Structural Parameterizations of k-Planarity

Tatsuya Gima
Hokkaido University, Sapporo, Japan

Yasuaki Kobayashi
Hokkaido University, Sapporo, Japan

Yuto Okada
Nagoya University, Japan

Abstract

The concept of k-planarity is extensively studied in the context of Beyond Planarity. A graph is k-planar if it admits a drawing in the plane in which each edge is crossed at most k times. The local crossing number of a graph is the minimum integer k such that it is k-planar. The problem of determining whether an input graph is 1-planar is known to be NP-complete even for near-planar graphs [Cabello and Mohar, SIAM J. Comput. 2013], that is, the graphs obtained from planar graphs by adding a single edge. Moreover, the local crossing number is hard to approximate within a factor $2 - \varepsilon$ for any $\varepsilon > 0$ [Urschel and Wellens, IPL 2021]. To address this computational intractability, Bannister, Cabello, and Eppstein [JGAA 2018] investigated the parameterized complexity of the case of $k = 1$, particularly focusing on structural parameterizations on input graphs, such as treedepth, vertex cover number, and feedback edge number. In this paper, we extend their approach by considering the general case $k \geq 1$ and give (tight) parameterized upper and lower bound results. In particular, we strengthen the aforementioned lower bound results to subclasses of constant-treewidth graphs: we show that testing 1-planarity is NP-complete even for near-planar graphs with *feedback vertex set number at most 3* and *pathwidth at most 4*, and the local crossing number is hard to approximate within *any constant factor* for graphs with *feedback vertex set number at most 2*.

2012 ACM Subject Classification Theory of computation → Parameterized complexity and exact algorithms

Keywords and phrases 1-planar graphs, local crossing number, beyond planarity, parameterized complexity, kernelization

Digital Object Identifier 10.4230/LIPIcs.GD.2025.16

Related Version *Full Version*: https://arxiv.org/abs/2506.10717

Funding *Tatsuya Gima*: JSPS KAKENHI Grant Number JP24K23847 and JP25K03077.
Yasuaki Kobayashi: JSPS KAKENHI Grant Numbers JP23K28034, JP24H00686, and JP24H00697.
Yuto Okada: Supported by JST SPRING, Grant Number JPMJSP2125.

Acknowledgements We would like to thank Alexander Wolff for valuable discussions in Würzburg and anonymous reviewers for insightful comments.

1 Introduction

Graph drawing is recognized as an important area in the research of graph theory and algorithms due to various real-world applications. In particular, efficient algorithms for drawing graphs on the plane without edge crossings have received considerable attention from various perspectives. However, to visualize real-world networks, it is necessary to consider drawing *non-planar* graphs in many cases. This research direction is positioned as Beyond Planarity [8, 11, 21], and a significant amount of effort has been dedicated to analyzing their graph-theoretic properties and designing algorithms for drawing these graphs. In this context, the problem of drawing an input graph on the plane with the minimum number of crossings is one of the most well-studied problems. In other words, the problem asks for

Table 1 The table summarizes our and known results. The columns "unbounded", "parameter", and "$k = 1$" indicate the results when k is taken as input, as parameter, and $k = 1$, respectively.

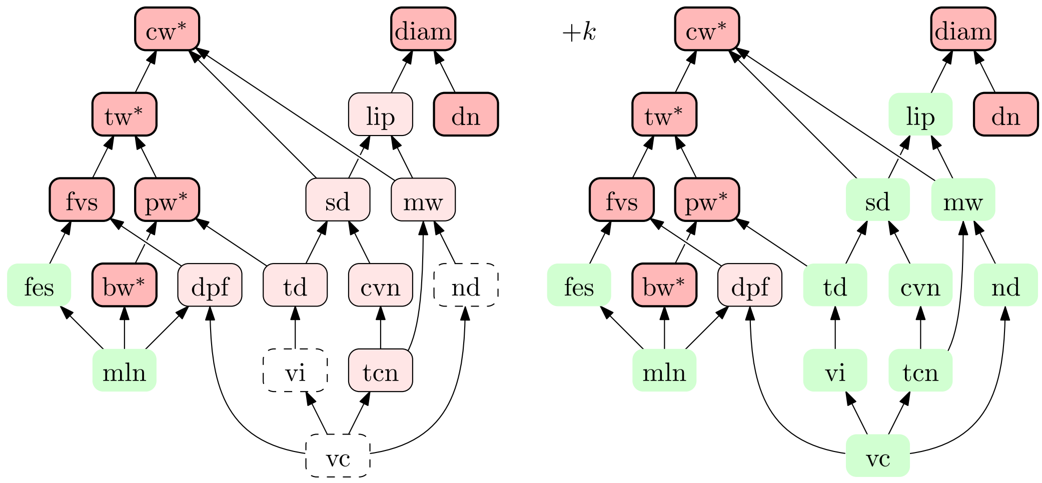
<table>
<tr><th></th><th>k: unbounded</th><th>k: parameter</th><th>$k = 1$</th></tr>
<tr><td>feedback edge set number</td><td colspan="2">FPT (Thm. 14)</td><td>FPT [2]</td></tr>
<tr><td>feedback vertex set number</td><td colspan="3">NP-complete when fvs = 2 (Thm. 5)</td></tr>
<tr><td>treedepth</td><td>W[1]-hard (Thm. 6)</td><td>FPT (Thm. 16) (non-uniform FPT [34])</td><td>FPT [2]</td></tr>
<tr><td>longest induced path</td><td rowspan="2">W[1]-hard (Thm. 13)</td><td rowspan="2" colspan="2">FPT (Cor. 18)</td></tr>
<tr><td>twin cover number</td></tr>
<tr><td>distance to path forest</td><td colspan="3">W[1]-hard (Thm. 9)</td></tr>
<tr><td>domination number</td><td colspan="3">NP-complete when dn = 2 (Cor. 10)</td></tr>
</table>

the *crossing number* of an input graph, which is known to be NP-hard [15]. Among various studies on this problem, the (recent) progress on the parameterized complexity of Crossing Number, where the goal is to determine whether an input graph G can be drawn in the plane with at most k crossings, is remarkable [7, 18, 23, 27]. In particular, this problem is *fixed-parameter tractable* (FPT) when parameterized by k, that is, it admits an algorithm with running time $f(k)n^{O(1)}$, where n is the number of vertices in the input graph and f is a computable function.

The *local crossing number* is one of the well-studied variations of the standard crossing number [1, 2, 4, 17, 20, 26, 31, 33], as witnessed by the fact that it is selected as the topic for the live challenge[1] held in conjunction with GD 2025. Let k be an integer. We say that a graph G is *k-planar* if it can be drawn in the plane so that each edge involves at most k crossings. The minimum integer k such that G admits a k-planar drawing is called the *local crossing number* of G, denoted by $\mathrm{lcr}(G)$. The problem of deciding if $\mathrm{lcr}(G) \leq k$ for a given graph G is called k-Planarity Testing. Unlike Crossing Number, it is already NP-complete to decide whether $\mathrm{lcr}(G) \leq 1$ [2, 4, 17, 26]. More strongly, this problem is NP-complete even when the input is restricted to planar graphs with an extra edge [4] or graphs with constant bandwidth [2]. Urschel and Wellens [33] later extended the NP-hardness by showing that k-Planarity Testing is NP-complete for any fixed $k \geq 1$, even if the input is restricted to graphs with local crossing number at most k or at least $2k$. This implies that, unless P = NP, there is no polynomial-time $(2 - \varepsilon)$-approximation for local crossing number for any $\varepsilon > 0$.

To overcome such an intractability, Bannister, Cabello, and Eppstein [2] pursued the parameterized complexity of the case of $k = 1$, namely 1-Planarity Testing, by focusing on *structural parameterizations*. They showed that 1-Planarity Testing is FPT parameterized by treedepth and by feedback edge number (see Section 2 for definitions). As mentioned above, 1-Planarity Testing is NP-complete even on the classes of graphs with bounded bandwidth. This intractability is indeed inherited by wider classes of graphs, such as those with bounded cliquewidth, treewidth, and pathwidth. In this direction, Münch, Pfister, and Rutter [28] recently considered k-Planarity Testing on special cases of pathwidth-bounded graphs and gave exact and approximation algorithms.

[1] `https://mozart.diei.unipg.it/gdcontest/2025/live/`



(a) **(b)**

Figure 1 A visualization of Table 1 of the cases where k is (a) unbounded and (b) parameter. No borderline, dotted borderline, normal borderline, bold borderline mean that the complexity parameterized by the parameter is FPT, unknown, W[1]-hard, paraNP-complete, respectively. The complexities for parameters with ($*$) are already known results and the others are our results. An arrow indicates that the upper parameter is bounded from above by a function of the lower parameter. See the footnote[2] for the abbreviations of parameter names.

In this paper, we extend results of [2] by considering the general case $k \geq 1$ and show a fine parameterized complexity landscape with respect to graph width parameters (see Table 1 and Figure 1 for a summary of our results[2]). We extend algorithms of [2] by showing that k-Planarity Testing is FPT parameterized by feedback edge set number (Theorem 14) and by treedepth plus k (Theorem 16). More generally, we show that k-Planarity Testing is FPT on P_t-free graphs when parameterized by $t+k$ (Corollary 18). We also give polynomial kernelizations for k-Planarity Testing with respect to vertex cover number (Theorem 23) and neighborhood diversity (Corollary 25). On the negative side, we show that k-Planarity Testing is W[1]-hard when parameterized solely by treedepth (Theorem 6), which indicates that Theorem 16 is tight in a certain sense, and by twin cover number (Theorem 13). We also show that 1-Planarity Testing is W[1]-hard parameterized by distance to path forest (Theorem 9). Similar to [4], the last result also proves that 1-Planarity Testing remains NP-complete even on planar graphs with a single additional edge. Our reduction shows that 1-Planarity Testing is NP-complete even when further adding restrictions that the input graph has feedback vertex set number at most 3. Using a completely different reduction, we show that 1-Planarity Testing is NP-complete even if the input is restricted to graphs with feedback vertex set number 2 (Theorem 5). This reduction also shows that there is no constant factor approximation for k-Planarity Testing unless P $=$ NP, which significantly strengthens the $(2-\varepsilon)$-inapproximability result of Urschel and Wellens [33].

Several proofs (marked with $\star$) are omitted and available in the full version.

[2] In Figure 1, the names are abbreviated as follows: cw is clique width, diam is diameter, tw is treewidth, lip is longest induced path, dn is domination number, fvs is feedback vertex set, pw is pathwidth, sd is shrub-depth, mw is modular-width, fes is feedback edge set, bw is bandwidth, dpf is distance to path forest, td is treedepth, cvn is cluster vertex deletion number, nd is neighborhood diversity, mln is max leaf number, vi is vertex integrity, tcn is twin cover number, and vc is vertex cover number.

2 Preliminaries

Let G be a graph. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex set $X \subseteq V(G)$ (resp. edge set $X \subseteq E(G)$), $G - X$ denotes the graph obtained from G by deleting all elements in X. The subgraph of G induced by $X \subseteq V(G)$ is denoted by $G[X]$. For $v \in V(G)$, $N_G(v)$ denotes the set of neighbors of v.

A (*topological*) *drawing* Γ of G is a representation in the plane that maps the vertices of G to distinct points in the plane and each edge of G to a non-self-intersecting (Jordan) curve connecting the points corresponding to the endpoints. In the rest of this paper, we may simply refer to these points as vertices and these curves as edges when no confusion is possible. A *crossing* in Γ is an intersection of distinct edges that is not a common endpoint. We assume that, in any drawing, an edge does not pass through any vertex other than its endpoints, and no three (or more) edges cross at a common point. The *crossing number* of G, denoted by $\mathrm{cr}(G)$, is the minimum number of crossings over all possible drawings of G. For an integer k, a drawing Γ is said to be *k-planar* if each edge is crossed at most k times. A graph is said to be k-planar if it admits a k-planar drawing. The *local crossing number* of G, denoted by $\mathrm{lcr}(G)$, is the minimum integer k such that G is k-planar. Throughout this paper, we use the following property of local crossing number.

▶ **Lemma 1** ($\star$). *Let G be a graph and k be a positive integer. Let G_k be the graph obtained from G by subdividing each edge $k-1$ times. Then,* $\mathrm{lcr}(G) \leq k$ *if and only if* $\mathrm{lcr}(G_k) \leq 1$.

In this paper, we present parameterized algorithms and hardness results with respect to several graph width parameters. However, some of them may not need to be defined precisely. Hence, we only provide the definitions that are directly needed to present our results. For basic concepts in parameterized complexity, we refer the reader to [5].

Let G be a graph. A *vertex cover* of G is a set of vertices S such that $G - S$ is edge-less, and the *vertex cover number* of G is the minimum size of a vertex cover of G, which is denoted by $\mathrm{vc}(G)$. The *feedback vertex set number* (resp. *feedback edge set number*) of G, denoted by $\mathrm{fvs}(G)$ (resp. $\mathrm{fes}(G)$), is the minimum cardinality of a vertex set (resp. edge set) X such that $G - X$ is acyclic. An *elimination forest* of G is a rooted forest F (i.e., a set of rooted trees) such that $V(F) = V(G)$ and for every edge in G, one of the endpoints is an ancestor of the other in F. The *treedepth* of G, denoted by $\mathrm{td}(G)$, is the minimum integer k such that G has an elimination forest of height k. Two vertices u and v are called *true twins* if they are adjacent and $N_G(u) = N_G(v)$; they are called *false twins* if they are non-adjacent and $N_G(u) = N_G(v)$. The *neighborhood diversity* of G is at most k if the vertex set of G can be partitioned into k sets $V_1, \dots, V_k$ such that either all the pairs in V_i are true twins or they are false twins. Each set V_i is called a *twin class*. Note that each true twin class induces a clique and each false twin class induces an independent set in G. The neighborhood diversity of G is denoted by $\mathrm{nd}(G)$. A vertex set $S \subseteq V(G)$ is called a *twin cover* of G if each connected component of $G - S$ consists of true twins in G. The *twin cover number* of G is the minimum size of a twin cover of G. The *max leaf number* of G is the maximum number of leaves of a spanning tree of G.

3 Hardness

In this section, we give several intractability results for k-Planarity Testing and 1-Planarity Testing. In particular, we show that k-Planarity Testing is hard to approximate within any constant factor in polynomial time, even on graphs with feedback vertex set number at most 2, and 1-Planarity Testing is NP-complete even on near-planar graphs with feedback vertex set number 3, where a graph is *near-planar* if it can be obtained from a planar graph by adding a single edge. These two results improve the previous results of [33] and [4]. To this end, we design two completely different reductions, one is given from TWO-SIDED k-PLANARITY and the other is given from UNARY BIN PACKING, which also yield several consequences other than these two results.

3.1 Inapproximability for Graphs with Feedback Vertex Set Number 2

In this subsection, we show that it is NP-hard to approximate the local crossing number of a given graph within any constant factor, even if the graph has feedback vertex set number 2.

Before describing our reductions, we start with a technical lemma, showing that by adding some vertices and edges to a graph, one can impose a certain form on its k-planar drawings.

Let G be a graph with m edges and let k be a positive integer. Let $X \subseteq V(G)$ be a nonempty vertex subset. We define a new graph G' by adding a vertex r and $km+1$ paths of length 2 between r and v for each $v \in X$ to G.[3] We refer to these paths of length 2 as *spokes*.

▶ **Lemma 2** ($\star$). *Suppose that G' has a k-planar drawing Γ. Then, there is a k-planar drawing Γ' of G' such that each spoke has no crossings and the subdrawings of Γ and Γ' induced by G are identical.*

Proof (sketch). Since the edges in G can cross other edges km times in total, for each $u \in X$, there is a spoke between r and u that does not cross any edges of G. Let P_u be such a spoke for each $u \in X$ and $\mathcal{P} \coloneqq \{P_u \mid u \in X\}$. Although spokes in $\mathcal{P}$ may cross, we can untangle them, preserving the subdrawing for G. Then we redraw the other spokes along $\mathcal{P}$. ◀

We will use this lemma in the following form.

▶ **Corollary 3.** *Let $S \subseteq V(G)$. Let G' be a graph obtained from G by adding a vertex s and $k|E(G)|+1$ spokes between s and each $u \in S$. Moreover, let $S' \subseteq V(G')$ and let G'' be a graph obtained from G' by adding a vertex s' and $k|E(G')|+1$ spokes between s' and each $u \in S'$. Suppose that G'' has a k-planar drawing. Then, there is a k-planar drawing of G'' such that all spokes have no crossings.*

Proof. By Lemma 2, G'' has a k-planar drawing Γ' in which all the spokes incident to s' have no crossings. Let Γ be the subdrawing of Γ' induced by G'. As shown in the proof of Lemma 2, for each $u \in S$, there is a spoke P_u between s and u that has no crossing with any edge of G in Γ. Since this spoke P_u does not cross any spoke incident to s', the uncrossing operation and redrawing spokes between s and u used in Lemma 2 never create a new crossing with spokes incident to s'. Thus, by applying Lemma 2 to G' and Γ, we have a k-planar drawing of G'' being claimed. ◀

We now turn to our reduction from TWO-SIDED k-PLANARITY, in which we are given a bipartite graph $G = (X \cup Y, E)$ with two independent sets X and Y, and an integer k. The goal is to determine whether G has a *2-layer k-planar drawing*, which is a special case of a k-planar drawing in which the vertices in X are drawn on a line, the vertices in Y are drawn on another line parallel to the first line, and each edge is drawn as a straight line.

[3] The length of a path is defined as the number of edges in it.

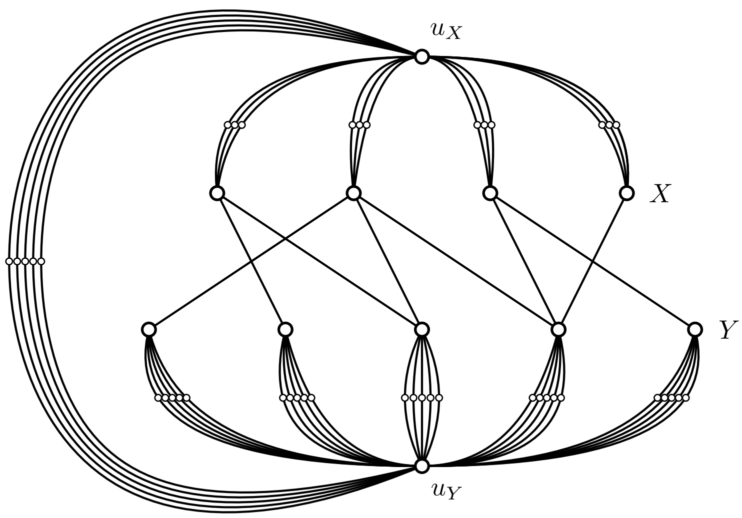


Figure 2 An illustration of the graph G' constructed from an instance $\langle G, k\rangle$ of Two-Sided k-Planarity. Note that the numbers of spokes are different from that of the actual construction.

The idea of our reduction is the same as that of Garey and Johnson [15] from Bipartite Crossing Number to Crossing Number. We remark that, however, the proof is more involved due to the difficulty of k-planarity.

Let $\langle G = (X, Y, E), k\rangle$ be an instance of Two-Sided k-Planarity, where $n_X = |X|$, $n_Y = |Y|$, and $m = |E|$. We construct a graph G' from G as follows (see Figure 2):

1. we introduce two vertices u_X, u_Y;
2. for each $x \in X$, add $\ell_1 := km + 1$ spokes between u_X and x;
3. for each $y \in Y \cup \{u_X\}$, add $\ell_2 := k(\ell_1 n_X + m) + 1$ spokes between u_Y and y.

Then, $\langle G', k\rangle$ is the instance we construct for k-Planarity Testing. It is easy to verify that the above construction can be done in polynomial time.

▶ **Lemma 4.** *The graph G admits a 2-layer k-planar drawing if and only if the graph G' admits a k-planar drawing.*

Proof. It is clear that we can obtain a k-planar drawing of G' from a 2-layer k-planar drawing of G by drawing u_X, u_Y, and spokes as Figure 2. Hence, we only show the other direction.

Suppose that the graph G' admits a k-planar drawing Γ'. By Corollary 3, we can assume that all the spokes in G' have no crossings in Γ'. We then replace the set of spokes connecting two vertices with a single edge between them and contract the edge between u_X and u_Y by identifying them into a new vertex u in Γ'. We let G^* denote the graph obtained in this way. As all spokes have no crossings, the obtained drawing of G^*, denoted by Γ^*, is still k-planar. Let us note that G^* is isomorphic to the graph obtained from G by adding a universal vertex u (which is a vertex adjacent to all the vertices in G). We claim that we can draw a closed curve C passing through all the vertices in $V(G^*) \setminus \{u\}$ such that

1. no edge of G^* intersects C (i.e., C is a noose in Γ^*);
2. all the edges incident to u are contained in one of the two regions R with boundary C;
3. all the edges in $E(G)$ are contained in the other region with boundary C;
4. all the vertices in X appear consecutively on C;
5. all the vertices in Y appear consecutively on C.

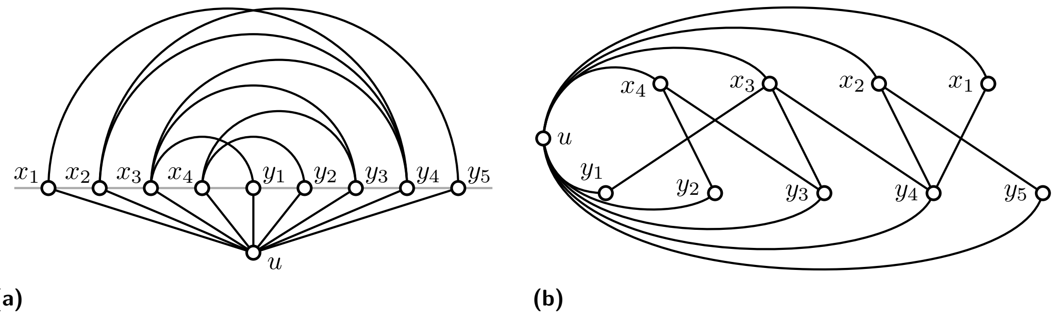


Figure 3 Two topologically equivalent drawings of G^*, where the subdrawing induced by G is (a) an arc diagram, and (b) a 2-layer drawing.

To this end, we draw a curve C while keeping track of the cyclic ordering of the edges incident to u (and hence the neighbors of u) in Γ^*. As these incident edges have no crossings, we can draw such a curve satisfying (1), (2), and (3). To see the properties (4) and (5), suppose for contradiction that there are $x, x' \in X$ and $y, y' \in Y$ such that x, y, x', y' appear in this order on C. Due to the property (2), C can be seen as a closed curve on Γ' such that the region corresponding to R contains all spokes of G'. Let P_x be a path in G' between x and x' via u_X consisting only of spokes. We define P_y similarly. In Γ' those paths are contained in R and C. However, since the vertices x, x', y, y' are all placed on C, such a drawing must yield a crossing, which contradicts the fact that spokes in G' have no crossings in Γ'.

Let $X = \{x_1, \ldots, x_{n_X}\}$ and let $Y = \{y_1, \ldots, y_{n_Y}\}$ such that $x_1, \ldots, x_{n_X}, y_1, \ldots, y_{n_Y}$ appear on C in this order. We now convert the drawing Γ^* into a drawing $\hat{\Gamma}$ such that

- all the vertices in $V(G^*) \setminus \{u\}$ are placed on a line ℓ in the order $x_1, \ldots, x_{n_X}, y_1, \ldots, y_{n_Y}$;
- all the edges incident to u are drawn as straight segments in a half plane separated by ℓ;
- all the other edges are drawn in the other half plane in such a way that the curve corresponding to an edge forms a semicircular arc between the endpoints.

See Figure 3a for an illustration. Observe that this drawing $\hat{\Gamma}$ is also k-planar as two edges of G^* cross in $\hat{\Gamma}$ only if they cross in Γ^* as well. Moreover, $\hat{\Gamma}$ is *simple*, meaning that no pair of edges crosses more than once and no two edges incident to a common vertex cross. This in turn can be converted into a k-planar drawing Γ as in Figure 3b. The subdrawing of Γ induced by $G^* - \{u\}$ is a 2-layer k-planar drawing such that two edges in $G^* - \{u\}$ cross if and only if they cross in $\hat{\Gamma}$. Hence, G $(= G^* - \{u\})$ admits a 2-layer k-planar drawing. ◀

This reduction immediately gives us the following consequence. Note that Theorem 5 is optimal in the sense that every graph with feedback vertex set number 1 is planar.

▶ **Theorem 5.** *1-Planarity Testing is NP-complete even if the given graph has feedback vertex set number* 2.

Proof. The NP-membership of 1-Planarity Testing is shown in [26]. It is known that Two-Sided k-Planarity is NP-complete even on trees [25, Theorem 11]. Thus, the graph G' constructed in the proof of Lemma 4 has feedback vertex set number at most 2. This in turn implies the claim by Lemma 1. ◀

The above reduction leads to further consequences. To see this, we first sketch the reduction used in [25, Theorem 11]. They performed a polynomial-time reduction from Bandwidth to Two-Sided k-Planarity. Let T be a tree, b be a positive integer, and ℓ be an arbitrary even number with $\ell \geq 2b^2$. For each edge in T, each edge $e \in E(T)$ is subdivided once by introducing a new vertex w_e, and we add ℓ leaves adjacent to each original vertex in T. They showed that there is an ordering $v_1, \ldots, v_{|V(T)|}$ of $V(T)$ such that $|i-j| \leq b$ for any edge $\{v_i, v_j\} \in E(T)$ if and only if the obtained graph T' has a 2-layer k-planar drawing with $k = (\ell+4)(b-1)/2$.

▶ **Theorem 6.** *k-Planarity Testing is W[1]-hard with respect to treedepth, even on graphs with feedback vertex set number 2.*

Proof. It is known that the problem Bandwidth is W[1]-hard with respect to treedepth, even on trees [16]. The above reduction we sketched [25, Theorem 11] also preserves the boundedness of treedepth. To see this, we convert an elimination forest F^* of T to that of T' as follows. Starting from F^*, we add the ℓ leaves attached to each vertex as leaf children of it. For each edge $e \in E(T)$, one of the end vertices, say v, is a descendant of the other in F^*. We then add w_e as a leaf child of v in F^*. It is easy to verify that the rooted forest obtained in this way is an elimination forest of T' and its height increases by at most 1 compared to the original elimination forest of T, yielding that $\mathrm{td}(T') \leq \mathrm{td}(T)+1$. Hence, Two-Sided k-Planarity is also W[1]-hard with respect to treedepth, even on trees. Moreover, the reduction used in Lemma 4 increases the treedepth by at most 3, which implies the claim. ◀

▶ **Lemma 7.** *Unless P = NP, for any constant $c \geq 1$, there is no polynomial-time c-approximation algorithm for Two-Sided k-Planarity on trees.*

Proof. For any constant $c \geq 1$, it is NP-hard to distinguish the following cases: the bandwidth of T is at most b or more than cb for a given tree T and an integer b [10]. We set $\ell = 2t-4$ for sufficiently large t and construct a tree T' according to the above reduction from T. If the bandwidth of T is at most b, then we have $\mathrm{lcr}(T') \leq t(b-1)$. Otherwise, we have $\mathrm{lcr}(T') \geq tcb$. This implies that a polynomial-time c-approximation algorithm for Two-Sided k-Planarity on trees distinguishes the above two cases. ◀

Combining Lemma 4 and Lemma 7, we obtain the following.

▶ **Theorem 8.** *Unless P = NP, for any constant $c \geq 1$, there is no polynomial-time c-approximation algorithm for k-Planarity Testing, even if the given graph has feedback vertex set number 2.*

3.2 Distance to Path Forest

In this subsection, we show that 1-Planarity Testing is W[1]-hard parameterized by distance to path forest, i.e., the minimum number of vertices required to make the graph into a disjoint union of paths. The underlying idea of the proof is similar to the one in [17]. We perform a reduction from Unary Bin Packing. An instance I of Unary Bin Packing consists of a (multi)set of positive integers S and positive integers B and b such that $\sum_{x \in S} x = bB$ and $B = O(\mathrm{poly}(|S|))$. Then the goal is to determine if it is possible to partition S into b sets so that the sum of each set is exactly B. Unary Bin Packing is NP-complete when b is given as input [15] and W[1]-hard when parameterized by b [22].

▶ **Theorem 9.** *1-Planarity Testing is W[1]-hard with respect to distance to path forest.*

Proof. Let $\langle S = \{x_1, \ldots, x_{|S|}\}, B, b\rangle$ be an instance of Unary Bin Packing. In the following, we assume that $b \geq 3$. We construct an instance $\langle G = (V,E)\rangle$ of 1-Planarity Testing as follows. Let G be a graph consisting of vertices $u_1, u_2, v_1, \ldots, v_b$, and paths $P_1, \ldots, P_{|S|}$ with $|V(P_i)| = x_i$ for $1 \leq i \leq |S|$. Let $\mathcal{P} = \{P_1, \ldots, P_{|S|}\}$. We add a path of

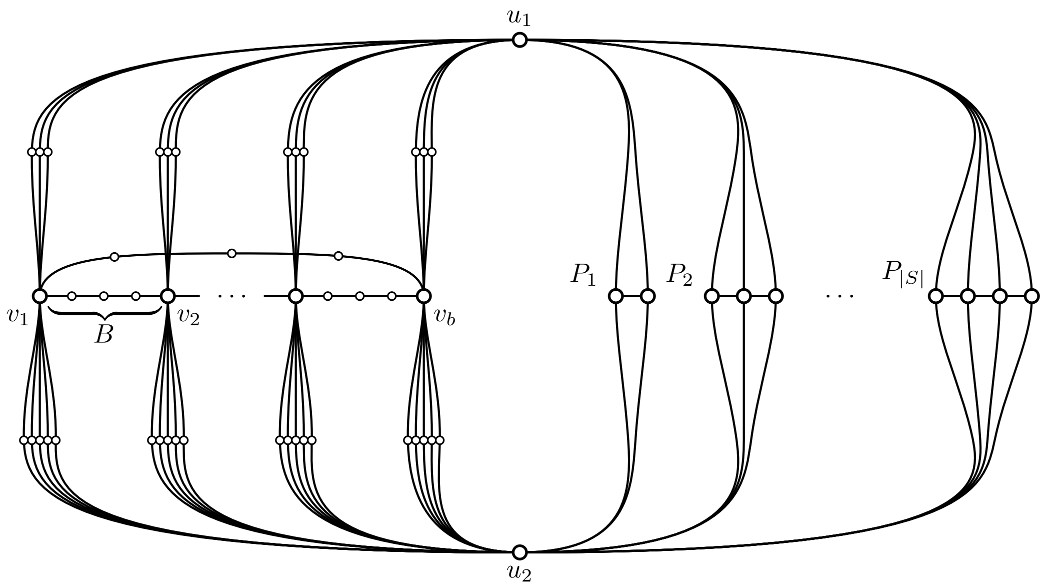


Figure 4 An illustration of the reduction for Theorem 9. Note that the number of depicted spokes between u_i and v_j is not accurate for aesthetic purposes.

length B between v_i and v_{i+1} for each $1 \leq i \leq b$, where $v_{b+1} \coloneqq v_1$. These paths form a cycle C of bB vertices, passing through v_i for all i. Next, we add an edge between u_i and each vertex in P_j for $1 \leq i \leq 2$ and $1 \leq j \leq |S|$. Let m be the number of edges in the graph constructed so far (namely, $m = 4bB - |S|$). Finally, for $1 \leq i \leq b$, we add $\ell_1 \coloneqq m+1$ spokes between u_1 and v_i and add $\ell_2 \coloneqq (\ell_1 b + m) + 1$ spokes between u_2 and v_i. See Figure 4 for an illustration of the constructed graph G. Observe that, by construction, removing $b+2$ vertices, say $u_1, u_2, v_1, \dots, v_b$, yields a path forest. The construction can be done in polynomial time.

From a solution of Unary Bin Packing, we can obtain a 1-planar drawing of G easily, as in Figure 5. To be more precise, let $\{S_1, \dots, S_b\}$ be a solution for Unary Bin Packing. We first draw the cycle C as a circle in such a way that the spokes between u_1 and v_i partition the interior of the circle into b regions $R_1, \dots, R_b$, where R_i is enclosed by a spoke between u_1 and v_i, a spoke between u_1 and v_{i+1}, and a path between v_i and v_{i+1} on C. For each region R_i, we draw a path $P_j \in \mathcal{P}$ inside R_i if $x_j \in S_i$. Since the sum of the numbers of vertices of the paths drawn in R_j is exactly B, we can draw those paths so that each edge in the path between v_i and v_{i+1} is crossed exactly once. Hence, we can draw G so that each edge is crossed at most once.

For the other direction, suppose that G has a 1-planar drawing. By Corollary 3, there exists a 1-planar drawing Γ of the graph G such that all the spokes in G have no crossings. Since the spokes have no crossings in Γ, the subdrawing induced by them is planar, and hence there are two vertices v_i and v_j that belong to the outer cycle of this subdrawing. As $b \geq 3$, all the other v_ℓ's are drawn inside the region bounded by this outer cycle. Moreover, since two consecutive v_ℓ and $v_{\ell+1}$ are connected by a path in C, the two distinguished vertices v_i and v_j must be consecutive, namely $j = (i+1) \bmod b$. By appropriately renaming those vertices, we can assume that $G - \bigcup_{1 \leq i \leq |S|} V(P_i)$ are drawn as in Figure 6a. Then, there are b internally vertex-disjoint paths $Q_1, \dots, Q_b$ between u_1 and u_2 such that Q_i is a concatenation of spokes between u_1 and v_i and between v_i and u_2 for each i. These paths separate the plane into b regions, $R_1, \dots, R_b$ as Figure 6b. As each path P_i cannot lie in two different regions,

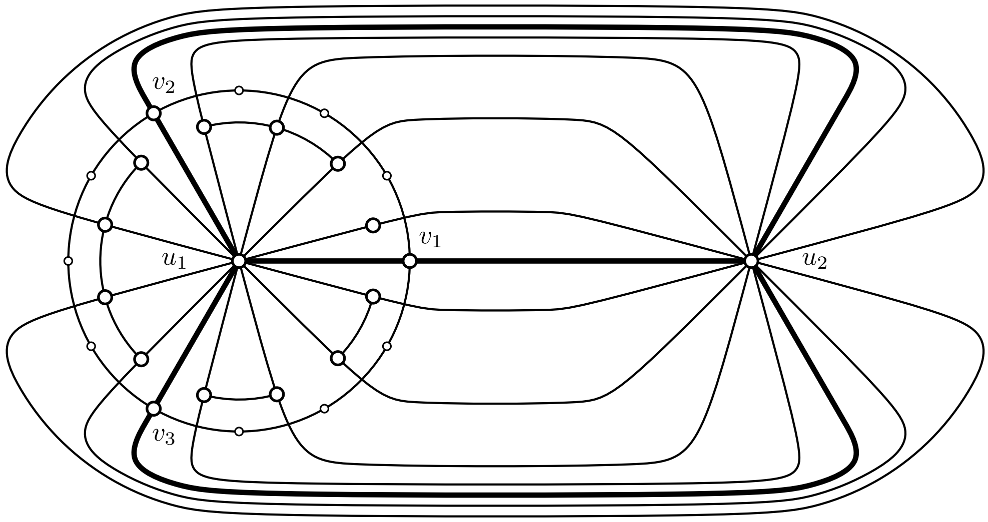


Figure 5 A 1-planar drawing of the graph constructed from an instance $\langle S = \{1, 2, 2, 3, 4\}, B = 4, b = 3\rangle$ of Unary Bin Packing. Each thick edge represents the spokes between its endpoints.

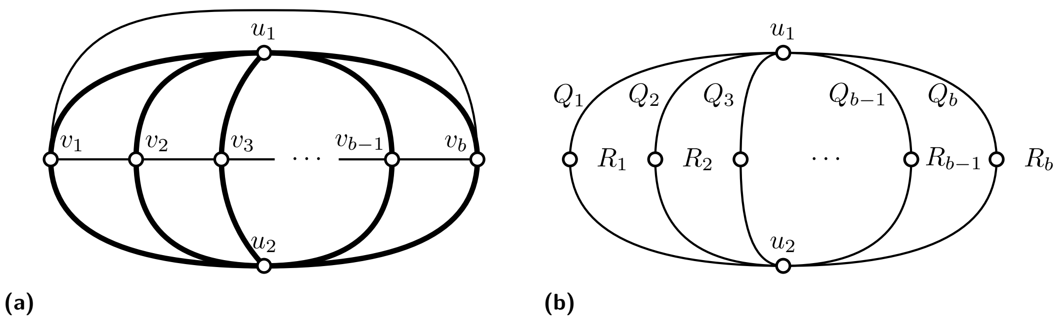


Figure 6 The subgraphs induced by (a) cycle C and spokes (boldlines) and (b) the paths $Q_1, \ldots, Q_b$, which defines b regions $R_1, \ldots, R_b$.

$\mathcal{P}$ can be partitioned into $\{\mathcal{P}_1, \dots, \mathcal{P}_b\}$, where $\mathcal{P}_i \subseteq \mathcal{P}$ is the collection of paths drawn inside R_i. Observe that, for each path $P_j \in \mathcal{P}_i$, there are $x_i = |V(P_i)|$ edge-disjoint paths between u_1 and u_2 that are drawn inside R_i. Each of them must cross the path between v_i and v_{i+1} on C, implying that there are at most B such paths inside R_i as Γ is 1-planar. Since there are exactly b regions, each $\mathcal{P}_i$ must satisfy $\sum_{P \in \mathcal{P}_i} |V(P)| = B$. Therefore, we can partition S into b sets such that the sum of each set is exactly B. ◀

Observe that the above reduction still works even if we add an edge between u_1 and each vertex in C, in which case the vertex set $\{u_1, u_2\}$ forms a dominating set.

▶ **Corollary 10.** *1-*Planarity Testing *is NP-complete even if the domination number is 2.*

As mentioned in Section 1, Cabello and Mohar [4] showed that 1-Planarity Testing is NP-hard even on near-planar graphs. Theorem 9 strengthens their hardness result as the graph G constructed in the reduction is a near-planar graph with the following properties.

▶ **Corollary 11.** *1-*Planarity Testing *is NP-complete on near-planar graphs with feedback vertex set number at most 3 and pathwidth at most 4.*

3.3 Twin Cover Number

In this subsection, we show that k-Planarity Testing is W[1]-hard with respect to twin cover number, which complements the positive result of the "$+k$ setting" to be presented in the subsequent section.

We give a similar reduction from Unary Bin Packing as in Section 3.2. However, in order to substitute cliques for paths, we first show the W[1]-hardness of a restricted version of Unary Bin Packing, where integers in S are sufficiently small that the constructed graph admits a B-planar drawing.

▶ **Lemma 12** ($\star$)**.** Unary Bin Packing *is W[1]-hard parameterized by the number b of bins, even if each integer $x \in S$ satisfies $x \le \sqrt{B} - 1$.*

Proof (sketch). We add $2bB$ copies of $B+1$ to S of an instance of Unary Bin Packing and increase the capacity of a bin by $2B(B+1)$ to obtain an instance with the claimed condition. We can show that there is no other solution than to distribute those $B+1$'s evenly to b bins, making it equivalent to the original instance. ◀

▶ **Theorem 13.** *k-*Planarity Testing *is W[1]-hard parameterized by twin cover number.*

Proof. The reduction is almost analogous to the one used in Theorem 9. Let $I = \langle S, B, b\rangle$ be an instance of Unary Bin Packing. We can assume that $b \ge 3$ and $x \le \sqrt{B} - 1$ for all $x \in S$ due to Lemma 12. While the basic construction of the graph G is the same as in Theorem 9, it differs in the following points:

- the cycle C contains exactly b vertices $v_1, \dots, v_b$;
- for each $x_i \in S$, G contains a clique C_i with x_i vertices, instead of a path P_i, such that each vertex in the clique is adjacent to both u_1 and u_2;
- for each v_i, we add $\ell_1 := Bm + 1$ spokes between u_1 and v_i and add $\ell_2 := B(b\ell_1 + m) + 1$ spokes between u_2 and v_2, where m is the number of edges in G that are not contained in the spokes.

When we draw G as in Figure 5, edges incident to a vertex in clique C_i may have crossings. Each edge e in C_i may cross other edges in C_i and all the edges between u_j and a vertex in C_i for $1 \le j \le 2$. This implies that the number of crossings that e involves is at most

$$|E(C_i)| - 1 + 2|V(C_i)| \le (\sqrt{B} - 1)(\sqrt{B} - 2)/2 - 1 + 2(\sqrt{B} - 1) < B.$$

Similarly, we can conclude that each edge between u_j and a vertex in C_i has at most $B - 1$ crossings. Since we can assume that all the spokes in G have no crossings due to Corollary 3, these spokes define exactly b regions, each of which contains exactly B vertices of cliques. Thus, we can show that I is a yes-instance if and only if G is B-planar as well.

As each connected component in $G - (V(C) \cup \{u_1, u_2\})$ consists of true twins, G has a twin cover of size at most $b + 2$. This completes the proof. ◀

4 FPT Algorithms

In contrast to the previous section, we mainly focus on the algorithmic aspects of k-Planarity Testing in this section and show that k-Planarity Testing is FPT when several well-known graph parameters (and k) are given as parameters.

4.1 Reducing to 1-Planarity Testing

There are several consequences from Lemma 1, combining known FPT algorithms of 1-Planarity Testing of [2].

Let G be a graph and let k be an integer. Let G_k be the graph obtained from G by subdividing each edge $k - 1$ times. Clearly, we have $\mathrm{fes}(G) = \mathrm{fes}(G_k)$. By Lemma 1, G is k-planar if and only if G_k is 1-planar. Since 1-Planarity Testing is FPT parameterized by feedback edge set number [2], the following theorem holds.

▶ **Theorem 14.** *k-Planarity Testing is FPT parameterized by feedback edge set number.*

We next observe that the treedepth of G_k is not much larger than the treedepth of G.

▶ **Lemma 15** ($\star$). *For an integer $k \ge 0$, it holds that $\mathrm{td}(G_k) \le \mathrm{td}(G) + \lceil \log_2 k \rceil$.*

Similarly to the above, we have the following theorem.

▶ **Theorem 16.** *k-Planarity Testing is FPT parameterized by treedepth $+ k$.*

We would like to note that a similar result is already mentioned in [34]. Since k-planarity is closed under vertex and edge deletions, it is known that the class of k-planar graphs with bounded treedepth is well-quasi-ordered by the induced-subgraph relation [30]. This implies a non-uniform version of Theorem 16: k-Planarity Testing can be solved by just checking whether G has an induced subgraph that belongs to a finite set $\mathcal{F}_{k,\mathrm{td}}$ of graphs, where the size of $\mathcal{F}_{k,\mathrm{td}}$ depends only on k and $\mathrm{td}(G)$.

Theorem 16 leads to FPT results for other parameters. To explain this, we introduce the notion of degeneracy. For an integer $d \ge 0$, a graph G is said to be *d-degenerate* if each nonempty subgraph of G contains a vertex of degree at most d. A graph class $\mathcal{C}$ is *degenerate* if there exists d such that every graph in $\mathcal{C}$ is d-degenerate.

▶ **Proposition 17** ([29, Proposition 6.4]). *A hereditary class of graphs $\mathcal{C}$ has bounded treedepth if and only if $\mathcal{C}$ is degenerate and does not contain a path P_t for some t as an induced subgraph.*

It is known that each k-planar graph with n vertices has at most $3.81\sqrt{k}n$ edges [1]. Since k-planarity is closed under taking a subgraph, the class of k-planar graphs is degenerate for every fixed k. Hence, combining with Proposition 17, k-Planarity Testing on P_t-free graphs parameterized by $k+t$ can be reduced to the one parameterized by treedepth $+\,k$. It is known that a graph class $\mathcal{C}$ does not contain a path P_t for some t as an induced subgraph if $\mathcal{C}$ has bounded shrub-depth [14] or bounded modular-width [24].

▶ **Corollary 18.** *Let $\mathcal{C}$ be a hereditary class of graphs that excludes some path P_t as an induced subgraph. Then, k-Planarity Testing over $\mathcal{C}$ is FPT parameterized by $k+t$. In particular, k-Planarity Testing is FPT parameterized by shrub-depth $+\,k$, and modular-width $+\,k$.*

4.2 Polynomial Kernels

In this subsection, we describe a kernelization algorithm for k-Planarity Testing parameterized by vertex cover number $+\,k$. The high-level ideas follow the polynomial kernel for 1-Planarity Testing due to Bannister, Cabello, and Eppstein [2].

Let G be a graph and let S be a vertex cover of G. We can find such a vertex cover S of size at most twice the optimum in linear time. First, we observe that the "diversity" of vertices outside of a vertex cover S is sufficiently small when G is k-planar. For a function f, let $\#_f(X)$ denote $|\{f(x) \mid x \in X\}|$, the number of possible images of X through f.

▶ **Lemma 19.** *Let $I_{\geq 2} \subseteq V(G) \setminus S$ be the set of vertices with degree at least 2 in $V(G) \setminus S$. Let π be a function that maps each vertex $v \in I_{\geq 2}$ to an unordered pair of its distinct neighbors (that is, $\pi\colon I_{\geq 2} \to \binom{N_G(v)}{2}$). If $\#_\pi(I_{\geq 2}) > |S| \cdot 3.81\sqrt{2k}$, then G is not k-planar.*

Proof. Let G_S be a graph with $V(G_S) = S$ and $E(G_S) = \pi(I_{\geq 2})$. Let G' be the graph obtained from G_S by subdividing each edge once. Then G' is a subgraph of G, and hence G' is k-planar. This implies that by Lemma 1, G_S is $2k$-planar. Hence, $\#_\pi(I_{\geq 2}) = |E(G_S)| \leq |S| \cdot 3.81\sqrt{2k}$, where the last inequality is shown in [1]. ◀

It is known that, for a positive integer k, if G contains $K_{7k+1,3}$ as a subgraph then G is not k-planar [6, 32]. Hence, the following also immediately holds.

▶ **Corollary 20.** *Let $u \in V(G) \setminus S$ be a vertex of degree at least 3. If there are at least $7k+1$ false twins of u (including u itself), then the graph G is not k-planar.*

Next, we consider vertices of degree 2. The following lemma is a key ingredient of our kernelization. It is worth mentioning that, although a similar reduction rule is used in the case $k = 1$ [2], the proof for the general case $k \geq 1$ is considerably more involved.

▶ **Lemma 21.** *Let $u \in V(G) \setminus S$ be a vertex of degree 2 with $N_G(u) = \{s_1, s_2\} \subseteq S$. If there are more than $16k^2|S|$ vertices in $V(G) \setminus S$ with the same neighborhood $\{s_1, s_2\}$, then G is k-planar if and only if $G - \{u\}$ is k-planar.*

Proof. The ($\Rightarrow$) direction is clear. Suppose that $G' := G - \{u\}$ is k-planar and let Γ be a k-planar drawing of G'. We can assume that G' has no vertex of degree 1.

Let $T_u = \{v \in V(G') \setminus S \mid N_{G'}(v) = \{s_1, s_2\}\}$. Then, we have $t_u = |T_u| \geq 16k^2|S|$. Let $\mathcal{P}$ be the set of paths between s_1 and s_2 via a vertex in T_u. Let us align the paths in $\mathcal{P}$ according to the cyclic order in Γ around s_1, as $P_0, P_1, \dots, P_{t_u - 1}$. For two paths P_i, P_j, the *cyclic difference* of P_i and P_j is defined as $\min\{j - i, i - j + t_u\}$. We then claim the following.

▷ Claim 22. Two paths $P_i, P_j \in \mathcal{P}$ with the cyclic difference at least $4k$ do not cross in Γ.

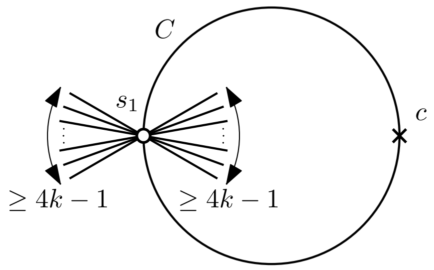


Figure 7 The Jordan curve C, the crossing point c, and the vertex s_1 with edges going out to each of the interior and exterior of C due to the cyclic difference condition.

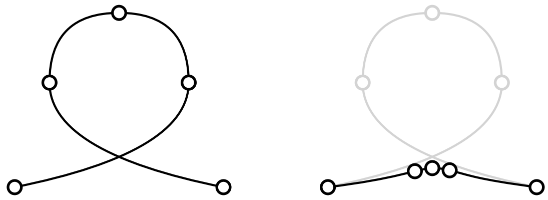


Figure 8 Let P be a path whose internal vertices have degree exactly 2. A "self-crossing" on the path can be removed by suppressing the "loop" formed by the edges in P.

Proof. Suppose that P_i and P_j do cross in Γ. Let c be a crossing point between P_i and P_j, and let C be a Jordan curve consisting of two arcs: the arc of P_i from s_1 to c and the arc of P_j from c to s_1. Such a crossing point c can be chosen so that C forms a closed curve. In both cases where s_2 is in the interior or in the exterior of regions bounded by C, from the condition that the cyclic difference of P_i and P_j is at least $4k$, there are at least $4k-1$ edges that must cross C as shown in Figure 7. As P_i, P_j can have at most $2k-1$ crossings other than c each, and hence $4k-2$ crossings in total, this leads to a contradiction. ◁

Let $\mathcal{P}' = \{P_{4ki} \mid 0 \leq i < 4k|S|\}$. As the cyclic difference of any pair of two paths in $\mathcal{P}'$ is at least $4k$, by Claim 22, no two paths in $\mathcal{P}'$ cross. We can also assume that each path in $\mathcal{P}'$ does not cross itself, as it can be resolved as shown in Figure 8. Hence, they separate the plane into $4k|S|$ regions similarly to Figure 6b. For each $0 \leq i \leq 4k|S|$, let R_i be the region that is bounded by two paths P_{4ki} and $P_{4k(i+1)}$, where $P_{4k \cdot 4k|S|} := P_0$. Observe that, for each vertex $v \in S \setminus \{s_1, s_2\}$, the number of paths in $\mathcal{P}'$ crossed by an edge incident to $N_{G'}(v)$ is at most $4k$: when v belongs to region R_i, each edge incident to a vertex in $N_{G'}(v)$ can only be in regions at most $2k$ apart from R_i, namely, $\{R_{j \bmod 4k|S|} \mid i-2k \leq j \leq i+2k\}$. Since S is a vertex cover of G', for every edge $e \in E(G')$, $e = \{s_1, s_2\}$ (if it exists); e is contained in a path in $\mathcal{P}$; or e is incident to some vertex in $S \setminus \{s_1, s_2\}$ due to the assumption that there is no degree-1 vertex. Since at most k paths in $\mathcal{P}'$ can be crossed by $e = \{s_1, s_2\}$ (if it exists) and at most $4k(|S|-2)$ paths in $\mathcal{P}'$ can be crossed by an edge incident to a vertex in $S \setminus \{s_1, s_2\}$, there are at least $4k|S| - (4k(|S|-2)) - k = 7k$ paths in $\mathcal{P}'$ that are crossed only by the paths in $\mathcal{P}$ or not crossed at all.

Let $P \in \mathcal{P}'$ be such a path. We remove all the paths in $\mathcal{P}$ but P from Γ. Observe that now P has no crossings at all. Hence, we can redraw all the removed paths along P without making a crossing. We can also add the path (s_1, u, s_2) to Γ along P in the same way and obtain a k-planar drawing of G. ◀

By summarizing the above lemmas, we can obtain a kernelization algorithm.

▶ **Theorem 23** (⋆). *k-Planarity Testing has a kernel with $O(\mathrm{vc}(G)^2 k^2 \sqrt{k})$ vertices. Moreover, such a kernel can be computed in linear time.*

We can obtain a similar result for the case of neighborhood diversity.

▶ **Lemma 24** (⋆). *The vertex cover number of a k-planar graph G is at most $O(\mathrm{nd}(G)\sqrt{k})$.*

By plugging Lemma 24 to Theorem 23, we have the following.

▶ **Corollary 25.** *k-Planarity Testing has a kernel with $O(\mathrm{nd}(G)^2 k^3 \sqrt{k})$ vertices.*

Finally, we remark that the results of Theorem 23 and Corollary 25 cannot be generalized to vertex integrity and modular-width parameterizations (see [13, 16] for their definitions) unless NP $\subseteq$ coNP/poly, respectively. Let $G_1, \ldots, G_t$ be graphs and G be the disjoint union of $G_1, \ldots, G_t$. Let $\mathrm{vi}(G)$ be the vertex integrity of G and let $\mathrm{mw}(G)$ be the modular-width of G. Then, it holds that (1) G is 1-planar if and only if G_i is 1-planar for all $1 \le i \le t$ and (2) $\mathrm{vi}(G) \le \max_i |V(G_i)|$ and $\mathrm{mw}(G) \le \max_i |V(G_i)|$. This implies that 1-Planarity Testing parameterized by vertex integrity and by modular-width is AND-cross-compositional [3, 9, 12], implying that unless NP $\subseteq$ coNP/poly, there is no polynomial kernelization for 1-Planarity Testing of size $\mathrm{poly}(\mathrm{vi}(G))$ or $\mathrm{poly}(\mathrm{mw}(G))$. As $\mathrm{td}(G) \le \mathrm{vi}(G)$, this kernelization lower bound also holds for treedepth parameterization.

5 Conclusion

In this paper, we study the parameterized complexity of k-Planarity Testing from the perspective of graph structural parameterizations and show several (tight) upper and lower bound results. We leave several interesting open problems relevant to our results.

When k is considered as input, we have only shown that k-Planarity Testing is FPT parameterized by feedback edge set number. It would be interesting to seek similar results using other graph parameters. In this direction, a highly related problem Crossing Number is known to be FPT parameterized by vertex cover number [19]. However, it seems that their approach cannot be directly applied to k-Planarity Testing, requiring new insights for this problem. Towards this, showing intractability with a general parameter of vertex cover number, such as vertex integrity or neighborhood diversity (see Figure 1a), would also be a nice open problem.

References

1 Eyal Ackerman. On topological graphs with at most four crossings per edge. *Comput. Geom.*, 85, 2019. doi:10.1016/J.COMGEO.2019.101574.

2 Michael J. Bannister, Sergio Cabello, and David Eppstein. Parameterized complexity of 1-planarity. *J. Graph Algorithms Appl.*, 22(1):23–49, 2018. doi:10.7155/JGAA.00457.

3 Hans L. Bodlaender, Rodney G. Downey, Michael R. Fellows, and Danny Hermelin. On problems without polynomial kernels. *J. Comput. Syst. Sci.*, 75(8):423–434, 2009. doi:10.1016/J.JCSS.2009.04.001.

4 Sergio Cabello and Bojan Mohar. Adding one edge to planar graphs makes crossing number and 1-planarity hard. *SIAM J. Comput.*, 42(5):1803–1829, 2013. doi:10.1137/120872310.

5 Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015. doi:10.1007/978-3-319-21275-3.

6 Július Czap and Dávid Hudák. 1-planarity of complete multipartite graphs. *Discrete Applied Mathematics*, 160(4):505–512, 2012. doi:10.1016/j.dam.2011.11.014.

7 Éric Colin de Verdière and Thomas Magnard. An FPT algorithm for the embeddability of graphs into two-dimensional simplicial complexes. In *29th Annual European Symposium on Algorithms, ESA 2021*, volume 204 of *LIPIcs*, pages 32:1–32:17. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021. doi:10.4230/LIPICS.ESA.2021.32.

8 Walter Didimo, Giuseppe Liotta, and Fabrizio Montecchiani. A survey on graph drawing beyond planarity. *ACM Comput. Surv.*, 52(1):4:1–4:37, 2019. doi:10.1145/3301281.

9 Andrew Drucker. New limits to classical and quantum instance compression. *SIAM J. Comput.*, 44(5):1443–1479, 2015. doi:10.1137/130927115.

10 Chandan Dubey, Uriel Feige, and Walter Unger. Hardness results for approximating the bandwidth. *Journal of Computer and System Sciences*, 77(1):62–90, 2011. Celebrating Karp's Kyoto Prize. doi:10.1016/j.jcss.2010.06.006.

11 Vida Dujmovic, Seok-Hee Hong, Michael Kaufmann, János Pach, and Henry Förster. Beyond-planar graphs: Models, structures and geometric representations (dagstuhl seminar 24062). *Dagstuhl Reports*, 14(2):71–94, 2024. doi:10.4230/DAGREP.14.2.71.

12 Fedor V. Fomin, Daniel Lokshtanov, Saket Saurabh, and Meirav Zehavi. *Kernelization: Theory of Parameterized Preprocessing.* Cambridge University Press, 2019. doi:10.1017/9781107415157.

13 Jakub Gajarský, Michael Lampis, and Sebastian Ordyniak. Parameterized algorithms for modular-width. In Gregory Z. Gutin and Stefan Szeider, editors, *Parameterized and Exact Computation - 8th International Symposium, IPEC 2013, Sophia Antipolis, France, September 4-6, 2013, Revised Selected Papers*, volume 8246 of *Lecture Notes in Computer Science*, pages 163–176. Springer, 2013. doi:10.1007/978-3-319-03898-8_15.

14 Robert Ganian, Petr Hlinený, Jaroslav Nesetril, Jan Obdrzálek, and Patrice Ossona de Mendez. Shrub-depth: Capturing height of dense graphs. *Log. Methods Comput. Sci.*, 15(1), 2019. doi:10.23638/LMCS-15(1:7)2019.

15 M. R. Garey and D. S. Johnson. Crossing number is NP-complete. *SIAM Journal on Algebraic Discrete Methods*, 4(3):312–316, 1983. doi:10.1137/0604033.

16 Tatsuya Gima, Tesshu Hanaka, Masashi Kiyomi, Yasuaki Kobayashi, and Yota Otachi. Exploring the gap between treedepth and vertex cover through vertex integrity. *Theor. Comput. Sci.*, 918:60–76, 2022. doi:10.1016/J.TCS.2022.03.021.

17 Alexander Grigoriev and Hans L. Bodlaender. Algorithms for graphs embeddable with few crossings per edge. *Algorithmica*, 49(1):1–11, 2007. doi:10.1007/S00453-007-0010-X.

18 Martin Grohe. Computing crossing numbers in quadratic time. *J. Comput. Syst. Sci.*, 68(2):285–302, 2004. doi:10.1016/J.JCSS.2003.07.008.

19 Petr Hliněný and Abhisekh Sankaran. Exact crossing number parameterized by vertex cover. In *Graph Drawing and Network Visualization - 27th International Symposium, GD 2019*, volume 11904 of *Lecture Notes in Computer Science*, pages 307–319. Springer, 2019. doi:10.1007/978-3-030-35802-0_24.

20 Michael Hoffmann, Chih-Hung Liu, Meghana M. Reddy, and Csaba D. Tóth. Simple topological drawings of k-planar graphs. In David Auber and Pavel Valtr, editors, *Graph Drawing and Network Visualization - 28th International Symposium, GD 2020, Vancouver, BC, Canada, September 16-18, 2020, Revised Selected Papers*, volume 12590 of *Lecture Notes in Computer Science*, pages 390–402. Springer, 2020. doi:10.1007/978-3-030-68766-3_31.

21 Seok-Hee Hong and Takeshi Tokuyama. *Beyond Planar Graphs, Communications of NII Shonan Meetings.* Springer, 2020. doi:10.1007/978-981-15-6533-5.

22 Klaus Jansen, Stefan Kratsch, Dániel Marx, and Ildikó Schlotter. Bin packing with fixed number of bins revisited. *J. Comput. Syst. Sci.*, 79(1):39–49, 2013. doi:10.1016/J.JCSS.2012.04.004.

23 Ken-ichi Kawarabayashi and Bruce A. Reed. Computing crossing number in linear time. In *Proceedings of the 39th Annual ACM Symposium on Theory of Computing, STOC 2007*, pages 382–390. ACM, 2007. doi:10.1145/1250790.1250848.

24 Minki Kim, Bernard Lidický, Tomás Masarík, and Florian Pfender. Notes on complexity of packing coloring. *Inf. Process. Lett.*, 137:6–10, 2018. doi:10.1016/J.IPL.2018.04.012.

25 Yasuaki Kobayashi, Yuto Okada, and Alexander Wolff. Recognizing 2-Layer and Outer k-Planar Graphs. In Oswin Aichholzer and Haitao Wang, editors, *Proc. 41st Annu. Sympos. Comput. Geom. (SoCG'25)*, volume 332 of *LIPIcs*, pages 65:1–65:16. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2025. doi:10.4230/LIPIcs.SoCG.2025.65.

26 Vladimir P. Korzhik and Bojan Mohar. Minimal obstructions for 1-immersions and hardness of 1-planarity testing. *J. Graph Theory*, 72(1):30–71, 2013. doi:10.1002/JGT.21630.

27 Daniel Lokshtanov, Fahad Panolan, Saket Saurabh, Roohani Sharma, Jie Xue, and Meirav Zehavi. Crossing number in slightly superexponential time (extended abstract). In *Proceedings of the 2025 Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2025*, pages 1412–1424. SIAM, 2025. doi:10.1137/1.9781611978322.44.

28 Miriam Münch, Maximilian Pfister, and Ignaz Rutter. Exact and approximate k-planarity testing for maximal graphs of small pathwidth. In *Graph-Theoretic Concepts in Computer Science - 50th International Workshop, WG 2024*, volume 14760 of *Lecture Notes in Computer Science*, pages 430–443. Springer, 2024. doi:10.1007/978-3-031-75409-8_30.

29 Jaroslav Nesetril and Patrice Ossona de Mendez. *Sparsity - Graphs, Structures, and Algorithms*, volume 28 of *Algorithms and combinatorics*. Springer, 2012. doi:10.1007/978-3-642-27875-4.

30 Jaroslav Nešetřil and Patrice Ossona de Mendez. On low tree-depth decompositions. *Graphs Comb.*, 31(6):1941–1963, 2015. doi:10.1007/S00373-015-1569-7.

31 János Pach and Géza Tóth. Graphs drawn with few crossings per edge. *Comb.*, 17(3):427–439, 1997. doi:10.1007/BF01215922.

32 Maximilian Pfister. *Algorithms and Combinatorics for Beyond Planar Graphs*. Phd thesis, Eberhard Karls Universität Tübingen, Tübingen, Germany, 2025. doi:10.15496/publikation-101469.

33 John C. Urschel and Jake Wellens. Testing gap k-planarity is NP-complete. *Inf. Process. Lett.*, 169:106083, 2021. doi:10.1016/J.IPL.2020.106083.

34 Meirav Zehavi. Parameterized analysis and crossing minimization problems. *Comput. Sci. Rev.*, 45:100490, 2022. doi:10.1016/J.COSREV.2022.100490.