

Stabbing Faces by a Convex Curve

David Eppstein 

Computer Science Department, University of California, Irvine, CA, USA

Abstract

We prove that, for every plane graph G and every smooth convex curve C not on a single line, there exists a straight-line drawing of G for which every face is crossed by C .

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1 Introduction

Every smooth convex curve C , not on a single line, contains arbitrarily many points in general position (its tangent points at distinct slopes within a suitable interval). It follows that every outerplanar graph has a straight-line drawing with all vertices on C [10]. A graph that can be drawn in this way must be outerplanar, as its vertices all belong to a common face outside C . But instead of asking for C to contain all vertices, here we ask: which planar graphs can be drawn so that C intersects all faces? The answer turns out to be all of them.

Our motivation for this question comes from the problem of characterizing universal point sets, sets of points U_n for each n having the property that every n -vertex planar graph has a straight-line drawing with its vertices in U_n . These sets have size $O(n^2)$ [1, 3, 6, 16], and $\geq cn$ for a constant $c > 1$ [5, 13, 15]. A recent line of research has investigated the combinatorial structure of universal point sets, in terms of the number of lines necessary to cover all points of such a set. Although point sets on $O(1)$ parallel lines can support only the planar graphs of bounded pathwidth [7], two crossed lines can support a much wider class of graphs, including all leveled planar graphs [2]. However, supporting all n -vertex planar graphs requires an unbounded number of lines, $\Omega(n^{1/3})$ [4, 9, 14]. Eppstein [8] asked: what if we replace lines in these results by convex curves? Is it possible that universal point sets can be supported by only $O(1)$ convex curves?

This question could be answered by finding a single planar graph $G^?$ that could not be drawn with all of its faces crossed by a single convex curve. If such a graph $G^?$ existed, we could assume without loss of generality that $G^?$ is a triangulation, and then recursively replace triangular faces of $G^?$ by more copies of $G^?$. This recursive replacement process would generate a family of graphs in which, for every drawing, every convex curve touches a number of faces bounded by a sublinear polynomial of the number of vertices. This would imply that universal point sets for graphs in this family could be supported only by a polynomially growing number of convex curves. However, the hope for a proof along these lines is dashed by our result: $G^?$ does not exist.

We remark that the corresponding question for edges rather than faces has an intermediate answer: the class of graphs that can be drawn with all edges touching a smooth convex curve C is broader than the outerplanar graphs but less broad than the planar graphs. In the full version of this paper, we provide an example of a planar graph that cannot be drawn in this way, regardless of the choice of C . It takes the form of the graph of nine regular octahedra,



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with the outer eight octahedra each glued to a separate face of a central octahedron. But unlike the case for faces crossed by C , we do not know how to leverage this example to prove anything about the number of convex curves needed to support a universal point set.

2 Preliminaries

Given a plane graph G ,¹ to be drawn with all faces crossed by a convex curve, we may assume without loss of generality that G is maximal planar, because if we complete any other graph to a maximal planar graph, and draw the result with C crossing all faces of the maximal planar completion, then C will also cross all faces of the original graph.

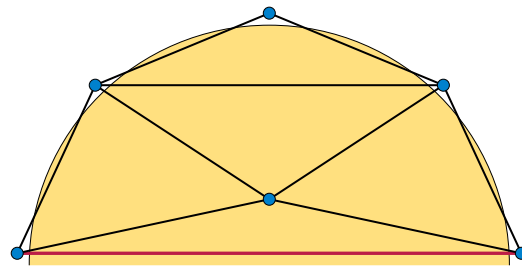
For a maximal plane graph G , with a fixed choice of outer face, we will draw G using a *canonical ordering* [12], as used by de Fraysseix, Pach, and Pollack to find grid drawings of planar graphs [6]. This is an ordering of the vertices of G such that the first three vertices in the ordering induce a triangle with one edge (the *base edge*) on the outer face, and such that each prefix of the ordering induces a triangulated disk² in the drawing of G , with the bounded faces of this triangulated disk also being faces of the full drawing of G . The next vertex in the ordering, after any prefix, must have as its earlier neighborhood a contiguous path along the boundary of the disk, not including the base edge, and the region between this path and the next vertex is triangulated in a fan of triangles meeting at the added vertex. This structure lends itself well to greedy incremental algorithms for planar graph drawing and to induction proofs of properties of these drawings.

For every positive integer k , there exist maximal planar graphs in which, for every drawing, the number of faces of the graph that can be crossed by a line is less than a $1/k$ fraction of all faces [11]. These graphs therefore cannot have all faces crossed by a k -gon. To avoid these examples, our results involve smooth convex curves. There are many ways of defining curves and their smoothness; the properties we require are that the curve must form a connected subset of the boundary of a convex set in the plane (for instance, its convex hull), it must have a unique tangent line at each point, and the slope of the tangent line must be continuous along the curve (C^1 smoothness, meaning that the first derivative is continuous). We do not require strict convexity: our curves may contain line segments (along which the tangent slope is constant). For instance, a *stadium*, a convex shape obtained by attaching semicircular end caps to two opposite sides of a rectangle, has a smooth convex boundary in this sense. However, a polygon is not smooth at its vertices. An *arc* of a convex curve is any connected proper subset. The *total curvature* of an arc can be measured by the change in slopes of the tangent lines along the arc, as an angle.

With a stronger smoothness assumption of being twice continuously differentiable (C^2 smoothness), the ability to cross all faces of a planar drawing would extend to non-convex smooth curves that do not lie on a line, because each such a curve has a smooth convex arc (within a neighborhood of any point of nonzero curvature) to which we can apply our proof. We omit the details.

¹ By a *plane graph* we mean a planar graph together with a combinatorial embedding, which should be respected in our drawing of the graph.

² By a *triangulated disk*, we mean a plane graph in which the outer face is a simple polygon and all interior faces are triangles.



■ **Figure 1** Adequate drawing of a triangulated disk (base edge shown in red) with respect to a semicircle: all boundary vertices are exterior to the semicircle and all non-base boundary edges are crossed, twice each, by disjoint arcs of the semicircle.

3 Adequate drawings

We will separate out into lemmas the steps of an induction proof that a given plane graph G has a drawing with all faces crossed by a given smooth convex curve C . If we reinterpret the proof as a greedy construction algorithm, each step of the construction has the following property.

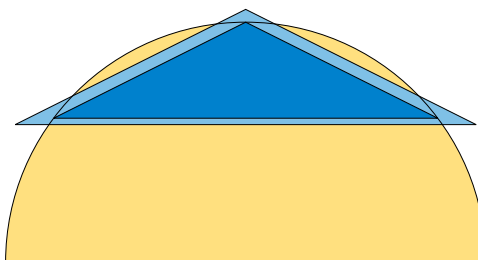
► **Definition 1.** *Given an abstract triangulated disk D with a specified base edge B , and a smooth convex curve C that is not on a single line, we define an adequate drawing of D to be a straight-line drawing with the following properties:*

- *Every boundary vertex of D is exterior to the convex hull of C .*
- *Every boundary edge e of D other than B is crossed twice by C . The two crossing points bound an arc of C that is disjoint from D and from the other arcs defined in the same way from other boundary edges of D .*
- *The crossings of C with D span an arc of C that bends through a total curvature of less than π .*

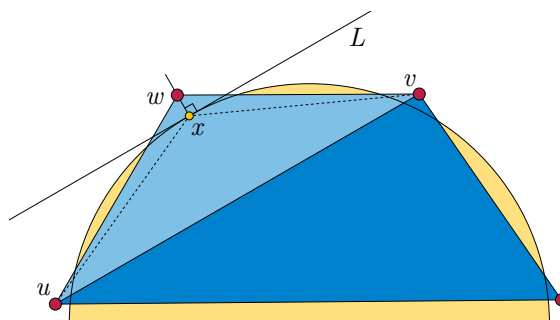
Figure 1 depicts an example. Not all faces in this example are crossed by the convex curve C (a semicircle), but the faces incident to the non-base boundary edges are crossed. As the name suggests, it will be adequate to prove that every triangulated disk formed by the canonical ordering has an adequate drawing, as every face will be adjacent to the boundary for the induced disk at some point in the ordering. This ensures the property we are really trying to prove, that the whole drawing of G has every face crossed by C :

► **Lemma 2.** *Let G be a maximal plane graph G with n vertices, with a straight-line drawing in the plane, and let C be a smooth curve that is not on a single line. Suppose that G has a canonical ordering with the property that, for each triangulated disk D_i (with $3 \leq i \leq n$) induced by the first i vertices in the canonical ordering, the drawing of G restricted to D_i is adequate. Then the drawing of G has all faces crossed by C .*

Proof. Each face Δ of G has at least one edge e_Δ that is a boundary edge of some D_i , but not the base edge of the canonical ordering. For the outer face, e_Δ can be any non-base edge of D_n ; for any other face, e_Δ is the edge of Δ induced by the first two vertices of Δ to appear in the canonical ordering. Because it is a non-base boundary edge of D_i , and because of the assumption that the drawing of D_i is adequate, e_Δ is crossed (twice) by C . Near these edge crossing points, Δ itself is crossed by C . ◀



■ **Figure 2** Base case (Lemma 3): constructing an adequate drawing of a triangle by scaling an inscribed triangle with three distinct tangent slopes.



■ **Figure 3** Ear case (Lemma 4): attaching triangle uvw to edge uv of an adequate drawing.

4 Induction proof

We will prove our main result by induction, using canonical orderings. The next lemma is the base case for this induction proof.

► **Lemma 3.** *For every smooth convex curve C that does not lie on a line, there exists an adequate drawing of a triangle with a designated base edge.*

Proof. Let A be any arc of C of nonzero curvature less than π , and inscribe a triangle in A so that the tangent lines to A at its three vertices have distinct slopes; this is possible because, by smoothness and nonzero curvature, A has tangents with a continuous range of slopes. Choose the base edge of this triangle to be the edge connecting the two extreme vertices in their ordering along A . Scale the triangle around its centroid by a factor $1 + \varepsilon$ for a parameter $\varepsilon > 0$ chosen sufficiently small,

By convexity, the inscribed triangle lies in the closed convex hull of C , and no point in the interior of one of its edges can lie on curve C itself: if such a point existed then the convexity of C would force C to contain that edge, violating the assumption of distinct tangent slopes at the three vertices. Scaling brings the triangle vertices exterior to C , but for sufficiently small choices of ε , some points interior to each non-base edge remain interior to the convex hull of C (Figure 2). Then each non-base edge of the scaled triangle will have two crossings with C separating its endpoints (exterior to the hull) with the points along the edge interior to the hull. These crossings will separate disjoint arcs of C , and together span an arc of C that is a sub-arc of A , so all the requirements of an adequate drawing are met. ◀

The simpler of the two inductive cases involves adding a single triangle as an ear of the triangulated disk, attached to it by a single boundary edge of the previous disk.

► **Lemma 4.** *Let D be a triangulated disk with an adequate drawing with respect to a smooth convex curve C , and let uv be any boundary edge of the drawing of D that is not the base edge of the drawing. Then it is possible to add a vertex w adjacent only to u and v , forming a triangle attached to D across edge uv , and to place w in the drawing so that it becomes an adequate drawing of the resulting augmented triangulated disk.*

Proof. For the notation used here, see Figure 3. By the assumption that the drawing of D is adequate, an arc of curve C crosses twice through edge uv . By smoothness, this arc has a tangent line L parallel to uv , touching the arc at a point x between the two crossings of C with uv . Polyline uxv starts outside C at u , crosses inside C to touch C at x from the interior of its convex hull, and then crosses C again to reach the point v exterior to C . Place w on a perpendicular line to L through x , outward from C at a sufficiently small distance ε from x .

If ε is sufficiently small, the polyline uwv crosses C from exterior to interior at a point near the crossing of C with line segment ux , and crosses again from interior to exterior at a point near the crossing of C with line segment xv . In order for polyline uwv to be exterior to C at w , between these two crossings, each of the two new edges uw and wv must have a second crossing near w . Thus, the requirements of an adequate drawing at the new boundary edges uw and wv are met, and the crossings of C with the other boundary edges of D remain unaffected, so the result is an adequate drawing. ◀

In the remaining inductive case, we add a fan of triangles, connecting an added vertex to a path of two or more edges of the triangulated disk.

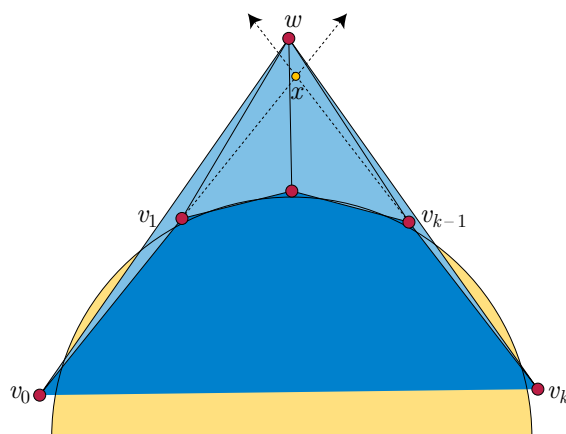
► **Lemma 5.** *Let D be a triangulated disk with an adequate drawing with respect to a smooth convex curve C , and let $P = v_0v_1 \dots v_k$ be a path of k boundary edges of the drawing of D (for $k \geq 2$) that does not pass through the base edge of the drawing. Then it is possible to add a vertex w adjacent to each vertex v_i of the path, forming a fan of triangles attached to P across edge uv , and to place w in the drawing so that it becomes an adequate drawing of the resulting augmented triangulated disk.*

Proof. For the notation used here, see Figure 4. Draw a ray from vertex v_0 through vertex v_1 , and a second ray from vertex v_k through vertex v_{k-1} . Each ray crosses C at the two crossing points of an edge of P . By the assumption that the drawing is adequate and therefore its crossings with C span an arc of C that bends through an angle less than π , the two rays cross rather than diverging. Let x be the crossing point of the two rays. At x , the two crossing rays subdivide the plane into four wedges. Place w interior to the wedge opposite from P , at a sufficiently small distance from w .

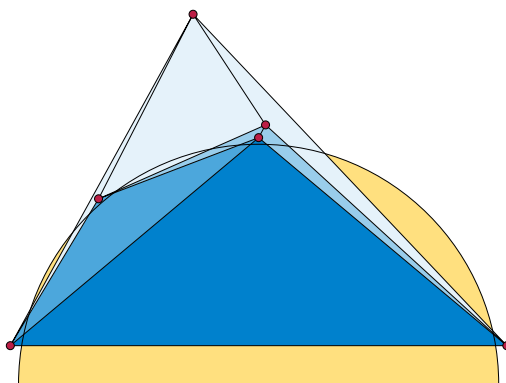
Vertices $v_1, \dots, v_{k-1}$ all lie in the wedge containing P , and are visible to x with respect to the given drawing of D ; however, the two endpoints v_0 and v_k of the path are blocked from visibility by v_1 and v_{k-1} respectively. Placing w anywhere in the wedge opposite from P preserves the visibilities with $v_1, \dots, v_{k-1}$ and allows w to also see the two path endpoints, producing a valid drawing of D . The segments v_0x and xv_k cross C twice, along edges v_0v_1 and $v_{k-1}v_k$ (which lie on these segments), and if w is sufficiently close to x then the segments v_0w and wv_k will also cross C twice, meeting the requirements of an adequate drawing at the two new boundary edges of the augmented drawing. The crossings of C with the other boundary edges of D remain unaffected, so the result is an adequate drawing. ◀

With the lemmas above, we are ready to prove our main result:

► **Theorem 6.** *Let G be a plane graph, and let C be a smooth convex curve that does not lie on a line. Then G has a straight-line drawing in which all faces are crossed by C .*



■ **Figure 4** Fan case (Lemma 5): attaching a fan of triangles to path $P = v_0 v_1 \dots v_k$ of an adequate drawing.



■ **Figure 5** Drawing of the graph of an octahedron with all faces crossed by a semicircle. The shading indicates the order in which triangles were added to the drawing in its canonical ordering, with darker triangles added earlier than lighter ones.

Proof. Augment G to a maximal plane graph, and construct a canonical ordering. We prove by induction that there exists a drawing D of the augmented graph such that, for each triangulated disk D_i (with $3 \leq i \leq n$) induced by the first i vertices in the canonical ordering, the drawing restricted to D_i is adequate. The base case $i = 3$ is Lemma 3, and the induction step for each $i > 4$ is either Lemma 4 (if vertex i has two neighbors in D_i) or Lemma 5 (if vertex i has three or more neighbors in D_i). This proves that D meets the hypotheses of Lemma 2, so every face of the augmented graph is crossed by C . Each face of G itself is a union of faces of the augmented graph, so it is also crossed by C . ◀

Our induction proof can be turned into an algorithm in which we add vertices one at a time to a drawing, using the construction of Lemma 3 to add the first three vertices, and either Lemma 4 or Lemma 5 to add each subsequent ear. An example of the resulting drawing, for the graph of an octahedron with its faces crossed by a semicircle, is shown in Figure 5. However, to analyze the numerical precision of vertex coordinates needed for this construction, and the angular resolution of the result, we would need additional assumptions about C . As the figure shows, even for a well-behaved curve such as a semicircle, the angular resolution can be quite low: this construction is more useful in producing counterexamples than as a method for producing readable graph drawings.

5 Conclusions and open problems

We have shown that for every planar graph and every smooth convex curve, there exists a drawing of the graph in which all faces are crossed by the curve. The most salient remaining open problem is the one that motivated this research: is there a constant k such that all planar graphs can be drawn with their vertices on $\leq k$ convex curves? It would also be of interest to more precisely characterize the graphs that can be drawn with all edges crossed by a convex curve (for which see the full version of this paper) and to extend these results to beyond-planar graphs.

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