

Internally-Convex Drawings of Outerplanar Graphs in Small Area

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Abstract

A well-known result by Kant [*Algorithmica*, 1996] implies that n -vertex outerplane graphs admit embedding-preserving planar straight-line grid drawings where the internal faces are convex polygons in $O(n^2)$ area. In this paper, we present an algorithm to compute such drawings in $O(n^{1.5})$ area. We also consider outerplanar drawings in which the internal faces are required to be strictly-convex polygons. In this setting, we consider outerplanar graphs whose weak dual is a path and give a drawing algorithm that achieves $\Theta(nk^2)$ area, where k is the maximum size of an internal facial cycle.

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1 Introduction

Drawing outerplanar graphs in small area is a core topic in Graph Drawing which has been the subject of intense work. The first one we are aware of dates back to 1980, when Leiserson provided an algorithm for the construction of linear-area VLSI layouts of bounded-degree outerplanar graphs [19]. In the early 90’s, de Fraysseix, Pach, and Pollack [10] and Schnyder [20] proved that n -vertex planar graphs, and thus n -vertex outerplanar graphs, admit $O(n^2)$ -area planar straight-line grid drawings. This bound is the best possible for planar graphs, however whether a subquadratic area upper bound can be achieved for outerplanar graphs has been an intriguing question asked repeatedly (see, e.g., [5, 7, 14, 17]). A positive answer was first provided by Di Battista and Frati [12]. They presented an algorithm which, given an n -vertex maximal outerplanar graph G with weak dual tree T , uses a structural decomposition of binary trees by Chan [8] in order to construct an $O(n^{1.48})$ area “star-shaped” drawing of T ; this is a planar straight-line grid drawing which can be



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augmented to an outerplanar straight-line drawing of G within the same area bound. The same strategy has later led to an almost-linear area bound, namely $O\left(n2^{\sqrt{2\log n}}\sqrt{\log n}\right)$ [16], and also to an $O(dn \log n)$ bound for outerplanar graphs of maximum degree d [15].

In this paper we study the area requirements of *convex drawings* of outerplanar graphs, in which the boundary of every face (including the outer face) is a convex polygon. It was surprising to us that this question has not been studied before, considering the prominent place that convex drawings occupy in the graph drawing literature. We only mention here the result on convex drawings that is most relevant to our paper. Namely, Chrobak and Kant [9, 18] and, independently, Felsner [13] have presented extensions of the algorithms by de Fraysseix, Pach, and Pollack [10] and Schnyder [20], proving that every 3-connected planar graph admits a convex drawing in quadratic area, which is asymptotically optimal. Specifically, Felsner’s upper bound is $(f - 1) \times (f - 1)$, where f denotes the number of faces.

Strictly-convex drawings, in which no three vertices on the boundary of a face are allowed to be collinear, require larger area. Indeed, an n -vertex cycle cannot be drawn as a strictly-convex polygon on a grid of size $o(n^3)$ [1]. This bound is not yet matched by a corresponding upper bound. Bárány and Rote [2] proved that n -vertex 3-connected planar graphs admit strictly-convex grid drawings in $O(n^4)$ area. Bekos, Gronemann, Montecchiani, and Symvonis [4] provided an alternative algorithm with the same asymptotic area bound but with significantly smaller hidden constants.

The construction of small-area convex grid drawings of outerplanar graphs encounters two natural barriers. First, a planar graph admits a convex drawing only if it is biconnected, hence one is required to consider biconnected outerplanar graphs only. Second, we observe that there exist n -vertex biconnected outerplanar graphs that require $\Omega(n^3)$ area in any convex grid drawing (Section 2); note that an $O(n^3)$ upper bound for (even strictly-)convex drawings of n -vertex biconnected outerplanar graphs can be achieved by drawing the cycle delimiting the outer face as a strictly-convex polygon [1] and drawing the internal edges as chords. We thus relax the convexity requirement and require only the internal faces to be convex, that is, we consider *internally-convex drawings*. For outerplanar graphs, this is a natural variation of the convex drawing style, taking into account that the outer face of an outerplanar graph is *special*: All the vertices are incident to it and this property defines the graph class. Chrobak and Kant’s algorithms [9, 18] allow us to obtain internally-convex drawings of n -vertex outerplanar graphs on a grid of size $O(n^2)$ (via a suitable augmentation of the input graph to a 3-connected planar graph, by adding vertices and edges in its outer face). This restores in the internally-convex setting the pursuit for a sub-quadratic area upper bound for grid drawings of outerplanar graphs. We also consider the, in our opinion equally interesting, question of determining the area requirements of *internally-strictly-convex drawings* of outerplanar graphs.

Our contributions. Our main result is an algorithm that constructs an internally-convex grid drawing with $O(n^{1.5})$ area for n -vertex outerplanar graphs (Section 3). We employ some ideas and tools from a recent paper [6]. The main geometric component of our algorithm consists of constructing a “leveled” drawing, where consecutive levels are drawn on consecutive horizontal grid lines, up to a level where there are sufficiently few vertices so that the drawing can “make a turn” and extend horizontally rather than vertically. We remark that ours is the first algorithm that constructs outerplanar straight-line grid drawings in sub-quadratic area by drawing directly the outerplanar graph, without passing through a star-shaped drawing of its weak dual tree. This aspect might be of independent interest.

For internally-strictly-convex grid drawings, the same arguments as in the strictly-convex setting prove that $\Theta(n^3)$ is a tight bound for the area requirements. However, the natural question to consider here is how the area relates not only to the number n of vertices but also to the maximum size k of an internal face. We show that there exist outerplanar graphs requiring $\Omega(nk^2)$ area in any internally-strictly-convex grid drawing and that the lower bound can be matched by a corresponding upper bound for outerpaths, that is, biconnected outerplanar graphs whose weak dual is a path (Section 4).

We conclude in Section 5 with several open problems. Full proofs of statements marked with a $(\star)$ can be found in the extended version of the paper [3].

2 Preliminaries

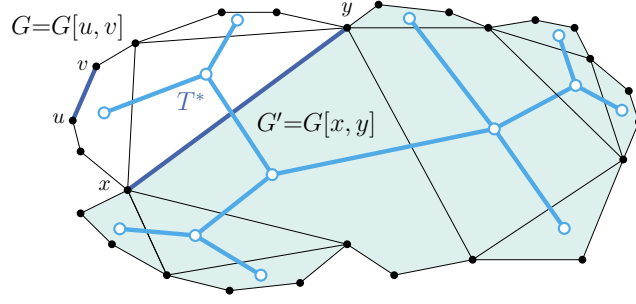
We assume familiarity with Graph Drawing, see, e.g. [11].

A *drawing* of a graph maps each vertex to a distinct point in the plane and each edge to a Jordan arc between its endpoints. In a *grid drawing* vertices have integer coordinates. Let Γ be a grid drawing of a graph. Let $\max_q(\Gamma)$ and $\min_q(\Gamma)$ be the maximum and minimum q -coordinate of a vertex in Γ , with $q \in \{x, y\}$, respectively. The *height* $h(\Gamma)$ and *width* $w(\Gamma)$ of Γ are the positive integers $h(\Gamma) = \max_y(\Gamma) - \min_y(\Gamma) + 1$ and $w(\Gamma) = \max_x(\Gamma) - \min_x(\Gamma) + 1$, respectively. In other words, $h(\Gamma)$ and $w(\Gamma)$ represent the number of horizontal and vertical grid lines intersecting the smallest axis-parallel rectangle enclosing Γ , respectively. The *area* of a drawing is the product of its width and height.

A drawing is *planar* if no two edges intersect, except at a common end-point. In a planar drawing, a vertex or an edge is *external* if it is incident to the outer face, and *internal* otherwise. A graph is *planar* if it admits a planar drawing. A graph is *outerplanar* if it admits a planar drawing in which each vertex is external. A planar drawing partitions the plane into topologically connected regions called *faces*. The unbounded face is the *outer face*, whereas the bounded faces are the *internal faces*. Two planar drawings of a connected graph are *equivalent* if they induce (i) the same counter-clockwise order of the edges incident to each vertex and (ii) the same counter-clockwise order of the edges along the boundaries of their outer faces. The equivalence classes of equivalent outerplanar drawings are called *outerplanar embeddings*, or in this paper just *embeddings*. An *outerplane graph* is a planar graph equipped with an outerplanar embedding. A biconnected outerplanar graph has a unique outerplanar embedding (up to an inversion of all the orders defining the embedding), hence we often talk about faces of a biconnected outerplane graph, referring to faces of its unique outerplanar embedding. Also, a drawing of a graph is *embedding-preserving* if it respects an embedding associated to the graph.

In a *straight-line drawing*, edges are drawn as straight-line segments. In a planar straight-line drawing, a face is *convex* (*strictly-convex*) if it is delimited by a polygon whose internal angles are not larger than 180° (resp. are smaller than 180°). A *convex drawing* (a *strictly-convex drawing*) is a straight-line planar drawing in which every face, including the outer face, is convex (resp., strictly-convex). An *internally-convex drawing* (an *internally-strictly-convex drawing*) is a straight-line planar drawing in which every internal face, but not necessarily the outer face, is convex (resp., strictly-convex). When dealing with outerplanar graphs, we do not require convex drawings to be outerplanar per se. Our results however hold in the strongest possible sense: The lower bounds on the area hold true even for drawings that are not outerplanar, while the upper bounds are proved by constructing outerplanar drawings.

The following theorem motivates the study of internally-convex drawings.



■ **Figure 1** An outerplanar graph $G[u, v]$ rooted at the external edge (u, v) and its weak dual T^* . The outerplanar graph $G' = G[x, y]$ is light-blue shaded.

► **Theorem 1.** *There exists an n -vertex biconnected outerplanar graph that requires $\Omega(n^3)$ area in every convex grid drawing.*

Proof. Let G be the n -vertex biconnected outerplanar graph obtained from a cycle $\mathcal{C}$ with $n/2$ vertices by attaching a degree-2 vertex incident to the end-vertices of each edge of $\mathcal{C}$. First, every convex drawing of G respects the unique outerplanar embedding of G , as moving a degree-2 vertex inside $\mathcal{C}$ would create an angle larger than 180° in an internal face at that vertex. Second, in every outerplanar convex drawing of G , the drawing of $\mathcal{C}$ is strictly-convex, as the angle at each vertex v of $\mathcal{C}$ in the interior of the cycle delimiting the outer face of G is at most 180° and part of this angle is used by the two triangular faces incident to v . Since every strictly-convex drawing of a cycle requires cubic area [1], the lower bound follows. ◀

Let G be a biconnected outerplane graph with $n \geq 3$ vertices; refer to Fig. 1. Note that G has at least one internal face. The *weak dual* of G , denoted by T^* , is the tree having a vertex for each internal face of G and an edge between any two vertices corresponding to internal faces sharing an edge. For simplicity, we use the same label to denote a vertex of T^* and the corresponding face of G . If T^* is a path, then G is an *outerpath*. Let (u, v) be an external edge of G such that v immediately precedes u in counter-clockwise order along the outer face of G . The outerplane graph G with the designated edge (u, v) , denoted by $G[u, v]$, is said to be *rooted at* (u, v) . Let (x, y) be an internal edge of $G[u, v]$ such that u, x, y, v appear in this counter-clockwise order along the outer face of G (possibly $u = x$ or $y = v$ may hold). Consider the outerplane subgraph G' of G induced by the vertices x, y , and the vertices that are encountered when traversing the outer face of G in counter-clockwise direction from x to y . Observe that the edge (x, y) is an external edge of G' such that y immediately precedes x in counter-clockwise order along the outer face of G' . Then we denote by $G[x, y]$ the outerplanar graph G' rooted at the edge (x, y) .

Let G be rooted at (u, v) and let f_1 be the internal face of G incident to (u, v) . The *extended weak dual tree* T of $G[u, v]$ is the tree with root f_1 obtained from the weak-dual tree T^* of G by inserting a new leaf ℓ_e for each external edge e of G different from (u, v) and an edge (f, ℓ_e) if f is the internal face of G incident to e . Let d be the number of vertices of T . Note that $d \geq n$, as T contains the root f_1 plus one leaf for each of the $n - 1$ external edges of G different from (u, v) . Also, $d \leq 2n - 3$, since G has $n - 1$ external edges different from (u, v) and at most $n - 2$ internal faces. Let $\pi = (f_1, f_2, \dots, f_p)$ be a root-to-leaf path in T . We denote by $G[\pi]$ the outerpath induced by the vertices incident to the faces $f_1, f_2, \dots, f_{p-1}$. We call $G[\pi]$ the *outerpath dual to* π . Denote by e_p the edge of G dual to (f_{p-1}, f_p) . Let $u = z_1, z_2, \dots, z_a, z_{a+1}, \dots, z_m = v$ be the counter-clockwise order of the

vertices along the outer face of $G[\pi]$, where $e_p = (z_a, z_{a+1})$. We call the rooted outerplanar graphs $G[z_1, z_2], \dots, G[z_{a-1}, z_a]$ *left subgraphs of G at π* and the rooted outerplanar graphs $G[z_{a+1}, z_{a+2}], \dots, G[z_{m-1}, z_m]$ *right subgraphs of G at π* .

3 Internally-Convex Drawings

In this section we prove the following theorem, which is our main result.

► **Theorem 2.** *Every n -vertex outerplane graph admits an embedding-preserving internally-convex grid drawing in $O(n^{1.5})$ area.*

Our proof is inspired by the proof that every outer-1-plane graph admits an embedding-preserving visibility representation in $O(n^{1.5})$ area [6]. In particular, we are going to use the following theorem, which generalizes a well-known result by Chan [8].

► **Theorem 3** ([6, Theorem 1]). *Let $p = 0.48$. Given any rooted tree with n vertices, there exists a root-to-leaf path π such that for any left subtree α and for any right subtree β of π , $|\alpha|^p + |\beta|^p \leq (1 - \delta)n^p$, for some constant $0 < \delta < 1$.*

The following lemma is similar to the combination of Definition 1 and Observation 3 from [6], although with different constants. Recall that $p = 0.48$ and $0 < \delta < 1$ are the constants from Theorem 3. We define the constant c as $c := 2/\delta$.

► **Lemma 4** ($\star$). *Let $f(d)$ be the recursive function defined on $\mathbb{N}_0$ as follows. First, $f(0) = 0$ and $f(1) = 1$. Second, for $d \in \mathbb{N}_0$ with $d \geq 2$, we have that*

$$f(d) = \max_{d_a, d_b \in \mathbb{N}_0 : d_a^p + d_b^p \leq (1-\delta)d^p} \{f(d_a) + f(d_b)\} + 2\sqrt{d}.$$

Then $f(d) \leq c\sqrt{d}$.

Let G be an n -vertex outerplane graph; refer to Fig. 2. In what follows, we assume G to be biconnected. Indeed, if it is not, it can be augmented to a biconnected outerplane graph G' by introducing edges in its outer face. Then the restriction of an embedding-preserving internally-convex drawing Γ' of G' yields an embedding-preserving internally-convex drawing of G that inherits the area bounds of Γ' . Let u and v be any two vertices such that v immediately precedes u in counter-clockwise order along the outer face of G . We root G at the edge (u, v) . Recall that $G[u, v]$ denotes the outerplanar graph G rooted at the edge (u, v) .

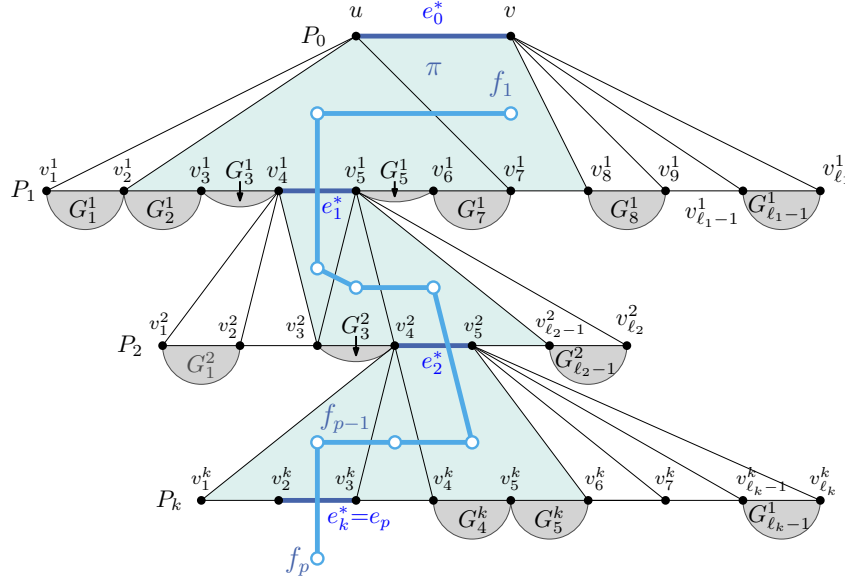
We show that $G[u, v]$ admits a (small-area) *uv-separated drawing*. This is an embedding-preserving internally-convex grid drawing Γ of G satisfying the following properties:

- P.1 u and v lie on a horizontal line h_0 ;
- P.2 all neighbors of u and v lie on the horizontal line h_1 one unit below h_0 ;
- P.3 all the other vertices of G lie not above h_1 ; and
- P.4 all vertices different from u (from v) are to the right of u (resp. to the left of v).

The following property is a direct consequence of the definition of *uv-separated drawing*.

► **Property 1.** *Let Γ be a *uv-separated drawing* of $G[u, v]$. Vertices u and v can be shifted arbitrarily by integer amounts to the left and right, respectively, while maintaining Γ an outerplanar internally-convex grid drawing.*

If $n \geq 3$, then let T be the extended weak dual tree of $G[u, v]$ and let d be the number of vertices of T . If $n = 2$, then T is not defined and we set $d = 1$. This choice is justified by the size of the subtrees of T we generate. Indeed, if $n \geq 3$, we will select a root-to-leaf path π



■ **Figure 2** An outerplanar graph $G[u, v]$ rooted at the external edge (u, v) . The outerpath $G[\pi]$ dual to the path π selected by Theorem 3 is light-blue shaded. Transition edges are drawn thick blue.

in T and consider left and right subgraphs of G at π . If any of such subgraphs is just an edge, it is associated with a subtree of T which consists of a single vertex, from which the correspondence between $n = 2$ and $d = 1$ stems.

We show an algorithm that, by induction on n , constructs a uv -separated drawing Γ of $G[u, v]$ satisfying the following two properties:

- *Property W*: every vertical grid line intersecting Γ contains at least one vertex of G (and thus the width of Γ is at most n); and
- *Property H*: every horizontal grid line intersecting Γ contains at least one vertex of G and the height of Γ is at most $f(d)$.

The statement implies Theorem 2, given that $f(d) \in O(\sqrt{d})$ by Lemma 4 and since $d \leq 2n - 3$.

Denote by $h(d)$ the maximum value of $h(\Gamma)$, over all drawings Γ constructed by our algorithm for rooted outerplane graphs whose extended weak dual has d vertices.

In the base case, we have $n = 2$ (and thus $d = 1$). Hence, G is the edge (u, v) and a drawing Γ satisfying *Properties W* and *H* is constructed by placing u on a grid point one unit to the left of v . In particular, note that $h(\Gamma) = 1$ and that $f(1) = 1$.

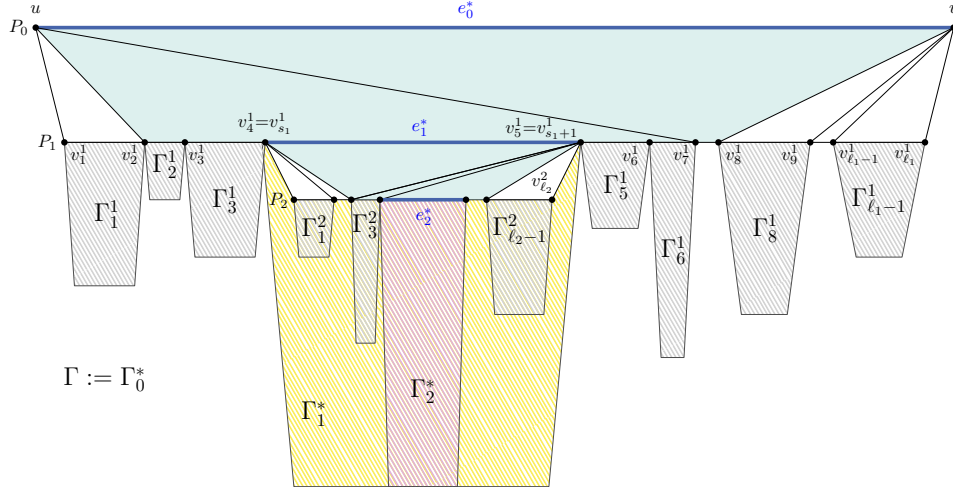
In the inductive case, we have $n > 2$, hence G has internal faces, since it is biconnected. Let f_1 be the root of T , which is the internal face of G incident to (u, v) . Let $\pi = (f_1, f_2, \dots, f_p)$ be the root-to-leaf path in T from Theorem 3, where e_p is the edge of G dual to (f_{p-1}, f_p) .

We introduce some notation and definitions. Call $e_0^* = (u, v)$ a *transition edge*; let $V_0 := \{u, v\}$, let the path $P_0 = (v_0^0, v_1^0)$ coincide with (u, v) , and let the rooted outerplane graph G_0^* coincide with $G[u, v]$. Suppose that, for some $i \geq 1$, a transition edge e_{i-1}^* has been defined that is part of a path P_{i-1} . Consider the set of vertices that do not belong to $V_0 \cup \dots \cup V_{i-1}$ and that belong to the faces of G incident to the end-vertices of e_{i-1}^* ; these vertices induce a path $P_i = (v_1^i, \dots, v_{\ell_i}^i)$, where the labels of the vertices are such that $u, v_1^i, v_{\ell_i}^i, v$ appear in this counter-clockwise order along the outer face of G . See again Fig. 2, where, for example, P_2 is the path induced by the vertices that do not belong to $V_0 \cup V_1$ (roughly speaking, by the vertices that are on the “correct side” of e_1^*) and that

belong to the faces of G incident to the end-vertices of e_1^* (only one of such faces is incident to both the end-vertices of e_1^* , the other ones are incident to a single end-vertex of e_1^*). Only one of the edges of P_i might be dual to an edge of π ; this edge of P_i is the transition edge e_i^* with end-vertices $v_{s_i}^i$ and $v_{s_i+1}^i$, for some $1 \leq s_i \leq \ell_i - 1$. The set V_i comprises the vertices $v_1^i, \dots, v_{\ell_i}^i$ of P_i , as well as the vertices of the subgraphs $G_j^i := G[v_j^i, v_{j+1}^i]$ of G , for $j = 1, \dots, \ell_i - 1$ with $j \neq s_i$. Observe that, for $j = 1, \dots, s_i - 1$, the graph G_j^i is a left subgraph of G at π or a proper subgraph of a left subgraph of G at π . Similarly, for $j = s_i + 1, \dots, \ell_i - 1$, the graph G_j^i is a right subgraph of G at π or a proper subgraph of a right subgraph of G at π . We denote by G_i^* the graph $G[v_{s_i}^i, v_{s_i+1}^i]$. For the last-defined path P_k , we have that either the transition edge e_k^* does not exist or, if it does, it is the edge e_p , which is incident to the outer face of G and thus G_k^* coincides with the edge e_k^* . An example of a situation in which e_k^* does not exist is the one in which P_k is a single vertex. This happens if f_{p-1} , the second-to-last vertex of π , is a triangular face delimited by the end-vertices of e_{k-1}^* and by one additional vertex, which coincides with P_k . In this case e_p , the edge of G dual to the last edge (f_{p-1}, f_p) of π , connects a vertex of e_{k-1}^* with the unique vertex of P_k and e_k^* does not exist.

Ideally, we would like the uv -separated drawing we construct to have the following features. The vertices $v_1^i, \dots, v_{\ell_i}^i$ of each path P_i should appear on a common horizontal line h_i in this left-to-right order, and h_i should lie one unit above h_{i+1} . The subgraphs G_j^i with $j \neq s_i$ should be recursively drawn and placed below h_i , except for the edge (v_j^i, v_{j+1}^i) which lies on h_i , in such a way that any two of these subgraphs are *horizontally disjoint*, that is, there exists a vertical line that keeps the drawings of the two subgraphs on different sides, except, possibly, for a single vertex shared by the two subgraphs which lies on the line. The subgraph $G_{s_i}^i$ is not drawn recursively, as it coincides with the subgraph G_i^* on which the level-by-level construction we are describing is further applied. Recall that k is the index of the path P_k defined last in the procedure discussed above. If $k \leq \sqrt{n}$ (**Case 1**), this plan can be accomplished. Otherwise (**Case 2**), by a simple vertex counting argument, there exists an index $2 \leq t \leq 1 + \sqrt{n}$ such that the set V_t has at most $\sqrt{n}$ vertices. In this case, we draw the vertices in V_t differently than before, so that the transition edge e_t^* is vertical. The small size of V_t allows us to “turn” from horizontal to vertical without increasing the height dramatically. From e_t^* towards the end of π , the construction proceeds by drawing the vertices of the outerpath dual to π on two horizontal lines, and by placing the recursively-constructed drawings of the right and left subgraphs of G at π above and below it, respectively.

We now formalize this argument. As sketched above, we distinguish two cases. In **Case 1**, we have $k \leq \sqrt{n}$; refer to Fig. 3. For $i = 1, \dots, k$ and for $j = 1, \dots, \ell_i - 1$ with $j \neq s_i$, we recursively construct a $v_j^i v_{j+1}^i$ -separated drawing Γ_j^i of $G_j^i[v_j^i, v_{j+1}^i]$. We construct, for $i = k - 1, k - 2, \dots, 0$, a $v_{s_i}^i v_{s_i+1}^i$ -separated drawing Γ_i^* of $G_i^*[v_{s_i}^i, v_{s_i+1}^i]$ as described in the following. Notice that the order of the indices ensures that, if G_{i+1}^* exists, then its drawing Γ_{i+1}^* has been already constructed once Γ_i^* is about to be constructed. In order to start the iteration, if e_k^* does not exist, then G_k^* does not exist either, and hence Γ_k^* does not need to be defined. If e_k^* exists, then G_k^* coincides with such an edge and Γ_k^* is constructed by placing $v_{s_k}^k$ on a grid point one unit to the left of $v_{s_k+1}^k$. In order to construct Γ_i^* , we place the recursively constructed drawings Γ_j^{i+1} with $j = 1, \dots, \ell_{i+1} - 1$ with $j \neq s_{i+1}$, as well as the drawing Γ_{i+1}^* (if G_{i+1}^* exists), side by side, so that the placement of a vertex v_j^{i+1} coincides in two drawings it belongs to; these are Γ_{j-1}^{i+1} and Γ_j^{i+1} if v_j^{i+1} is neither $v_{s_{i+1}}^{i+1}$ nor $v_{s_{i+1}+1}^{i+1}$, or are $\Gamma_{s_{i+1}-1}^{i+1}$ and Γ_{i+1}^* if v_j^{i+1} is $v_{s_{i+1}}^{i+1}$, or are Γ_{i+1}^* and $\Gamma_{s_{i+1}+1}^{i+1}$ if v_j^{i+1} is $v_{s_{i+1}+1}^{i+1}$. This ensures that the vertices of P_{i+1} all lie on the same horizontal line h_{i+1} . Further, we



■ **Figure 3** Construction for a uv -separated drawing of $G[u, v]$ when $k \leq \sqrt{n}$. To avoid cluttering, vertical and horizontal proportions are not respected, but vertices are assumed to lie on the grid.

place $v_{s_i}^i$ one unit above and one unit to the left of the leftmost vertex of P_{i+1} , and $v_{s_i+1}^i$ one unit above and one unit to the right of the rightmost vertex of P_{i+1} . This results in the desired $v_{s_i}^i v_{s_i+1}^i$ -separated drawing Γ_i^* of $G[v_{s_i}^i, v_{s_i+1}^i]$. When $i = 0$, the resulting drawing $\Gamma := \Gamma_0^*$ is a uv -separated drawing of $G[u, v]$, as established by the following lemma.

► **Lemma 5** (*). *In Case 1, the constructed drawing Γ is a uv -separated drawing of $G[u, v]$ satisfying Properties W and H.*

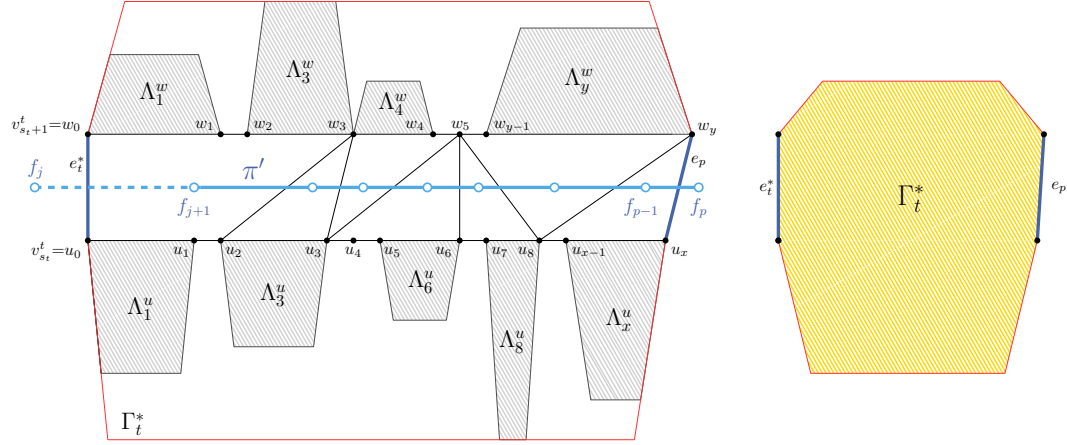
We now describe the construction of **Case 2**. Recall that, in this case, we have $k \geq 1 + \sqrt{n}$. We start the description with the following simple combinatorial lemma.

► **Lemma 6**. *There exists an index t with $2 \leq t \leq 1 + \sqrt{n}$ such that $|V_t| \leq \sqrt{n}$.*

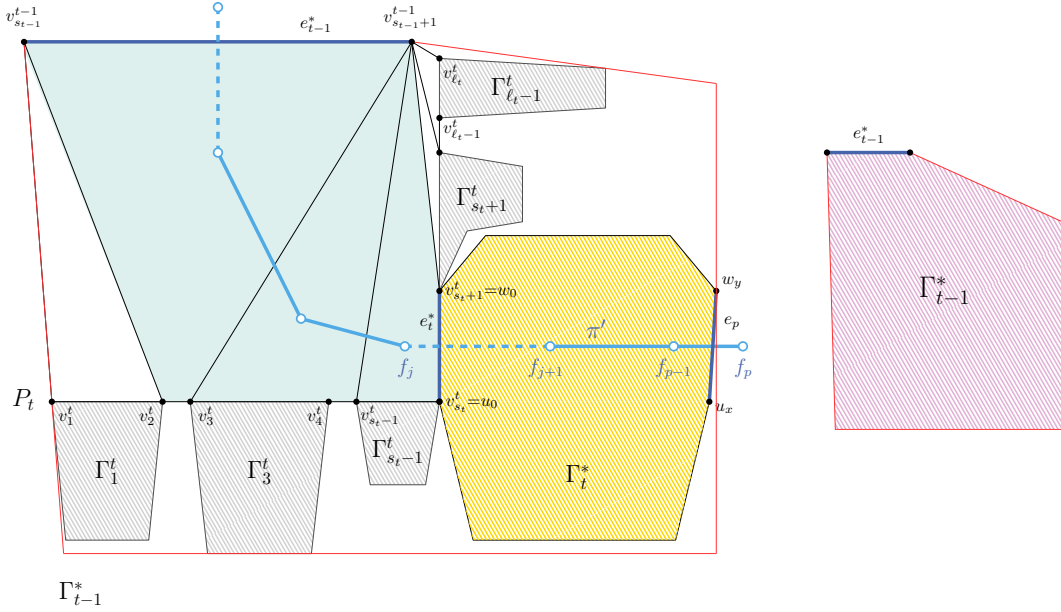
Proof. Since the sets $V_2, V_3, \dots, V_{1+\sqrt{n}}$ are disjoint and since there are $\sqrt{n}$ of them, the smallest among them contains at most $\sqrt{n}$ vertices. ◀

The construction for Case 2 proceeds in three steps. In **Step 1**, we construct a drawing Γ_t^* of G_t^* ; in **Step 2**, we augment Γ_t^* to a drawing Γ_{t-1}^* of G_{t-1}^* ; finally, in **Step 3**, we augment Γ_{t-1}^* to the desired drawing Γ of G .

Step 1. We construct a drawing Γ_t^* of G_t^* ; refer to Fig. 4. Recall that the transition edge e_t^* has end-vertices $v_{s_t}^t$ and $v_{s_t+1}^t$ and that G_t^* is the graph $G[v_{s_t}^t, v_{s_t+1}^t]$. Let (f_j, f_{j+1}) be the edge of π dual to e_t^* , let π' be the subpath $(f_{j+1}, f_{j+2}, \dots, f_p)$ of π , and let $(u_0 = v_{s_t}^t, u_1, \dots, u_x, w_y, w_{y-1}, \dots, w_1, w_0 = v_{s_t+1}^t)$ be the counter-clockwise order of the vertices along the outer face of $G[\pi']$, where (u_x, w_y) is the edge e_p . We initialize Γ_t^* by drawing e_t^* as a vertical segment of height 1. For $j = 1, \dots, x$, we recursively construct a drawing Λ_j^u of $G[u_{j-1}, u_j]$. We place the drawings $\Lambda_1^u, \dots, \Lambda_x^u$ side by side, so that the placement of a vertex u_j coincides in the drawings Λ_j^u and Λ_{j+1}^u where it appears, for $j = 1, \dots, x-1$, and so that the placement of u_0 coincides in Λ_1^u and in the drawing of e_t^* . Analogously, for $j = 1, \dots, y$, we recursively construct a drawing Λ_j^w of $G[w_j, w_{j-1}]$. Unlike for $\Lambda_1^u, \dots, \Lambda_x^u$, we rotate the drawings $\Lambda_1^w, \dots, \Lambda_y^w$ by 180° , and place them side by side, so that the placement of a vertex w_j coincides in Λ_j^w and Λ_{j+1}^w , for $j = 1, \dots, y-1$, and so that the placement of w_0 coincides in Λ_1^w and in the drawing of e_t^* . This completes the construction of Γ_t^* .

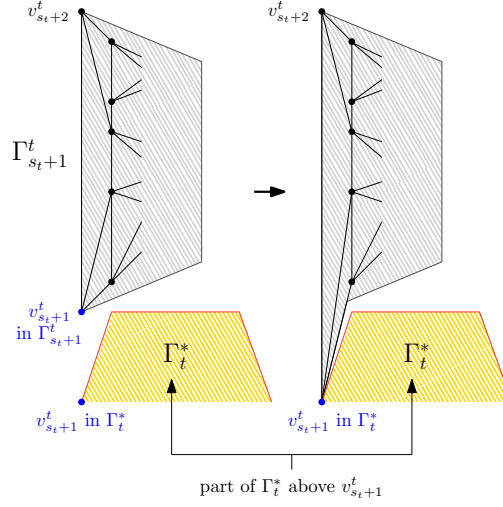


■ **Figure 4** (Left) Construction of the drawing Γ_t^* of G_t^* . Labels Λ_j^u and Λ_j^v are omitted for single edges. (Right) Polygon used to represent the drawing of Γ_t^* in the subsequent figures.

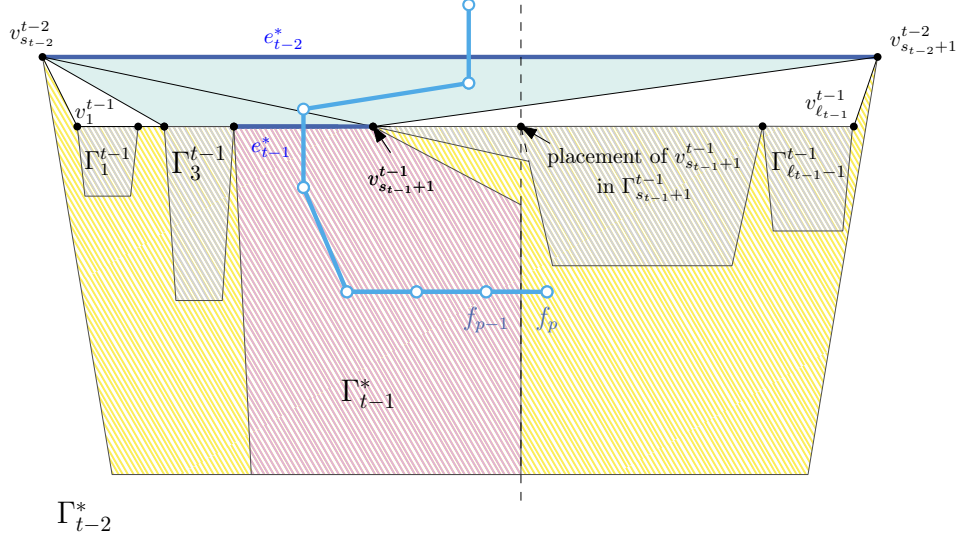


■ **Figure 5** (Left) Construction of Γ_{t-1}^* from Γ_t^* , whose boundary is depicted by the yellow-tiled region. (Right) Polygon used to represent Γ_{t-1}^* in the subsequent figures.

Step 2. We now augment Γ_t^* to a drawing Γ_{t-1}^* of G_{t-1}^* ; refer to Fig. 5. Recall that the path P_t has vertices $(v_1^t, v_2^t, \dots, v_{s_t}^t, v_{s_t+1}^t, \dots, v_{\ell_t}^t)$. For $j = 1, \dots, \ell_t - 1$ with $j \neq s_t$, we recursively construct a $v_j^t v_{j+1}^t$ -separated drawing Γ_j^t of $G[v_j^t, v_{j+1}^t]$. We place the recursively constructed drawings $\Gamma_1^t, \dots, \Gamma_{s_t-1}^t$ side by side, so that the placement of a vertex v_j^t coincides in Γ_{j-1}^t and Γ_j^t , for $j = 2, \dots, s_t - 1$, and so that the placement of $v_{s_t}^t$ coincides in $\Gamma_{s_t-1}^t$ and in Γ_t^* . Next, we rotate the drawings $\Gamma_{s_t+1}^t, \dots, \Gamma_{\ell_t-1}^t$ in counter-clockwise direction by 90° , and we stack them one on top of the other, so that the placement of a vertex v_j^t coincides in Γ_{j-1}^t and Γ_j^t , for $j = s_t + 2, \dots, \ell_t - 1$, and so that the placement of $v_{s_t+1}^t$ in $\Gamma_{s_t+1}^t$ is on the same vertical line as in Γ_t^* and on the same horizontal line as the highest vertex in Γ_t^* (refer to the left side of Fig. 6). This might provide a double placement for $v_{s_t+1}^t$ (unless $v_{s_t+1}^t$ is the highest vertex in Γ_t^*). The issue is resolved by keeping for $v_{s_t+1}^t$ the position it



■ **Figure 6** Transformation of $\Gamma_{s_t+1}^t$ shifting downward $v_{s_t+1}^t$ while avoiding overlapping with Γ_t^* .



■ **Figure 7** Construction of Γ_{t-2}^* from Γ_{t-1}^* , whose boundary is depicted by the red-tiled region. The illustration shows the modification of the placement of $v_{s_t-1+1}^{t-1}$ from the one it has in $\Gamma_{s_t-1+1}^{t-1}$.

has in Γ_t^* ; note that, in $\Gamma_{s_t+1}^t$, this amounts to shifting $v_{s_t+1}^t$ downwards (refer to the right side of Fig. 6). The drawing of Γ_{t-1}^* is completed by placing $v_{s_t-1}^{t-1}$ and $v_{s_t-1+1}^{t-1}$ on the same horizontal line, one unit higher than the highest vertex in Γ_{t-1}^* so far (this is either $v_{\ell_t}^t$ if $s_t \leq \ell_t - 2$ or the highest vertex in Γ_t^* otherwise), so that $v_{s_t-1+1}^{t-1}$ is one unit to the left of the vertical line through e_t^* and $v_{s_t-1}^{t-1}$ is one unit to the left of the leftmost vertex in Γ_{t-1}^* so far (this is either v_1^t if $s_t \geq 2$ or $v_{s_t-1+1}^{t-1}$ otherwise).

Step 3. Finally, we show how to augment Γ_{t-1}^* to a uv -separated drawing of $G[u, v]$. This is done similarly to Case 1. Namely, we construct, for $i = t-2, t-3, \dots, 0$, a $v_{s_i}^i v_{s_i+1}^i$ -separated drawing Γ_i^* of $G[v_{s_i}^i, v_{s_i+1}^i]$ from an already constructed $v_{s_{i+1}}^{i+1} v_{s_{i+1}+1}^{i+1}$ -separated drawing Γ_{i+1}^* of $G[v_{s_{i+1}}^{i+1}, v_{s_{i+1}+1}^{i+1}]$. Recall that, in Step 2, we constructed Γ_{t-1}^* . As in Case 1, we place the

recursively constructed drawings Γ_j^{i+1} with $j = 1, \dots, \ell_{i+1} - 1$ with $j \neq s_{i+1}$, as well as the drawing Γ_{i+1}^* , side by side, so that the placement of a vertex v_j^{i+1} coincides in two drawings it belongs to, with one exception in the case $i = t - 2$; refer to Fig. 7. Namely, the drawing $\Gamma_{s_{t-1}+1}^{t-1}$ of $G_{s_{t-1}+1}^{t-1}$ is embedded so that $v_{s_{t-1}+1}^{t-1}$ is on the same horizontal line as in Γ_{t-1}^* and on the same vertical line as the rightmost vertex in Γ_{t-1}^* . This provides a double placement for $v_{s_{t-1}+1}^{t-1}$. The issue is resolved by keeping for $v_{s_{t-1}+1}^{t-1}$ the position it has in Γ_{t-1}^* ; note that, in $\Gamma_{s_{t-1}+1}^{t-1}$, this amounts to shifting $v_{s_{t-1}+1}^{t-1}$ leftwards. The desired $v_{s_i}^i v_{s_{i+1}}^i$ -separated drawing Γ_i^* of $G[v_{s_i}^i, v_{s_{i+1}}^i]$ is completed by placing $v_{s_i}^i$ one unit above and one unit to the left of the leftmost vertex of P_{i+1} , and $v_{s_{i+1}}^i$ one unit above and one unit to the right of the rightmost vertex of P_{i+1} . When $i = 0$, the obtained drawing $\Gamma := \Gamma_0^*$ is a uv -separated drawing of $G[u, v]$, as proved in the following.

► **Lemma 7** ($\star$). *In Case 2, the constructed drawing Γ is a uv -separated drawing of $G[u, v]$ satisfying Property W and H.*

Sketch. First, it trivially comes from the construction that Γ is a straight-line grid drawing.

Most of the arguments used to prove that Γ is planar and satisfies Properties P.1–P.4 of a uv -separated drawing are easy geometric considerations. For example, each graph $G[u_i, u_{i+1}]$ does not cross any graph $G[u_j, u_{j+1}]$, as it is horizontally disjoint from it, does not cross any subgraph $G[w_j, w_{j+1}]$ as it is vertically disjoint from it, and does not cross $G[\pi']$ as it is vertically disjoint from it, except for the edge (u_i, u_{i+1}) which is shared by the two graphs. The most interesting part in the proof of the planarity of Γ consists of showing that the drawings $\Gamma_{s_{t+1}}^t$ and Γ_t^* do not cross each other. This proof uses the fact that $\Gamma_{s_{t+1}}^t$ satisfies the properties of a $v_{s_{t+1}}^t v_{s_{t+2}}^t$ -separated drawing. In fact, the avoidance of crossings between $\Gamma_{s_{t+1}}^t$ and Γ_t^* is the driving force behind the definition of our drawing invariant. Namely, by Property P.4 and by the 90° counter-clockwise rotation of $\Gamma_{s_{t+1}}^t$, we have that $v_{s_{t+1}}^t$ is the lowest vertex in $\Gamma_{s_{t+1}}^t$. Since $v_{s_{t+1}}^t$ is on the same horizontal line as the highest vertex in Γ_t^* , the only edges of $G_{s_{t+1}}^t$ that might cross G_t^* in Γ are those incident to $v_{s_{t+1}}^t$. However, since all the vertices of $G_{s_{t+1}}^t$ different from $v_{s_{t+1}}^t$ lie above Γ_t^* and all the neighbors of $v_{s_{t+1}}^t$ in $G_{s_{t+1}}^t$ lie on the vertical line one unit to the right of $v_{s_{t+1}}^t$, by Property P.2 and by the rotation of $\Gamma_{s_{t+1}}^t$, it follows that the edges of $G_{s_{t+1}}^t$ incident to $v_{s_{t+1}}^t$ lie above every edge of G_t^* with which they have a horizontal overlap. Note that, by Property 1, placing $v_{s_{t+1}}^t$ at the point where it is placed in Γ_t^* maintains the planarity of $\Gamma_{s_{t+1}}^t$.

The convexity of each internal face f of G in Γ follows by induction if f is internal to some recursively drawn subgraph G_j^i of G . If f is an internal face of $G[\pi']$ or has vertices on two distinct paths among $P_0, P_1, \dots, P_{t-1}$, then f is drawn as a triangle or a trapezoid in Γ , hence it is convex. The only faces that do not fit in any of these two categories are those incident to $v_{s_{t-1}}^{t-1}$ and/or $v_{s_{t-1}+1}^{t-1}$ and incident to at least one vertex in P_t . The convexity of these faces follows from two facts. First, the polygon Q delimited by the edge $(v_{s_{t-1}}^{t-1}, v_{s_{t-1}+1}^{t-1})$ and by the path P_t is a convex pentagon $(v_{s_{t-1}}^{t-1}, v_{s_{t-1}+1}^{t-1}, v_{\ell_t}^t, v_{s_t}^t, v_1^t)$ (if $s_t > 1$) or a convex quadrilateral $(v_{s_{t-1}}^{t-1}, v_{s_{t-1}+1}^{t-1}, v_{\ell_t}^t, v_1^t)$ (if $s_t = 1$). Second, the faces incident to $v_{s_{t-1}}^{t-1}$ and/or $v_{s_{t-1}+1}^{t-1}$ and incident to at least one vertex in P_t form a convex subdivision of Q .

Property W follows by induction and since every vertical grid line intersecting Γ intersects a recursively constructed drawing or a vertex directly placed by the algorithm.

Concerning *Property H*, we prove that $h(\Gamma)$ is at most t , plus $\sqrt{n}$, plus the maximum height of a recursively constructed drawing of a subgraph of a left subgraph of G at π , plus the maximum height of a recursively constructed drawing of a subgraph of a right subgraph of G at π . The first term accounts for the horizontal grid lines $h_0, h_1, \dots, h_{t-1}$ on which $P_0, P_1, \dots, P_{t-1}$ are placed. The $\sqrt{n}$ term accounts for the horizontal grid lines

between the highest line intersecting $\Gamma_{\ell_t-1}^t$ (which is one unit below h_{t-1}) and the lowest line intersecting $\Gamma_{s_t+1}^t$ before $v_{s_t+1}^t$ is moved to the position it has in Γ_t^* . Indeed, since the drawings $\Gamma_{s_t+1}^t, \Gamma_{s_t+2}^t, \dots, \Gamma_{\ell_t-1}^t$ are constructed recursively, and hence satisfy *Property W*, and are rotated by 90° , and since the graphs $G_{s_t+1}^t, G_{s_t+2}^t, \dots, G_{\ell_t-1}^t$ have a total of at most $\sqrt{n}$ vertices, by Lemma 6, it follows that the drawings $\Gamma_{s_t+1}^t, \Gamma_{s_t+2}^t, \dots, \Gamma_{\ell_t-1}^t$ intersect, before moving $v_{s_t+1}^t$, at most $\sqrt{n}$ grid lines. The third and fourth terms in the upper bound for $h(\Gamma)$ account for the total height of the recursively drawn subgraphs different from $G_{s_t+1}^t, G_{s_t+2}^t, \dots, G_{\ell_t-1}^t$. The key observation here is that, apart for the drawings of $\Gamma_{s_t+1}^t, \Gamma_{s_t+2}^t, \dots, \Gamma_{\ell_t-1}^t$, which are already accounted for, any two recursively constructed drawings which are stacked one on top of the other are drawings of subgraphs of a left subgraph of G at π and of a right subgraph of G at π (and not of two left subgraphs or of two right subgraphs); in fact, only the right subgraphs $G[w_j, w_{j+1}]$ of G at π are forced to be stacked on top of the left subgraphs $G[u_j, u_{j+1}]$ of G at π (and on top of the graphs $G[v_j^t, v_{j+1}^t]$ with $j < s_t$ which are also subgraphs of left subgraphs of G at π).

Note that $t \leq \sqrt{n}$ by construction and hence $t \leq \sqrt{d}$, given that $d \geq n$. Assume that i and j are indices such that G_j^i is a recursively drawn subgraph of a left subgraph of G at π and $h(\Gamma_j^i)$ is maximum. Indices i' and j' are defined analogously for the right subgraphs of G at π . Let d_j^i and $d_{j'}^{i'}$ be the number of vertices in the extended weak dual trees of G_j^i and $G_{j'}^{i'}$, respectively. By the upper bound proved above, we have $h(\Gamma) \leq 2\sqrt{d} + h(\Gamma_j^i) + h(\Gamma_{j'}^{i'})$ and thus, by induction, $h(\Gamma) \leq 2\sqrt{d} + f(d_j^i) + f(d_{j'}^{i'})$. By Theorem 3, it holds true that $(d_j^i)^p + (d_{j'}^{i'})^p \leq (1 - \delta)d^p$. By the definition of $f(d)$ in Lemma 4 (with $d_a = d_j^i$ and $d_b = d_{j'}^{i'}$), we have that $2\sqrt{d} + f(d_j^i) + f(d_{j'}^{i'}) \leq f(d)$, hence $h(\Gamma) \leq f(d)$, which proves *Property H*. ◀

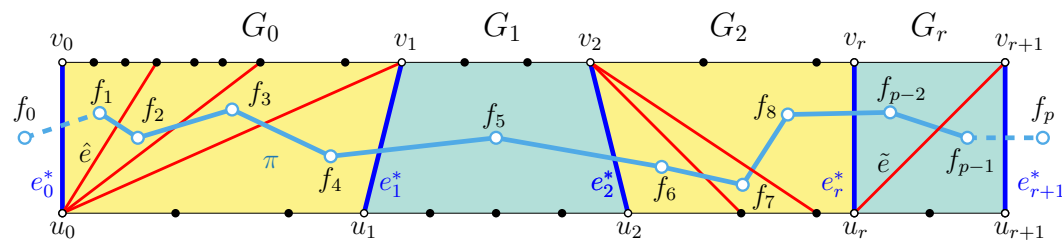
4 Internally-Strictly-Convex Drawings of Outerpaths

In this section, we prove a tight bound on the area required by internally-strictly-convex grid drawings of outerpaths. More precisely, we prove that every n -vertex outerpath whose internal faces have size at most k admits an internally-strictly-convex grid drawing in area $O(nk^2)$, which we also show to match a general lower bound for the internally-strictly-convex grid drawings of outerplanar graphs. We start by proving the lower bound.

► **Theorem 8.** *Every n -vertex outerplane graph with $\Omega(\frac{n}{k})$ internal faces of size k requires $\Omega(nk^2)$ area in any internally-strictly-convex grid drawing.*

Proof. Let G be an n -vertex outerplane graph with $\Omega(\frac{n}{k})$ internal faces of size k . First, note that any internally-strictly-convex grid drawing of G must be outerplanar. Indeed, any embedding of an outerplanar graph which is not an outerplanar embedding contains a degree-2 vertex as an internal vertex. However, such a vertex would create an angle larger than or equal to 180° in an internal face. The area lower bound is obtained by a simple packing argument. Since any internally-strictly-convex grid drawing of G is outerplanar, it contains $\Omega(\frac{n}{k})$ internal faces of size k . Each of these faces occupies $\Omega(k^3)$ area [1], hence the total area of the drawing is $\Omega(\frac{n}{k} \cdot k^3)$, which gives the bound claimed by the theorem. ◀

In the rest of the section, we prove a matching upper bound for the case of outerpaths. Note that there exist n -vertex outerpaths with $\Omega(\frac{n}{k})$ internal faces of size k , hence the lower bound of Theorem 8 applies to outerpaths as well. We introduce some notation and definitions. Let G be an outerpath and let $\pi = (f_1, f_2, \dots, f_{p-1})$ be the weak-dual of G , which is a path; refer to Fig. 8. Let $\hat{e}$ be the edge of G dual to the edge (f_1, f_2) of π and let $\tilde{e}$ be the edge of G dual to the edge (f_{p-2}, f_{p-1}) of π . Moreover, let $\hat{g} = (u_0, v_0)$ be any of



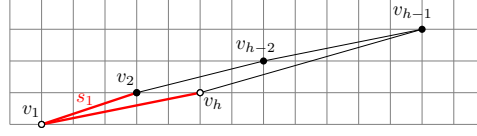
the two external edges incident to $\hat{e}$ that belong to the face f_1 of G , where u_0 immediately follows v_0 when traversing the outer face of G in counter-clockwise order. Also, let $\tilde{g}$ be any of the two external edges incident to $\tilde{e}$ that belong to the face f_{p-1} of G . We augment π with two new vertices f_0 and f_p , and edges (f_0, f_1) and (f_{p-1}, f_p) . We interpret (f_0, f_1) and (f_{p-1}, f_p) as edges dual to $\hat{g}$ and $\tilde{g}$, respectively.

We will show that each fan graph G_i with gate edges e_i^* and e_{i+1}^* admits a *block-drawing*. This is an outerplanar internally-strictly-convex grid drawing Γ_i of G_i satisfying the following properties:

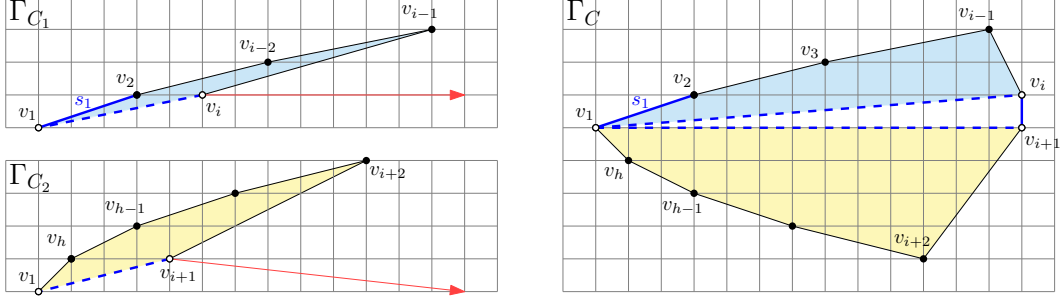
- B.1** The two gate edges e_i^* and e_{i+1}^* are drawn as vertical segments of unit length such that $y(u_i) = y(u_{i+1}) = 0$ and $y(v_i) = y(v_{i+1}) = 1$;
- B.2** the vertices u_i and v_i are the leftmost vertices of Γ_i and no other vertex of G_i has the same x -coordinate as these vertices; and
- B.3** the vertices u_{i+1} and v_{i+1} are the rightmost vertices of Γ_i and no other vertex of G_i has the same x -coordinate as these vertices.

In the following, by *half-plane of a line* we mean an open half-plane bounded by the line.

1.1 s_1 represents e_1 in Γ_C ;



■ **Figure 9** Illustrations for the proof of Lemma 9.



■ **Figure 10** Illustration for the proof of Lemma 10.

- I.2 the vertex v_h is placed at $(x(v_2) + h - 2, 1)$;
- I.3 let ℓ be the line that passes through s_1 oriented in the direction from v_1 to v_2 . Then, all the vertices $v_3, \dots, v_h$ lie in the right half-plane of ℓ ;
- I.4 the width of Γ_C is $x(v_2) \cdot (h - 2) + \frac{(h-3)(h-2)}{2} + 1$; and
- I.5 the height of Γ_C is $h - 1$.

Sketch. The drawing Γ_C can be obtained as follows; see Fig. 9. Vertices v_1 and v_2 are at $(0, 0)$ and $(x(v_2), 1)$ in Γ_C . For $j = 3, \dots, h - 1$, we place v_j at $(x(v_2) \cdot (j - 1) + \frac{(j-2)(j-1)}{2}, j - 1)$. Finally, we place vertex v_h at $(x(v_2) + h - 2, 1)$. This completes the construction of Γ_C . We defer the proof that Γ_C satisfies Properties I.1–I.5 to the full version [3]. ◀

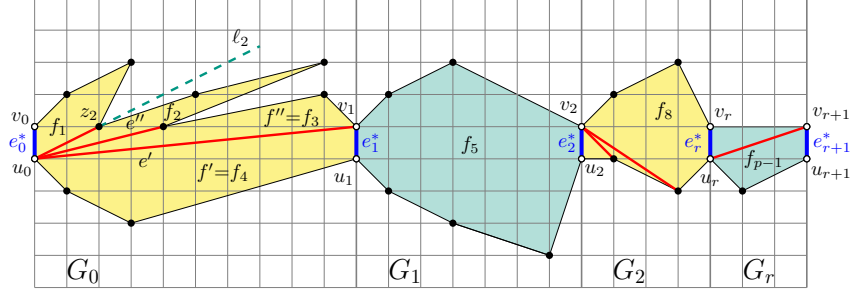
The following property is a consequence of the constraints of the drawings of Lemma 9.

► **Property 2.** Let Γ_C be the drawing of a cycle $C = (v_1, v_2, \dots, v_h)$ produced by Lemma 9. Then the drawing obtained by shifting vertex v_h in Γ_C by any integer amounts to the right and/or to the bottom is an internally-strictly-convex grid drawing of C .

► **Lemma 10 (*)**. Let $C = (v_1, v_2, \dots, v_h)$ with $h > 3$ be a cycle. Let $e_1 = (v_1, v_2)$ and $e_2 = (v_i, v_{i+1})$ be two non-adjacent edges of C . Let s_1 be a segment that represents (v_1, v_2) such that v_1 is at the origin $(0, 0)$, $x(v_2)$ is a non-negative integer, and $y(v_2) = 1$. Then C admits an internally-strictly-convex grid drawing Γ_C such that:

- J.1 s_1 represents e_1 in Γ_C ;
- J.2 the edge e_2 is represented by a vertical segment such that $x(v_i) > x(v_2)$, $y(v_i) = 1$, and $y(v_{i+1}) = 0$;
- J.3 the width of Γ_C is $1 + \max\{x(v_2) \cdot (i - 2) + \frac{(i-3)(i-2)}{2}, \frac{(h-i-1)(h-i)}{2}\}$; and
- J.4 the height of Γ_C is $h - 2$.

Sketch. We show how to construct the drawing Γ_C ; refer to Fig. 10. First, we define two cycles $C_1 = (v_1, v_2, \dots, v_i)$ and $C_2 = (v_1, v_h, v_{h-1}, \dots, v_{i+1})$. We apply Lemma 9 to obtain a drawing Γ_{C_1} of C_1 in which the edge (v_1, v_2) is represented by s_1 and to obtain a drawing Γ_{C_2} of C_2 in which the edge (v_1, v_h) is represented by the segment connecting points $(0, 0)$



■ **Figure 11** Illustration for Lemma 11 and Theorem 12. The drawing represents the graph in Fig. 8.

and $(1, 1)$. Let M be the maximum x -coordinate of a vertex in Γ_{C_1} and Γ_{C_2} . We leverage Property 2 to obtain a drawing Γ'_{C_1} of C_1 by shifting the vertex v_i horizontally and rightward in Γ_{C_1} so that its x -coordinate is $1 + M$. Analogously, we leverage Property 2 to obtain a drawing Γ'_{C_2} of C_2 by shifting the vertex v_{i+1} downward and rightward in Γ_{C_2} so that its x -coordinate is $1 + M$ and its y -coordinate is 0. To obtain the drawing Γ_C we then proceed as follows. We first initialize Γ_C to Γ'_{C_1} . Then we construct a vertically-mirrored copy Γ''_{C_2} of Γ'_{C_2} , that is, we flip the sign of the y -coordinate of each vertex, and we insert Γ''_{C_2} in Γ_C so that the placement of v_1 is the same in both drawings. Finally, we remove from Γ_C the drawing of the edges (v_1, v_i) and (v_1, v_{i+1}) and draw the edge (v_i, v_{i+1}) as a vertical segment of unit length. This concludes the construction of Γ_C . We defer the proof that Γ_C satisfies Properties J.1–J.4 to the full version [3]. ◀

► **Property 3.** *Let Γ_C be the drawing of a cycle $C = (v_1, v_2, \dots, v_i, v_{i+1}, \dots, v_h)$ produced by Lemma 10. Then the drawing obtained by shifting vertices v_i and v_{i+1} in Γ_C by the same integer amount to the right is an internally-strictly-convex grid drawing of C .*

We next show how to construct a block-drawing of a fan graph with appropriate area bounds.

► **Lemma 11** (★). *Every n_i -vertex fan graph G_i with internal faces of size at most k_i admits a block-drawing whose width is $O(n_i k_i)$ and whose height is $O(k_i)$.*

Sketch. We show how to construct a block drawing Γ_i of an n_i -vertex fan graph G_i with gate edges $e_i^* = (u_i, v_i)$ and $e_{i+1}^* = (u_{i+1}, v_{i+1})$ in the more interesting case in which one of u_i and v_i , say u_i , is incident to at least two internal edges (see, e.g., G_0 in Fig. 11). Let $g_1, g_2, \dots, g_{t+2}$ be the internal faces of G_i in clockwise order around u_i , where g_1 is incident to e_i^* . Then, for $j = 1, \dots, t$, we draw the cycle bounding face g_j by applying Lemma 9 with e_1 being e_i^* and s_1 being the segment between $(0, 0)$ and $(0, 1)$, if $j = 1$, and e_1 being the unique edge (u_i, z_j) shared by g_{j-1} and g_j , and s_1 being the segment representing (u_i, z_j) in the drawing constructed so far, otherwise. Finally, we draw the cycle C , obtained by merging the boundaries of g_{t+1} and g_{t+2} and by removing their shared edge (u_i, z_{t+2}) , by applying Lemma 10 with e_1 being the edge (u_i, z_{t+1}) , s_1 being the segment representing the edge (u_i, z_{t+1}) , and e_2 being e_{i+1}^* . Then, by Property 3, we shift to the right vertices u_{i+1} and v_{i+1} so that they lie one unit to the right of the rightmost of the remaining vertices of G_i . Finally, we obtain the desired block drawing Γ_i of G_i by simply drawing the removed edge (u_i, z_{t+2}) as a straight-line chord of the strictly-convex polygon representing C . We defer the proof that Γ_C is a block drawing with the desired width and height bounds, as well as the remaining cases of the proof, to the full version [3]. ◀

We are now ready to prove the main theorem of this section.

► **Theorem 12.** *Every n -vertex outerpath whose internal faces have size at most k admits an internally-strictly-convex grid drawing in $O(nk^2)$ area.*

Proof. We compute block drawings $\Gamma_0, \dots, \Gamma_r$ of the fan graphs $G_0, \dots, G_r$ of the outerpath G so that each block drawing has height $O(k_i)$ and width $O(n_i k_i)$ by Lemma 11, where n_i is the number of vertices of G_i and k_i is the maximum size of any face of G_i ($0 \leq i \leq r$). We iteratively construct Γ by “gluing” together the block drawings $\Gamma_0, \dots, \Gamma_r$ at their common gate edges. More precisely, we proceed as follows. We initialize Γ to Γ_0 . Then, for $i = 1, \dots, r$, we augment Γ with a copy of Γ_i so that the drawing of the gate edge e_i^* in Γ_i coincides with the drawing of e_i^* in the drawing constructed so far. It follows that the height of Γ is $O(\max_{0 \leq i \leq r} \{k_i\})$, which is in $O(k)$, while its width is $O(\sum_{i=0}^r n_i k_i)$, which is in $O(nk)$. ◀

5 Conclusions

In this paper we have proved the first sub-quadratic area upper bound for internally-convex grid drawings of outerplanar graphs. We also presented algorithms to construct internally-strictly-convex grid drawings of outerpaths whose area is asymptotically optimal, with respect to the number of vertices of the graph and the maximum size of an internal face.

Several intriguing questions are left open by our research:

- A first, natural, research direction is to narrow the gap between the $O(n^{1.5})$ upper bound and the $\Omega(n)$ lower bound for internally-convex grid drawings of n -vertex outerplanar graphs. In particular, any super-linear lower bound would be an important step towards proving an analogous result for (not necessarily convex) planar straight-line grid drawings.
- Concerning internally-strictly-convex grid drawings of (general) n -vertex outerplane graphs whose faces have size at most k , for which we proved an $\Omega(nk^2)$ lower bound, we are not aware of any upper bound better than $O(n^3)$, which comes from the literature on strictly-convex polygons [1]. Tightening this gap is a nice goal.
- The case $k = 4$, for which there is an $O(n^2)$ upper bound (a consequence of results in [2]), is especially interesting. Indeed, any upper bound would translate to an upper bound for the area requirements of straight-line drawings of outer-1-planar graphs.
- Finally, we have observed that, for convex grid drawings of n -vertex outerplane graphs (in which the outer face is also required to be convex), $\Theta(n^3)$ is a tight bound for the area requirements. However, if the maximum size of an internal face is bounded, then our lower bound does not hold anymore. We believe that an $\Omega(n \log n)$ lower bound can be proved, given that the outer face would require a dimension to have $\Omega(n)$ length in order to be convex and given that there exist n -vertex trees that require $\Omega(\log n)$ length in both dimensions in any planar straight-line drawing (see, e.g., [14]). However, this lower bound is far from the $O(n^3)$ upper bound and it would be interesting to close this gap.

We conclude by mentioning that we believe that the techniques we developed in order to construct internally-strictly-convex grid drawings of outerpaths could lead to non-trivial area bounds for outerplanar graphs whose weak dual tree has bounded diameter. We plan to investigate the truth of this intuition in the near future.

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