

Separability of Witness Gabriel Drawings

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Abstract

A witness Gabriel drawing Γ is a straight-line drawing of a graph in which any two vertices of Γ are adjacent if and only if the disk having these vertices as antipodal points contains no element of a special set of points called witnesses. A witness Gabriel drawing is linearly separable if the vertices and the witnesses lie in opposite half-planes. We prove that every outerplanar graph has a linearly separable witness Gabriel drawing by introducing and studying a new type of drawing that we call a border parabola drawing. We then use border parabola drawings to characterize those triangle-free graphs that admit a linearly separable witness Gabriel drawing. We also consider witness Gabriel drawings where no witness lies in the interior of the convex hull of the vertex set, which we call convexly separable drawings. We construct witness Gabriel drawable graphs for which any witness Gabriel drawing must be convexly separable and that do not admit any linearly separable witness Gabriel drawing.

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1 Introduction

A proximity graph of a set of distinct points S in the Euclidean plane is a geometric graph G whose vertex set is S . Any two vertices u and v of G are adjacent if and only if no other vertex w of G is in the “neighborhood” of u and v . Depending on the definition of neighborhood, we have different types of proximity graphs. For example, in a *Gabriel graph* uv is an edge if and only if the *Gabriel disk*, i.e., the disk having u and v as antipodal points, does not contain any other vertex w . In a *rectangle of influence drawing* u and v are adjacent if and only if the *rectangle of influence* of u and v , i.e., the axis-parallel rectangle having u and v at opposite corners, does not contain any other vertex w .

For a given definition of proximity, different graph topologies correspond to distinct distributions of their vertex sets, making these graphs effective descriptors of the morphological properties of point configurations (see, e.g., [18, 19]). As a consequence, proximity graphs find applications in a variety of contexts, including machine learning, wireless networks, computer vision, and business analytics (see, e.g., [5, 16, 19, 21]). In information visualization, proximity graphs are used to measure the faithfulness of network layouts (see, e.g., [6, 7, 10, 11, 17]).



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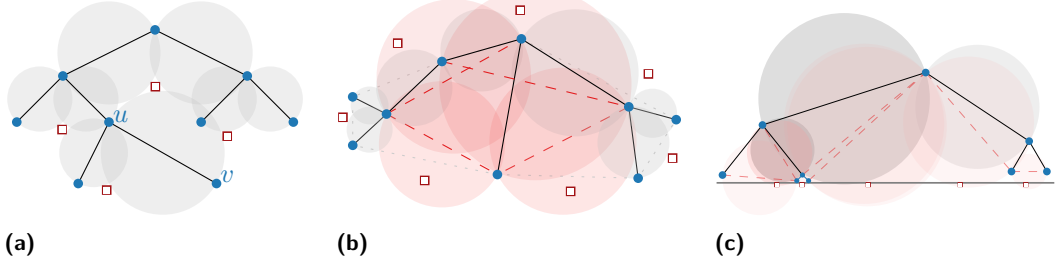
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■ **Figure 1** (a) A witness Gabriel drawing, (b) a convexly separable witness Gabriel drawing, and (c) a linearly separable witness Gabriel drawing of a tree. The vertices are drawn as blue filled disks and the witnesses are drawn as red empty squares.

From the graph drawing perspective, a proximity graph is a straight-line drawing where adjacent vertices are close to one another while pairs of non-adjacent vertices are far from each other according to a given definition of closeness. Partly motivated by the applications above, the study of which graph topologies can be realized for some definition of proximity is a well studied question that can be stated as the following graph drawing problem: Given a graph G and a definition of proximity, does G admit a straight-line drawing Γ such that the proximity graph of the vertex set of Γ is isomorphic to G ? In the affirmative case, Γ is a *proximity drawing* of G and G is said to be *proximity drawable*. See [15] for a survey on proximity drawings and the proximity drawability problem.

This paper studies *witness proximity drawable* graphs, i.e., graphs that can be realized as the *witness proximity graph* of some point sets. In a witness proximity graph we are given two sets of points: the *vertices*, and the *witnesses* that act as obstacles to the existence of the edges. Any two vertices u and v are adjacent if and only if there is no witness in the neighborhood of u and v . Hence, a *witness Gabriel drawing* of a given graph G is computed by defining a set of points that act as vertices and a set of witnesses that act as obstacles; uv is an edge in the drawing if and only if the Gabriel disk of u and v does not contain any witnesses. Note that the Gabriel disk of an edge uv may contain other vertices and that a witness Gabriel drawing coincides with a Gabriel drawing if the witness set and the vertex set coincide. Figure 1a shows a witness Gabriel drawing Γ of a tree where the Gabriel disk of edge uv contains another vertex of Γ .

Motivated by the design of classifiers in pattern recognition applications, Ichino and Sklansky [12] use witness rectangle of influence graphs¹ to study the correlations among different features, represented by points in the plane. Witness Gabriel graphs are introduced in a seminal paper by Aronov, Dulieu, and Hurtado [2] who also study the witness Gabriel drawability problem for different graph families, including trees and complete bipartite graphs. The same authors extend the results of Ichino and Sklansky about witness rectangle of influence graphs in [3] and study witness Delaunay graphs in [1]. It is worth remarking that both the early work of Ichino and Sklansky [12] and the subsequent papers of Aronov, Dulieu, and Hurtado [1, 2, 3] stress the importance of the geometric separability between the vertex set and the witness set in a witness proximity graph, since this property provides useful information about the interclass structure of the two point sets.

¹ These graphs are called interclass rectangular influence graphs in [12]. The term witness proximity graph and drawing was first used by Aronov, Hurtado and Dulieu in [1].

This has motivated recent studies about witness Gabriel drawings where the witnesses are either linearly or convexly separable from the vertex set (see, e.g., [9, 13, 14]); the witness set is convexly separable from the vertex set when no witness is in the convex hull of the vertices. At first sight, one might think that the geometric separability of the witness set from the vertex set cannot give rise to sparse topologies. For example, the algorithm of Aronov, Dulieu, and Hurtado [2] to construct witness Gabriel drawings of trees places witnesses well in the interior of the convex hull of the vertex set. However, many sparse topologies are in fact realizable even with linear or convex separability constraints; furthermore, we will show that the geometric separability is actually required for some graph instances that include sparse graphs. Figure 1b is a convexly separable witness Gabriel drawing of the tree of Figure 1a, while Figure 1c is a linearly separable one.

An outline of our results is as follows.

- We show that every outerplanar graph admits a linearly separable witness Gabriel drawing. This extends the result of Aronov, Dulieu, and Hurtado about the witness Gabriel drawability of trees by both enlarging the family of representable graphs and by proving the linear separability of the drawings. The linear separability property for witness Gabriel drawings of outerplanar graphs was previously known only for caterpillars [9].
- We prove that any triangle-free graph has a linearly separable witness Gabriel drawing if and only if it does not have $K_{2,3}$ as a minor; that is, if and only if the triangle-free graph is outerplanar [20]. This extends a result of [13, 14] where it is shown that the complete bipartite graphs admitting linearly separable witness Gabriel drawings are exactly those not containing $K_{2,3}$ as an (induced) subgraph.
- We investigate the relationship between convexly separable and linearly separable witness Gabriel drawable graphs. We construct infinitely many witness Gabriel drawable graphs which admit convexly separable drawings but do not have a linearly separable witness Gabriel drawing. In fact, we show that for any of these graphs, any witness Gabriel drawing must be convexly separable. We believe that identifying graphs whose witness Gabriel drawings require convex separability offers a novel perspective on the long-standing open problem of characterizing witness Gabriel drawable graphs [2].

Our construction of witness Gabriel drawings of outerplanar graphs depends on a new type of drawing which we call a *border parabola drawing* and that may be of interest in its own right. It is a type of drawing in which every vertex is associated with a convex region that contains it and such that there is an edge between two vertices if and only if one end-vertex of an edge is in the region of the other. Any border parabola drawing of a graph can be converted to a linearly separable witness Gabriel drawing with the addition of appropriately chosen witness locations.

2 Preliminaries

We assume familiarity with graph drawing terms and concepts (refer, for example, to [4]). We recall that the Gabriel disk of two distinct points p, q is the disk whose diameter is segment pq . We denote the Gabriel disk of p and q as $\mathcal{O}[p, q]$. In the rest of the paper $\mathcal{O}[p, q]$ is a closed set, which is the standard definition of Gabriel disks [8]. We also denote by $\triangle[u, v, w]$ the triangle whose corners are points u, v , and w , and by $\overline{pq}$ the line through points p and q . The following lemma can be proved by elementary geometry.

► **Lemma 1.** *Let p, q_1 and q_2 be three non-collinear points and let L be the line through q_1 and q_2 . Then the Gabriel disks $\mathcal{O}[p, q_1]$ and $\mathcal{O}[p, q_2]$ intersect at a point on L and their union contains all points in $\triangle[p, q_1, q_2]$.*

The lemma has some immediate consequences.

► **Corollary 2** ([2] Lemma 6). *Let Γ be a witness Gabriel drawing, let uv and vw be two adjacent edges of Γ . Then: (i) No witness is contained in $\triangle[u, v, w]$, and (ii) any vertex in the interior of $\triangle[u, v, w]$ is adjacent to v in Γ .*

► **Corollary 3** ([2] Proposition 1). *Let Γ be a witness Gabriel drawing, and let uv , vw , and wu be edges of Γ forming a three-cycle. Any vertex $r \in \triangle[u, v, w]$ is adjacent to all of $\{u, v, w\}$.*

3 Witness Gabriel Drawings of Outerplanar Graphs

We prove here that every outerplanar graph admits a linearly separable witness Gabriel drawing. The drawings we produce have the additional property that for any edge, its Gabriel disk does not intersect the (open) half-plane containing the witnesses. To do this, we introduce and study a new type of graph drawing, which we call a *border parabola drawing*.

3.1 Border Parabola Drawings

► **Definition 4.** *Given a point $p \in \mathbb{R}^2$ with $y(p) > 0$, the border parabola of p is the set of points $q \in \mathbb{R}^2$ such that $\odot[p, q]$ is tangent to the x -axis. The border parabola of p can be expressed by the function $f_p(x) = \frac{(x-x(p))^2}{4y(p)}$.*

Note that we use x above both as a variable and as a projection function and that the border parabola of p has p as its focus and $(x(p), 0)$ as its vertex. The following property is an immediate consequence of Definition 4.

- **Property 1.** *For any points p and q in the upper half-plane:*
- *if $y(p) > f_q(x(p))$, then the Gabriel disk $\odot[p, q]$ lies above the x -axis;*
 - *if $y(p) = f_q(x(p))$, then the x -axis is tangent to $\odot[p, q]$;*
 - *if $y(p) < f_q(x(p))$, then the Gabriel disk $\odot[p, q]$ crosses the x -axis.*

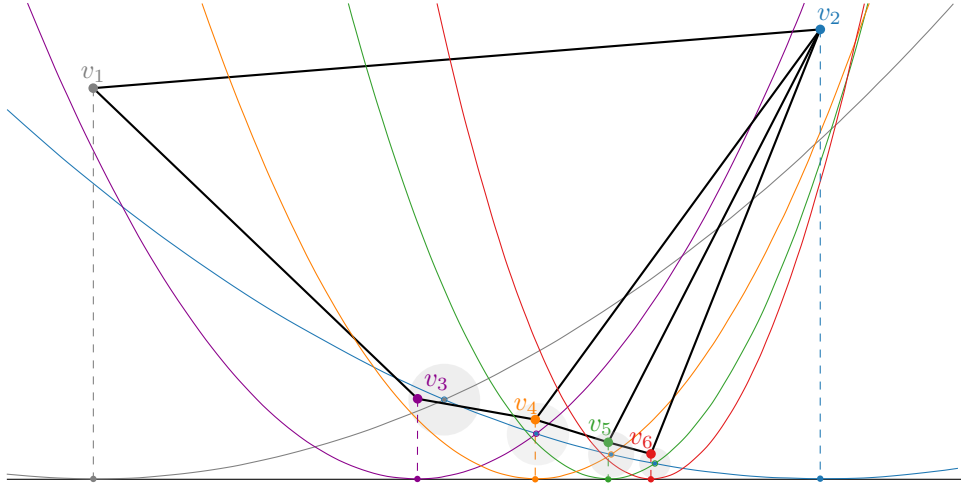
For any point q in the upper half-plane, let $B_q = \{p : y(p) \geq f_q(x(p))\}$; if $p \in B_q$, we say that p lies within the border parabola of q . Property 1 immediately implies that p lies within the border parabola of q if and only if q lies within the border parabola of p , and in particular p is a point of the border parabola of q if and only if q is a point of the border parabola of p .

► **Definition 5.** *Given a set P of points in the upper half-plane, the graph $G = (P, E)$, where $E = \{(p, q) : q \in B_p\}$ is called the border parabola (or PB) graph of P . The associated straight-line drawing, denoted by $\Gamma(P)$, is called the border parabola drawing of P .*

We say that G admits a *border parabola drawing* if G is the border parabola graph for some set P ; if G admits a border parabola drawing, we call G a *border parabola graph* or say that G is *border parabola drawable*. Figure 2 shows a border parabola drawing. The following properties will be useful:

► **Property 2.** *Let $\Gamma(P)$ be a border parabola drawing and $\{p_1, p_2, p_3\} \subset P$ be such that $x(p_1) < x(p_2) < x(p_3)$. If $\{p_1, p_3\}$ forms an edge and p_2 lies above $\overline{p_1 p_3}$, then $\{p_1, p_2\}$ and $\{p_2, p_3\}$ also form edges.*

Proof. Since the segment $\overline{p_1 p_3}$ is contained in both B_{p_1} and B_{p_3} , the portion of the vertical strip determined by $\{p_1, p_3\}$ that lies above $\overline{p_1 p_3}$ is completely contained in $B_{p_1} \cap B_{p_3}$ and so p_2 must lie within both parabolas. Hence $\{p_1, p_2\}$ and $\{p_2, p_3\}$ also form edges. ◀



■ **Figure 2** A border parabola drawing of an outerplanar graph.

► **Property 3.** Let $\Gamma(P)$ be a border parabola drawing and let $\{p_0, p_1, p_2\}$ form a triangle in $\Gamma(P)$. Then any point $p \in P \setminus \{p_0, p_1, p_2\}$ contained in the triangle $\triangle(p_0, p_1, p_2)$ is adjacent to each of $\{p_0, p_1, p_2\}$.

Proof. It suffices to note that $\{p_0, p_1, p_2\}$ all lie in $B_{p_0} \cap B_{p_1} \cap B_{p_2}$, which is a convex set and so contains every point in $\triangle(p_0, p_1, p_2)$. ◀

3.2 Border Parabola Drawings of Outerplanar Graphs

A witness Gabriel drawing $\Gamma(P, W)$ with vertex set P and witness set W of a graph G is *strongly linearly separable* if there is a line separating P from W such that no Gabriel disk of an edge crosses the separating line. We shall say that Γ is a SLSWG drawing of graph G . We assume without loss of generality that the separating line is the x -axis.

Property 1 implies that in a SLSWG drawing $\Gamma(P, W)$, for any $p \in P$, the border parabola of p contains p and its neighbors, but no other point of P . That is, the drawing can be computed from P alone, without reference to W . Hence, we have the following.

► **Theorem 6.** A graph G admits a SLSWG drawing $\Gamma(P, W)$ if and only if G is the border parabola graph of P .

The question of which graphs admit a SLSWG drawing is therefore the question: Which graphs G are border parabola graphs? We are going to show that outerplanar graphs are border parabola graphs. We start with some definitions.

Let p and q be points in the plane such that $x(p) < x(q)$. The (open) *vertical strip* for p and q is the set $S(p, q) = \{r : x(p) < x(r) < x(q)\}$. The (open) *lower half-strip* for p and q $S^-(p, q)$ is the portion of $S(p, q)$ lying strictly below the line determined by p and q . Note that neither p nor q is contained in $S^-(p, q)$ or $S(p, q)$ because of the strict inequality used in the definitions.

Let $\Gamma(P)$ be a border parabola drawing and let $r \in \mathbb{R}^2$ be any point in the upper half-plane. We define the *neighborhood* of r in P to be the set $N_P(r) = B_r \cap P$; that is $N_P(r)$ is the set of points of P that would be adjacent to r in $\Gamma(P \cup \{r\})$. Finally, for $\{p, q\} \subseteq P$, we say that a subset R of the upper half-plane is an *extension zone* for $\{p, q\}$ if

- for all $r \in R$, $N_P(r) \subseteq \{p, q\}$,
- R contains points $r_1, \dots, r_4$ such that $N_P(r_1) = \emptyset$, $N_P(r_2) = \{p\}$, $N_P(r_3) = \{q\}$, and $N_P(r_4) = \{p, q\}$.

The purpose of an extension zone is to identify, given points $\{p, q\}$, a set of locations such that a point r from the zone could be added to P so that $\Gamma(P \cup \{r\})$ would contain the edges of $\Gamma(P)$ along with both, either, or neither of $\{p, r\}, \{q, r\}$ but would contain no other additional edges.

Finally, let $\Gamma(P)$ be a border parabola drawing and let $\{p, q\} \subseteq P$. The *meet* of p and q , denoted $m_{p,q}$, is the point of $B_p \cap B_q$ having minimum y -coordinate. The meet of p and q can be equivalently defined to be the point of intersection of f_p and f_q that lies in the vertical strip determined by p and q . Note that the portion of $B_{m_{p,q}}$ below the line determined by p and q is always a subset of $S^-(p, q)$ since p and q both lie on $f_{m_{p,q}}$; thus $m_{p,q} \in S^-(p, q)$.

► **Lemma 7.** *Let $\Gamma(P)$ be a border parabola drawing and let $\{p, q\} \subseteq P$ be such that $x(p) < x(q)$, $S^-(p, q) \cap P = \emptyset$ and $N_P(m_{p,q}) = \{p, q\}$. Then there exists a disk D_s centered at $s = m_{p,q}$ such that D_s is an extension zone for $\{p, q\}$, $D_s \subset S^-(p, q)$, and every $r \in D_s$ satisfies*

1. $S^-(p, r) \cap P = \emptyset$ and $S^-(r, q) \cap P = \emptyset$, and
2. $N_{P \cup \{r\}}(m_{p,r}) = \{p, r\}$ and $N_{P \cup \{r\}}(m_{r,q}) = \{r, q\}$,

In other words, if the pair $\{p, q\}$ satisfies the hypotheses of Lemma 7 for P , then any point r that is close enough to $m_{p,q}$ has the property that both $\{p, r\}$ and $\{r, q\}$ satisfy the hypotheses of Lemma 7 for $P \cup \{r\}$; see Figure 3 for an illustration of the Lemma.

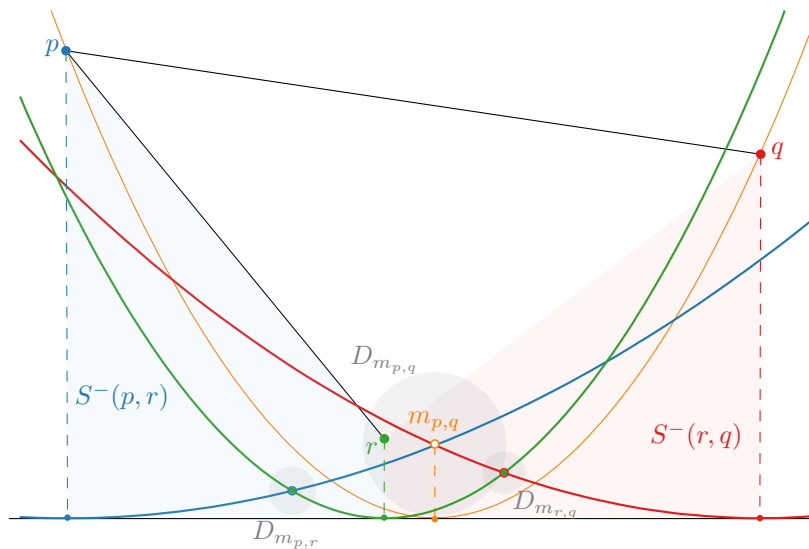
Proof. Since $x(p) < x(q)$ and B_s is a closed region, there exists an $\varepsilon > 0$ such that the disk $D_\varepsilon(s)$ with radius ε and center s has the property that for any $s' \in D_\varepsilon(s)$, $N_P(s') \subseteq N_P(s) = \{p, q\}$. Consider any ε' such that $0 < \varepsilon' \leq \varepsilon$. Since the border parabolas f_p and f_q intersect at s , f_p is strictly increasing at s , and f_q is strictly decreasing at s , the points $p_1 = (x(s), y(s) + \varepsilon')$, $p_2 = (x(s) + \varepsilon', y(s))$, $p_3 = (x(s), y(s) - \varepsilon')$, and $p_4 = (x(s) - \varepsilon', y(s))$ are all in $D_\varepsilon(s)$ and will satisfy $N_P(p_1) = \{p, q\}$, $N_P(p_2) = \{q\}$, $N_P(p_3) = \emptyset$, and $N_P(p_4) = \{p\}$ respectively, so $D_\varepsilon(s)$ is an extension zone for $\{p, q\}$. Also, if $S^-(p, q) \cap P = \emptyset$, then for *any* $r \in S^-(p, q)$, $S^-(p, r) \cap P = \emptyset$ and $S^-(r, q) \cap P = \emptyset$, so the first condition is satisfied.

We now show that every $r \in D_\varepsilon(s)$, satisfies $N_{P \cup \{r\}}(m_{p,r}) = \{p, r\}$; the proof that $N_{P \cup \{r\}}(m_{r,q}) = \{r, q\}$ is analogous. First note that since $S^-(p, q) \cap P = \emptyset$, any point $t \in P \cap N_{P \cup \{r\}}(m_{p,r})$ other than p lies in the portion of $N_{P \cup \{r\}}(m_{p,r})$ above the line L determined by p and q . But that portion of $N_{P \cup \{r\}}(m_{p,r})$ is completely contained in $N_P(s)$, since f_s intersects $f_{m_{p,r}}$ at p and at a point whose x -coordinate is between those of p and r , and between p and r , $f_{m_{p,r}}$ lies below f_s . This, and an analogous argument that $N_{P \cup \{r\}}(m_{r,q}) = \{r, q\}$, establish the second condition. Letting $D_s = D_\varepsilon(s)$ completes the proof. ◀

► **Theorem 8.** *Every outerplanar graph $G = (V, E)$ admits a border parabola drawing and hence it has a linearly separable witness Gabriel drawing.*

Proof. We incrementally construct a set P and a function $g : P \rightarrow V$ such that at all times the border parabola graph $\Gamma(P)$ of P is isomorphic to the subgraph of G induced by $g(P) \subseteq V$. We assume G is connected; otherwise the construction is applied to each component of G separately, translating successive components far enough to the right of previously drawn components so that if $\{u, v\}$ lie in different components, the Gabriel disk of u and v intersects the negative half-plane.

Assume that we have a topological embedding of G . If G is not a maximal outerplanar graph, add new edges E' to G until $G' = (V, E \cup E')$ is maximal outerplanar. Note that this can be done so that the original embedding of G is preserved. We use the geometric



dual of G' to impose an order on the vertices of G' . Let $\{v_1, v_2\}$ be an edge on the outer face of G' and let v_3 be their common neighbor. Now consider the tree whose vertices are the triangular faces of G' and whose edges connect faces of G' that share an edge. Root this tree at $\{v_1, v_2, v_3\}$. We now order the remaining vertices so that the following invariant holds: For each $i > 3$, v_i forms a triangle with some pair $\{v_j, v_k\}$ such that $\{j, k\}$ are both less than i (note that this also holds for $i = 3$ by construction). Any such ordering will serve our purposes. We call $\{v_j, v_k\}$ the *parents* of v_i .

At any point in the construction of P , there will be pairs $\{p_j, p_k\} \subset P$ such that $\{v_j, v_k\}$ is an edge of G' and there exists a (unique) m such that $\{v_j, v_k, v_m\}$ is a triangle of G' but p_m has not yet been constructed. We call such a pair $\{p_j, p_k\}$ *pending*.

We now describe how to add points to P so that, for all i with $1 \leq i \leq n$, after point p_i has been added to P , the following invariants hold:

1. $\Gamma(P)$ is isomorphic to the subgraph of G induced by $\{v_1, \dots, v_i\}$
2. For every j with $3 \leq j \leq i$, p_j is contained in the lower half-strip of its parents
3. The lower half-strips of any two pending pairs in P are disjoint, and

4. For every pending pair $\{p_j, p_k\}$
 - a. $\{p_j, p_k\}$ appear as consecutive elements in $C(P)$
 - b. $S^-(p_j, p_k) \cap P = \emptyset$,
 - c. $m_{p_j, p_k} \in S^-(p_j, p_k)$
 - d. $N_P(m_{p_j, p_k}) = \{p_j, p_k\}$

Note that the invariants hold (mostly vacuously) for $i = 1$ and $i = 2$. So assume now that, for some value $i \geq 3$, the invariants hold for $i - 1$. Let $\{v_j, v_k\}$ be the parents of v_i ; then $j, k : 1 \leq \{j, k\} \leq i - 1$ and so $\{p_j, p_k\}$ have already been added to P , and so by invariant 4, $\{p_j, p_k\}$ satisfy the hypotheses of Lemma 7. That is, letting $s = m_{p_j, p_k}$, there exists a disk D_s with center s such that D_s is an extension zone for $\{p_j, p_k\}$, $D_s \subset S^-(p_j, p_k)$, and such that every $r \in D_s$ satisfies

1. $S^-(p_j, r) \cap P = \emptyset$ and $S^-(r, p_k) \cap P = \emptyset$.
2. $m_{p_j, r} \in S^-(p_j, r)$ and $m_{r, p_k} \in S^-(r, p_k)$, and
3. $N_{P \cup \{r\}}(m_{p_j, r}) = \{p_j, r\}$ and $N_{P \cup \{r\}}(m_{r, p_k}) = \{r, p_k\}$,

We now choose a location for $p_i \in D_s$ as follows

- If $\{v_j, v_i\} \in E$ and $\{v_k, v_i\} \in E$, choose p_i so that $N_P(p_i) = \{p_j, p_k\}$,
- If $\{v_j, v_i\} \in E$ and $\{v_k, v_i\} \in E'$, choose p_i so that $N_P(p_i) = \{p_j\}$,
- If $\{v_j, v_i\} \in E'$ and $\{v_k, v_i\} \in E$, choose p_i so that $N_P(p_i) = \{p_k\}$,
- If $\{v_j, v_i\} \in E'$ and $\{v_k, v_i\} \in E'$, choose p_i so that $N_P(p_i) = \emptyset$.

Add p_i to P and let $g(p_i) = v_i$, and observe that by virtue of the selection rule for p_i , $\Gamma(P)$ is still isomorphic to the subgraph of G induced by $\{v_1, \dots, v_i\}$. Also, p_i is contained in the lower half-strip of its parents. In addition, since $S^-(p_j, p_i)$ and $S^-(p_i, p_k)$ are disjoint and contained in $S^-(p_j, p_k)$, and $\{p_j, p_k\}$ is no longer pending, the lower half-strips of all pending pairs are still disjoint, since p_i can't form a pending pair with any point of P other than its parents p_j and p_k (the only neighbors in G' of v_i in $\{v_1, \dots, v_{i-1}\}$ are v_j and v_k). We thus just need to verify that conditions 4a-4d are still satisfied.

- 4a** Before p_i was added, Condition 4a held. After p_i is added, $\{p_j, p_k\}$ is no longer a pending pair so $\{p_j, p_k\}$ no longer need to be consecutive in $C(P)$. The only possible new pending pairs are $\{p_j, p_i\}$ and $\{p_i, p_k\}$; both of these pairs occur as consecutive points in $C(P)$. and so Condition 4a is satisfied.
- 4b** Similarly, Condition 4b is clearly satisfied since p_i was added inside $S^-(p_j, p_k)$ and so after it has been added to P , p_i is not contained in the lower vertical strip of any pending pair since all such pairs have disjoint strips.
- 4c** Condition 4c is still satisfied for any pending pair not including p_i , so it only needs to be checked for $S^-(p_j, p_i)$ and $S^-(p_i, p_k)$ (which may or may not be pending pairs). But the choice of location of p_i along with Lemma 7 guarantee that m_{p_j, p_i} and m_{p_i, p_k} are in $S^-(p_j, p_i)$ and $S^-(p_i, p_k)$ respectively.
- 4d** Assume that Condition 4d was satisfied before the addition of p_i to P . We need to ensure that, after p_i is added to P , the condition still holds. Note that p_i is selected to lie below $C(P)$. As noted earlier, for any pair of points $\{p, q\} \in P$, the portion of $B_{m_{p, q}}$ below the line determined by $\{p, q\}$ is completely contained in $S^-(p, q)$. So for each pending pair $\{p, q\}$ in $C(P)$, the portion of $B_{m_{p, q}}$ below $C(P)$ is disjoint from the portion of $B_{m_{p', q'}}$ below $C(P)$ for any other pending pair $\{p', q'\}$ in $C(P)$. Thus when p_i is added to P , it will not lie in $B_{m_{p, q}}$ for any pending pair $\{p, q\}$ (other than, perhaps, $\{p_j, p_i\}$ or $\{p_i, p_k\}$), and so the neighborhoods for those pending pairs are unchanged. But by Lemma 7, $N_P(m_{p_j, p_i}) = \{p_j, p_i\}$ and $N_P(m_{p_i, p_k}) = \{p_i, p_k\}$, so Condition 4d is satisfied. ◀

Figure 2 depicts a border parabola drawing constructed with the procedure described in the proof of Theorem 8.

It is worth remarking that Theorem 8 extends a previous result by Aronov, Dulieu, and Hurtado [2] about the witness Gabriel drawability of trees. We also note that the witness Gabriel drawings of trees computed with the algorithm in [2] are neither linearly nor convexly separable.

4 Linearly Separable Triangle-free Witness Gabriel Drawings

As proved in the previous section, any graph that admits a border parabola drawing also admits a linearly separable witness Gabriel Drawing. In this section we will show that, for the family of triangle-free graphs, the converse is also true.

► **Lemma 9.** *Let $\Gamma(P, W)$ be a witness Gabriel drawing, and let $\{p_1, p_2, q_1, q_2\} \subseteq P$ be four distinct points such that*

- $\{p_1, p_2\}$ and $\{q_1, q_2\}$ are crossing edges of $\Gamma(P, W)$, and
- $\{p_1, q_1\}$ and $\{p_2, q_2\}$ are non-edges of $\Gamma(P, W)$.

Then for any witnesses $\{w_1, w_2\} \subseteq W$ for $\{p_1, q_1\}$ and $\{p_2, q_2\}$ respectively, the segment $\overline{w_1 w_2}$ crosses edges $\{p_1, p_2\}$ and $\{q_1, q_2\}$.

Proof. Note that for the hypotheses of the lemma to be satisfied, it cannot be that an endpoint of one of the two edges lies on the other edge, since then the Gabriel disk of one of the non-edges would lie inside the Gabriel disk of one of the edges, making witness placement impossible. Thus the intersection point of the edges is in the interior of each edge.

Let L_1 be the line through $\{p_1, q_2\}$, L_2 be the line through $\{p_2, q_1\}$, and, if the lines are not parallel, let c be their intersection point. By applying a suitable rotation, translation, and/or reflection, we can assume, w.l.o.g., that

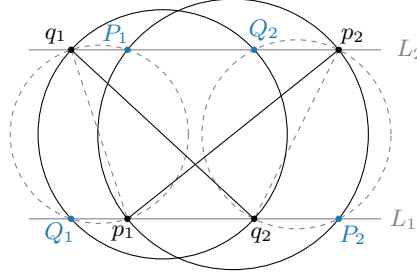
- L_1 and L_2 are parallel with slope 0 and are equidistant from the x -axis, or c lies at the origin and L_1 and L_2 have equal but oppositely signed slopes with all of $\{p_1, p_2, q_1, q_2\}$ appearing to the left of c ,
- $\{p_1, p_2, q_1, q_2\}$ all lie on or above L_1 ,
- p_1 lies to the left of q_2 on L_1 , and so q_1 lies to the left of p_2 on L_2 ,

Let $\odot[p_1, p_2]$, $\odot[q_1, q_2]$, $\odot[p_1, q_1]$, and $\odot[p_2, q_2]$ be the Gabriel disks of the two edges and two non-edges. By Lemma 1, each of $\odot[p_1, p_2]$, $\odot[q_1, q_2]$ intersect each of $\odot[p_1, q_1]$, $\odot[p_2, q_2]$ at two points and all eight such points of intersection lie on L_1 and L_2 . Four of the eight points are $\{p_1, p_2, q_1, q_2\}$; denote the other four by $\{P_1, P_2, Q_1, Q_2\}$ as follows: If $r \in \{p_1, p_2, q_1, q_2\}$, then R denotes the other point of intersection of the two Gabriel disks that intersect at r . Note that if $r \in \{p_1, p_2, q_1, q_2\}$ lies on L_i , for $i = 1, 2$, then R lies on L_{3-i} . Figure 4 and 5 show the two possible configurations.

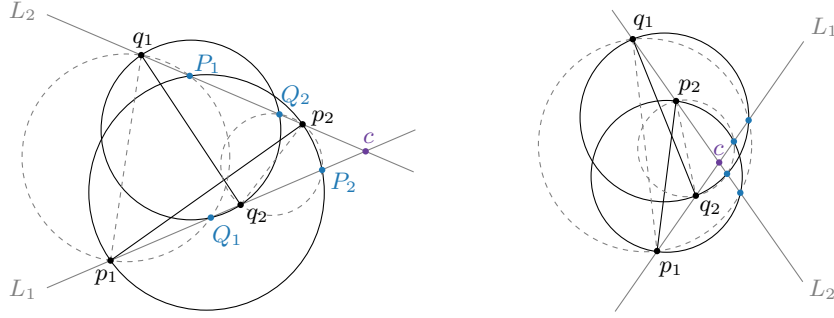
Note that since p_1 is to the left of q_2 on L_1 , P_1 must be to the left of Q_2 on L_2 ; similarly Q_1 must be to the left of P_2 on L_1 , since q_1 is to the left of p_2 on L_2 .

Consider first the case where L_1 and L_2 are parallel and consider L_2 and the region above it. Now consider the portions of the Gabriel disks for the two edges that lie above L_2 . One portion connects q_1 to Q_2 and the other connects p_2 to P_1 . But since q_1 must be to the left of p_2 and P_1 must be to the left of Q_2 , the projections of those portions onto L_2 intersect and so either the portions cross or one is contained below the other. Since all of the eight intersection points of the four Gabriel disks lie on L_1 and L_2 either every point on the portions of the Gabriel disks of the two non-edges lying above L_2 lie below some point on a portion of one (or both) of the Gabriel disks for the two edges, or none do. But near

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■ **Figure 4** Illustration for Lemma 9 – parallel case.



■ **Figure 5** Illustration for Lemma 9 – skew case.

the left-most of the four intersection points on L_2 (q_1 in Figure 4), the portion of $\odot[q_1, q_2]$ above L_2 contains the portion of $\odot[q_1, p_1]$ above L_2 . Similarly, near the right-most of the four intersection points on L_2 (p_2 in Figure 4), the portion of $\odot[p_2, p_1]$ above L_2 contains the portion of $\odot[p_2, q_2]$ above L_2 . Thus no portion of either of the Gabriel disks for the non-edges lies both above L_2 and outside the union of the Gabriel disks for the two edges.

A similar argument holds for the region below L_1 . Therefore any witnesses for $\odot[p_1, q_1]$ and $\odot[p_2, q_2]$ must lie in the horizontal strip between L_1 and L_2 . Note now that the entire quadrilateral bounded by $\{p_1, p_2, q_1, q_2\}$ is contained in the union of the Gabriel disks of the two edges. Note also that the only points to the right of that quadrilateral and between L_1 and L_2 that disk $\odot[q_1, p_1]$ can contain are also contained in $\odot[p_1, p_2] \cup \odot[q_1, q_2]$: for any point p' that lies in this region, the angle $q_1 p' q_2$ is larger than the angle $q_1 p' p_1$, so if p' lies in $\odot[q_1, p_1]$, then it also lies in $\odot[q_1, q_2]$. Similarly, the only points to the left of that quadrilateral and between L_1 and L_2 that disk $\odot[p_2, q_2]$ can contain are also contained in $\odot[p_1, p_2] \cup \odot[q_1, q_2]$. Thus a witness for $\odot[q_1, p_1]$ must lie in the strip between L_1 and L_2 to the left of the quadrilateral and a witness for $\odot[q_2, p_2]$ must lie in the strip between L_1 and L_2 to the right of the quadrilateral, implying that the segment between them crosses both $\{p_1, p_2\}$ and $\{q_1, q_2\}$.

The proof in the case that L_1 and L_2 intersect is similar but slightly more involved. Instead of three regions of interest (below L_1 , between L_1 and L_2 , and above L_2), there are now four regions that we call the *north*, *east*, *south*, and *west* regions:

- north: the intersection of the regions above both L_1 and L_2 ,
- south: the intersection of the regions below both L_1 and L_2 ,
- west: the intersection of the region above L_1 and below L_2 , and
- east: the intersection of the region below L_1 and above L_2 .

Note that by construction, $\{p_1, p_2, q_1, q_2\}$ are contained in the west region.

The analogous proof for this case uses the following observations:

- Any point within either of the Gabriel disks for the two non-edges that lies in the north or south regions is also within the union of the Gabriel disks for the two edges. Thus any witnesses for the two non-edges must lie in the west and/or east regions.
- The Gabriel disks of the two edges completely contain the quadrilateral formed by $\{p_1, p_2, q_1, q_2\}$,
- The only points to the left of that quadrilateral and within the west region that disk $\mathcal{O}[q_2, p_2]$ can contain are also contained in $\mathcal{O}[p_1, p_2] \cup \mathcal{O}[q_1, q_2]$.

However, it is not always the case that the disk $\mathcal{O}[p_1, q_1]$ does not contain any points to the right of that quadrilateral. There are two situations, depending upon whether c , the intersection point of L_1 and L_2 , lies inside the four Gabriel disks or not. (It either lies in all four of the disks if $\angle q_1 c q_2 = \angle p_1 c p_2 \geq \pi/2$ or in none of them if not.) If c lies outside of the four disks, then, in fact, the only points to the right of that quadrilateral and within the west region that disk $\mathcal{O}[p_1, q_1]$ can contain are also contained in $\mathcal{O}[p_1, p_2] \cup \mathcal{O}[q_1, q_2]$. Moreover, in this case, neither of the Gabriel disks of the non-edges intersects the east region, and so witnesses for the two non-edges must both lie in the west region, but on opposite sides of the quadrilateral. Thus the segment between them will cross both $\{p_1, p_2\}$ and $\{q_1, q_2\}$.

If c lies inside the four Gabriel disks, then the portion of $\mathcal{O}[q_2, p_2]$ that lies in the east region (and, in fact, all of $\mathcal{O}[q_2, p_2]$) will be contained in the union of the Gabriel disks of the two edges, so there will be no location at which a witness for $G[q_2, p_2]$ can be placed. ◀

As a final ingredient, we will prove the following generalization of Property 2:

► **Property 4.** *Let $\Gamma(P)$ be a linearly separable witness Gabriel drawing and $\{p_1, p_2, p_3\} \subset P$ be such that $x(p_1) < x(p_2) < x(p_3)$. If $\{p_1, p_3\}$ forms an edge and p_2 lies above $\overline{p_1 p_3}$, then $\{p_1, p_2\}$ and $\{p_2, p_3\}$ also form edges.*

Proof. It suffices to note that the portion of the Gabriel disk $\mathcal{O}[p_1, p_3]$ below segment $\overline{p_1 p_3}$ contains the portions of the Gabriel disks $\mathcal{O}[p_1, p_2]$ and $\mathcal{O}[p_2, p_3]$ that lie below $\overline{p_1 p_3}$. ◀

We are now ready to state the main result of this section.

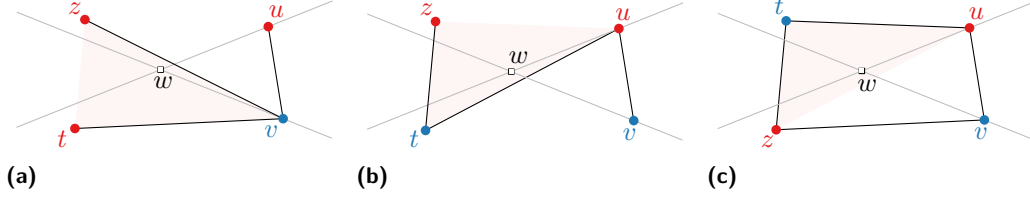
► **Theorem 10.** *The following statements are equivalent for a triangle-free graph G :*

- G is outerplanar
- G admits a linearly separable witness Gabriel drawing
- G admits a border parabola drawing.

Proof. By Theorem 8, every outerplanar graph admits a border parabola drawing, and by definition every border parabola drawing is also a linearly separable witness Gabriel drawing. To show the equivalency, we still have to argue that, if a triangle-free graph admits a linearly separable witness Gabriel drawing, then it must be outerplanar.

An immediate consequence of Lemma 9 is that if $\Gamma(P, W)$ is a linearly separable witness Gabriel drawing, then P cannot contain four points satisfying the hypotheses of the lemma: P and W are contained in disjoint half-planes and so no segment between two points in P can cross a segment between two points in W . In particular, if $\Gamma(P, W)$ is a linearly separable witness Gabriel drawing of a triangle-free graph, then $\Gamma(P, W)$ must be a planar drawing, since any pair of crossing edges would satisfy the hypotheses of Lemma 9. Moreover, no point of P in such a drawing could lie vertically above any edge, since, otherwise, by Property 4, the drawing would contain a triangle, and so the drawing must, in fact, be outerplanar. ◀

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■ **Figure 6** Illustration for the proof of Lemma 11.

We recall that results of [13,14] show that the complete bipartite graphs admitting linearly separable witness Gabriel drawings are exactly those not containing $K_{2,3}$ as an induced subgraph. Following a characterization of triangle-free outerplanar graphs by Syslo [20], Theorem 10 extends these results showing that any triangle-free graph has a linearly separable witness Gabriel drawing if and only if it does not have $K_{2,3}$ as a minor.

5 Convexly Separable Witness Gabriel Drawings

A witness Gabriel drawing Γ is said to be *convexly separable* if none of its witnesses lie within the interior of the convex hull of Γ . Clearly, if a family of graphs admits linearly separable witness Gabriel drawings, then it also admits convexly separable ones. This holds, for example, for outerplanar graphs (see Theorem 8). However, two natural questions arise: (i) If a graph admits a convexly separable witness Gabriel drawing, does it always admit a linearly separable one? (ii) Are there graphs that are witness Gabriel drawable, but for which all such drawings must be convexly separable?

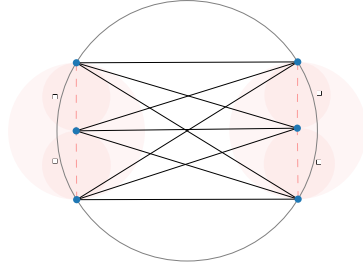
In this section, we answer the first question negatively and the second affirmatively by constructing witness Gabriel drawable graphs that require convex separability in any witness Gabriel drawing, but for which linear separability is not possible.

► **Lemma 11.** *Every $K_{m,n}$ such that $m \geq 2$ and $n \geq 2$ admits a witness Gabriel drawing. Also, if $n > 2$ any such drawing must be convexly separable but cannot be linearly separable.*

Proof. Every $K_{m,n}$ is witness Gabriel drawable by Theorem 5 of [2]. Also, for any $n > 2$ $K_{m,n}$ contains $K_{2,3}$ as a subgraph and hence, by Theorem 10, it does not admit a linearly separable witness Gabriel drawing. It remains to show that every witness Gabriel drawing of $K_{m,n}$ is convexly separable. We color red every vertex of one partition set and blue every vertex of the other partition set of $K_{m,n}$.

Let Γ be a witness Gabriel drawing of $K_{m,n}$ and assume that there is a witness w in the interior of the convex hull and let uv be an edge of $K_{m,n}$ such that u is a red vertex and v is a blue vertex. Consider the line L_1 passing through u and w and the line L_2 passing through v and w . Lines L_1 and L_2 divide the plane into four wedges each having w as their apex. We call these wedges the *top wedge*, *bottom wedge*, *left wedge*, and *right wedge* of w , respectively. The top (left) wedge of w is opposite to its bottom (right) wedge. We assume without loss of generality that edge uv is in the right wedge of w . See also Figure 6.

Observe that no vertex z of Γ can be in the left wedge of w . Namely, depending on its color, vertex z would be adjacent to either u or v and we would have that w is a point of the triangle whose corners are u, v , and z , violating Corollary 2. Also, no edge zt of the convex hull of Γ can cross the left wedge of w : If z and t have the same color, let s be whichever of u and v has the other color. Now $\triangle[z, t, s]$ contains w , violating Corollary 2. If z and t have different colors, they form an edge of Γ and $\triangle[z, t, u]$ contains w , which again violates Corollary 2.



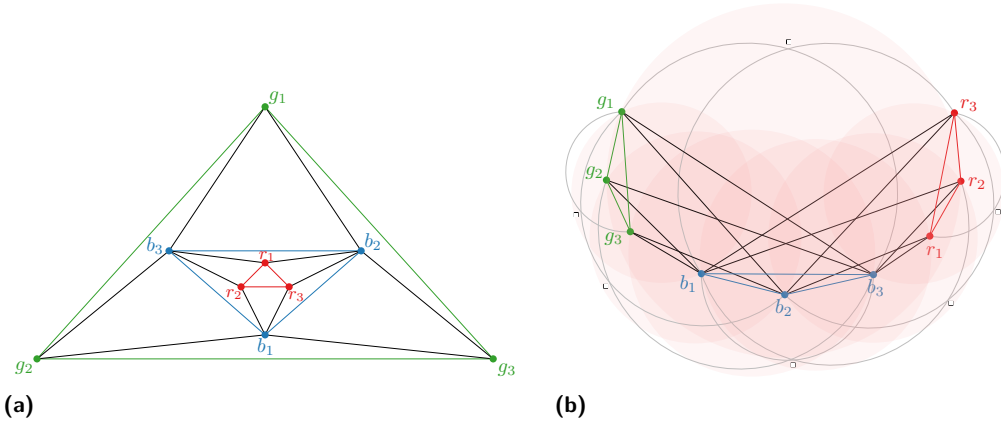
■ **Figure 7** A witness Gabriel drawing of $K_{3,3}$.

Since no vertex and no edge of the convex hull of Γ can intersect the left wedge of w , it follows that w is not in the interior of the convex hull of Γ . ◀

Figure 7 shows a convexly separable witness Gabriel drawing of $K_{3,3}$.

The graphs of Lemma 11 are triangle-free and dense. We now prove a similar result for a planar triangulation consisting of just nine vertices.

Let $C = \{r, g, b\}$ and $I = \{1, 2, 3\}$. Then we define the graph G_9 to have vertex set $V = \{c_i : c \in C, i \in I\}$ and edge set $E = \{\{r_i, r_j\}, \{b_i, b_j\}, \{g_i, g_j\}, \{b_i, r_j\}, \{b_i, g_j\} : 1 \leq i, j \leq 3, i \neq j\}$. Note that G_9 can also be described as the complete graph on V minus all edges $\{r_i, g_j\}, i, j \in I$ and $\{c_i, c'_i\}, c, c' \in C, i \in I$; see Figure 8a. One can also obtain the graph G_9 by taking two octahedra and gluing them together at a triangular face. The vertex $v = c_i \in V$, $c \in C, i \in I$, is said to have *color* $c(v) = c$ and *index* $i(v) = i$; the *index set* of $V' \subseteq V$ is given by $I(V') = \{i(v) : v \in V'\}$. We will refer to vertices having color r as *red* vertices, color b as *blue* vertices, and color g as *green* vertices.



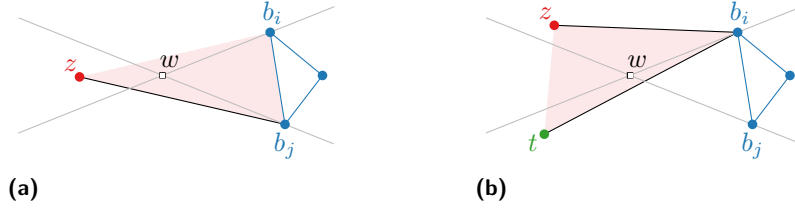
■ **Figure 8** Graph G_9 and a convexly separable witness Gabriel drawing of G_9 .

By means of Lemma 9, Corollary 2 and Corollary 3, we can now prove the following.

► **Lemma 12.** *Graph G_9 admits a witness Gabriel drawing. Also any such drawing must be convexly separable but cannot be linearly separable. Any proper subgraph of G_9 admits a linearly separable witness Gabriel drawing.*

Proof. It is possible to construct a convexly separable witness Gabriel drawing of G_9 . See Figure 8b for such an example.

We now show that no witness Gabriel drawing of G_9 can be linearly separable. Assume that there existed a linearly separable witness Gabriel Drawing Γ of G_9 . By Lemma 9, Corollary 3, and Corollary 2, Γ must be such that: (1) Any $V' \subset V$ having $|I(V')| < |I| = 3$



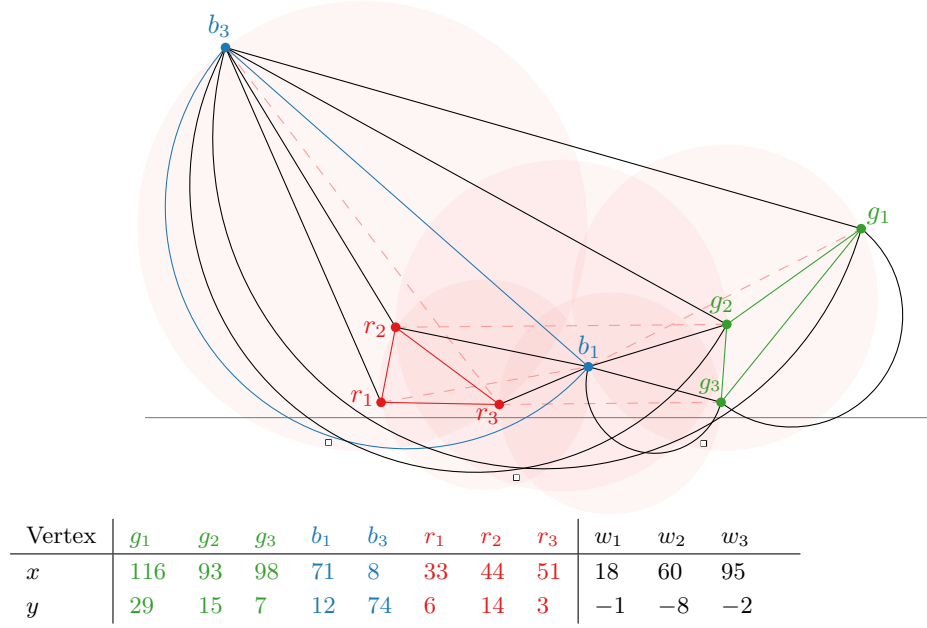
■ **Figure 9** Illustration for the proof of Lemma 12: A witness Gabriel drawing Γ of G_9 must be convexly separable.

induces a plane subgraph of Γ . Note that since V' does not contain vertices of all three indices, the subgraph induced by V' contains no triangles. Thus, by Lemma 9, no two of its edges can cross. (2) Any induced 4-cycle of Γ must form a convex quadrilateral: By (1), the 4-cycle forms a plane subgraph of Γ . If it were not convex, one of the four vertices would be contained in the wedge formed by the other three and so, by Corollary 2, a diagonal would exist. (3) For any $\{i, j\} \subset I$, the subgraph $G_{i,j}$ of G_9 induced by all vertices having indices in $\{i, j\}$ is a plane subgraph of Γ consisting of the two convex quadrilaterals $R_{i,j} = \{b_i, b_j, r_i, r_j\}$ and $G_{i,j} = \{b_i, b_j, g_i, g_j\}$. Further, these two quadrilaterals lie on opposite sides of the line $L_{i,j}$ determined by $\{b_i, b_j\}$. Both $R_{i,j}$ and $G_{i,j}$, which we will call the *red* and *green* quadrilaterals, respectively, are induced 4-cycles of G_9 , and so by (2), each forms a convex quadrilateral in Γ . If both were on the same side of $L_{i,j}$, then either some edge of $R_{i,j}$ intersects some edge of $G_{i,j}$, which is forbidden by (1), or one of the two quadrilaterals is contained in the other. Suppose that $R_{i,j}$ contains $G_{i,j}$; in particular $R_{i,j}$ contains the vertex g_i . Now b_k ($k \neq i, j$) is adjacent to all of $\{b_i, b_j, r_i, r_j\}$ and the triangles $\triangle(b_k, b_i, b_j)$, $\triangle(b_k, b_j, r_i)$, $\triangle(b_k, r_i, r_j)$, and $\triangle(b_k, r_j, b_i)$ completely cover $R_{i,j}$. This means, however, that one of the four triangles contains g_i , violating Corollary 3, thus $R_{i,j}$ and $G_{i,j}$ must lie on opposite sides of $L_{i,j}$.

Now consider the triangle $\Delta = \triangle(b_1, b_2, b_3)$ and the three lines $L_{1,2}, L_{2,3}, L_{3,1}$ determined by its edges. We will refer to the half-plane of $L_{i,j}$ not containing Δ as the *outer half-plane* of $L_{i,j}$. Since each outer half-plane $L_{i,j}$ contains one of $R_{i,j}$ and $G_{i,j}$, then by the pigeon-hole principle, there are either at least two red quadrilaterals or two green quadrilaterals contained in their respective outer half-planes. So assume that, w.l.o.g., $R_{i,j}$ and $R_{i,k}$ are contained in the outer half-planes of $L_{i,j}$ and $L_{i,k}$, respectively. Then, in particular, r_i is contained in the intersection of these two outer half-planes. But this implies that b_i is contained in the triangle $\triangle(r_i, b_j, b_k)$, violating Corollary 3. It follows that no witness Gabriel drawing of G_9 can be linearly separable.

With a similar reasoning as in the proof of Lemma 11, we now show that every witness Gabriel drawing Γ of G_9 must be convexly separable. Suppose that there existed a witness w in the interior of the convex hull of Γ . Consider the smallest infinite wedge with apex w containing the blue vertices of Γ and extend the rays of the wedge to two lines that intersect at w . These two lines define four wedges each of them having w as an apex. We call these wedges the *top wedge*, *bottom wedge*, *left wedge*, and *right wedge* of w , respectively. The top (left) wedge of w is opposite to its bottom (right) wedge. We assume without loss of generality that the three blue vertices lie in the right wedge of w and denote as b_i and b_j the two blue vertices along the boundary of this wedge; see Figure 9 for an illustration.

We show that no edge of the convex hull of Γ can intersect the left wedge of w . First observe that there cannot be a vertex z in the left wedge of w : If vertex z existed it would be adjacent to one of $\{b_i, b_j\}$ and therefore w would be a point in the interior of the triangle whose corners are b_i , b_j , and z , which contradicts Corollary 2. Furthermore, no edge zt of



■ **Figure 10** Coordinates of the vertices and witnesses in a linearly separable witness Gabriel drawing of G_9 minus a blue vertex.

the convex hull of Γ can cross the left wedge of w : Neither z nor t is a blue vertex; also $I(\{z, t\})$ has size at most 2, so there is some blue vertex b adjacent to both z and t . The wedge formed by these three vertices contains w , violating Corollary 2.

Since no edge of the convex hull of Γ can intersect the left wedge of w , it follows that w is not in the interior of the convex hull of Γ .

To conclude the proof, we need to show that every proper subgraph of G_9 has a linearly separable Gabriel drawing. It is sufficient to show that deleting any vertex of G_9 results in a graph that has a linearly separable witness Gabriel drawing. Figures 10 and 11 depict linearly separable witness Gabriel drawings (and exact coordinates) of G_9 minus a green vertex and of G_9 minus a blue vertex, respectively. ◀

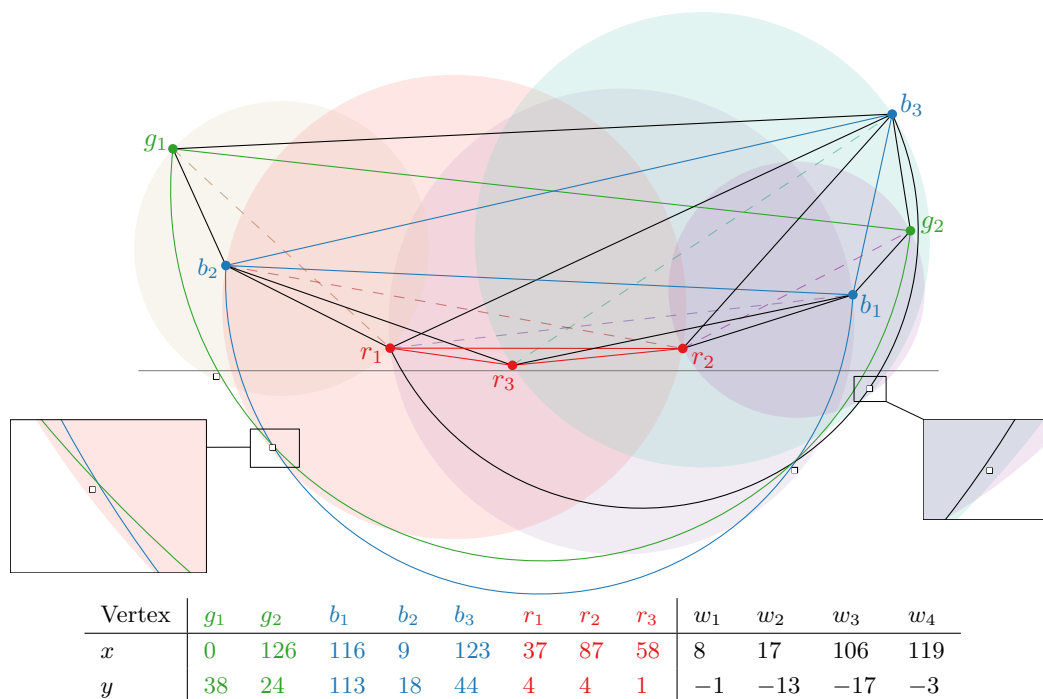
Figure 8b depicts a convexly separable witness Gabriel drawing of G_9 , while Figures 10 and 11 depict linearly separable witness Gabriel drawings and exact coordinate of G_9 minus a green vertex and of G_9 minus a blue vertex, respectively. Lemma 12 complements Lemma 11 by providing an example which is a triangulated planar graph.

If we denote as $\mathcal{CSWGD}$ the set of those witness Gabriel drawable graphs whose witness Gabriel drawings must be convexly separable, and as $\mathcal{LSWGD}$ the set of those witness Gabriel drawable graphs that admit a linearly separable witness Gabriel drawing, then the results of this section can be summarized as follows.

► **Theorem 13.** $\mathcal{LSWGD} \subsetneq \mathcal{CSWGD}$. Also, $\mathcal{CSWGD} \setminus \mathcal{LSWGD}$ contains both bipartite graphs and planar triangulations.

6 Concluding Remarks and Open Problems

In this paper we have studied witness Gabriel drawings where the sets of vertices and the set of witnesses are either linearly or convexly separable. We showed that every outerplanar graph admits a linearly separable witness Gabriel drawing. We remark that the witness



■ **Figure 11** Drawing and exact coordinates of the vertices and witnesses in a linearly separable witness Gabriel drawing of G_9 minus a green vertex.

Gabriel drawability of outerplanar graphs was known only for trees and, in addition, the known construction was not giving rise to separable realizations. We then characterized the triangle-free graphs that admit a linearly separable witness Gabriel drawing. Finally we proved that the family of linearly separable witness Gabriel drawable graphs is a proper subset of the convexly separable ones.

Several open problems naturally arise from our results. We conclude the paper by listing some of those that, in our opinion, are among the most interesting.

- Further investigate the relationship between border parabola graphs and linearly separable witness Gabriel drawable graphs. Specifically, are there graphs that admit a linearly separable Gabriel drawing but do not have a border parabola drawing?
- Extend the characterization of Theorem 10 to graphs that have 3-cycles.
- Characterize the convexly separable witness Gabriel drawings.
- Finally, an immediate consequence of Lemma 12 is that there are two planar triangulations (namely two copies of G_9) which do not admit a mutual witness Gabriel drawing. A mutual witness Gabriel drawing consists of a pair of witness Gabriel drawings where the vertex set of one drawing acts as the witness set for the other and vice versa. Clearly, if every witness Gabriel drawing of G_9 is convexly separable, any mutual witness Gabriel drawing of two copies of G_9 should be linearly separable, which is however impossible by Lemma 12. It would be interesting to further study pairs of planar graphs that admit or do not admit a mutual witness Gabriel drawing. Results in this direction are known for pairs of isomorphic trees and for bipartite graphs [9, 13, 14].

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