Characterizing and Recognizing Twistedness

Oswin Aichholzer
Graz University of Technology, Austria

Alfredo García
Departamento de Métodos Estadísticos and IUMA, University of Zaragoza, Spain

Javier Tejel
Departamento de Métodos Estadísticos and IUMA, University of Zaragoza, Spain

Birgit Vogtenhuber
Graz University of Technology, Austria

Alexandra Weinberger
FernUniversität in Hagen, Germany

Abstract

In a simple drawing of a graph, any two edges intersect in at most one point (either a common endpoint or a proper crossing). A simple drawing is *generalized twisted* if it fulfills certain rather specific constraints on how the edges are drawn. An abstract rotation system of a graph assigns to each vertex a cyclic order of its incident edges. A realizable rotation system is one that admits a simple drawing such that at each vertex, the edges emanate in that cyclic order, and a generalized twisted rotation system can be realized as a generalized twisted drawing. Generalized twisted drawings have initially been introduced to obtain improved bounds on the size of plane substructures in any simple drawing of K_n. They have since gained independent interest due to their surprising properties. However, the definition of generalized twisted drawings is very geometric and drawing-specific.

In this paper, we develop characterizations of generalized twisted drawings that enable a purely combinatorial view on these drawings and lead to efficient recognition algorithms. Concretely, we show that for any $n \geq 7$, an abstract rotation system of K_n is generalized twisted if and only if all subrotation systems induced by five vertices are generalized twisted. This implies a drawing-independent and concise characterization of generalized twistedness. Besides, the result yields a simple $O(n^5)$-time algorithm to decide whether an abstract rotation system is generalized twisted and sheds new light on the structural features of simple drawings. We further develop a characterization via the rotations of a pair of vertices in a drawing, which we then use to derive an $O(n^2)$-time algorithm to decide whether a realizable rotation system is generalized twisted.

2012 ACM Subject Classification Mathematics of computing → Combinatorics; Mathematics of computing → Graph theory

Keywords and phrases generalized twisted drawings, simple drawings, rotation systems, recognition, combinatorial characterization, efficient algorithms

Digital Object Identifier 10.4230/LIPIcs.GD.2025.25

Related Version *ArXiv Version*: `https://arxiv.org/abs/2508.16178v1`

Funding *Javier Tejel*: Partially supported by project E41-23R, funded by Gobierno de Aragón, and project PID2023-150725NB-I00, funded by MICIU/AEI/10.13039/501100011033.

1 Introduction

Simple drawings are drawings of graphs in which the vertices are drawn as distinct points in the plane, the edges are drawn as Jordan arcs connecting their end-vertices without passing through other vertices, and each pair of edges share at most one point (a proper crossing or a common endpoint). We consider two simple drawings the same if they are strongly isomorphic, that is, if there is a homeomorphism of the plane transforming one drawing

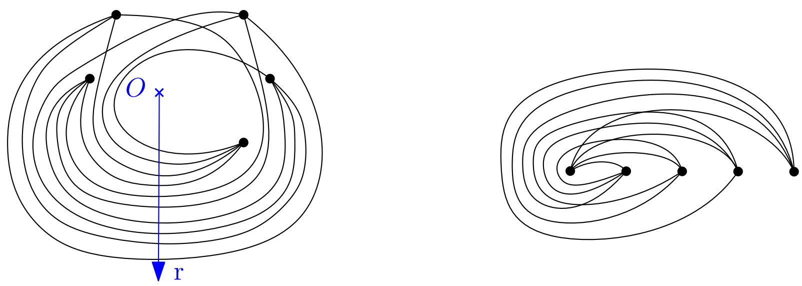

Figure 1 Two realizations of the generalized twisted rotation system of K_5. Left: Generalized twisted drawing where the origin O and ray r are depicted. Right: The classical twisted drawing.

into the other. In this work we focus on simple drawings of K_n. Unless explicitly stated otherwise, all considered drawings are of K_n and simple. A simple drawing D is *generalized twisted* if it is strongly isomorphic to a simple drawing in which there is a point O such that each ray emanating from O crosses each edge of D at most once and there exists a ray r emanating from O such that all edges of D cross r. For an example see Figure 1(left), which has been reproduced from [15].

Generalized twisted drawings have been used to find plane substructures in all simple drawings of K_n: The currently best bounds on the size of the largest plane matching and longest plane cycle and path in any simple drawing of K_n have been obtained via generalized twisted drawings [5]. Moreover, generalized twisted drawings are the largest class of drawings for which it has been shown that each drawing has exactly $2n-4$ empty triangles [14, 15], which is conjectured to be the minimum over all simple drawings [6]. There are several more interesting properties of generalized twisted drawings that have been shown already [4, 5, 15]. For example, all generalized twisted drawings contain plane cycles of length at least $n-1$ [5].

Despite the usefulness of generalized twisted drawings, their definition is very geometrical and drawing-specific. In this paper, we obtain combinatorial characterizations that no longer depend on the specific drawing, which ease the work with them and also allow for efficient recognition. Note that ad hoc it is not clear that such a combinatorial characterization needs to exist. The study of special classes of simple drawings [3, 11, 12, 13] and their combinatorial characterizations [8, 9, 10, 22, 23] has a strong tradition in the research of simple drawings and contributes significantly to the understanding simple drawings in general.

A widely used combinatorial abstraction of drawings are rotation systems. The *rotation of a vertex* in a drawing is the cyclic order of its incident edges. The *rotation system of a simple drawing* is the collection of the rotations of all vertices. Rotation systems are especially useful for describing simple drawings of complete graphs, since two simple drawings of K_n have the same crossing edge pairs if and only if they have the same or inverse rotation system [7, 16, 17, 19]. Rotation systems of graphs can also be considered without a concrete drawing. An *abstract rotation system* of a graph G gives, for every vertex of G, a cyclic order of its adjacent vertices. An abstract rotation system is called *realizable* if there exists a simple drawing with that rotation system. For abstract rotation systems of complete graphs, realizability can be checked in polynomial time. Specifically, an abstract rotation system of K_n is realizable if and only if every subrotation system induced by five vertices is realizable. This follows from a combinatorial characterization via subrotation systems of size six [20, Theorem 1.1.] together with a complete enumeration of all realizable rotation systems of K_n for $n \leq 9$ [2].

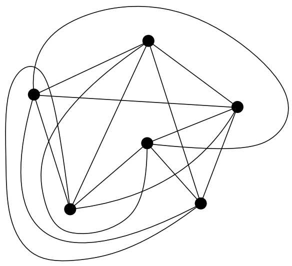

Figure 2 A simple drawing of K_6 whose rotation system is not generalized twisted, even though all subrotation systems on five vertices are.

In this work, we use rotation systems to characterize generalized twisted drawings. An abstract rotation system of a graph is *generalized twisted* if there is a generalized twisted drawing realizing this rotation system. As our first main result, we show that generalized twisted rotation systems of K_n can also be characterized via subrotation systems of size five. The main benefit of this result is to have a drawing-independent (and concise) characterization of generalized twistedness. The substructures to be considered are of constant size and straightforward to be checked, which improves the insight into generalized twisted drawings.

▶ **Theorem 1.** *Let R be an abstract rotation system of K_n with $n \geq 7$. Then R is generalized twisted if and only if every subrotation system induced by five vertices is generalized twisted.*

This characterization of generalized twisted rotation systems nicely complements the characterization of all realizable simple drawings in general, which can also be done by considering subrotation systems of size five [2, 20]. Per se, such characterizations of bounded complexity do not always need to exist. For example, if the combinatorics of an (abstract) set of points is given by their triple orientations – so-called abstract order types – no simple characterization to decide their realizability as a point set in the plane can exist, as this decision problem is known to be $\exists\mathbb{R}$-complete; see the recent survey [24] for details.

The main part of Theorem 1 is the "if" part. The "only-if" part follows from the fact that subdrawings of generalized twisted drawings are generalized twisted by definition. As mentioned in the article introducing generalized twisted drawings [4], there is exactly one generalized twisted rotation system T_5 of K_5, namely, the one of the *twisted* drawing [18, 21] of K_5, which gave rise to the name "generalized twisted"; see Figure 1(right) for an illustration as in [21]. We remark that Theorem 1 cannot be extended to $n \leq 6$. Figure 2 depicts a drawing from [4], where it has been shown that the rotation system of this drawing is not generalized twisted, although all its subrotation systems are (generalized) twisted.

Our second main result is a characterization of generalized twisted drawings via rotation systems and crossings as stated in Theorem 2 below and depicted on an example in Figure 3.

▶ **Theorem 2.** *Let R be the rotation system of a simple drawing D of K_n and let V be the set of vertices of D. Then R is generalized twisted if and only if there exist two vertices v_1 and v_2 in V and a bipartition $A \cup B$ of the vertices in $V \setminus \{v_1, v_2\}$, where some of A or B can be empty, such that:*

(i) *For every pair of vertices a and a' in A, the edge aa' crosses v_1v_2.*
(ii) *For every pair of vertices b and b' in B, the edge bb' crosses v_1v_2.*
(iii) *For every vertex $a \in A$ and every vertex $b \in B$, the edge ab does not cross v_1v_2.*
(iv) *Beginning at v_2, in the rotation of v_1, all vertices in B appear before all vertices in A.*
(v) *Beginning at v_1, in the rotation of v_2, all vertices in B appear before all vertices in A.*

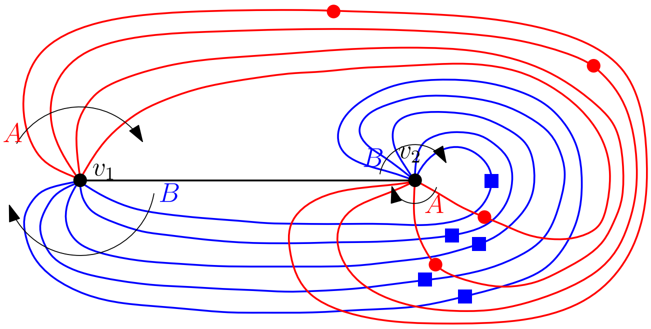

Figure 3 Illustration of Theorem 2: Possible rotations of v_1 and v_2. The vertices of partition set A are drawn as red disks and the vertices of partition set B as blue squares.

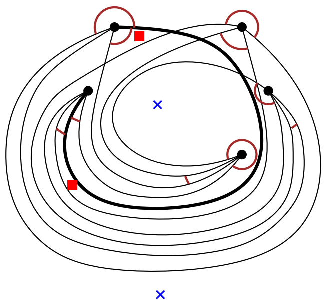

Figure 4 A realization of the generalized twisted rotation system T_5 of K_5. The pairs of antipodal vi-cells are marked with red squares and blue crosses, the maximum crossing edge is bold, and empty triangles are indicated by (brown) circular arcs.

The main implication of Theorem 2 is that it leads to fast algorithms for recognizing generalized twisted rotation systems. If it is already known that a rotation system is realizable (which needs $O(n^5)$ time to be checked [2, 20]), we can even use it to obtain a quadratic time recognition algorithm.

▶ **Theorem 3.** *Let R be an abstract rotation system of K_n. Then it can be decided in $O(n^5)$ time whether R is generalized twisted. If R is known to be realizable, then deciding whether it is also generalized twisted can be done in $O(n^2)$ time.*

Outline. We start by introducing some notation and showing first properties in Section 2. Sections 3 and 4 are devoted to proving the two characterizations, Theorem 1 and Theorem 2, respectively. In Section 5, we prove Theorem 3 by providing the algorithms. We conclude in Section 6 with some open questions.

2 Preliminaries

Given a simple drawing D, the *star at v*, denoted by $S(v)$, is the set of edges incident to v in D. For a set V of vertices of D, we denote by $D \setminus V$ the drawing D without the vertices in V and without all edges incident to a vertex in V. A *triangle* Δ in D is the simple closed curve formed by three vertices and the three edges connecting them. Every triangle partitions the plane and hence D into two regions which we call the *sides* of Δ. The whole drawing D partitions the plane into regions called *cells*.

In our proofs, we heavily rely on antipodal vi-cells, empty star triangles, and maximum crossing edges, all of which we describe in the following paragraphs.

Antipodal vi-cells. A cell that is incident to a vertex (that is, has the vertex on its boundary) is called vertex-incident cell or *vi-cell*. We observe that if there is a vi-cell at v containing the initial parts of vu and vw in one drawing of a rotation system, then vu and vw are neighboring in the rotation of v. Since the latter property depends only on the rotation system of the drawing, there is a vi-cell at v in every drawing of this rotation system with initial parts of vu and vw on the boundary. Note that the remainder of the boundary of the vi-cell is not necessarily identical across different drawings of the rotation system. Two cells are *antipodal* if for every triangle, the two cells are on different sides. A vi-cell that is antipodal to some other vi-cell is called a *via-cell*. (In Figure 4, the two pairs of via-cells are marked.) Since the set of crossing edge pairs determines the rotation system and vice versa [17, 19], also the existence of via-cells is a property of the rotation system. A vertex incident to a via-cell is called a *via-vertex*.

We will need the following known theorem [4, Theorem 15][1].

▶ **Theorem 4** ([4]). *The rotation system of a simple drawing D of K_n is generalized twisted if and only if D contains a pair of antipodal vi-cells.*

Note that the characterization in Theorem 4 implies that a rotation system is generalized twisted if and only if *all its realizations as simple drawings* contain a pair of antipodal vi-cells.

Empty star triangles. A triangle Δ with vertices x, y and z is a *star triangle* at x if the star $S(x)$ does not cross the edge yz; it is *empty* if one side of Δ contains no vertices of $D \setminus \Delta$ (and hence the other side contains all of them). (In Figure 4, each circular arc between two edges at a vertex indicates that the triangle containing those two edges is an empty star triangle at that vertex.) We call two star triangles at x *adjacent* if they are both incident to some edge xy and span disjoint angles at x. We will use the following results on empty star triangles in simple and generalized twisted drawings.

▶ **Lemma 5** ([6]). *Let v be a vertex of a simple drawing D of K_n. Then D contains at least two empty star triangles at v. If vxy is a star triangle at v then it is empty if and only if the vertices x and y are consecutive in the rotation of v.*

▶ **Lemma 6** ([15]). *Let D be a generalized twisted drawing of K_n and let v be a vertex of D. Then there are exactly two empty star triangles at v. If (C_1, C_2) is a pair of antipodal vi-cells in D, then one of the two empty star triangles at v contains C_1 on the empty side and the other triangle contains C_2 on the empty side.*

Maximum crossing edges. An edge v_1v_2 is *maximum crossing* if v_1v_2 crosses every edge not in $S(v_1) \cup S(v_2)$. (For example, the bold edge in Figure 4 is maximum crossing.) Any simple drawing has at most one maximum crossing edge, as the following lemma states.

▶ **Lemma 7.** *Any simple drawing of K_n with $n \geq 5$ has at most one maximum crossing edge.*

[1] The original formulation uses drawings and weak isomorphism. Our formulation is equivalent due to the equivalence of rotation systems and weak isomorphism [17, 19].

Lemma 7 holds for $n = 5$ because none of the five simple drawings of K_5 (up to strong isomorphism) [6] contains more than one maximum crossing edge. Using this fact, the lemma can be proven for $n > 5$ by contradiction, assuming that a simple drawing contains several maximum crossing edges.

We next observe in Lemma 8 that maximum crossing edges imply generalized twisted rotation systems. Figure 5 illustrates the lemma.

▶ **Lemma 8.** *Let D be a simple drawing of K_n that has a maximum crossing edge, v_1v_2. Let C_1 and C_1' be the two cells incident to v_1 along the edge v_1v_2 and let C_2 and C_2' be the two cells incident to v_2 along the edge v_1v_2 such that when going along the edge from v_1 to v_2, the cell C_1 is on the other side of v_1v_2 than the cell C_2 (and consequently C_1' is on the other side of v_1v_2 than C_2'). Then D is generalized twisted and (C_1, C_2) and (C_1', C_2') are two pairs of antipodal vi-cells.*

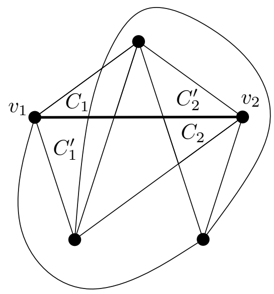

Figure 5 A maximum crossing edge v_1v_2 (in bold) and the two pairs, (C_1, C_2) and (C_1', C_2'), of antipodal vi-cells.

Lemma 8 can be shown by taking an arbitrary triangle of the generalized twisted drawing and considering where the cells adjacent to the edge v_1v_2 and its endpoints must lie with respect to this triangle in order to allow the edge v_1v_2 to cross all non-adjacent edges.

3 Characterization via subrotation systems induced by five vertices

In this section, we prove Theorem 1 by induction. To this end, we use the following theorem, whose proof we sketch in Section 3.1.

▶ **Theorem 9.** *Let R be an abstract rotation system of K_n with $n \geq 12$ such that every induced subrotation system on at most $n - 1$ vertices is generalized twisted. Then R is generalized twisted.*

Proof of Theorem 1. By definition, if R is generalized twisted, then any subrotation system induced by five vertices is generalized twisted. We prove the converse: if every subrotation system induced by five vertices is generalized twisted, then R is generalized twisted.

For $7 \leq n \leq 11$, the theorem is verified by computer. Following the lines of [1, 5], we compute all generalized twisted rotation systems of K_n for n up to 15 in the following way. First, we use the method of [2] to generate all realizable rotation systems, but discard those with subrotation systems induced by five vertices different to T_5, thus computing all candidate rotation systems for generalized twisted drawings. In each candidate rotation system, we then search for a pair of antipodal vi-cells. By Theorem 4, this identifies all generalized twisted rotation systems. Any rotation system contradicting Theorem 1 does not have antipodal vi-cells. The computational results give only one rotation system of size $n = 6$ (drawn in Figure 2) that is not generalized twisted.

For $n \geq 12$, we use induction. Every induced subrotation system of R on at most $n-1$ vertices is generalized twisted by induction. Thus, R is generalized twisted by Theorem 9. ◀

3.1 Proof sketch of Theorem 9

Let R be as in Theorem 9 and let D be a realization of R, which exists since any abstract rotation system of K_n is realizable if all subrotation systems induced by five vertices are [2, 20]. Since each induced and proper subdrawing of D has a generalized twisted rotation system, it also has a pair of antipodal vi-cells by Theorem 4. We use this to show that also D has such a pair and thus, by Theorem 4, R is generalized twisted. The idea is to locate antipodal vi-cells of D using those in its induced subdrawings and empty triangles. The following lemma can be shown using Lemma 5 (for a lower bound) and Lemma 6 (for an upper bound).

▶ **Lemma 10.** *Let D be a drawing of K_n with $n \geq 5$ such that every induced and proper subdrawing has a generalized twisted rotation system. Then at any vertex of D, there are exactly two empty star triangles.*

We separately consider the case where D contains a vertex u with two adjacent empty star triangles at u and the case that there is no such vertex.

Case 1: Two adjacent empty star triangles at a vertex u.
This is the easy case. We first show the following lemma on generalized twisted rotation systems of K_n. Its proof is based on considering two adjacent empty star triangles at a vertex v_1 and analyzing all possibilities of how to extend these triangles to a drawing of T_5 and then to a drawing of K_6 with a generalized twisted rotation system.

▶ **Lemma 11.** *Let v_1 and v_2 be two vertices of a simple drawing of K_n with generalized twisted rotation system such that the two empty star triangles at vertex v_1 are sharing the edge v_1v_2. Then the following properties hold.*

1. *The edge v_1v_2 is maximum crossing.*
2. *The two empty star triangles at v_1 are also empty star triangles at v_2.*
3. *For each of the two empty star triangles at v_1, the vertices v_1 and v_2 are adjacent in the rotation of the third vertex of the triangle in such a way that all edges emanate from the third vertex in the empty side of the triangle.*

Then we consider the drawing D and the two adjacent empty star triangles at u in D. As every proper subdrawing of D has a generalized twisted rotation system, Lemma 11 implies that the edge shared by the two adjacent empty star triangles at u is maximum crossing. Thus, the rotation system of D is generalized twisted by Lemma 8.

Case 2: No vertex with two adjacent empty star triangles.
This is the involved case. Let v_1 be an arbitrary vertex of D and let $\Delta_1 = v_1u_1u_1'$ and $\Delta_2 = v_1u_2u_2'$ be the (nonadjacent) empty star triangles at v_1. Our goal is to show that using via-cells of some induced and proper subdrawings of D, we can find a pair (C_1, C_2) of vi-cells in D such that C_1 lies in Δ_1 and C_2 in Δ_2, and prove that C_1 and C_2 are antipodal.

Let V_0 be the set of vertices of Δ_1 and Δ_2, and let D_0 be the subdrawing induced by V_0. Since the rotation system of D_0 is generalized twisted and has only five vertices, we have all the information about that drawing. This includes in particular the maximum crossing edge of D_0, which we denote by m, the two pairs of antipodal vi-cells of D_0, and the empty star triangles. As Δ_1 and Δ_2 are not adjacent, v_1 is one of the vertices $\{a, b, d\}$ in Figure 6.

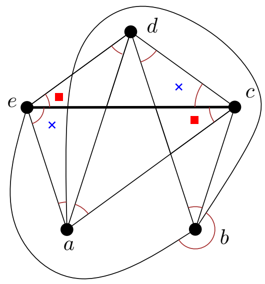

Figure 6 A drawing of the unique generalized twisted rotation system T_5 of K_5, which is strongly isomorphic to the one in Figure 1, but redrawn to make empty star triangles and the relevant cells more visible. Empty star triangles at a vertex are marked with brown arcs at the angle that spans the triangle. The two pairs of antipodal vi-cells are marked with red squares and blue crosses.

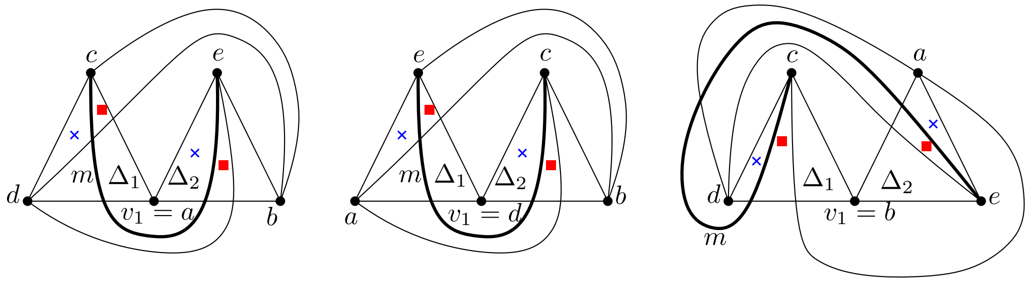

Figure 7 Three drawings realizing T_5 in such a way that v_1 is drawn in the center. The underlying unlabeled drawings are strongly isomorphic. The labels correspond to those in Figure 6 with $v_1 = a$ (left), $v_1 = d$ (middle), or $v_1 = b$ (right). The maximum crossing edge m is drawn bold, the two pairs of antipodal vi-cells are marked with red squares and blue crosses.

Observe that the vertices a and d behave analogously to each other while vertex b behaves differently (see Figure 7). We argue via the triangles Δ_1 and Δ_2 separately but simultaneously, because we will consider only one triangle but require that the other triangle has the same properties. We sketch the case where v_1 is a or d and focus mostly on Δ_1. The properties for Δ_2 can be shown analogously and the case of v_1 behaving like b then follows immediately.

For identifying all pairs of antipodal vi-cells in subdrawings of D, we state in Lemma 12 properties of simple drawings with more than one such pair.

▶ **Lemma 12.** *Let D be a simple drawing of K_n with different antipodal vi-cell pairs (C_1, C_2) and (C'_1, C'_2). Then*

1. *For one of the vertices on the boundary of a cell in $\{C_1, C_2, C'_1, C'_2\}$, there are two empty star triangles at that vertex that share an edge.*
2. *There is a maximum crossing edge which is the shared edge of the two empty star triangles at one of the endpoints of the edge.*
3. *The via-cells are adjacent to the maximum crossing edge (as described in Lemma 8).*
4. *D has only those two (and no more) pairs of antipodal vi-cells.*

Proof Sketch. The proof of (1) is by contradiction. Assuming that the two empty star triangles at any vertex on the boundary of a cell in $\{C_1, C_2, C'_1, C'_2\}$ are not adjacent, we can find an induced subdrawing on 5 or 6 vertices that is not generalized twisted, a contradiction. (2) and (3) follow from Lemma 11, as by (1) there are two empty star triangles that are adjacent. (4) follows from the fact that if there is a third pair of antipodal vi-cells, then D would contain two maximum crossing edges, contradicting Lemma 7. ◀

Let V' be a nonempty subset of $V \setminus V_0$. By Lemma 12 and Lemma 6, one of the following holds for the subdrawing $D \setminus V'$ of D:

1. $D \setminus V'$ contains exactly one pair of antipodal vi-cells. One of those cells is in the empty side of triangle Δ_1 and the other one is in the empty side of triangle Δ_2.
2. $D \setminus V'$ contains exactly two pairs of antipodal vi-cells. For each pair, one cell is in the empty side of Δ_1 and the other one is in the empty side of Δ_2. Further, for each of Δ_1 and Δ_2, the two vi-cells in its empty side are adjacent along the maximum crossed edge of $D \setminus V'$ and incident to the endpoints of this edge.

The next lemma states the possibilities of how *arbitrary antipodal vi-cell pairs* of two subdrawings $D\setminus\{v\}$ and $D\setminus\{w\}$ of D can be contained in *all pairs* of the subdrawing $D\setminus\{v,w\}$. The proof uses observations on the containment of cells of a drawing D' in cells of its subdrawings and consequences of two cells being antipodal. We denote by $(C_1^{v_1v_2\cdots}, C_2^{v_1v_2\cdots})$ an antipodal vi-cell pair in the drawing $D\setminus\{v_1, v_2, \ldots\}$.

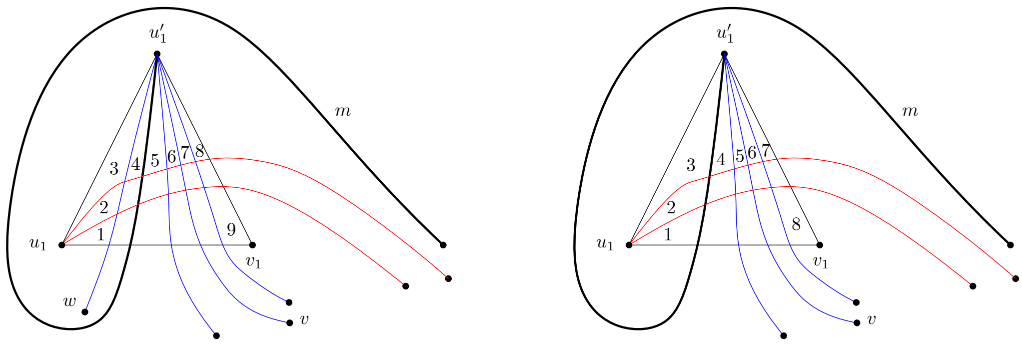

Figure 8 The grid cells of D and $D\setminus\{w\}$ inside triangle Δ_1.

▶ **Lemma 13.** *For any pair (C_1^v, C_2^v) of antipodal vi-cells in $D\setminus\{v\}$ and any pair (C_1^w, C_2^w) of antipodal vi-cells in $D\setminus\{w\}$, then*

1. *If $D\setminus\{v,w\}$ has* ***only one antipodal vi-cell pair****, (C_1^{vw}, C_2^{vw}), then (C_1^v, C_2^v) and (C_1^w, C_2^w) both are contained in (C_1^{vw}, C_2^{vw}).*
2. *If $D\setminus\{v,w\}$ has* ***two antipodal vi-cell pairs*** *(C_1^{vw}, C_2^{vw}) and $(C_1'^{vw}, C_2'^{vw})$, then either (C_1^v, C_2^v) and (C_1^w, C_2^w) both are contained in one of (C_1^{vw}, C_2^{vw}) and $(C_1'^{vw}, C_2'^{vw})$, or (C_1^v, C_2^v) is contained in one of (C_1^{vw}, C_2^{vw}) and $(C_1'^{vw}, C_2'^{vw})$, and (C_1^w, C_2^w) in the other.*

We identify grid structures in the empty sides of Δ_1 and Δ_2 in D that we will work with to locate candidates for antipodal vi-cells of D. As Δ_1 is an empty star triangle at v_1 and as D is a simple drawing, no edge incident to v_1 can cross an edge of Δ_1. However, every generalized twisted drawing of K_4 contains a crossing [5]. Hence, for every vertex $v \notin \Delta_1$, exactly one of vu_1 or vu_1' crosses Δ_1 (on v_1u_1' or v_1u_1, respectively). We denote this edge by r_v. The set of edges r_v for every v form a grid in the empty side of Δ_1 in D. We denote the cells of this grid that have some vertex (of Δ_1) on the boundary as *grid cells*; see Figure 8. Grid cells in Δ_2 are defined analogously. Note that grid cells may consist of several cells of D, as edges of D can cross through Δ_1 and/or Δ_2.

Next, we obtain an order on the grid cells of Δ_1 (and analogously on Δ_2). Consider the dual graph of the grid cells (where the vertices are the grid cells and two vertices share an edge if the corresponding grid cells are adjacent). This dual graph is either a path or a path plus an isolated vertex (e.g. grid cell 9 in Figure 8 left). We label the grid cells $G_1, \ldots, G_l$, along this dual path, beginning with the grid cell with vertex u_1 and adjacent to edge u_1v_1, and finishing with the cell that has v_1 on its boundary; see Figure 8 left for an example. The *distance* between two grid cells G_i, G_j is the number of edges r_v separating G_i and G_j. If edge r_v is incident to both G_i and G_{i+1}, we say that G_i and G_{i+1} are *glued along* r_v. We denote the two grid cells that are glued along the edge m (which is the maximum crossed edge of D_0) by G_c and G_{c+1} (cells $4, 5$ in Figure 8 left). We say that grid cell G_i is *placed before* m if $i \leq c$; and *placed after* m otherwise.

When removing a vertex $v \in V \setminus V_0$, we obtain a grid in $D\backslash\{v\}$ whose only difference to the grid in D is that two grid cells of D that are adjacent at the edge r_v become one grid cell in $D\backslash\{v\}$, and that some grid cells may enlarge their size, depending on the removed vertex; see Figure 8 right for an illustration. For each via-cell C^v of $D\backslash\{v\}$ in the empty side of Δ_1, we denote by G^v the grid cell of $D\backslash\{v\}$ containing C^v. Observe that G^v is either an (enlarged) grid cell of D or two such grid cells glued along an edge r_v. With the following lemma, we obtain that the grid cells containing via-cells of the subdrawings have common intersections in the grid cells of D.

▶ **Lemma 14.** *Let $w_1, w_2, \ldots, w_s$ be the set of vertices in $V \backslash V_0$ such that for every w_i, the grid cell G^{w_i} in $D\backslash\{w_i\}$ is placed before m. Let $x_1, x_2, \ldots, x_t$ be the set of vertices in $V \backslash V_0$ such that for every x_i, the grid cell G^{x_i} in $D\backslash\{x_i\}$ is placed after m. Then*

1. *If $s \geq 2$, the common intersection of $G^{w_1}, G^{w_2}, \ldots, G^{w_s}$ is a grid cell of D placed before m.*
2. *If $t \geq 2$, the common intersection of $G^{x_1}, G^{x_2}, \ldots, G^{x_t}$ is a grid cell of D placed after m.*
3. *It $s \geq 2$ and $t \geq 2$, then the two obtained grid cells of D are G_c and G_{c+1}, respectively.*

Proof Sketch. (1) Given w_i and w_j, we show that $G^{w_i} \cap G^{w_j}$ is a grid cell of D by analyzing the intersections among G^{w_i}, G^{w_j} and $G^{w_i w_j}$, for the different possibilities of the positions of r_{w_i} and r_{w_j}. Then we show by contradiction that G^{w_1}, $G^{w_2}, \ldots, G^{w_s}$ have a common intersection, which is a grid cell of D placed before m. Assuming that $G^{w_i} \cap G^{w_j} \neq G^{w_j} \cap G^{w_k}$, we show that G^{w_i} and G^{w_k} cannot be contained in $G^{w_i w_k}$, a contradiction. (2) is proved in a similar way. To prove (3), we show by contradiction that G_c (respectively G_{c+1}) is contained in G^{w_i} (respectively G^{x_i}) for any i, so G_c and G_{c+1} must be the common grid cells. ◀

Next we show that with enough vertices in total, we also obtain vi-cells of D in the (shared) grid cells we found in D.

▶ **Lemma 15.** *Suppose that for the via-cells $C^{w_1}, C^{w_2}, \ldots, C^{w_s}$ with $s \geq 2$, the corresponding grid cells $G^{w_1}, G^{w_2}, \ldots, G^{w_s}$ in $D\backslash\{w_1\}, D\backslash\{w_2\}, \ldots, D\backslash\{w_s\}$, respectively, contain a common grid cell G of D. Then $C^{w_1}, C^{w_2}, \ldots, C^{w_s}$ also contain a common vi-cell C of D.*

Proof Sketch. We first prove that if G^{w_i} and G^{w_j} contain a common grid cell G, then the interior of C^{w_i} (or C^{w_j}) and the interior of G are not disjoint. Using this result, if G has only one vertex on its boundary, then the lemma trivially holds. If G contains two vertices on its boundary, we partition $C^{w_1}, C^{w_2}, \ldots, C^{w_s}$ into three sets S_1, S_2, and S_3, where S_1 contains the cells C^{w_i} that are incident to only one of the vertices, S_2 contains the cells C^{w_i} that are incident to only the other vertex, and S_3 contains the cells C^{w_i} that are incident to both vertices. We then prove that one of S_1 and S_2 must be empty, so $C^{w_1}, C^{w_2}, \ldots, C^{w_s}$ must all be incident to the same vertex of G and share a common vi-cell C of D. ◀

The following lemma shows that D contains a pair of antipodal vi-cells.

▶ **Lemma 16.** *For $n \geq 12$, there exists a pair (C, C') of antipodal vi-cells in D.*

Proof Sketch. For $i = 1, \ldots, 7$, let w_i be a vertex not in V_0 and let $(C_1^{w_i}, C_2^{w_i})$ be a pair of antipodal vi-cells in the subdrawing $D\backslash\{w_i\}$. For any pair of antipodal vi-cells of V_0, one of the cells is in Δ_1 and the other in Δ_2. Hence, by Lemma 13, $C_1^{w_i}$ must lie in Δ_1 and $C_2^{w_i}$ in

Δ_2 for any i. Using Lemmas 14 and 15, we show that at least four of the vi-cells $C_1^{w_i}$ share a common vi-cell C in Δ_1 and a common vi-cell C' in Δ_2 for the corresponding vi-cells $C_2^{w_i}$. By the antipodality of the pairs $(C_1^{w_i}, C_2^{w_i})$ that contain (C, C'), we prove the antipodality of (C, C') by showing for all triangles of D that C and C' are on different sides. ◀

4 Characterization via bipartition

This section is devoted to the proof of Theorem 2. Before starting with the proof, we introduce the concept of compatibility of vertices. Given a realizable rotation system of K_n, let $R_{v_1} = \{v_2 = w_1, w_2, \ldots, w_{n-1}\}$ and $R_{v_2} = \{v_1 = t_1, t_2, \ldots, t_{n-1}\}$ be the (clockwise) rotations of v_1 and v_2, respectively. We say that v_1 and v_2 are *compatible* (or that v_1 is compatible with v_2) if $\{v_1, t_2, \ldots, t_{n-1}\} = \{v_1, w_i, w_{i-1}, \ldots, w_2, w_{n-1}, w_{n-2}, \ldots, w_{i+1}\}$, for some $i \in \{1, 2, \ldots, n\}$. Note that if v_1 and v_2 are compatible and i is 1 or n, then $\{v_1, t_2, \ldots, t_{n-1}\} = \{v_1, w_{n-1}, w_{n-2}, \ldots, w_2\}$.

Next we consider pairs of antipodal vi-cells in simple drawings (and thus in generalized twisted drawings) and show that each such pair contains a pair of compatible vertices. For any antipodal vi-cell pair (C_1, C_2) in a simple drawing of K_n, there exist two vertices $v_1 \neq v_2$ such that v_1 and v_2 are incident to C_1 and C_2, respectively [4]. The following lemma implies that v_1 and v_2 are compatible.

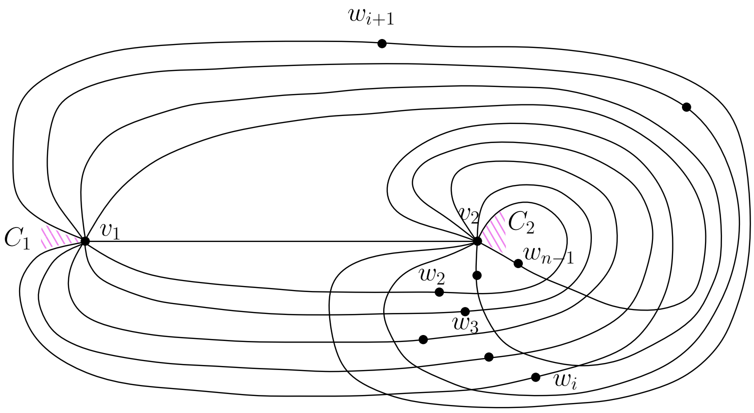

Figure 9 The rotations of v_1 and v_2.

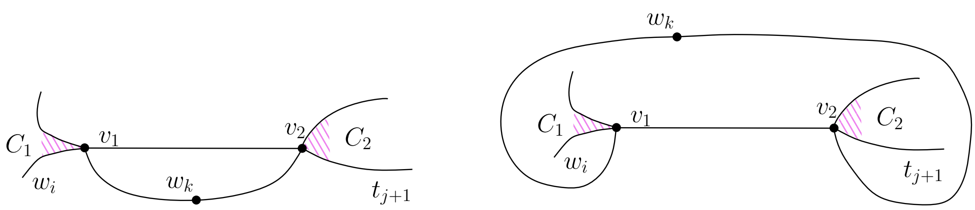

Figure 10 w_k cannot appear after C_2 around v_2.

▶ **Lemma 17.** *Let D be a realization of a generalized twisted rotation system of K_n with $n \geq 3$. Let (C_1, C_2) be a pair of antipodal vi-cells of D, and let v_1 and v_2 be vertices of D incident to C_1 and C_2, respectively, with $v_2 \neq v_1$. Assume that $R_{v_1} = \{v_2 = w_1, w_2, \ldots, w_{n-1}\}$ is the (clockwise) rotation of v_1, and that the edges v_1w_i and v_1w_{i+1} define part of the boundary of C_1, for some $i \in \{1, \ldots, n-1\}$. Then the following statements hold.*

(i) *If $i \in \{2, 3, \ldots, n-2\}$ then the (clockwise) rotation of v_2 is $\{v_1, w_i, w_{i-1}, \ldots, w_2, w_{n-1}, w_{n-2}, \ldots, w_{i+1}\}$ and the edges v_2w_2 and v_2w_{n-1} define part of the boundary of C_2.*

(ii) *If $i = 1$ then the (clockwise) rotation of v_2 is $\{v_1, w_{n-1}, w_{n-2}, \ldots, w_2\}$ and the edges v_2v_1 and v_2w_{n-1} define part of the boundary of C_2.*

(iii) *If $i = n-1$ then the (clockwise) rotation of v_2 is $\{v_1, w_{n-1}, w_{n-2}, \ldots, w_2\}$ and the edges v_2v_1 and v_1w_2 define part of the boundary of C_2.*

Proof Sketch. We sketch the proof of (i); the proofs of (ii) and (iii) are very similar. Figure 9 shows an example of the rotations of v_1 and v_2. Assume that the clockwise rotation of v_2 is $\{v_1 = t_1, t_2, \ldots, t_{n-1}\}$ and that the edges v_2t_j and v_2t_{j+1}, for some $j \in \{1, \ldots, n-1\}$, define part of the boundary of C_2. To place w_k in the rotation of v_1 and v_2 with respect to C_1 and C_2, we consider for w_k with $2 \leq k \leq i$ the triangle spanned by v_1, v_2 and w_k. As C_1 and C_2 are antipodal, they must lie on different sides of the triangle. This is not possible if w_k corresponds to a vertex $t_{k'}$ with $j+1 \leq k' \leq n-1$ (see Figure 10 for a depiction). Thus, if a vertex appears before (via analogous arguments after) C_1 around v_1 clockwise from v_2, then it also appears before (after) C_2 around v_2 clockwise from v_1.

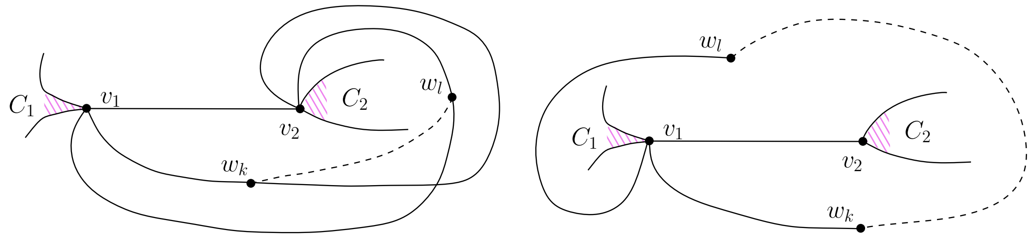

Figure 11 w_k cannot appear before w_l around v_2.

For two vertices w_k and w_l with $2 \leq k < l \leq i$, we show that w_l appears before w_k in the clockwise rotation of v_2 from v_1 by contradiction. Consider the edges v_1w_l and v_2w_l. If w_k appears before w_l in the clockwise rotation of v_2 from v_1, then the edges v_1w_l and v_2w_l emanate in different sides of the triangle $v_1v_2w_k$ from v_1 and v_2. From simplicity of the drawing and the fact that each realization of a generalized twisted rotation system of K_4 contains exactly one crossing [5], we can determine how that edge w_lw_k must behave (see Figure 11 for a depiction). We then show that with this drawing, C_1 and C_2 must lie on the same side of the triangle $v_1w_kw_l$, which contradicts that they are antipodal cells. ◀

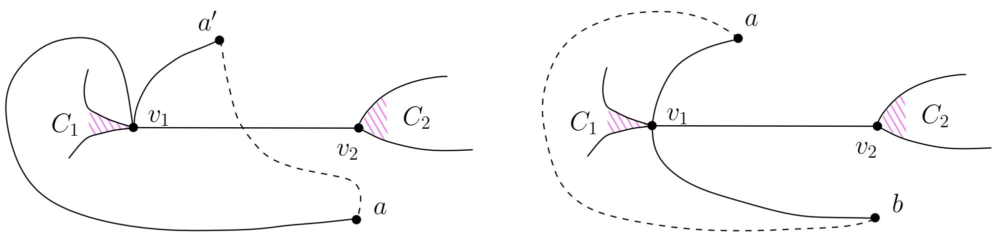

Figure 12 Illustrating the proof of Theorem 2 (i) and (iii).

Using Lemma 17 we can prove Theorem 2.

Proof Sketch of Theorem 2. If R is generalized twisted, then there exists a pair (C_1, C_2) of antipodal vi-cells in D, and we show that a bipartition of the vertices satisfies (i)–(v). Let v_1 be a vertex of V incident to C_1 and let $v_2 \neq v_1$ be a vertex of V incident to C_2.

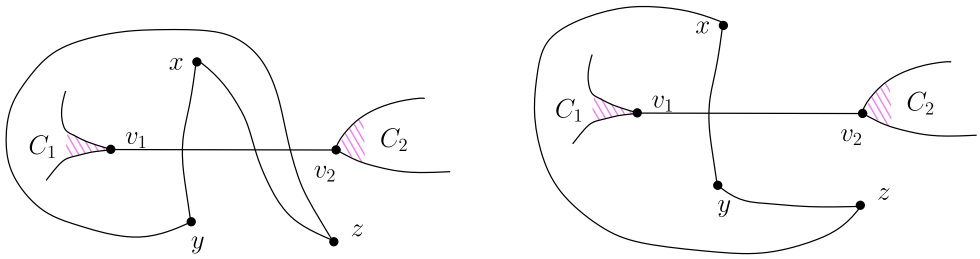

Figure 13 C_1 and C_2 are on different sides of Δxyz. Left: x, y, and z belong to the same set of the bipartition. Right: x and y belong to one of the sets and z to the other.

Suppose that $\{v_2 = w_1, w_2, \ldots, w_{n-1}\}$ is the (clockwise) rotation of v_1 such that v_1w_i and v_1w_{i+1} define part of the boundary of C_1, for some $i \in \{1, \ldots, n-1\}$. Let A and B be defined as follows: if $i \notin \{1, n-1\}$, set $A := w_{i+1}, ..., w_{n-1}$, $B := w_2, ..., w_i$; if $i = 1$, set $A := \{v_1, w_{n-1}, w_{n-2}, \ldots, w_2\}$, $B := \emptyset$; if $i = n-1$, set $A := \emptyset$, $B := \{v_1, w_{n-1}, w_{n-2}, \ldots, w_2\}$. We show that this bipartition satisfies (i)–(v).

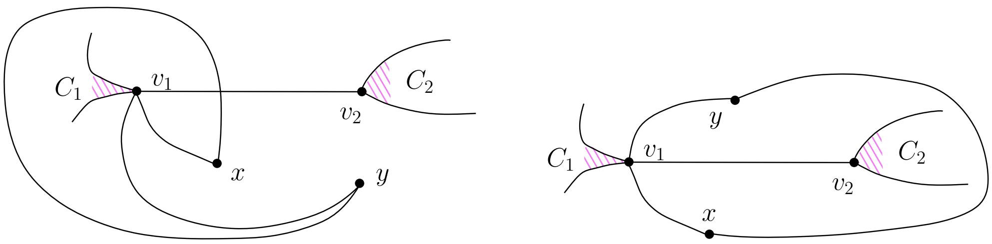

Figure 14 C_1 and C_2 are on different sides of Δv_1xy. Left: x and y belong to the same set of the bipartition. Right: x and y belong to different sets of the bipartition.

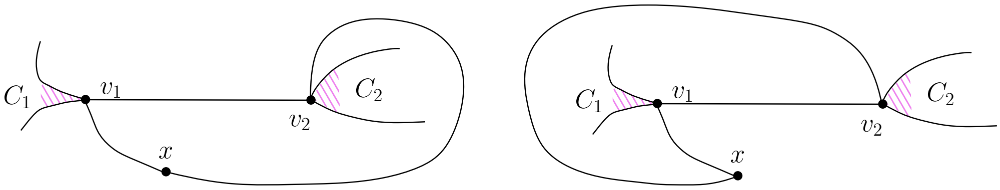

Figure 15 C_1 and C_2 are on different sides of Δv_1v_2x.

To prove that (i) holds, we take two vertices a and a' in A. (If A contains at most one vertex, (i) is vacuously true.) We use Lemma 17 to obtain the rotation of v_1 and v_2 and information on the bounding edges of C_1 and C_2. We conclude that the edge aa' crosses v_1v_2 because otherwise C_1 and C_2 would be on the same side of the triangle v_1aa' (see Figure 12(left) for an example) contradicting that C_1 and C_2 are antipodal. Thus, (i) follows and, by analogous arguments for B, (ii) follows.

To prove that (iii) holds for A and B, we take vertices $a \in A$ and $b \in B$. By Lemma 17, b is before C_1 and a is after C_1 around v_1. Thus, the edge ab cannot cross v_1v_2 so that C_1 and C_2 are on different sides of Δv_1ab (see Figure 12(right) for an example). Hence, (iii) follows. Finally, (iv)–(v) follow directly from the definitions of A and B.

To prove the other direction, we assume that R and D are a rotation system and drawing fulfilling (i)–(v), and show that R is generalized twisted. Let $\{v_2, b_1, \ldots, b_{n_1}, a_1, \ldots, a_{n_2}\}$ be the (clockwise) rotation of v_1 beginning at v_2, and $\{v_1, b'_1, \ldots, b'_{n_1}, a'_1, \ldots, a'_{n_2}\}$ be the (clockwise) rotation of v_2 beginning at v_1. Let C_1 be the vi-cell of D defined by the edges $v_1b_{n_1}$ and v_1a_1, and C_2 be the vi-cell of D defined by the edges $v_2b'_{n_1}$ and $v_2a'_1$. We show that C_1 and C_2 are antipodal and thus R is generalized twisted by showing for all possible triangles in D that C_1 and C_2 are on different sides of that triangle. We analyze the triangles and show the following. If v_1 and v_2 are not incident to Δ, the number of crossings give that v_1 and v_2 – and thus C_1 and C_2 – are on different sides of Δ (see Figure 13). If v_1 or v_2 are incident to Δ, we use C_1 and C_2 being on different ends of v_1v_2 to show that C_1 and C_2 are on different sides of Δ (see Figures 14 and 15). ◀

5 Efficient algorithms

In this section, we turn to the algorithmic aspect and prove Theorem 3.

To check whether an abstract rotation system of K_n is generalized twisted, by Theorem 1, only the subrotation systems induced by five vertices need checking. This can be done in $O(n^5)$ time. (The only exception where subdrawings are not sufficient are abstract rotation systems of K_6, but for $n = 6$ all three generalized twisted rotation systems are known.) Alternatively, one can first check in $O(n^5)$ time whether the rotation system is realizable [2, 20]. If it is not, it cannot be generalized twisted. If it is, we can use the following faster algorithm for realizable rotation systems of K_n, which we developed using Theorem 2. The algorithm runs in $O(n^2)$ time and in two steps, which we describe in the remainder of this section.

Step 1. In the first step, the empty star triangles at an arbitrary vertex x are computed.

▶ **Lemma 18.** *Given a realizable rotation system R of K_n and a vertex x of R, the set of all empty star triangles at x is the same in each realization of R and can be computed from R in $O(n^2)$ time.*

Proof Sketch. By Lemma 5, a star triangle xyz at x is empty if and only if y and z are consecutive in the rotation of x. Hence, there are at most $n-1$ empty star triangles at x. Whether a triangle xyz is a star triangle at x is determined by whether there exists a vertex w such that xw crosses yz. Since deciding for two edges if they cross each other can be done in constant time via the rotation system, deciding if a star triangle is empty requires $O(n)$ time. Thus, computing all empty star triangles at x can be done in $O(n^2)$ time. ◀

Step 2. In the second step, the previously computed empty star triangles at x and Theorem 2 are used to decide whether R is generalized twisted.

▶ **Lemma 19.** *Given a realizable rotation system R of K_n and the set of empty star triangles of a vertex x of R, it can be determined in $O(n^2)$ time whether R is generalized twisted.*

Proof Sketch. By Lemma 17, every drawing with generalized twisted rotation system has a pair (v_1, v_2) of compatible vertices. If (C_1, C_2) is a pair of antipodal vi-cells, then v_1 and v_2 lie on the boundary of C_1 and C_2 (or vice versa). We look for compatible vertices using empty triangles. Let x be the vertex with given empty star triangles. If there are not exactly two empty star triangles at x, then R is not generalized twisted by Lemma 6. Otherwise, we denote by $\Delta = xyz$ and $\Delta' = xy'z'$ the two empty star triangles at x. (Possibly, y or z coincides with y' or z'.) Using Lemma 6, we can show that if there is a compatible pair of vertices it must be in $S = \{(y, y'), (y, z'), (y, x), (z, y'), (z, z'), (z, x), (x, y'), (x, z')\}$. Checking compatibility of any pair in S by comparing their rotation systems takes linear time.

For every such potentially compatible pair (v_1, v_2), we check properties (i)–(v) of Theorem 2. Let $\{v = w_1, w_2, \ldots, w_{n-1}\}$ be the rotation of v_1. We define A and B as follows. If the rotation of v_2 is $\{v_1, w_{n-1}, \ldots, w_2\}$, then $B = \{w_2, \ldots, w_{n-1}\}$ and $A = \emptyset$ (or vice versa), and if the rotation of v_2 is $\{v_1, w_i, \ldots, w_2, w_{n-1}, \ldots, w_{i+1}\}$, for some i different from 1 and $n-1$, then $B = \{w_2, \ldots, w_i\}$ and $A = \{w_{i+1}, \ldots, w_{n-1}\}$. Then A and B clearly satisfy (iv) and (v) of Theorem 2. Checking if (i), (ii), and (iii) of Theorem 2 are satisfied requires $O(n^2)$ time.

Therefore, since $|S|$ is a constant, deciding if there is a pair (v_1, v_2) in S such that v_1 and v_2 are compatible and the properties (i)–(v) of Theorem 2 are satisfied for this pair can be done in $O(n^2)$ time. ◀

6 Conclusion and open problems

We presented two new characterizations for generalized twisted drawings based solely on their rotation systems. From these, we derived an $O(n^5)$-time algorithm that decides whether an abstract rotation system of K_n is generalized twisted and an $O(n^2)$-time algorithm to decide whether a realizable rotation system is generalized twisted. The obvious open question is whether the decision can also be done faster for abstract rotation systems. The quadratic time algorithm we give does not directly apply to them. There is a non-realizable rotation system of K_5 for which the output of the algorithm depends on the choice of vertex x and might be 'generalized twisted' or 'not generalized twisted'.

Generalized twisted drawings have been useful before, for example to find large plane substructures in simple drawings [5, 15]. Overcoming their rather geometrical definition, our new combinatorial characterizations might ease work with generalized twisted drawings and foster further results with this flavor. For example, Pach, Solymosi, and Tóth [21] showed that every simple drawing of a complete graph on n vertices contains $O(\log^{\frac{1}{8}} n)$ vertices inducing a complete subdrawing that is either convex or twisted. Recently, Suk and Zeng [25, 26] improved this result to $(\log n)^{\frac{1}{4}-o(1)}$ vertices. As both convex and twisted drawings contain large plane paths and cycles, this provides lower bounds on the size of plane paths and cycles that every simple drawing contains. It would be interesting to generalize this to show that every simple drawing contains a large generalized twisted or generalized convex subdrawing. Generalized convex drawings are – like generalized twisted drawings – a family of drawings of graphs that are well investigated [9, 11, 12]. Similarly to generalized twisted drawings, they admit characterizations via small constant sized subrotation systems [9]. Thus, a result guaranteeing the existence of a large generalized convex or generalized twisted drawing would help to extend our knowledge on simple drawings and, for example, to improve the lower bounds on the size of plane paths and cycles. We expect that this and several other questions on simple drawings can be tackled using the characterizations of this paper – independently or in combination. Especially, since the two given characterizations are very different from each other, with Theorem 1 being on all small subrotation systems and Theorem 2 being on two specific points and their rotation in the whole rotation system, they might foster the application of different other tools to obtain new results.

References

1 URL: `https://figshare.com/projects/g-twisted_rotation_systems/184474`.

2 Bernardo M. Ábrego, Oswin Aichholzer, Silvia Fernández-Merchant, Thomas Hackl, Jürgen Pammer, Alexander Pilz, Pedro Ramos, Gelasio Salazar, and Birgit Vogtenhuber. All good drawings of small complete graphs. In *Proc. 31st European Workshop on Computational*

Geometry EuroCG '15, pages 57–60, 2015. URL: http://www.ist.tu-graz.ac.at/files/publications/geometry/aafhpprsv-agdsc-15.pdf.

3 Bernardo M. Ábrego, Oswin Aichholzer, Silvia Fernández-Merchant, Pedro Ramos, and Gelasio Salazar. Shellable drawings and the cylindrical crossing number of K_n. *Discret. Comput. Geom.*, 52(4):743–753, 2014. doi:10.1007/S00454-014-9635-0.

4 Oswin Aichholzer, Alfredo García, Javier Tejel, Birgit Vogtenhuber, and Alexandra Weinberger. Twisted ways to find plane structures in simple drawings of complete graphs. In Xavier Goaoc and Michael Kerber, editors, *38th International Symposium on Computational Geometry, SoCG 2022*, volume 224 of *LIPIcs*, pages 5:1–5:18, 2022. doi:10.4230/LIPICS.SOCG.2022.5.

5 Oswin Aichholzer, Alfredo García, Javier Tejel, Birgit Vogtenhuber, and Alexandra Weinberger. Twisted ways to find plane structures in simple drawings of complete graphs. *Discret. Comput. Geom.*, 71(1):40–66, 2024. doi:10.1007/S00454-023-00610-0.

6 Oswin Aichholzer, Thomas Hackl, Alexander Pilz, Pedro Ramos, Vera Sacristán, and Birgit Vogtenhuber. Empty triangles in good drawings of the complete graph. *Graphs Comb.*, 31(2):335–345, 2015. doi:10.1007/S00373-015-1550-5.

7 Alan Arroyo, Dan McQuillan, R. Bruce Richter, and Gelasio Salazar. Drawings of $\mathrm{k_n}$ with the same rotation scheme are the same up to reidemeister moves (gioan's theorem). *Australas. J Comb.*, 67:131–144, 2017. URL: http://ajc.maths.uq.edu.au/pdf/67/ajc_v67_p131.pdf.

8 Alan Arroyo, Dan McQuillan, R. Bruce Richter, and Gelasio Salazar. Levi's lemma, pseudolinear drawings of K_n, and empty triangles. *J. Graph Theory*, 87(4):443–459, 2018. doi:10.1002/JGT.22167.

9 Alan Arroyo, Dan McQuillan, R. Bruce Richter, and Gelasio Salazar. Convex drawings of the complete graph: topology meets geometry. *Ars Math. Contemp.*, 22(3):3, 2022. doi:10.26493/1855-3974.2134.AC9.

10 Martin Balko, Radoslav Fulek, and Jan Kynčl. Crossing numbers and combinatorial characterization of monotone drawings of K_n. *Discret. Comput. Geom.*, 53(1):107–143, 2015. doi:10.1007/S00454-014-9644-Z.

11 Helena Bergold, Stefan Felsner, Meghana M. Reddy, Joachim Orthaber, and Manfred Scheucher. Plane Hamiltonian cycles in convex drawings. In *40th International Symposium on Computational Geometry, SoCG 2024, June 11-14, 2024, Athens, Greece*, volume 293 of *LIPIcs*, pages 18:1–18:16, 2024. doi:10.4230/LIPICS.SOCG.2024.18.

12 Helena Bergold, Joachim Orthaber, Manfred Scheucher, and Felix Schröder. Holes in convex and simple drawings. In Stefan Felsner and Karsten Klein, editors, *32nd International Symposium on Graph Drawing and Network Visualization, GD 2024*, volume 320 of *LIPIcs*, pages 5:1–5:9, 2024. doi:10.4230/LIPICS.GD.2024.5.

13 Stefan Felsner, Sandro Roch, and Manfred Scheucher. Arrangements of pseudocircles: On digons and triangles. In Patrizio Angelini and Reinhard von Hanxleden, editors, *30th International Symposium on Graph Drawing and Network Visualization, GD 2022*, volume 13764 of *Lecture Notes in Computer Science*, pages 441–455, 2022. doi:10.1007/978-3-031-22203-0_32.

14 Alfredo García, Javier Tejel, Birgit Vogtenhuber, and Alexandra Weinberger. Empty triangles in generalized twisted drawings of K_n. In Patrizio Angelini and Reinhard von Hanxleden, editors, *30th International Symposium on Graph Drawing and Network Visualization, GD 2022*, volume 13764, pages 40–48, 2022. doi:10.1007/978-3-031-22203-0_4.

15 Alfredo García, Javier Tejel, Birgit Vogtenhuber, and Alexandra Weinberger. Empty triangles in generalized twisted drawings of K_n. *J. Graph Algorithms Appl.*, 27(8):721–735, 2023. doi:10.7155/JGAA.00637.

16 Emeric Gioan. Complete graph drawings up to triangle mutations. In Dieter Kratsch, editor, *31st International Workshop on Graph-Theoretic Concepts in Computer Science, WG 2005*, volume 3787 of *Lecture Notes in Computer Science*, pages 139–150, 2005. doi:10.1007/11604686_13.

17 Emeric Gioan. Complete graph drawings up to triangle mutations. *Discret. Comput. Geom.*, 67(4):985–1022, 2022. doi:10.1007/S00454-021-00339-8.

18 Heiko Harborth and Ingrid Mengersen. Drawings of the complete graph with maximum number of crossings. *Congressus Numerantium*, pages 225–225, 1992.
19 Jan Kynčl. Improved enumeration of simple topological graphs. *Discret. Comput. Geom.*, 50(3):727–770, 2013. doi:10.1007/S00454-013-9535-8.
20 Jan Kynčl. Simple realizability of complete abstract topological graphs simplified. *Discret. Comput. Geom.*, 64(1):1–27, 2020. doi:10.1007/S00454-020-00204-0.
21 János Pach, József Solymosi, and Géza Tóth. Unavoidable configurations in complete topological graphs. *Discrete Comput Geometry*, 30:311–320, 2003. doi:10.1007/s00454-003-0012-9.
22 Yan Alves Radtke, Stefan Felsner, Johannes Obenaus, Sandro Roch, Manfred Scheucher, and Birgit Vogtenhuber. Flip graph connectivity for arrangements of pseudolines and pseudocircles. In David P. Woodruff, editor, *Proceedings of the 2024 ACM-SIAM Symposium on Discrete Algorithms, SODA 2024*, pages 4849–4871, 2024. doi:10.1137/1.9781611977912.172.
23 Marcus Schaefer. Taking a detour; or, Gioan's theorem, and pseudolinear drawings of complete graphs. *Discret. Comput. Geom.*, 66(1):12–31, 2021. doi:10.1007/S00454-021-00296-2.
24 Marcus Schaefer, Jean Cardinal, and Tillmann Miltzow. The existential theory of the reals as a complexity class: A compendium, 2024. doi:10.48550/arXiv.2407.18006.
25 Andrew Suk and Ji Zeng. Unavoidable patterns in complete simple topological graphs. In Patrizio Angelini and Reinhard von Hanxleden, editors, *30th International Symposium on Graph Drawing and Network Visualization, GD 2022*, volume 13764 of *Lecture Notes in Computer Science*, pages 3–15, 2022. doi:10.1007/978-3-031-22203-0_1.
26 Andrew Suk and Ji Zeng. Unavoidable patterns in complete simple topological graphs. *Discret. Comput. Geom.*, 73(1):79–91, 2025. doi:10.1007/S00454-024-00658-6.