Treewidth of Outer k-Planar Graphs

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Abstract

Treewidth is an important structural graph parameter that quantifies how closely a graph resembles a tree-like structure. It has applications in many algorithmic and combinatorial problems. In this paper, we study the treewidth of *outer k-planar* graphs, that is, graphs admitting a *convex drawing* (a straight-line drawing where all vertices lie on a circle) in which every edge crosses at most k other edges. We also consider the more general class of *outer min-k-planar* graphs, which are graphs admitting a convex drawing where for every crossing of two edges at least one of these edges is crossed at most k times.

Firman, Gutowski, Kryven, Okada and Wolff [GD 2024] proved that every outer k-planar graph has treewidth at most $1.5k + 2$ and provided a lower bound of $k + 2$ for even k. We establish a lower bound of $1.5k + 0.5$ for every odd k. Additionally, they showed that every outer min-k-planar graph has treewidth at most $3k + 1$. We improve this upper bound to $3 \cdot \lfloor k/2 \rfloor + 4$.

Our approach also allows us to upper bound the *separation number*, a parameter closely related to treewidth, of outer min-k-planar graphs by $2 \cdot \lfloor k/2 \rfloor + 4$. This improves upon the previous bound of $2k + 1$ and achieves a bound with an optimal multiplicative constant.

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1 Introduction

In this paper, we study classes of graphs admitting a *convex drawing* with a bounded number of edge crossings. A convex drawing is a straight-line drawing with all vertices drawn on a common circle. Bannister and Eppstein [1, 2] proved that the treewidth of graphs admitting a convex drawing with at most k crossings in total is bounded by a linear function of $\sqrt{k}$. For fixed k, they also provided a linear-time algorithm deciding whether a given graph admits such a drawing (using Courcelle's theorem [5]). Another well-studied class of graphs in this context is the class of *outer k-planar* graphs, that is, graphs that admit a convex drawing in which every edge crosses at most k other edges. These graphs have treewidth bounded by a linear function of k, which was first proven by Wood and Telle [13, Proposition 8.5]. The authors of [4], also using Courcelle's theorem, presented, for any fixed k, a linear-time algorithm that tests whether a given graph is maximal outer k-planar. Recently, Kobayashi, Okada and Wolff [10], for any fixed k, provided a polynomial-time algorithm to test whether a given graph is outer k-planar and proved that recognising outer k-planar graphs is XNLP-hard.

For disambiguation, we recall the definition of k-*outerplanar* graphs. A graph is *outerplanar* if it has a planar drawing with all vertices lying on the outer face. A graph is 1-*outerplanar* when it is outerplanar. A graph is k-*outerplanar* for $k > 1$ when it has a planar drawing such that after removing the vertices of the outer face, each of the remaining components is $(k-1)$-outerplanar.

We mainly study the treewidth of outer k-planar graphs and *outer min-k-planar* graphs. A graph is outer min-k-planar if it admits a convex drawing in which, for every crossing of two edges, at least one of these edges is crossed at most k times. Firman, Gutowski, Kryven, Okada and Wolff [7] proved that outer k-planar graphs have treewidth at most $1.5k + 2$ and outer min-k-planar graphs have treewidth at most $3k + 1$. To obtain these results, they showed that every outer k-planar graph admits a *triangulation* of the outer cycle such that every edge of the triangulation is crossed at most k times by the edges of the graph. A similar property was proven for outer min-k-planar graphs.

Another property closely related to treewidth is the separation number of a graph. A *separation* of a graph G is a pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$ and there are no edges between the sets $A \setminus B$ and $B \setminus A$. The *order* of a separation is $|A \cap B|$. A separation is *balanced* if $|A \setminus B| \le \frac{2}{3}|V(G)|$ and $|B \setminus A| \le \frac{2}{3}|V(G)|$. The *separation number* of a graph G, denoted $\operatorname{sn}(G)$, is the minimum integer a such that every subgraph of G has a balanced separation of order at most a. Robertson and Seymour [11] proved that $\operatorname{sn}(G) \le \operatorname{tw}(G)+1$ for every graph G. From the other side, Dvořák and Norin [6] showed that $\operatorname{tw}(G) \le 15\operatorname{sn}(G)$. Recently, Houdrouge, Miraftab and Morin [8] provided a more constructive proof of an analogous inequality, but with a worse multiplicative constant.

Our contribution. The authors of [7] proved that every outer k-planar graph has treewidth at most $1.5k + 2$. They also presented a lower bound of $k + 2$ for every even k. We present an infinite family of outer k-planar graphs with treewidth at least $1.5k + 0.5$, showing that the multiplicative constant 1.5 in the upper bound cannot be improved; see Section 3.

We also improve the upper bounds for the treewidth and separation number of outer min-k-planar graphs. It was previously known that the treewidth of such graphs is at most $3k+1$ and the separation number is at most $2k+1$ [7]. We give an upper bound of $3\cdot\lfloor k/2\rfloor+4$ for the treewidth (see Section 4) and an upper bound of $2 \cdot \lfloor k/2\rfloor + 4$ for the separation number (see Section 5). Both multiplicative constants are optimal, as the lower bounds for outer k-planar graphs also hold for outer min-k-planar graphs – namely, our lower bound of $1.5k+0.5$ for the treewidth and the lower bound of $k+2$ for the separation number presented in [7].

2 Preliminaries

Let G be a graph. By $V(G)$ and $E(G)$ we denote the set of vertices and edges of G, respectively. For an edge that connects vertices u and v, we use the compact notation uv, instead of $\{u, v\}$. For a directed edge, we use the standard notation (u, v). Let $\deg(v)$ denote the degree of a vertex v, and let $\Delta(G)$ denote the maximum degree of a vertex of G.

For a graph G, a subgraph *induced* by a set $U \subseteq V(G)$, denoted $G[U]$, is a subgraph with vertex set U and all edges of G between the vertices of U. A *spanning tree* of a graph G is a subgraph of G containing all the vertices of G that is a tree. By $\operatorname{dist}_G(v, w)$, we denote the distance (i.e. the length of the shortest path) between v and w in a graph G. For any tree T rooted at vertex r, we define the depth of a vertex v as $\operatorname{depth}_T(v) = \operatorname{dist}_T(r, v)$. We may omit subscripts if they are clear from the context.

A *tree decomposition* $\mathcal{T} = (T, B)$ of a graph G is a collection of *bags*, $\{B_x : x \in V(T)\}$, indexed by the vertices of a tree T. The bags are subsets of $V(G)$ and satisfy the following properties:

1. for every vertex $v \in V(G)$, the set $\{x : v \in B_x\}$ induces a non-empty subtree of T;
2. for every edge $uv \in E(G)$, there exists a bag containing both u and v.

The *width* of a given tree decomposition is the size of the largest bag minus one. The *treewidth* of a graph G, denoted by $\text{tw}(G)$, is the minimum width of any tree decomposition of G.

A set $\mathcal{B}$ of non-empty subsets of $V(G)$ is a *bramble* if:

1. for every $X \in \mathcal{B}$, the induced subgraph $G[X]$ is connected;
2. for every $X_1, X_2 \in \mathcal{B}$, the induced subgraph $G[X_1 \cup X_2]$ is connected. In other words, each $X_1, X_2 \in \mathcal{B}$ either share a common vertex or there exists an edge of G incident to both X_1 and X_2.

A *hitting set* of a bramble is a set of vertices with non-empty intersection with every element of $\mathcal{B}$. The *order* of a bramble is the size of its smallest hitting set. The *bramble number* of a graph G, denoted by $\text{bn}(G)$, is the maximum order of any bramble of G.

The following result by Seymour and Thomas shows the relation between the bramble number and treewidth.

▶ **Theorem 1** (Seymour and Thomas, [12]). *For every graph G, $\text{tw}(G) = \text{bn}(G) - 1$.*

We say that a graph G is a *minor* of a graph H if G can be obtained from H by a sequence of vertex deletions, edge deletions or edge contractions. The contraction of an edge uv is an operation that replaces vertices u and v with a new vertex adjacent to every vertex other than u and v that was adjacent to u or v. It is a well-known fact that if G is a minor of H, then $\text{tw}(G) \leq \text{tw}(H)$. A proof of this fact can be found in [3].

In the remainder of this section, we introduce some notation and simple observations regarding drawings. A *convex drawing* of a graph G is a straight-line drawing where the vertices of G are placed on distinct points of a circle. Given a cyclic order $(v_1, \ldots, v_n)$ of vertices, we say that an edge v_iv_j with $i < j$ *crosses* an edge $v_{i'}v_{j'}$ with $i' < j'$ if either $1 \leq i < i' < j < j' \leq n$ or $1 \leq i' < i < j' < j \leq n$. We only consider convex drawings where no three pairwise non-adjacent edges pass through the same point. An *outer k-planar drawing* of a graph is a convex drawing such that every edge crosses at most k other edges. An *outer min-k-planar drawing* of a graph is a convex drawing such that for every crossing of two edges, at least one of these edges crosses at most k other edges.

An outer min-k-planar graph G is *maximal outer min-k-planar* if for every $\{u, v\} \subseteq V(G)$ with $uv \notin E(G)$, the graph $G + uv$ is not outer min-k-planar.

▶ **Observation 2.** *Let G be a maximal outer min-k-planar graph with at least three vertices. Then, in every outer min-k-planar drawing of G, the outer face is bounded by a cycle.*

Proof. Consider an outer min-k-planar drawing Γ of G. Let u and v be consecutive vertices in the cyclic order defined by Γ. Suppose, for contradiction, that $uv \notin E(G)$. Note that the graph $G + uv$ has an outer min-k-planar drawing defined by the same cyclic order as Γ, contradicting the maximality of G. ◀

A graph G is *expanded outer min-k-planar* if G is an outer min-k-planar graph with $\Delta(G) \leq 3$ and its outer face is bounded by a cycle in some outer min-k-planar drawing of G.

▶ **Observation 3.** *Every outer min-k-planar graph G is a minor of an expanded outer min-k-planar graph G'.*

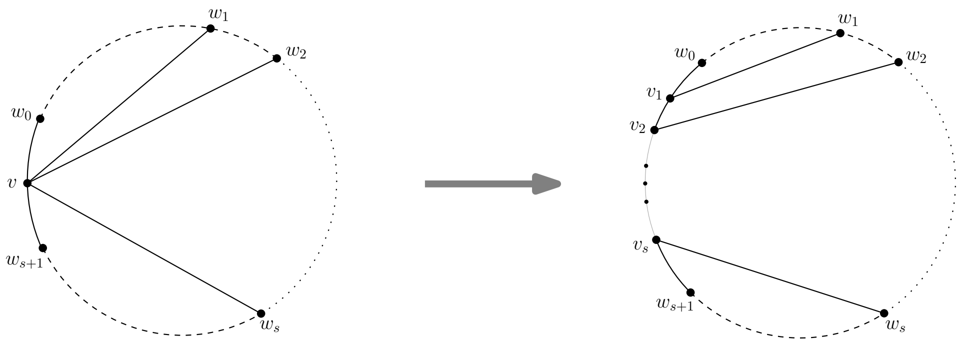


Figure 1 The transformation described in Observation 3.

Proof. Let us assume that G is maximal outer min-k-planar. Now, in order to obtain G' from G, we perform the following transformation to every vertex v of G with $\deg(v) \geq 4$. The transformation is depicted in Figure 1. Let $w_0, w_1, \ldots, w_s, w_{s+1}$ be all neighbors of v in clockwise order, with edges vw_0 and vw_{s+1} incident to the outer face of G. We replace v with a path $v_1, \ldots, v_s$, put it on the outer face of G in counterclockwise order, in the place of v. We connect this path to vertices w_0 and w_{s+1} by adding edges v_1w_0 and v_sw_{s+1}. Finally, for every $1 \leq i \leq s$, we add an edge v_iw_i that corresponds to an edge vw_i in the original graph. It is easy to see that G is a minor of G' and the ordering of corresponding edges in G' matches the one in G. Moreover, the crossings in the resulting graph naturally correspond to the crossings in the original graph. ◀

The vertices $v_1, \ldots, v_s$ obtained in the proof as a replacement of v are called *images* of v. The vertex v is the *origin* of these vertices, which we denote as $\mathrm{org}(v_i) = v$. If the transformation was not performed for some vertex v of G, i.e. $\deg(v) \leq 3$, then v is an image and origin of itself.

Since removing edges increases neither the treewidth nor the separation number, we are interested in the properties of maximal outer min-k-planar graphs. Also, taking a minor does not increase treewidth, so we work with expanded graphs when establishing upper bounds on treewidth.

3 Lower bound on the treewidth of outer k-planar graphs

In this section, we construct an infinite family of outer k-planar graphs with treewidth at least $1.5k + 0.5$. This improves the previous lower bound of $k + 2$ that was presented in [7]. We begin by defining the necessary graphs.

For positive integers m and n, let $X_{m,n}$ denote the grid of m rows and n columns, i.e. a graph with

$$V(X_{m,n}) = \{x_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} \text{ and } E(X_{m,n}) = \{x_{i,j}x_{k,l} : |i-k| + |j-l| = 1\}.$$

For a positive integer k, let Q_k be a copy of the grid $X_{2k,2k}$ and let R_k be a copy of $X_{2k(k+1),k}$. Denote by $v_{i,j}$, for $1 \leq i,j \leq 2k$, the vertex in the i-th row and j-th column of Q_k, and by $u_{i,j}$, for $1 \leq i \leq 2k(k+1)$, $1 \leq j \leq k$, the vertex in the i-th row and j-th column of R_k. Let G_k be a graph such that $V(G_k) = V(Q_k) \cup V(R_k)$ and

$$E(G_k) = E(Q_k) \cup E(R_k) \cup \left\{v_{i,2k}u_{(i-1)(k+1)+j,1} : 1 \leq i \leq 2k, 1 \leq j \leq k+1\right\};$$

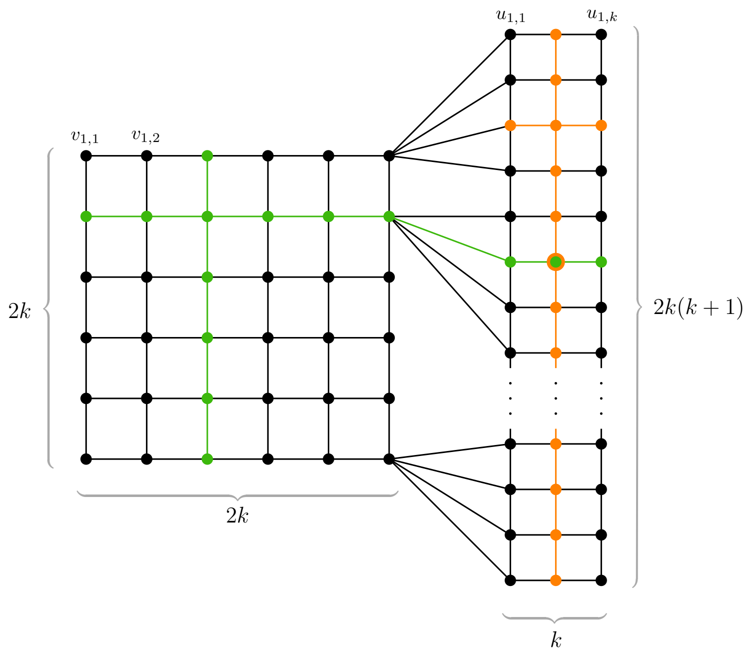


Figure 2 The graph G_k, for $k = 3$, with a subgraph of $\mathcal{B}_1$ colored green and a subgraph of $\mathcal{B}_2$ colored orange.

see Figure 2. For $1 \le i \le 2k(k+1)$, let the i-th *extended row* of G_k be the union of the i-th row of R_k and the $\lceil \frac{i}{k+1} \rceil$-th row of Q_k. Note that each row of Q_k is contained in $k+1$ extended rows and the graph induced by each extended row is a path.

The graph G_k was previously introduced by Kammer and Tholey [9] as an example of tightness of the upper bound on the treewidth of k-outerplanar graphs. They used the *cops and robber game* to establish a lower bound on the treewidth of G_k. Below, we present a proof using brambles.

▶ **Theorem 4** (Kammer and Tholey, [9]). *For every $k \ge 1$, $\mathrm{tw}(G_k) = 3k - 1$.*

Proof. Notice that the drawing of G_k in Figure 2 is k-outerplanar. By the fact that k-outerplanar graphs have treewidth at most $3k-1$ [3, Theorem 83], we get $\mathrm{tw}(G_k) \le 3k-1$.

To prove that $\mathrm{tw}(G_k) \ge 3k-1$, we construct a bramble of order $3k$. Then, using Theorem 1, we get $\mathrm{tw}(G_k) \ge 3k-1$. Let $\mathcal{B}_1$ be a family consisting of every subset of $V(G_k)$ that is a union of an extended row of G_k and a column of Q_k. Let $\mathcal{B}_2$ be a family consisting of every subset of $V(G_k)$ that is a union of a row of R_k and a column of R_k. The set $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ forms a bramble of G_k, since each subgraph induced by an element of $\mathcal{B}$ is connected and every two such subgraphs have at least one common vertex.

Consider any hitting set S of $\mathcal{B}$. Let q and r be the number of vertices of S in $V(Q_k)$ and in $V(R_k)$, respectively. We would like to show that $|S| = q + r \ge 3k$. Note that $r \ge k$, as otherwise there is a row and a column of R_k not containing any element of S, and thus there is an element of $\mathcal{B}_2$ not hit by S.

If $q \ge 2k$, then $q + r \ge 3k$. Otherwise, let $q = 2k - l$ for some positive integer l. Now, we can find at least l columns and at least l rows of Q_k not intersecting S. These l rows are contained in $l(k+1)$ extended rows. Each of them has to intersect S at some vertex of R_k, because otherwise we can find a column of Q_k and an extended row not intersecting S that

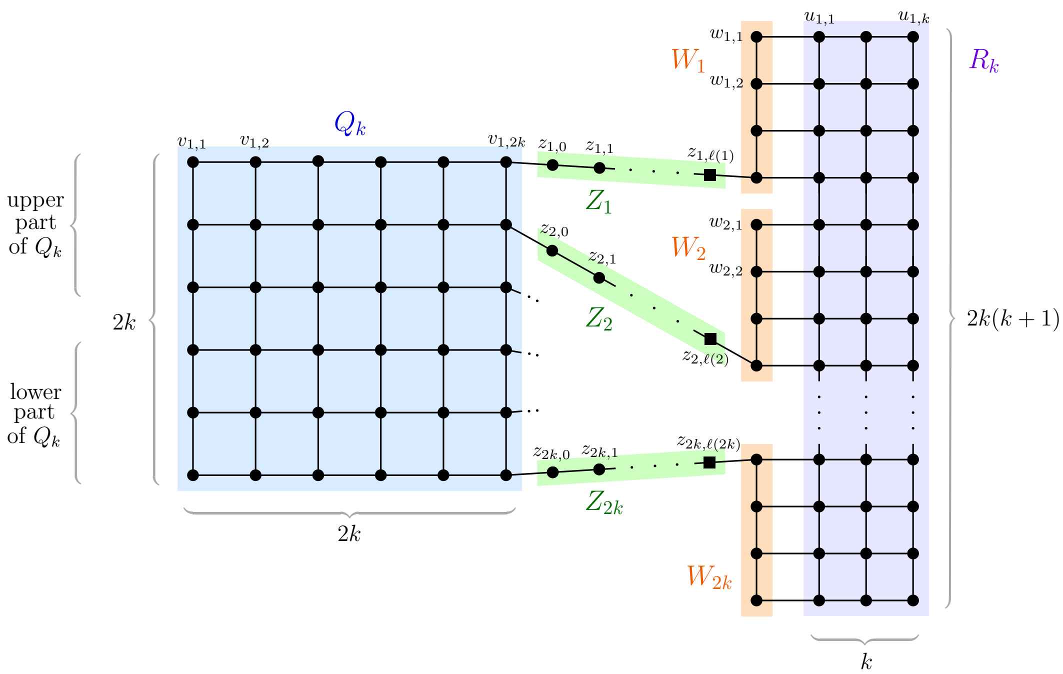


Figure 3 The graph F_k, which is a modification of the graph G_k, for $k = 3$.

form an element of $\mathcal{B}_1$. The extended rows restricted to R_k are pairwise disjoint, so we have $r \geq l(k+1)$. Summing up, we get $q + r \geq 2k - l + l(k+1) = 2k + lk \geq 2k + k = 3k$, which concludes the proof. ◀

Let F_k be the following modification of G_k depicted in Figure 3. We set $\ell(i) = (k-i)(k+1)$ for $1 \leq i \leq k$, and $\ell(i) = (i-k-1)(k+1)$ for $k+1 \leq i \leq 2k$. We remove every edge between the grids Q_k and R_k. For every $1 \leq i \leq 2k$, we add a path Z_i of length $\ell(i)$ on new vertices $z_{i,0}, z_{i,1}, \ldots, z_{i,\ell(i)}$. Note that

$$k^2 - 1 = |V(Z_1)| = |V(Z_{2k})| > |V(Z_2)| = |V(Z_{2k-1})| > \ldots > |V(Z_k)| = |V(Z_{k+1})| = 1.$$

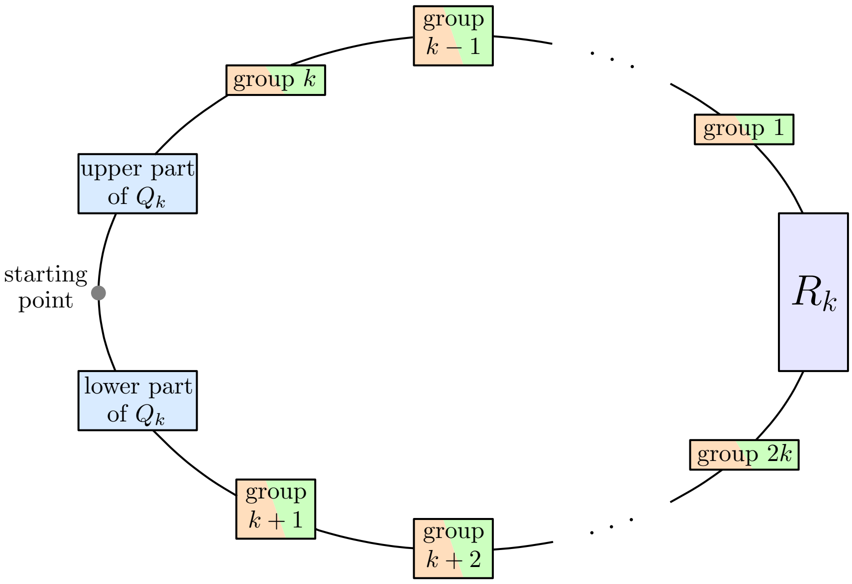
We also add a path W_i of length k on new vertices $w_{i,1}, w_{i,2}, \ldots, w_{i,k+1}$. We connect $v_{i,2k}$ with Z_i by adding the edge $v_{i,2k}z_{i,0}$. Next, we connect Z_i with W_i by adding the edge $z_{i,\ell(i)}w_{i,k+1}$ for $1 \leq i \leq k$, or the edge $z_{i,\ell(i)}w_{i,1}$ for $k+1 \leq i \leq 2k$. Finally, we connect W_i with R_k by adding the edges $w_{i,j}u_{(i-1)(k+1)+j,1}$ for every $1 \leq j \leq k+1$.

To see that G_k is a minor of F_k, it is enough to contract, for every $1 \leq i \leq 2k$, vertex $v_{i,2k}$ with all vertices of the paths Z_i and W_i. Since taking a minor does not increase the treewidth, we obtain the following corollary.

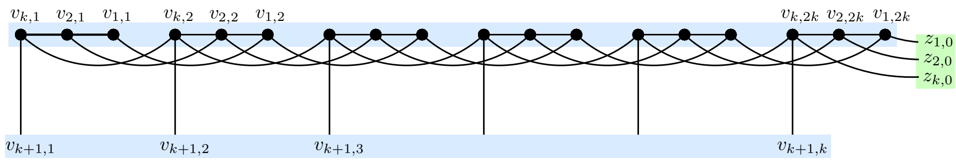
▶ **Corollary 5.** *For every $k \geq 1$, $\mathrm{tw}(F_k) \geq 3k - 1$.*

▶ **Theorem 6.** *For every $k \geq 1$, The graph F_k has an outer $(2k-1)$-planar drawing.*

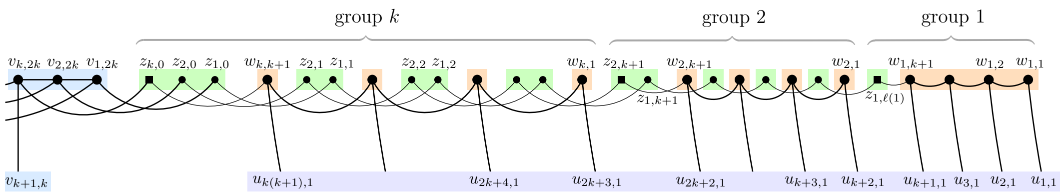
Proof. We describe an outer $(2k-1)$-planar drawing of F_k, as depicted in Figure 4. We call the set of vertices $\{v_{i,j} : 1 \leq i \leq k, 1 \leq j \leq 2k\}$ the *upper part* of Q_k. The other vertices of Q_k are called the *lower part* of Q_k. We define a cyclic order of the vertices of F_k by arranging them in clockwise order from some selected starting point on a circle.



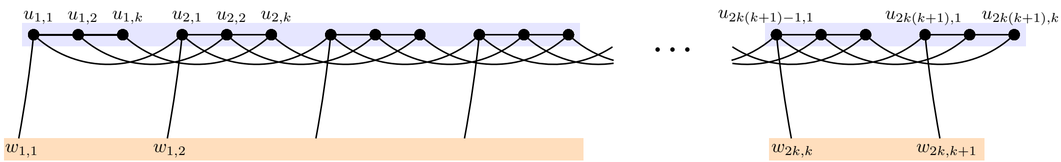
(a) Overview of the outer $(2k-1)$-planar drawing of F_k.



(b) Drawing of the upper part of the grid Q_k.



(c) Drawing of the groups $k, k-1, \dots, 1$ connected to the upper part of Q_k.



(d) Drawing of the grid R_k.

Figure 4 Fragments of the outer $(2k-1)$-planar drawing of F_k, where $k = 3$. The drawings are not shown as straight-line drawings, but illustrate the order in which the vertices are placed. A straight-line drawing can be easily obtained by placing vertices in this order on the circle.

First, we place the vertices of the upper part of Q_k in column-by-column order (see Figure 4b):

$$v_{k,1}, \dots, v_{2,1}, v_{1,1}, \quad v_{k,2}, \dots, v_{2,2}, v_{1,2}, \quad \dots \quad v_{k,2k}, \dots, v_{2,2k}, v_{1,2k}.$$

We divide the vertices of paths Z_i and W_i, for $1 \le i \le k$, into k groups, as follows. The i-th group contains the vertices of W_i and the vertex $z_{i,\ell(i)}$. If $i \ge 2$, the i-th group also contains vertices $z_{a,b}$ for $1 \le a < i$ and $(k-i)(k+1) \le b < (k-i+1)(k+1)$. In the drawing, we place the groups of indices $k, k-1, \dots, 1$, in this order. We arrange the vertices in the i-th group, for $2 \le i \le k$, in the order (see Figure 4c):

$$\begin{aligned}
&z_{i,\ell(i)},\\
&z_{i-1,(k-i)(k+1)}, z_{i-2,(k-i)(k+1)}, \dots, z_{1,(k-i)(k+1)}, w_{i,k+1},\\
&z_{i-1,(k-i)(k+1)+1}, z_{i-2,(k-i)(k+1)+1}, \dots, z_{1,(k-i)(k+1)+1}, w_{i,k},\\
&\vdots\\
&z_{i-1,(k-i+1)(k+1)-1}, z_{i-2,(k-i+1)(k+1)-1}, \dots, z_{1,(k-i+1)(k+1)-1}, w_{i,1}.
\end{aligned}$$

The group of index 1 has vertices arranged in the order: $z_{1,\ell(1)}, w_{1,k+1}, w_{1,k}, \dots, w_{1,1}$.

Next, we put the vertices of R_k in row-by-row order (see Figure 4d):

$$u_{1,1}, u_{1,2}, \dots, u_{1,k}, \quad u_{2,1}, u_{2,2}, \dots, u_{2,k}, \quad \dots \quad u_{2k(k+1),1}, u_{2k(k+1),2}, \dots, u_{2k(k+1),k}.$$

The vertices of F_k that are not placed yet are in the lower part of Q_k or in the paths Z_i, W_i with $k+1 \le i \le 2k$. We arrange them in counterclockwise order from the starting point and place them between the starting point and the vertices of R_k. The order is symmetric, with respect to the starting point, to the one used to arrange the upper part of Q_k and the paths Z_i, W_i with $1 \le i \le k$. Every vertex $v_{i,j}$, where $k+1 \le i \le 2k$ and $1 \le j \le 2k$, is placed symmetrically to $v_{2k-i+1,j}$. Vertex $z_{i,j}$, where $k+1 \le i \le 2k$ and $0 \le j \le \ell(i)$, is placed symmetrically to $z_{2k-i+1,j}$, and vertex $w_{i,j}$, where $k+1 \le i \le 2k$ and $1 \le j \le k+1$, is placed symmetrically to $w_{2k-i+1,k-j+2}$. The symmetrical drawing of the i-th group, for every $1 \le i \le k$, forms the group of index $2k-i+1$.

Now, we partition the edges into several numbered types. For the edges of each type, we show that they cross at most $2k-1$ other edges.

Edges of Q_k:

1. The "column" edges of the upper or lower part of Q_k, that is, the edges $v_{i,j}v_{i+1,j}$, for $1 \le i \le 2k-1, i \ne k$ and $1 \le j \le 2k$. They cross no other edges.
2. The "column" edges between the upper part and the lower part of Q_k, that is, the edges $v_{k,j}v_{k+1,j}$, for $1 \le j \le 2k$. Each of these edges crosses $k-1$ edges of type 3 of the upper part of Q_k, and $k-1$ edges of type 3 of the lower part of Q_k. The edge $v_{k,1}v_{k+1,1}$ does not cross any edges.
3. The "row" edges of Q_k, that is, the edges $v_{i,j}v_{i,j+1}$, for $1 \le i \le 2k$ and $1 \le j \le 2k-1$. Each of these edges crosses at most $2(k-1)$ edges of type 3 (and type 4, for $j = 2k-1$), and additionally at most one edge of type 2.

Edges between Q_k and the groups:

4. Each edge $v_{i,2k}z_{i,0}$, for $1 \le i \le k$, crosses $2(k-1)$ edges in total: $2(i-1)$ edges of types 3, 4 incident to vertices $v_{j,2k}$, for all $j \in \{1, \dots, i-1\}$; and $2(k-i)$ edges incident to vertices $z_{j,0}$, for all $j \in \{i+1, \dots, k\}$. By symmetry, each edge $v_{i,2k}z_{i,0}$, for $k+1 \le i \le 2k$, also crosses exactly $2(k-1)$ edges.

Edges of the groups:

5. Each edge $z_{i,\ell(i)}w_{i,k+1}$, for $1 \le i \le k$, crosses exactly $2(i-1)$ edges incident to the vertices $z_{i-1,\ell(i)}, \dots, z_{1,\ell(i)}$. By symmetry, each edge $z_{i,\ell(i)}w_{i,1}$, for $k+1 \le i \le 2k$, also crosses exactly $2(2k-i)$ edges.

6. Each edge of the path Z_i, for $1 \le i \le 2k$, crosses at most $2(k-2)$ edges incident to at most $k-2$ vertices of some paths Z_j with $j \in \{1, \dots, 2k\}$, and at most three edges incident to some vertex $w_{a,b} \in V(W_j)$ with $j \in \{1, \dots, 2k\}$.

7. Each edge of the path W_i, for $1 \le i \le k$, crosses $2(i-1)$ edges incident to $i-1$ vertices of some paths Z_j with $j \in \{1, \dots, k\}$. By symmetry, each edge of the path W_i, for $k+1 \le i \le 2k$, also crosses $2(2k-i)$ edges.

Edges between the groups and R_k:

8. Each edge $w_{i,j}u_{(i-1)(k+1)+j,1}$, for $1 \le i \le 2k$ and $1 \le j \le k+1$, crosses at most $k-1$ edges of some paths Z_a with $a \in \{1, \dots, 2k\}$, and at most $k-1$ edges of R_k of type 10.

Edges of R_k:

9. The "row" edges of R_k, that is, the edges $u_{i,j}u_{i,j+1}$, for $1 \le i \le 2k(k+1)$ and $1 \le j \le k-1$. They cross no other edges.

10. The "column" edges of R_k, that is, the edges $u_{i,j}u_{i+1,j}$, for $1 \le i \le 2k(k+1)-1$ and $1 \le j \le k$. Each of these edges crosses at most $2(k-1)$ other edges of this type and at most one edge of type 8. ◀

▶ **Theorem 7.** *For every odd positive integer k, there exists an outer k-planar graph G with* $\operatorname{tw}(G) \ge 1.5k + 0.5$.

Proof. By Theorem 6, the graph $F_{\frac{k+1}{2}}$ is outer k-planar, and by Corollary 5, it has treewidth at least $3 \cdot \frac{k+1}{2} - 1 = 1.5k + 0.5$. ◀

4 Upper bound on the treewidth of outer min-k-planar graphs

In this section, we establish an upper bound on the treewidth of outer min-k-planar graphs. We improve the previous bound of $3k+1$ presented in [7] to $3 \cdot \lfloor k/2 \rfloor + 4$. We begin by introducing required notation.

For an outer min-k-planar graph G with a given drawing Γ, we define the *planarization* G_P (with respect to Γ) as the graph whose vertex set is the union of $V(G)$ and all crossing points of edges of G. We say that a vertex $w \in V(G_P)$ *lies* on an edge $uv \in E(G)$ if w is an endpoint of uv, or if the crossing point corresponding to w belongs to the segment representing the drawing of uv in Γ. Graph G_P contains an edge between two vertices if and only if they are consecutive vertices lying on the drawing of some edge of G. Observe that G_P is a planar graph. We say that an edge $xy \in E(G_P)$ *lies* on an edge $uv \in E(G)$ if both x and y lie on uv in Γ. Furthermore, we say that a vertex $v \in V(G_P)$ is *outer* if it is incident to the outer face of G_P. Otherwise, v is an *inner* vertex. As we consider only maximal outer min-k-planar graphs G, the outer vertices of G_P are exactly the vertices of G, while the inner vertices of G_P are exactly the crossing points of edges of G. By f_o we denote the outer face of G_P.

For a planar graph G, let G^* denote its dual graph. Let $f^* \in V(G^*)$ denote the vertex dual to the face f of G, and let $e^* \in E(G^*)$ denote the edge dual to the edge $e \in E(G)$. We remark that G^* can be drawn on the drawing of G in a way that f^* is on the face f and the drawing of e^* is a curve that crosses the edge e exactly once and passes only through the faces corresponding to the endpoints of e^*.

The following lemma shows a bijection between a spanning tree T of a planar graph G and a spanning tree T^* of G^*, where $T^* = \mathrm{dual}(T)$. We also use the notation $\mathrm{dual}(T^*)$ to denote T.

▶ **Lemma 8** (Folklore). *Let T be a spanning tree of a planar graph G. Then T^* with $V(T^*) = V(G^*)$ and $E(T^*) = \{e^* : e \in E(G) \setminus E(T)\}$ is a spanning tree of G^*.*

The next lemma proves that there exists a spanning tree preserving shortest paths from a given vertex. Such a tree can be found using a breadth-first search.

▶ **Lemma 9** (Folklore). *Let G be a graph and let r be a vertex of G. Then there exists a spanning tree T of G rooted at r such that $\mathrm{depth}_T(v) = \mathrm{dist}_G(r, v)$ for every vertex v of G.*

▶ **Lemma 10.** *Let G be an expanded outer min-k-planar graph with its planarization G_P. Then $\mathrm{dist}(f^*, f_o^*) \leq \lfloor k/2 \rfloor + 1$ for every vertex $f^* \in V(G_P^*)$.*

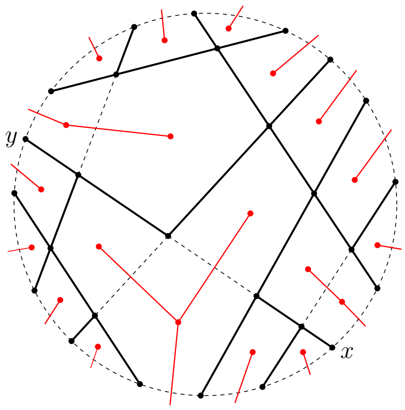
Proof. Let f be an inner face of G_P. If f is adjacent to f_o, then $\mathrm{dist}(f^*, f_o^*) = 1$. Otherwise, let v be a vertex of G_P incident to f. Since G is expanded, the vertex v is inner, and hence it lies on an edge e of G that crosses at most k other edges. Let $v_0, v_1, \ldots, v_s, v_{s+1}, \ldots, v_{s+t+1}$ be all vertices lying on e, listed in the order along e, where $v_s = v$ and v_{s+1} is a neighbor of v that is incident to f. We may assume that $s \leq t$, i.e. v_s is closer to an endpoint of the edge e than v_{s+1} to the other endpoint of e. Note that at most $k + 2$ vertices lie on e (two endpoints and at most k crossing points), so $s + t + 2 \leq k + 2$. Together with the previous inequality, this implies $s \leq k/2$. The number s is an integer so $s \leq \lfloor k/2 \rfloor$.

We inductively define a sequence $w_s, w_{s-1}, \ldots, w_0$ of vertices. Vertex w_s is the neighbor of v_s that is incident to f and does not lie on e. For every $i = s - 1, \ldots, 0$, the vertex w_i is one of the two neighbors of v_i not lying on e, chosen so that w_i, v_i, v_{i-1} and w_{i-1} are incident to the same face of G_P. Let e_i denote the edge $v_i w_i$. Note that the path formed by the edges $e_s^*, \ldots, e_0^*$ connects f^* with f_o^* (since e_0 is incident to f_o). Hence, $\mathrm{dist}(f^*, f_o^*) \leq s + 1 \leq \lfloor k/2 \rfloor + 1$. ◀

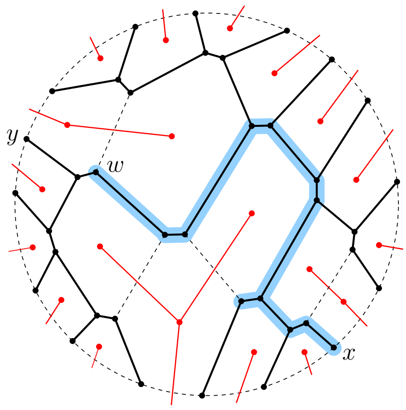
Let v be an inner vertex of G_P. Since we forbid common crossing points of three edges of G, the vertex v has four neighbors, which we denote by w_1, w_2, w_3, w_4 in clockwise order. The *splitting* of the vertex v replaces it with two vertices v_1 and v_2 connected by an edge, and adds edges $v_1 w_1$, $v_1 w_2$ and $v_2 w_3$, $v_2 w_4$. We fix a planar embedding of the new graph by placing v_1, v_2 very close to where v was drawn. We say that vertices v_1 and v_2 *lie* on the same edges as the vertex v. Moreover, we call the edge $v_1 v_2$ an *auxiliary* edge. After splitting v, every edge of G_P has an edge corresponding to it, and every face of G_P corresponds to a new one in a natural way. Additionally, the dual graph has one new edge, which is dual to $v_1 v_2$.

Let G_S denote the *split planarization* of the outer min-k-planar graph G, that is, the graph G_P with all inner vertices split. See Figure 5 for an example. Observe that there is a one-to-one correspondence between $V(G_P^*)$ and $V(G_S^*)$. Further, every edge of G_P^* has a corresponding edge of G_S^*. The following lemma shows how the properties of a spanning tree T_P of G_P and a spanning tree $\mathrm{dual}(T_P)$ of the dual graph G_P^* are preserved after splitting the vertices of G_P.

▶ **Lemma 11.** *Let G be an expanded outer min-k-planar graph with its split planarization G_S. Then there exists a spanning tree T_S of G_S and a spanning tree $T_S^* = \mathrm{dual}(T_S)$ of G_S^* rooted at f_o^*, such that $\mathrm{depth}(f^*) \leq \lfloor k/2 \rfloor + 1$ for every vertex $f^* \in V(G_S^*)$ and $E(T_S)$ contains all auxiliary edges of G_S.*



(a) The graph G_P with its spanning tree T_P colored black and a spanning tree T_P^* of G_P^* in red.



(b) The graph G_S with its spanning tree T_S colored black and a spanning tree T_S^* of G_S^* in red.

Figure 5 Drawings of an example graphs with their spanning trees and spanning trees of the dual graphs. An example walk, constructed in the proof of Theorem 12, that is connecting w and x, is marked in blue. The vertex f_o^* is missing in both drawings.

Proof. Let G_P be a planarization of G. By Lemmas 9 and 10, there exists a spanning tree T_P^* of G_P^* that is rooted at f_o^* whose vertices have depth at most $\lfloor k/2 \rfloor + 1$. Let $T_P = \mathrm{dual}(T_P^*)$ denote the spanning tree of G_P.

Let G_S denote the graph obtained from G_P by splitting each of its inner vertices. After this transformation, let T_S^* be a tree constructed of edges corresponding to those of T_P^*. Clearly, T_S^* is a spanning tree of the graph G_S^*. The spanning tree $T_S = \mathrm{dual}(T_S^*)$ of G_S contains all auxiliary edges of G_S, because none of the duals of auxiliary edges are in $E(T_S^*)$, as they were not in $E(T_P^*)$. ◀

Now, we are ready to prove the main result of this section.

▶ **Theorem 12.** *Let G be an outer min-k-planar graph. Then* $\mathrm{tw}(G) \leq 3 \cdot \lfloor k/2 \rfloor + 4$.

Proof. By Observation 3 we may assume that G is an expanded outer min-k-planar graph. Let G_S be the split planarization of G. By Lemma 11 there exists a spanning tree T_S of G_S and a spanning tree $T_S^* = \mathrm{dual}(T_S)$ of G_S^* rooted at f_o^* such that $\mathrm{depth}(f^*) \leq \lfloor k/2 \rfloor + 1$ for every vertex $f^* \in V(G_S^*)$ and $E(T_S)$ contains all auxiliary edges of G_S.

We orient the edges of G as follows. Edges incident to the outer face are oriented clockwise, while all other edges are oriented arbitrarily. Observe that every vertex of G has at most two incoming edges.

Now, we construct the tree decomposition $\mathcal{T} = (T_S, B)$ of the graph G. The bags of $\mathcal{T}$ are indexed by the vertices of the spanning tree T_S. We place vertices of G into bags using the following rules.

1. For every outer vertex $v \in V(G_S)$, we place v into the bag B_v.
2. For every oriented edge (x, y) of G, we place x into the bag B_y.
3. For every oriented edge (x, y) of G, and for every inner vertex $v \in V(G_S)$ lying on (x, y), we place x into the bag B_v.
4. For every inner face f and every edge e^* on the path from f^* to f_o^* in T_S^*, where e lies on the edge (x, y) of G, we place x into the bag B_v for each vertex v incident to f.

Note that, in rule 4, the edge e is not in $E(T_S)$, so it is not an auxiliary edge, which implies that it is lying only on a single edge of G.

For every edge (x, y) of G, by rules 1 and 2, the bag B_y contains both x and y. Moreover, every vertex of G is present in some bag. So, to prove that $\mathcal{T}$ is a tree decomposition of G, it suffices to show that for every vertex $x \in V(G)$, the set $\{w : x \in B_w\}$ induces a connected subtree in T_S.

Fix a vertex x and an edge (x, y) of G. Let $e = uv$ be an edge lying on (x, y) such that $e \notin E(T_S)$. Assume that $e^* = f_1^* f_2^*$ and $\text{depth}_{T_S^*}(f_2^*) = \text{depth}_{T_S^*}(f_1^*) + 1$. Let T_e^* be a subtree of T_S^* induced by the set of all descendants of f_2^*, including f_2^*. Define $F_e = \{f : f^* \in V(T_e^*)\}$ and let boundary(F_e) be the set of vertices incident to some face in F_e. Observe that, by rule 4, x is placed into all bags indexed by boundary(F_e). The set boundary(F_e) induces a connected subgraph of T_S, i.e. $T_S[\text{boundary}(F_e)]$ is connected, containing both u and v. Note that the bags of $\mathcal{T}$, into which we placed x by rule 4, are exactly the bags of vertices of T_S that are in boundary(F_e) for some edge $e \notin E(T_S)$ lying on some edge (x, y) of G.

By rules 1, 2 and 3, x is contained in all bags B_w such that w lies on (x, y) for some edge (x, y) of G. We claim that the vertices indexing these bags, together with the vertices indexing bags we placed x into by rule 4, form a connected subgraph of T_S. To see that, we show that for every vertex w with $x \in B_w$, w is connected to x by a walk in T_S such that bags indexed by the vertices of this walk contain x.

If w lies on an edge (x, y) of G then, in order to construct this walk, we start at vertex w. We iterate over consecutive edges lying on (x, y) between w and x, starting at the edge incident to w. If the current edge e is in $E(T_S)$, then we extend the walk by e. Otherwise, $e \notin E(T_S)$. As $T_S[\text{boundary}(F_e)]$ is connected and every bag of a vertex in boundary(F_e) contains x, we can extend the walk by some path in $T_S[\text{boundary}(F_e)]$ connecting the endpoints of e. See Figure 5b for an example of such a walk.

If w is in the set boundary(F_e) for some edge e lying on (x, y), then we begin the walk with a path contained in boundary(F_e) between w and an endpoint v of e. We extend this walk by a walk between v and x, whose existence we have already proven.

Next, we bound the size of the bags in $\mathcal{T}$. Consider an inner vertex v of T_S. It lies on exactly two edges of G, so by rule 3 we placed two vertices into B_v. Also, v is incident to three non-outer faces of G_S. For every such face f and every edge e^* on the path from f^* to f_o^* in T_S^*, by rule 4 we placed one vertex into B_v. By Lemma 11 we have $\text{depth}(f^*) \leq \lfloor k/2 \rfloor + 1$, so each such path has at most $\lfloor k/2 \rfloor + 1$ edges. Thus, $|B_v| \leq 2 + 3 \cdot (\lfloor k/2 \rfloor + 1)$. Now, consider an outer vertex v of T_S. By rules 1 and 2, the bag B_v contains v and at most two other endpoints of edges incoming to v in G. Also, v is incident to two non-outer faces of G_S. Hence, we derive a bound $|B_v| \leq 3 + 2 \cdot (\lfloor k/2 \rfloor + 1)$. The width of the constructed tree decomposition is at most

$$\max\{2 + 3 \cdot (\lfloor k/2 \rfloor + 1), 3 + 2 \cdot (\lfloor k/2 \rfloor + 1)\} - 1 = 2 + 3 \cdot (\lfloor k/2 \rfloor + 1) - 1 = 3 \cdot \lfloor k/2 \rfloor + 4. \blacktriangleleft$$

5 The separation number of outer min-k-planar graphs

The inequality $\text{sn}(G) \leq \text{tw}(G) + 1$, that bounds the separation number, holds for every graph G. We remark that Theorem 12 directly implies that $\text{sn}(G) \leq 3 \cdot \lfloor k/2 \rfloor + 5$ for every outer min-k-planar graph G. By carefully choosing some bag B_x of a tree decomposition, we can construct a balanced separation (C, D) satisfying $C \cap D = B_x$. To establish a better upper bound, we first prove a general lemma showing how we can obtain a balanced separation (C, D) such that $C \cap D = B_x \cap B_y$ for some two neighboring vertices x, y of a tree decomposition, which needs to satisfy some additional properties.

▶ **Lemma 13.** *Let $\mathcal{T} = (T, B)$ be a tree decomposition of a graph G. Assume that $\Delta(T) \leq 3$ and that every vertex $v \in V(G)$ is in at least two bags of $\mathcal{T}$. Let a be an integer such that $|B_x \cap B_y| \leq a$ for every edge $xy \in E(T)$. Then G has a balanced separation of order at most a.*

Proof. For every edge $xy \in E(T)$, after removing it from T, we obtain two connected components C_x and C_y of T such that $x \in V(C_x)$ and $y \in V(C_y)$. We define $S_{x,y} = \bigcup_{v \in V(C_x)} B_v$ and $S_{y,x} = \bigcup_{v \in V(C_y)} B_v$. It is a well known fact that the pair $(S_{x,y}, S_{y,x})$ is a separation of G of order $|S_{x,y} \cap S_{y,x}| = |B_x \cap B_y| \leq a$.

We claim that there exists an edge $xy \in E(T)$ such that $(S_{x,y}, S_{y,x})$ is a balanced separation of G. Suppose the contrary. Then for every $xy \in E(T)$ it holds that $|S_{x,y} \setminus S_{y,x}| > \frac{2}{3}n$ or $|S_{y,x} \setminus S_{x,y}| > \frac{2}{3}n$, where $n = |V(G)|$. Now, we orient every edge of T. If the first inequality holds then we orient xy as (y, x), in the other case as (x, y). Also, note that $|S_{x,y} \setminus S_{y,x}| > \frac{2}{3}n$ implies $|S_{y,x}| < \frac{1}{3}n$.

The tree T with oriented edges is an acyclic graph, so there exists a sink, that is, a vertex in T such that all edges incident to x are oriented towards x. Let $\{y_1, \ldots, y_d\}$, where $d \leq 3$, be the set of neighbors of x in T. We have $|S_{y_i,x}| < \frac{1}{3}n$. Moreover,

$$\bigcup_{1 \leq i \leq d} S_{y_i,x} = \bigcup_{v \in V(T) \setminus \{x\}} B_v = V(G),$$

since every vertex of G is in at least two bags of $\mathcal{T}$. We obtain the following inequalities

$$|V(G)| = \left| \bigcup_{1 \leq i \leq d} S_{y_i,x} \right| \leq \sum_{1 \leq i \leq d} |S_{y_i,x}| < d \cdot \frac{1}{3}n \leq n,$$

which is a contradiction. ◀

Now, we are ready to establish an upper bound on the separation number of outer min-k-planar graphs.

▶ **Theorem 14.** *Let G be an outer min-k-planar graph. Then $\mathrm{sn}(G) \leq 2 \cdot \lfloor k/2 \rfloor + 4$.*

Proof. The class of outer min-k-planar graphs is closed under taking subgraphs. Therefore, it suffices to find a balanced separation of order at most $2 \cdot \lfloor k/2 \rfloor + 4$ for every maximal outer min-k-planar graph G. Let H be an expanded outer min-k-planar graph obtained from G by Observation 3. By Theorem 12, there exists a tree decomposition $\mathcal{T} = (T_S, B)$ of H, where T_S is a spanning tree of the split planarization of H. From the proof of Theorem 12, it follows that $\Delta(T_S) \leq 3$ and that every vertex $v \in V(H)$ is in at least two bags of $\mathcal{T}$ (since there is an oriented edge (v, w) in H, so $v \in B_v$ and $v \in B_w$).

We construct a tree decomposition $\mathcal{T}' = (T_S, B')$ of G with $B'_x = \{\mathrm{org}(v) : v \in B_x\}$, where $\mathrm{org}(v)$ denotes the original vertex that v replaced in the transformation described in Observation 3. Every vertex $v \in V(G)$ is in at least two bags of $\mathcal{T}'$, since every image of v is in at least two bags of $\mathcal{T}$. Every edge $vw \in E(G)$ is realized in some bag of $\mathcal{T}'$, because in H there is an edge corresponding to vw between an image of v and an image of w. To prove that, for every vertex v of G, the bags of $\mathcal{T}'$ containing v are spanning a connected subtree of T_S, we denote the images of v by $v_1, \ldots, v_s$, ordered along the outer face. Since H is maximal, for every $i \in \{1, \ldots, s-1\}$, there is an edge $v_i v_{i+1}$ in $E(H)$. Thus, the two subtrees of T_S induced by the bags of $\mathcal{T}$ containing v_i and those containing v_{i+1} share a common vertex. Bags containing v in $\mathcal{T}'$ are spanning a connected subtree of T_S, because this subtree is a union of subtrees spanned by the images of v. Hence, $\mathcal{T}'$ is a valid tree decomposition of G.

We say that a vertex v was placed into a bag B_x of $\mathcal{T}$ due to rule 4 of constructing the tree decomposition being applicable to the vertex v and a face f if:

- the face f is incident to x;
- there exists an edge e^* on the path between f_o^* and f^* in T_S^* such that e lies on an edge (v, w) of H, for some $w \in V(H)$.

Now, we want to show that, for every edge $xy \in E(T_S)$, we have $|B'_x \cap B'_y| \leq 2 \cdot \lfloor k/2 \rfloor + 4$. Let f_1 and f_2 be the faces of H_S incident to xy.

▷ Claim. If $v \in B'_x \cap B'_y$ then there exists an image v_t of v such that either

- xy lies on an edge (v_t, w) of H, for some $w \in V(H)$; or
- v_t was placed into both B_x and B_y due to rule 4 of constructing $\mathcal{T}$ being applicable to v_t and face f_1 or face f_2.

Proof. If x has degree 3 in H_S, then let $f_x \notin \{f_1, f_2\}$ be the face of H_S incident to vertex x. Similarly, if y has degree 3 in H_S, then let $f_y \notin \{f_1, f_2\}$ be the face of H_S incident to y. Assume that $v \in B'_x \cap B'_y$, but no image of v was placed into B_x and B_y due to the reasons stated in the claim. So there exist v_{i_x} and v_{i_y} that are, not necessarily distinct, images of v such that $v_{i_x} \in B_x$ and $v_{i_y} \in B_y$. For $t \in \{x, y\}$, vertex v_{i_t} was placed into B_t because either

1. t lies on an edge (v_{i_t}, w_t) of H such that xy does not lie on (v_{i_t}, w_t); or
2. when t has degree 3 in H_S, rule 4 of constructing $\mathcal{T}$ is applicable to the vertex v_{i_t} and face f_t, i.e. there exists an edge (v_{i_t}, w_t) of H and an edge e_t of H_S lying on (v_{i_t}, w_t) such that e_t^* is on the path between f_t^* and f_o^* in T_S^*.

Now, we draw a curve $\mathcal{C}$ on the drawing of H_S. The curve $\mathcal{C}$ consists of the drawing of the edge xy and the drawing of an arc of the outer face between v_{i_x} and v_{i_y} that contains only images of v (the images of v are spanning a single arc of the outer face). Next, we add to $\mathcal{C}$ curves connecting v_{i_x} with x and v_{i_y} with y. For $t \in \{x, y\}$, the way we draw these curves depends on the reason v_{i_t} was placed into B_t, considered in the same order as above.

1. If t lies on an edge (v_{i_t}, w_t) of H, we draw along (v_{i_t}, w_t), starting at vertex v_{i_t} and ending at vertex t.
2. If rule 4 of constructing $\mathcal{T}$ is applicable to the vertex v_{i_t} and face f_t, let p_t be a path between f_t^* and f_o^* in T_S^*. We draw along (v_{i_t}, w_t), starting at vertex v_{i_t} and ending at the crossing point with the drawing of p_t. We continue along p_t till vertex f_t^*. Finally, we connect vertices f_t^* and t with a segment.

Note that if $x = v_{i_x}$ and $y = v_{i_y}$, then $\mathcal{C}$ is degenerated to the arc between v_{i_x} and v_{i_y}, implying that xy connects two consecutive images of v – contradiction. Otherwise, we claim that one of the closed regions induced by $\mathcal{C}$ contains f_1 or f_2. Indeed, $\mathcal{C}$ follows edges of H_S and edges of T_S^*, but cannot contain f_1^* nor f_2^*, because then rule 4 of constructing $\mathcal{T}$ would be applicable to vertex v_{i_x} or v_{i_y} and face f_1 or f_2. The segments between f_t^* and t does not intersect f_1 nor f_2. We may assume that f_1 is contained inside a closed region induced by $\mathcal{C}$. Consider a path p_1 between f_1^* and f_o^* in T_S^*. Since f_1^* is inside $\mathcal{C}$ and f_o^* is outside $\mathcal{C}$, drawing of p_1 has to intersect $\mathcal{C}$. We consider where the first intersection point is located.

- Path p_1 cannot intersect e nor the segments between f_t^* and t.
- If p_1 intersects an edge (v_{i_t}, w_t) then rule 4 of constructing $\mathcal{T}$ is applicable to v_{i_t} and f_1.
- If p_1 intersects p_t, then p_1 follows along p_t up to the intersection point with (v_{i_t}, w_t), so the previous case applies.
- If p_1 intersects the arc of the outer face between v_{i_x} and v_{i_y} then it has to intersect an edge (v_r, v_{r+1}), where v_r and v_{r+1} are consecutive images of v on the outer face. Hence, rule 4 of constructing $\mathcal{T}$ is applicable to v_r and f_1.

In all cases, we obtain a contradiction. ◁

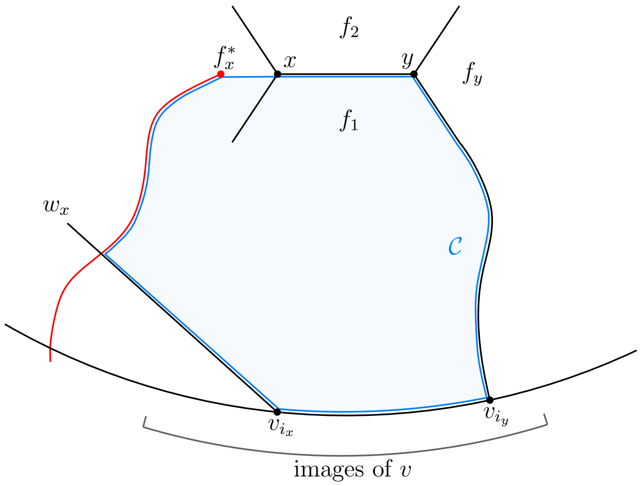


Figure 6 Drawing of an example curve $\mathcal{C}$.

We proved that if $v \in B'_x \cap B'_y$ then there is an image v_t of v such that either

- xy lies on an edge (v_t, w) of H, for some $w \in V(H)$; or
- v_t was placed into both B_x and B_y due to rule 4 of constructing the tree decomposition $\mathcal{T}$ being applicable to vertex v_t and face f_1 or f_2.

Note that xy lies on at most two edges of H (two if xy is an auxiliary edge, one otherwise). Moreover, the two paths from f_o^* to f_1^* and from f_o^* to f_2^* in T_S^* each have at most $\lfloor k/2 \rfloor + 1$ edges. Therefore, $\left|B'_x \cap B'_y\right| \leq 2 + 2 \cdot (\lfloor k/2 \rfloor + 1) = 2 \cdot \lfloor k/2 \rfloor + 4$. By applying Lemma 13 to $\mathcal{T}'$, we obtain that G has a balanced separation of order at most $2 \cdot \lfloor k/2 \rfloor + 4$. ◀

To give a lower bound we define a graph called *stacked prism*. A stacked prism $Y_{m,n}$ is an $m \times n$ grid with additional edges connecting the vertices of the first and the last row that are in the same column. The $Y_{m,n}$ has an outer $(2n-2)$-planar drawing, thus also an outer min-$(2n-2)$-planar drawing. In the cyclic order of the drawing, we place rows consecutively, one after another. The edges from rows cross no other edges and the edges from columns cross exactly $2n-2$ other edges. The authors of [7] showed that for every number n and for every sufficiently large even number m, $\mathrm{sn}(Y_{m,n}) = 2n$. This leads to the following theorem.

▶ **Theorem 15.** *For every even number k, there exists an outer min-k-planar graph G such that* $\mathrm{sn}(G) = k + 2$.

We remark that the multiplicative constant of 1 in the upper bound given in Theorem 14 is tight, as it matches that of the lower bound in Theorem 15.

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