a plane odd matching on the complete graph with vertex set S .

In the *combinatorial setting*, we consider odd matchings on a fixed given graph G . A *matching* M in a simple graph $G = (V, E)$ is a set of pairwise disjoint edges, that is, no two edges in M share a common vertex. It is called *odd* if exactly one vertex of G is not incident to an edge in M . A *combinatorial flip* in an odd matching M removes one edge and replaces it with another from G , preserving the property of being an odd matching. In both settings, the new edge has to be incident to the isolated vertex of the previous odd matching.

The *flip graph* of a graph G (or point set S) has as its vertices the odd matchings of G (resp., S), with edges connecting odd matchings that differ in a single flip. We say that an odd matching M can be *reconfigured* to another odd matching M' if there exists a flip sequence transforming M into M' , i.e., there is a path between them in the flip graph.

A graph or point set is said to be *reconfigurable* if its flip graph is connected, meaning any two odd matchings can be reconfigured into one another. The *flip distance* $d(M, M')$ between two matchings M and M' is the length of a shortest path between them in the flip graph. The *diameter* of the flip graph is the maximum flip distance over all pairs of odd matchings. Its *radius* is defined as $\min_M \max_{M'} d(M, M')$, and this minimum is attained by odd matchings that minimize their farthest distance to any other odd matching. Note that the diameter is at least the radius and at most twice the radius.

Union and symmetric difference of matchings in both settings. Let M and M' be odd matchings. An M -alternating path is a path that alternates edges in M and not in M ; M -alternating cycles are defined analogously. Then $M \cup M'$ is a disjoint union of

- a path of even length (possibly zero) that connects the isolated vertex in M to the isolated vertex in M' and alternates edges of M' and M ,
- cycles of even length alternating edges of M and M' , and
- edges that lie in both M and M' , which we call *happy edges*.

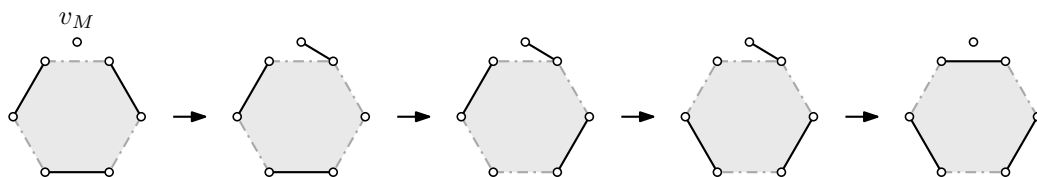
The *symmetric difference* $M \triangle M'$ is the set of edges that appear in only one of the two odd matchings. In other words, it is obtained from $M \cup M'$ by removing all happy edges.

Flipping along alternating paths. We consider the combinatorial setting. Let P be an M -alternating path from the isolated vertex v_M to some vertex v , where the last edge belongs to M . We can flip the matching along P , hence replacing $M \cap P$ by the edges in $P \setminus M$; the new isolated vertex is v . Next, we consider M -alternating paths to cycles in $M \triangle M'$.

► **Lemma 1.** *Let M and M' be odd matchings of a graph G . Suppose M can be reconfigured to M' . For every cycle C in $M \triangle M'$, there exists a vertex $v \in V(C)$ such that v has an M -alternating path to the isolated vertex v_M .*

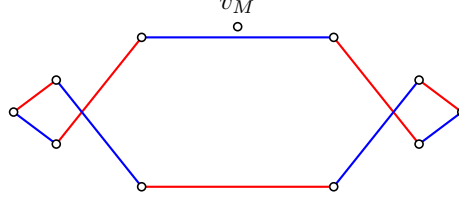
Proof. Let C be a cycle in $M \triangle M'$. The edges in $M \cap C$ are eventually flipped to reconfigure M into M' . After flipping such an edge $e = vw$, one of its endpoints, say v , becomes isolated in the corresponding intermediate matching. Consequently, $G - v$ has a perfect matching. Hence, $M \setminus \{e\}$ is not maximal in $G - v$. By Berge's theorem [5], there exists an M -alternating path from w to v_M . ◀

Consider a cycle C in $M \triangle M'$ and assume that there exists an M -alternating path P from v_M to $v \in C$. Then $M \cap C$ can be replaced by $M' \cap C$ by flipping the edges along P , around C , and then restoring the edges on P . After this, v_M is again the isolated vertex. When applying the described flipping procedure in the special case that v_M is adjacent to a vertex $v \in C$, we say that we *switch* C . If C has k edges, then it takes $k + 1$ flips to switch C . The additional flip occurs because we need to first place the isolated vertex on the cycle. This also holds for the geometric setting if C is planar; see Figure 2 for an illustration.



■ **Figure 2** How to switch an alternating cycle.

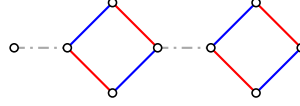
However, this does not hold in general in the geometric setting. Suppose we wish to switch the cycle illustrated in Figure 3. The only feasible flips initially remove the blue edge “closest” to v_M . However, regardless of how this edge is removed, a red edge cannot be added immediately in the next flip, as doing so would cause it to cross a remaining blue edge.



■ **Figure 3** A non-plane alternating cycle in the geometric setting.

3 Reachability

We consider the question of whether two odd matchings of a given graph can be reconfigured into each other. For the geometric setting this has been proven to be always possible [2]. In contrast, the situation in the combinatorial setting is fundamentally different, as there exist simple examples where reconfiguration is not possible, as illustrated in Figure 4.



■ **Figure 4** A connected graph with two odd matchings (red and blue, respectively) that cannot be reconfigured into each other. Specifically, the rightmost cycle cannot be reconfigured. This example can be extended for any odd n by increasing the size of the rightmost cycle.

In this section, we characterize the graphs that are reconfigurable, that is, for which the flip graph is connected. Further, we give a simple polynomial-time algorithm to decide whether a graph is reconfigurable.

► **Lemma 2.** *Let G be a graph of odd order, and consider an edge e .*

1. *Suppose that e is not contained in any odd matching of G . Then G is reconfigurable if and only if $G - e$ is reconfigurable.*
2. *Suppose that $e = uv$ is contained in every odd matching of G . Then G is reconfigurable if and only if $G - u - v$ is reconfigurable.*

Proof. If e is not contained in any odd matching of G , then clearly G is reconfigurable if and only if $G - e$ is, as e never appears in a flip sequence in G . Now suppose that $e = uv$ is contained in every odd matching of G . As no reconfiguration sequence in G ever flips e , it is easy to see that G is reconfigurable if and only if $G - u - v$ is reconfigurable. ◀

► **Theorem 3.** *Let G be a graph of odd order. Then G is reconfigurable if and only if for every edge $e = uv \in E(G)$, one of the following holds:*

1. *the edge e is either contained in all or in no odd matchings of G , or*
2. *at least one of $G - u$ and $G - v$ contains a perfect matching.*

Proof. By Lemma 2, we may assume that every edge of G is contained in some, but not all odd matchings of G . First assume that for every edge $e = uv \in E(G)$, $G - u$ or $G - v$ contains a perfect matching. Let M and M' be odd matchings in G with isolated vertices

v_M and $v_{M'}$, respectively. Flipping edges along the path in $M \triangle M'$ maps v_M to $v_{M'}$, so assume $v_M = v_{M'}$. Then $M \triangle M'$ is a collection of cycles. Let C be a cycle in $M \triangle M'$ and let $e = uv \in M$ be an edge on C . By assumption, $G - u$ or $G - v$ contains a perfect matching, say $G - u$. By applying Berge's theorem [5] to $M \setminus \{e\}$ in $G - u$, there exists an M -alternating path from v_M to v . Using this alternating path to switch C yields a matching M_1 with $|M_1 \triangle M'| < |M \triangle M'|$. Inductively, this yields a flip sequence from M to M' .

Now assume that G contains an edge $e = uv$ such that $G - u$ and $G - v$ do not have a perfect matching. In other words, u and v are covered in all odd matchings. By assumption, there exists an odd matching M_1 containing e and an odd matching M_2 not containing e . To reconfigure M_1 to M_2 , we eventually need to perform a flip involving e . However, this isolates u or v , which is a contradiction. Hence, G is not reconfigurable. ◀

► **Remark 4.** Let G be a factor-critical graph, i.e., $G - v$ has a perfect matching for every $v \in V(G)$. Clearly, G then satisfies condition (2) in Theorem 3, and hence G is reconfigurable.

► **Remark 5 (*)**. For any odd values $w, h \in \mathbb{N}$ the $w \times h$ square grid graph, i.e., the Cartesian product $P_w \square P_h$, is reconfigurable.

There is a straightforward polynomial-time algorithm to check the conditions of the characterization (Theorem 3), i.e., deciding whether a graph is reconfigurable.

► **Theorem 6 (*)**. *Given a graph G , we can determine whether the flip graph of odd matchings in G is connected in $\mathcal{O}(n^{4.5})$ time. In other words, the reconfigurability of a graph can be checked in polynomial time.*

4 Radius and diameter of the flip graph

We now study the diameter of the flip graphs in both combinatorial and geometric settings.

4.1 Linear diameter of the flip graph in general graphs

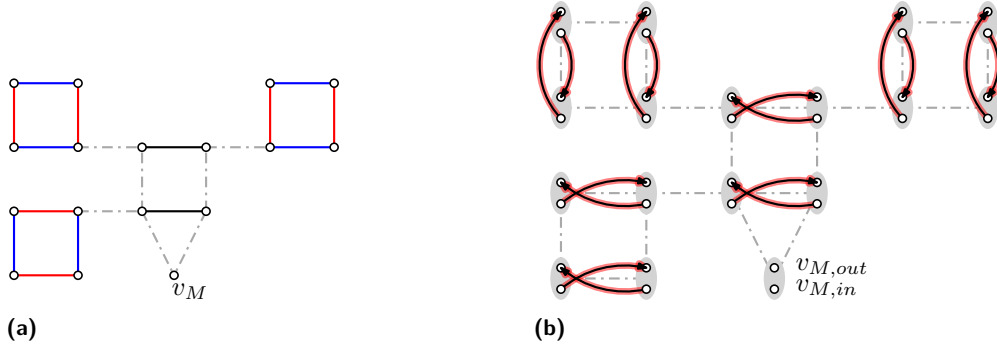
In this section, we show that the diameter of the flip graph in the combinatorial setting is linear in the order n of the input graph. Given an input odd matching M and a target matching M' in a graph G of order n , such that it is possible to reconfigure M to M' , the idea is to switch the cycles in $M \triangle M'$, traversing them in a tree-like fashion. In order to do so, we use an auxiliary tree structure. The proof proceeds in two steps. In the first part, we construct an auxiliary graph for the cycle traversal. In the second part, we prove that there exists a flip sequence of length linear in n reconfiguring M into M' . In the following, we always assume that the isolated vertices of M and M' coincide. This will later be achieved by first flipping the path in $M \triangle M'$ between these two vertices.

Construction of the auxiliary graph

We construct an auxiliary directed graph $\vec{H}$ that will be used to construct the flip sequence. At any step of the construction, let H denote the underlying undirected graph of $\vec{H}$.

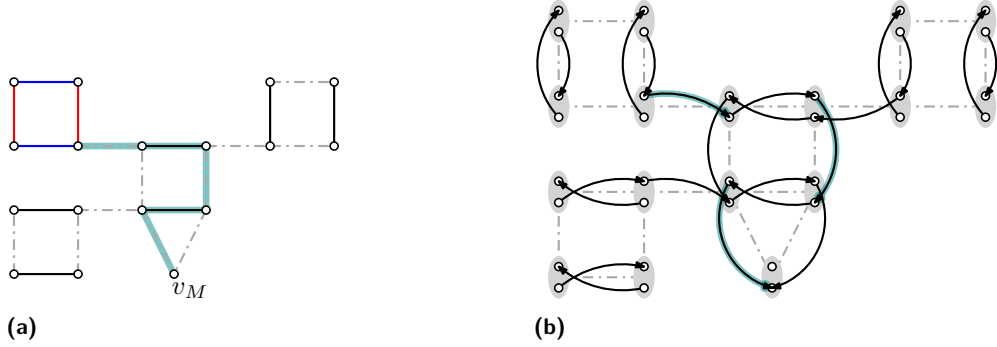
We initialize $\vec{H}$ by introducing, for each vertex v of G , two corresponding vertices in $\vec{H}$: an *in-vertex* v_{in} and an *out-vertex* v_{out} ; for an illustration see Figure 5. For every edge uv in M' , we add the arcs $(u_{\text{in}}, v_{\text{out}})$ and $(v_{\text{in}}, u_{\text{out}})$ to $\vec{H}$. Note that $v_{M, \text{out}}$ remains isolated throughout the process, so we refer to the in-vertex $v_{M, \text{in}}$ simply as v_M .

Let t be the number of cycles in $M \triangle M'$. In the following, we modify $\vec{H}$ by adding arcs. To this end, we iteratively construct auxiliary odd matchings $M_0, \dots, M_t$ in G . Set $M_0 := M$. Suppose M_i , for some $i \in \{0, \dots, t-1\}$, is the last constructed matching. We now process M_i



■ **Figure 5** (a) A graph G with the initial and target matching M and M' , respectively, and (b) the initialization of the auxiliary graph $\vec{H}$.

to construct M_{i+1} . To this end, we choose an arbitrary cycle K in $M_i \triangle M'$. By Lemma 1, G contains an M_i -alternating path P from v_M to K . Let u_{C_i} be the first vertex on a cycle C_i in $M_i \triangle M'$ that we encounter when traversing P starting from v_M . We now traverse P from u_{C_i} towards v_M . Suppose that $e = ww'$ is the current edge on the path and assume that w' is closer to v_M on P than w . If $e \in M_i$, we simply move on to the next edge in P . If $e \notin M_i$, we add the arc $(w_{\text{out}}, w'_{\text{in}})$ to $\vec{H}$. If there is no directed path from w'_{in} to v_M in $\vec{H}$, we traverse the next edge on P . Otherwise, we stop the traversal and define M_{i+1} as the matching obtained from M_i by replacing $M_i \cap C_i$ by $M' \cap C_i$. One such iteration is visualized in Figure 6.



■ **Figure 6** Iteration of the construction of $\vec{H}$, at the moment of inclusion of the top-left cycle. (a) The auxiliary matching M_i with the highlighted alternating path connecting the cycle to v_M . (b) The corresponding state of $\vec{H}$ and the highlighted corresponding arcs added to $\vec{H}$.

We now collect some observations on the structure of $\vec{H}$ and the matchings $M_0, \dots, M_t$.

- **Lemma 7** (*). ■ We have $M_t = M'$ and for all $i \in \{0, \dots, t\}$, v_M is isolated in M_i .
- The arcs of $\vec{H}$ always join an in-vertex and an out-vertex. In particular, H is bipartite.
- Each out-vertex in $\vec{H}$ has at most one outgoing arc.

In particular, Lemma 7 implies the following:

- **Corollary 8.** The graph H is a forest. Further, let $\mathcal{C}$ be the connected component of H containing v_M . In $\vec{H}$, there is a directed path from every vertex in $\mathcal{C}$ to v_M . Moreover, for every cycle K in $M \triangle M'$, $\mathcal{C}$ contains the vertex $u_{K, \text{out}}$.

► **Lemma 9** ($\star$). If (u_{in}, v_{out}) is an arc in $\vec{H}$, then $uv \in M'$. If (u_{out}, v_{in}) is an arc in $\vec{H}$, then $uv \in E(G) \setminus M'$. Hence, every directed path in $\vec{H}$ defines an M' -alternating path in G .

Description of the flipping procedure

Using the graph H above, we now describe a flip sequence that transforms M to M' .

Let $\mathcal{C}$ be the connected component of H containing $v_M = v_{M,in}$. We traverse $\mathcal{C}$ in a depth-first search fashion, starting and ending at v_M . In parallel, we perform certain flip operations in G . We maintain the following property: whenever a vertex v_{in} is visited in the traversal, the corresponding vertex v is isolated in G .

Let v_{in} be the currently visited vertex. Let $v_{1,out}, \dots, v_{k,out}$ be all vertices of $\vec{H}$ such that $(v_{i,out}, v_{in})$ is an arc. Two cases can occur:

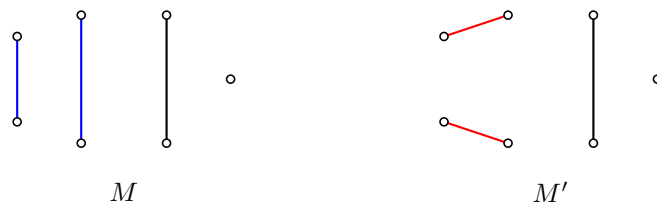
- (1) All vertices $v_{1,out}, \dots, v_{k,out}$ have already been visited. Then we backtrack. That is, if $v_{in} = v_M$, we terminate. Otherwise, let $w_{1,out}, w_{2,in}$ be the next vertices on the directed path from v_{in} to v_M . We will show that the edge $w_1 w_2$ is present in G . We flip it to $v w_1$, so that the isolated vertex becomes w_2 , and next visit $w_{2,in}$.
- (2) There is an unvisited vertex $v_{i,out}$. In this case, we check whether v_i is contained in a cycle C in $M \triangle M'$ that has not yet been switched, and if so, switch C . Subsequently, we consider the second neighbor v'_{in} of $v_{i,out}$ in H . We will show that the edge $v' v_i$ is present in G . We flip it to $v_i v$, so that the isolated vertex becomes v' . We mark $v_{i,out}$ as visited and then visit v'_{in} .

► **Theorem 10** ($\star$). Let M and M' be odd matchings of a graph G such that M can be reconfigured to M' . Using the above procedure, we can reconfigure M to M' using $\mathcal{O}(n)$ flips.

4.2 Convex point sets

We start with a refinement of the notion of happy edges in the case of convex point sets. *Good happy edges* are defined by the following recursive definition:

- (1) Happy edges on the convex hull are necessarily good happy edges.
- (2) If a happy edge e splits the convex point set into two parts such that one part contains the isolated vertex (possibly along with some edges) and all edges in the other part are happy edges (implying that they all are good happy edges), then e is a good happy edge.



■ **Figure 7** Example of a convex point set in which a happy edge needs to be flipped.

A happy edge that is not good is a *bad* happy edge. Figure 7 provides an example in which a happy edge needs to be flipped. However, this edge is not a good happy edge. On the contrary, good happy edges are preserved in shortest flip sequences.

► **Lemma 11** ($\star$). No shortest flip sequence between any two odd matchings on a convex point set flips good happy edges.

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Our recursive definition of good happy edges allows us to construct an equivalent instance without happy edges in linear time.

► **Remark 12** (★). Let M and M' be two matchings. We can remove good happy edges and their incident vertices in time that is linear in the number of vertices.

► **Theorem 13.** *The flip distance k between two odd matchings of the same convex point set S is fixed-parameter tractable in k .*

Proof. We remove all good happy edges using Remark 12. As all remaining edges need to be flipped at least once, the instance is a “no” instance, if there are more than k edges left. In each step, there are at most $2k$ endpoints of matching edges. Adding an edge between the isolated vertex and an endpoint determines which edge gets removed. Thus, we need to check at most $2k$ possible flips per step and at most $(2k)^k$ sequences. ◀

► **Remark 14.** Actually, Theorem 13 shows a slightly stronger property: The problem of determining whether the flip distance between two odd matchings is at most k admits a polynomial size kernel with at most k edges and at most $2k + 1$ vertices. That is, for every instance consisting of two matchings M and M' along with a parameter k , there exists a corresponding instance whose size is polynomial in k , such that this new instance is a YES instance if and only if the original instance is a YES instance.

Consider the union of an initial matching M and a target matching M' . With respect to this, let A be the set of all good happy edges, B the set of all bad happy edges, C the set of edges in M that lie on even alternating cycles, c the number of such cycles, and D the set of edges in M that lie on the alternating path. We have observed that all edges in C and D are flipped at least once, and all edges in B are flipped at least twice. Additionally, each even alternating cycle requires one extra flip to place the isolated vertex onto the cycle. The final flip in a cycle returns the isolated vertex to the component it originally came from. This yields a lower bound on the flip distance, which is tight when the union of M and M' is crossing-free.

► **Theorem 15** (★). *Let M , M' , A , B , C , c , and D be as defined above, and let $M \cup M'$ be crossing-free. Then the flip distance from M to M' is $2|B| + |C| + c + |D|$.*

We only give an intuition for the proof of Theorem 15. The idea is to place the isolated vertex once on every alternating cycle in $M \cup M'$ without making any unnecessary flips. In this step, it helps that in a convex point set if a vertex sees a connected component, it always sees two consecutive vertices. Therefore, we do not have to worry about parity constraints. Then, since the two matchings are together crossing-free, we can greedily switch every cycle in the optimal number of flips.

► **Corollary 16.** *The radius of the flip graph of odd matchings for $n = 2m + 1$ points in convex point sets is at most $\frac{3m}{2} - \mathcal{O}(1)$. As a consequence, the diameter of the flip graph is at most $3m - \mathcal{O}(1) = \frac{3n}{2} - \mathcal{O}(1)$.*

Proof. For any given matching M we show that we can flip M into a given matching M' that has all its edges on the convex hull. Observe that all happy edges of $M \cup M'$ are good happy edges, and that it contains at most $\frac{m}{2}$ alternating cycles and one alternating path. Plugging the parameters into Theorem 15 we obtain the desired upper bound. Since the diameter of a graph is bounded by twice its radius, we obtain an upper bound on the diameter of $3m - \mathcal{O}(1) = \frac{3n}{2} - \mathcal{O}(1)$. ◀

► **Remark 17** (★). For any given odd matching M there exists a matching M' on $n = 2m + 1$ points in a convex point set such that $d(M, M') = \frac{3m}{2} - \mathcal{O}(1)$.

With this result, we improve the previously known upper bound on the diameter from $4m - \mathcal{O}(1) = 2n - \mathcal{O}(1)$, as given in [2], to $3m - \mathcal{O}(1) = \frac{3n}{2} - \mathcal{O}(1)$. By also establishing a matching lower bound on the radius, we demonstrate that this upper bound is best possible among all approaches that proceed via a canonical intermediate structure. Any further improvement to the upper bound on the diameter will likely require more sophisticated techniques, potentially exploiting structural similarities between M and M' .

5 Hardness results

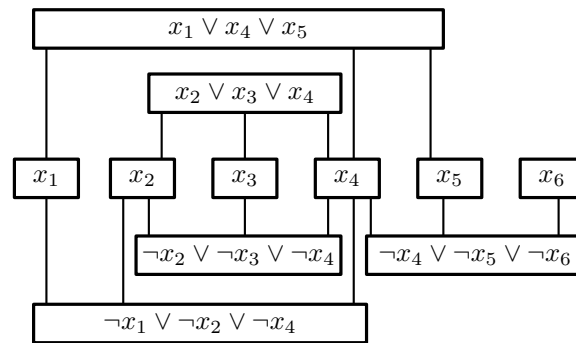
In this section, we analyze the computational complexity of the problem in various settings, specifically for point sets in general position, grid graphs, and more broadly, planar graphs. We prove that deciding whether there exists a flip sequence of length at most k between two odd matchings is NP-complete in each of these settings.

5.1 NP-completeness in point sets

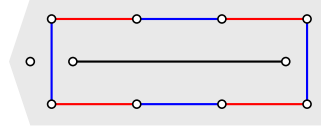
► **Theorem 18.** *Let M and M' be two odd matchings on a set of n points in the plane. Deciding whether there is a flip sequence of length k transforming M into M' is NP-complete.*

We reduce from the NP-complete problem PLANAR MONOTONE 3SAT [9]. Membership in NP follows easily from the fact that a valid flip sequence of length at most k serves as a certificate. As the flip graph of odd matchings in this setting has diameter $\mathcal{O}(n^2)$ as shown in [2], a valid flip sequence has polynomial length.

Planar Monotone 3SAT. This is a variant of 3SAT where each clause has at most three literals, either all positive or all negative. The variable-clause incidence graph is planar and must be embedded with variables on a horizontal line, positive clauses above, and negative clauses below. Moreover, every such instance can be represented in a rectilinear form [20], where each variable and clause corresponds to a rectangle, and their relationship is depicted by a vertical line segment connecting them; we refer to Figure 8 for an illustration.



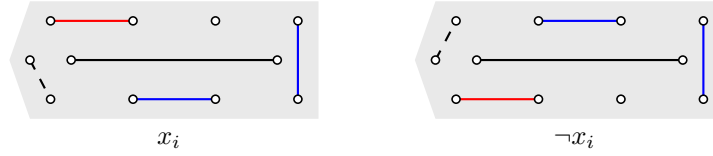
■ **Figure 8** The clause-variable incidence graph of the Boolean formula Φ in conjunctive normal form, $\Phi = (x_1 \vee x_4 \vee x_5) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_4) \wedge (\neg x_2 \vee \neg x_3 \vee \neg x_4) \wedge (\neg x_4 \vee \neg x_5 \vee \neg x_6)$ as an instance of PLANAR MONOTONE 3SAT.



■ **Figure 9** The variable gadget with the isolated vertex on the left.

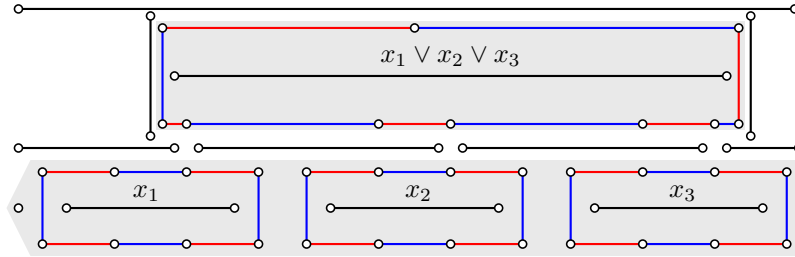
Variable gadget. The variable gadget is shown in Figure 9. While the black edge is happy, the blue and red edges are only contained in the initial and target configuration, respectively.

There are two optimal flip sequences that transform the variable gadget from its initial state to its target state. Two representative intermediate configurations are shown in Figure 10. The direction of traversal encodes the truth value assigned to the variable.



■ **Figure 10** The variable gadget with a truth assignment: true on the left, false on the right.

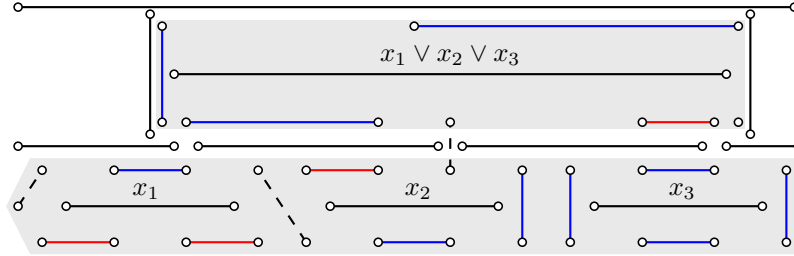
Clause gadget. The clause gadget is depicted in Figure 11; note that the figure also includes three variable gadgets to illustrate the overall embedding. As before, blue and red edges indicate the initial and target state, respectively. If a clause contains only positive literals, we place it above the corresponding variable gadgets, and if it contains only negative literals, we place it below. Additionally, we introduce edges, shown in black, which are part of both matchings; they serve to obstruct visibility between the vertices of the variable and clause gadgets, permitting interaction only through a single designated pair.



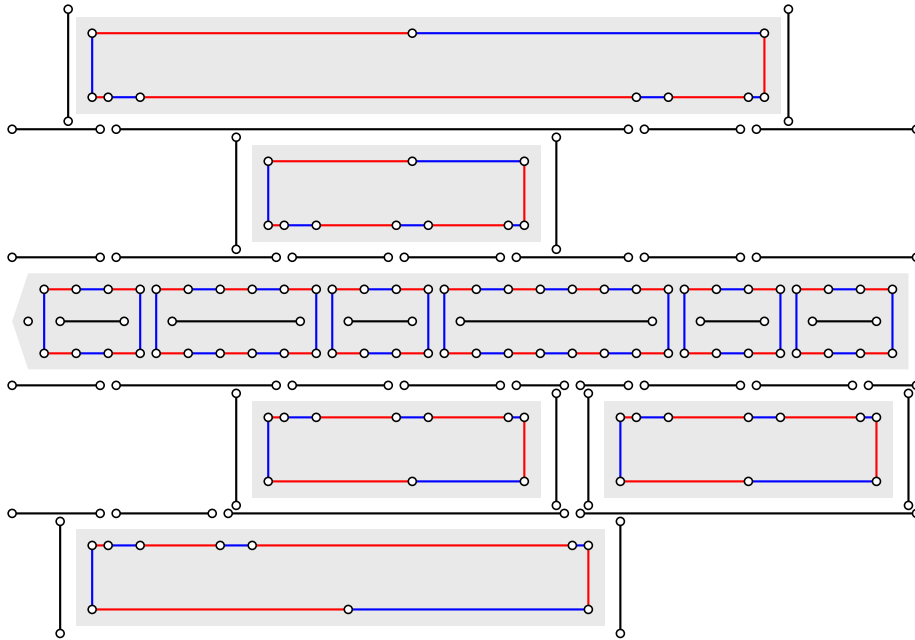
■ **Figure 11** Clause gadget corresponding to $(x_1 \vee x_2 \vee x_3)$, with the associated variable gadgets included for clarity.

Figure 12 illustrates how a valid truth assignment enables interaction with a clause gadget. During an optimal reconfiguration of a variable gadget, there is a point in the sequence where an isolated vertex becomes visible to a vertex in the clause gadget if and only if the chosen sequence reflects a valid truth assignment. Moreover, when the clause gadget is reconfigured optimally, no isolated vertex within the clause gadget will ever have visibility to a vertex in any other variable gadget.

Rectilinear representation. We complete the construction using the rectilinear representation of the PLANAR MONOTONE 3SAT instance. Each variable and clause rectangle is replaced by the corresponding gadget. Happy edges are added to block unintended visibility, while vertical segments in the rectilinear layout define visibility gaps between designated vertices of variable and clause gadgets. We refer to Figures 8 and 13 for illustrations.



■ **Figure 12** Clause gadget in an intermediate state. The clause $(x_1 \vee x_2 \vee x_3)$ is satisfied because the variable x_2 is assigned **TRUE**. In contrast, x_1 is assigned **FALSE** and therefore does not contribute to reconfiguring the clause.



■ **Figure 13** An instance of the odd matching reconfiguration problem, derived from the PLANAR MONOTONE 3SAT instance depicted in Figure 8.

► **Proposition 19 (★).** *Let Φ be an instance of PLANAR MONOTONE 3SAT with C clauses and V variables. There exists a flip sequence from M to M' (the corresponding odd matching instances) of length $\frac{1}{2}|M \triangle M'| + V + C$ flips if and only if Φ has a valid truth assignment.*

Theorem 18 follows from Proposition 19. In our reduction, we construct point sets that are not in general position, as illustrated in the figures. To address this, Lemma 20 shows how to transform the point set into general position while preserving the essential structure.

5.2 Generalizing to point sets in general position

This section is dedicated to show the following lemma.

► **Lemma 20.** *Let S be a point set that can be embedded in an $h \times w$ grid. Then there exists a mapping ϕ that maps each point of S to a point on an $f(h, w) \times g(h, w)$ grid where f and g are polynomial functions in h and w such that*

1. *for every three points p, q, r of S , $\phi(p), \phi(q)$, and $\phi(r)$ are not collinear,*
2. *if p, q , and r are three non-collinear points of S , then the orientation of the triplet (p, q, r) is the same as that of $(\phi(p), \phi(q), \phi(r))$, and*
3. *for every two points p and q of S , $\phi(p)$ and $\phi(q)$ have different x - and y -coordinates.*

An ε -square is a square with side length ε . Our main idea is to place small ε -squares $B_\varepsilon(v)$ centered at each vertex v . These ε -squares are small enough such that for any triple of non-collinear points $p, q, r \in S$, a line $p'q'$ with $p' \in B_\varepsilon(p)$ and $q' \in B_\varepsilon(q)$ does not cross $B_\varepsilon(r)$. Further, we blow up our grid such that there are enough points inside each ε -square, and hence, there is at least one grid point that shares no x - or y -coordinate with any other point and that is not on a line spanned by two points of S .

► **Lemma 21** (*). *Let ℓ be a line through two points of an $h \times w$ square grid G . For every point p of G not on ℓ , let p^x and p^y be the point on ℓ that shares the same x - and y -coordinate with p , respectively. Then the distance from p to each of p^x and p^y is at least $\frac{1}{2hw}$.*

The lemma above is used to show the following lemma.

► **Lemma 22** (*). *Let p, q be two points of an $h \times w$ grid G . Let p' and q' be points in the ε -squares centered at p and q , respectively. If $\varepsilon \leq \frac{1}{16w^3h^3}$, then for every grid point r that does not lie on the line through p and q , the orientation of the triplet (p, q, r) is the same as (p', q', r') for every point r' in the ε -square centered at r .*

We are now ready to prove Lemma 20.

Proof of Lemma 20. Let S be a set of n points, and denote them by $p_1, \dots, p_n$. Let $\varepsilon = \frac{1}{16h^3w^3}$. We construct the new grid G' such that for every vertex p of the point set, the ε -square $B_\varepsilon(p)$ centered at p contains $(n^4 + n) \times (n^4 + n)$ grid points. The resulting grid G' has now size $h \cdot 16h^3w^3 \cdot (n^4 + n) \times w \cdot 16h^3w^3 \cdot (n^4 + n)$ which is polynomial in h and w .

We inductively define the mapping ϕ . We maintain that for $i = 1, \dots, n$, the set $S_i := \{\phi(p_1), \dots, \phi(p_i)\}$ satisfies the conditions (1), (2), and (3) of the lemma.

We define $\phi(p_1) = p_1$. The set S_1 trivially satisfies (1)–(3). Suppose we have constructed the set S_i for some $i \in \{1, \dots, n-1\}$. We now construct the set S_{i+1} . Consider $\binom{i}{2}$ lines spanned by any two points in S_i and the i vertical lines and i horizontal lines passing through a point in S_i . The former $\binom{i}{2}$ lines intersect in less than $(i^2)^2 < n^4$ grid points in $B_\varepsilon(p_{i+1})$. Hence, even when we avoid the i vertical and i horizontal lines, we can choose one point in $B_\varepsilon(p_{i+1})$ to be $\phi(p_{i+1})$ such that S_{i+1} satisfies (1) and (3). It remains to prove that S_{i+1} also satisfies (2). It is sufficient to consider a triplet (p_a, p_b, p_{i+1}) for $1 \leq a < b \leq i$. By construction, $\phi(p_a) \in B_\varepsilon(p_a)$ and $\phi(p_b) \in B_\varepsilon(p_b)$. Hence, by Lemma 22, the orientation of (p_a, p_b, p_{i+1}) is the same as that of $(\phi(p_a), \phi(p_b), \phi(p_{i+1}))$. The lemma then follows. ◀

5.3 NP-completeness in square grid graphs

A *rectilinear Steiner tree* is a tree that connects a given set K of points in the plane by only using horizontal and vertical line segments. The RECTILINEAR STEINER TREE PROBLEM [10] asks whether there exists a tree that connects a given set of vertices using only horizontal and vertical line segments, such that the total edge length does not exceed a specified bound.

The problem is known to be strongly NP-hard; that is, it remains NP-hard even when all vertices lie on a grid of polynomial size. Moreover, there always exists an optimal rectilinear Steiner tree contained within the Hanan grid [12], which is constructed by extending horizontal and vertical lines from each vertex in the input set. This result also implies that the grid can be arbitrarily refined if needed, without affecting the edges of an optimal solution.

We reduce from the RECTILINEAR STEINER TREE PROBLEM to obtain the following.

► **Theorem 23.** *Deciding whether there exists a flip sequence of length ℓ between two odd matchings in grid graphs is NP-hard (and consequently in planar and general graphs as well).*

High-level overview. Let K be a set of k points in an $n \times n$ grid. We assume, without loss of generality, that $n > k$. We aim to construct two odd matchings on the grid graph such that any short flip sequence between them corresponds to a rectilinear Steiner tree of short total edge length. Our reduction works in two phases: (1) By leveraging the structure of the Hanan grid, we restrict the input to point sets whose coordinates are all congruent to 1 mod $8n$, and (2) by carefully constructing two odd matchings on a sufficiently large grid graph, we ensure that short flip sequences between them correspond to short trees.

Construction. Let v_{low} be a point in K with the smallest y -coordinate (break ties arbitrarily). Without loss of generality, assume that its y -coordinate is 0. We embed the point set K into an $8n \cdot (n + 4n^2) \times 8n \cdot (n + 4n^2)$ grid graph G' ; in particular, each point $v_i = (x_i, y_i) \in K$ is mapped to the vertex $(8n \cdot x_i + 1, 8n \cdot y_i + 1)$. We obtain a new set of points denoted by K' .

► **Lemma 24** ($\star$). *K can be connected by an RST of length ℓ if and only if K' can be connected by an RST of length $8n\ell$.*

Next, we replace every vertex in G' as follows: if a vertex belongs to K , we replace it by eight vertices arranged in an alternating cycle of four edges from M and M' ; otherwise, we replace it with eight vertices and four happy edges; see Figure 14 for an illustration.

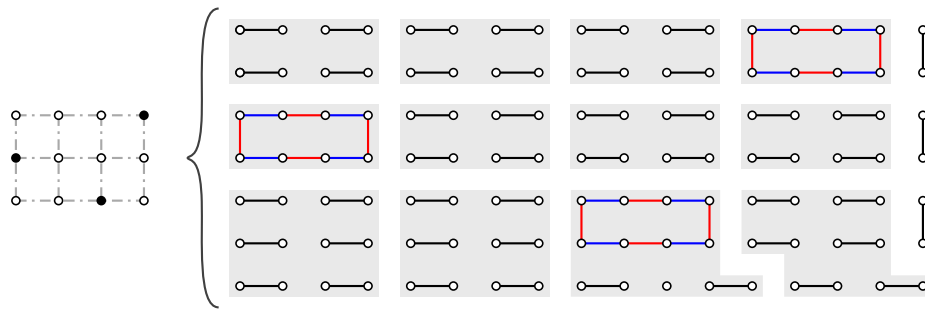


■ **Figure 14** Vertex gadgets for vertices in K (left), and not in K (right).

Then we add an additional row and an additional column to the bottom and to the right. We place the isolated vertex of both M and M' below the third column of the gadget that corresponds to v_{low} . For the remaining points there is a unique way to connect them to an odd matching. The construction is illustrated in Figure 15. We remark that in order to keep the instance smaller, we do not use the refined version of the grid (i.e., G') in the illustration, we also do not add the additional $4n^2$ vertices to the right and top of the relevant area.

► **Proposition 25** ($\star$). *For any instance of RSTP with k vertices on an $n \times n$ grid, there exists an RST of total edge length at most ℓ if and only if there exists a flip sequence for two corresponding odd matchings M, M' transforming M into M' of length at most $4 \cdot 8n \cdot \ell + 8k$.*

With this, Theorem 23 can be easily obtained from Proposition 25.



■ **Figure 15** Reduction from rectilinear Steiner tree, massively scaled down for illustration.

6 Conclusions and future work

In this work, we investigated the reconfiguration of odd matchings and provided a complete characterization of when the associated flip graph is connected. This complements a recent result that showed connectedness of the flip graph in the geometric setting [2]. We further showed that, in the combinatorial setting, each connected component of the flip graph has diameter linear in the cardinality of the matchings. We established that the problem is NP-complete, both in general graphs (in particular, even in rectangular grid graphs) and in geometric settings. For the geometric setting, we developed a framework to embed partial order types into general position. We believe that our framework can be applied in several similar proofs. Lastly, for point sets in convex position, we provided a fixed-parameter tractable algorithm and improved the upper bound on the diameter of the flip graph.

The following questions are related to our research.

- Considering that planarity constraints make things harder: Can the linear bound for the diameter of the flip graph in the combinatorial setting be generalized to the geometric setting? Or can it be used to derive bounds on the diameter of the flip graph in the geometric setting?
- What is the complexity of finding shortest flip sequences between odd matchings in convex point sets? Closing the gap between the upper bound and the lower bound of the diameter can be seen as a first promising direction.
- Now that we established hardness of the flip distance problem: What can be said about fixed-parameter tractability and approximability of flip distances in both the combinatorial and the general geometric setting?

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