From Local Pair-Crossing Number to Local Crossing Number

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Abstract

We prove that if a graph can be drawn in the plane such that each edge crosses at most k other edges, then it can be redrawn so that each edge participates in at most $k^3 + O(k^2)$ crossings. This improves the previous exponential bound that follows from a result of Schaefer and Štefankovič and answers a question of Ackerman and Schaefer.

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1 Introduction

In much of the early literature about Turán's brick factory problem and other questions related to graph drawing, the precise problem setting was ambiguous. The basic notions were thought to be so self-evident, that it was often felt that there was no need for definitions. This anomaly was pointed out by Pach and G. Tóth [11, 12], who made a distinction between several different crossing numbers. The two most important among them were the *crossing number*, $\mathrm{cr}(G)$, and the *pairwise crossing number*, pair-$\mathrm{cr}(G)$, of a graph G. The first one is the minimum possible number of *crossing points* between the edges of G, in a drawing of G, where

1. two edges, including edges that have a common endpoint, are allowed to cross any finite number of times,
2. no three edges meet at the same point,
3. the edges are drawn as simple non-self-intersecting Jordan curves.

On the other hand, pair-$\mathrm{cr}(G)$, is the minimum number of *crossing pairs of edges*, that is, if a pair of edges cross multiple times, it still contributes only 1 to this quantity. Obviously, we have pair-$\mathrm{cr}(G) \leq \mathrm{cr}(G)$ for every graph G. It was often assumed with no explanation whatsoever that these two parameters coincide. It is one of the most annoying open problems in the area to decide if this is indeed the case.

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▶ **Problem 1** ([11, 12]). *Is it true that* pair-cr$(G) =$ cr(G) *for every graph* G*?*

It was proved in [12] that $\mathrm{cr}(G) \leq \binom{\text{pair-cr}(G)}{2}$ holds for every graph G. This result was subsequently improved several times [6, 7, 18, 20, 21, 22]. The best presently known bound, $\mathrm{cr}(G) = O((\text{pair-cr}(G))^{3/2})$ is due to O. Solé Pi [13].

The separation of the corresponding two parameters can also be observed at a local level. It was probably Ringel [14, 15], who first studied loc-cr(G), the *local crossing number*, of a graph G. It is defined as the minimum number k such that G can be drawn in such a way that the number of crossings along every edge is at most k. As above, we can also define loc-pair-cr(G), the *local pair-crossing number* of G, by counting the maximum number of different *edges* that cross a given edge, without multiplicities. In [2], Ackerman and Schaefer showed that if loc-pair-cr$(G) \leq 2$, then loc-pair-cr$(G) =$ loc-cr(G). Slightly modifying the example depicted on Figure 1 of [9] (see also Exercise (7.1) in [16]), we can construct a graph G with $4 =$ loc-pair-cr$(G) <$ loc-cr$(G) = 5$, showing that these two parameters are not always equal.

Graphs with local crossing number at most k are often called k*-planar*. For getting the best constant in the crossing lemma of Ajtai, Chvátal, Newborn, Szemerédi [3] and Leighton [8], it was useful to estimate the maximum number of edges that a k-planar graph with n vertices can have, for $k \leq 4$; see [1, 10, 9]. The crossing lemma, in turn, has many applications in discrete geometry, in additive combinatorics and elsewhere [19, 4].

A drawing of a graph is called *simple* (or a *simple topological graph*) if any two edges cross at most once. Hoffmann, Liu, Reddy, and C. D. Tóth [5] proved that there exists a function $f(k) \leq (3 + o(1))^k$ such that, if loc-cr$(G) \leq k$, then G also has a simple drawing in which every edge has at most $f(k)$ crossings.

In their seminal paper [17], Schaefer and Štefankovič proved the following (see also Lemma 9.2 in [16]). Suppose that the local pair-crossing number of G satisfies loc-pair-cr$(G) \leq k$. Then one can redraw G in such a way that the total number of crossings along any edge is at most 2^k. This implies that we have loc-cr$(G) \leq 2^k$; see Remark 1 in the last section. In [2], Ackerman and Schaefer asked whether or not this bound can be improved. The aim of this note is to replace this by a polynomial upper bound.

Let $G = (V, E)$ be a graph drawn in the plane. For each vertex $v \in V$, consider the cyclic order of the edges around v. The system of these cyclic permutations is called the *rotation system* of the drawing.

▶ **Theorem 2.** *Let* G *be a graph that can be drawn in the plane such that every edge crosses at most* k *other edges, that is,* loc-pair-cr$(G) \leq k$*. Then we have*

$$\text{loc-cr}(G) \leq k(k+1)(k+2).$$

That is, G *can be redrawn in such a way that along every edge, the number of crossings is at most* $k(k+1)(k+2)$*. We can also ensure that the rotation systems of the initial and final drawings are the same.*

Combining this result with the theorem of Hoffmann et al. [5] mentioned above, we obtain the following.

▶ **Corollary 3.** *Suppose that the local pair-crossing number of a graph* G *is at most* k*. Then* G *has a simple drawing such that along every edge there are at most* $3^{k^3+O(k^2)}$ *crossings.*

2 Proof of Theorem 2

We use standard terminology; see, e.g., [16]. Let $G = (V, E)$ be a graph, which may have loops and multiple edges. A *drawing* of G, denoted by $\mathcal{D}(G)$, is a representation of G in which the vertices are distinct points in the plane, and the edges are Jordan arcs connecting the corresponding pairs of points. Whenever there is no danger of confusion, in notation and terminology, we make no distinction between the vertices and the points representing them, and the edges and the corresponding Jordan arcs. No edge is allowed to pass through a vertex, no three edges pass through the same point, and any two edges have only finitely many points in common. We may assume that no two edges "touch," that is, if they have an interior point in common, then they must properly cross at this point. Otherwise we can locally pull apart the two edges near their touching point to get rid of the touching. We may also assume that there is no edge that crosses itself, otherwise we can shorten it by deleting the resulting loops.

Proof of Theorem 2. Let G be a graph on n vertices and m edges, $e_1, \ldots, e_m \in E(G)$, and let $\mathcal{D}(G)$ be a drawing of G in the plane such that each edge in G crosses at most k other edges. Let $\mathcal{R}$ be the rotation system of the drawing $\mathcal{D}(G)$. If G is not connected, we can pull apart the drawing $\mathcal{D}(G)$ so that the connected components do not interact with one another. So, we may assume without loss of generality that G is a connected graph, and let T be a spanning tree of G. Furthermore, let us assume that $T = \{e_1, \ldots, e_{n-1}\}$, and for each $i \in [n-1]$, the edges $e_1, \ldots, e_i$ form an edge-induced connected subgraph of G. We will need the following lemma.

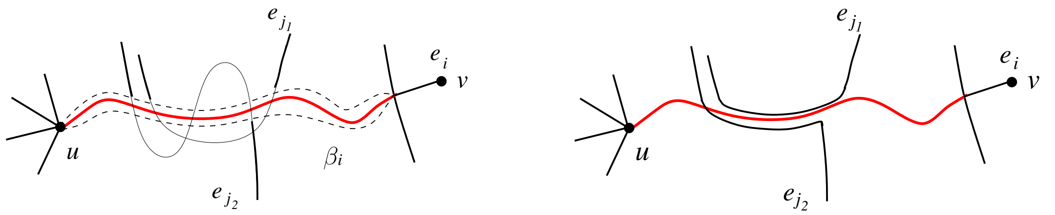
▶ **Lemma 4.** *Let $T \subset G$ and $\mathcal{D}(G)$ be as described above. Then for each $e_i \in T$, there is a subcurve $\alpha_i \subset e_i$ in the plane such that*

1. *no two subcurves α_i, α_j cross each other properly (but one may have an endpoint on the other),*
2. *every vertex in G is an endpoint of some curve α_i, and*
3. *the union $\mathcal{T} = \bigcup_{i=1}^{n-1} \alpha_i$ is a simply connected set in the plane.*

Proof. For each $i \in [n-1]$, we will greedily construct the subcurves $\alpha_1, \ldots, \alpha_i$ with the properties described above, such that for $\mathcal{T}_i = \bigcup_{j=1}^{i} \alpha_j$, $\mathcal{T}_i$ is simply connected and contains all of the endpoints of the edges $e_1, \ldots, e_i$.

We start with $\alpha_1 = e_1$. Having defined $\alpha_1, \ldots, \alpha_{i-1}$ with the desired properties, we obtain α_i as follows. Consider edge $e_i \in T$. Since the edges $e_1, \ldots, e_{i-1}$ form an edge-induced connected subgraph of G, and the drawing $\mathcal{T}_{i-1} = \bigcup_{j=1}^{i-1} \alpha_j$ contains the endpoints of $e_1, \ldots, e_{i-1}$, exactly one of the endpoints of e_i lies in $\mathcal{T}_{i-1}$. Let u be the endpoint of e_i that does lie in $\mathcal{T}_{i-1}$ and v be the other endpoint that does not lie in $\mathcal{T}_{i-1}$. Then we define α_i by starting at vertex v and tracing along the curve e_i until we reach a point on $\mathcal{T}_{i-1}$, which is either a crossing point or a vertex. Clearly, $\mathcal{T}_i = \mathcal{T}_{i-1} \cup \alpha_i$ is simply connected and contains the endpoints of the edges $e_1, \ldots, e_i$. ◀

Let $\alpha_i \subset e_i$, $1 \leq i \leq n-1$, be the subcurves from Lemma 4, and let $\mathcal{T} = \bigcup_{i=1}^{n-1} \alpha_i$. We then define γ to be a simple closed curve drawn very close to $\mathcal{T}$, such that $\mathcal{T}$ lies inside of γ, that is, $\mathcal{T}$ lies in the closed connected component of $\mathbb{R}^2 \setminus \gamma$. Moreover, γ will not contain any crossing points in our drawing $\mathcal{D}(G)$, and apart from the crossing points on $\mathcal{T}$, all other crossing points will lie outside of γ. For each subcurve $\alpha_i \in \mathcal{T}$, let β_i be a simple closed curve drawn very close to α_i, such that β_i lies inside of γ, the endpoints of α_i lie on β_i, and all crossing points along α_i lie inside of β_i. Moreover, for $j \neq i$, the crossing points along the



(a) Before redrawing. **(b)** After redrawing.

Figure 1 For edge $e_i = uv$, α_i drawn in red, and β_i drawn dashed, we redraw edges e_{j_1} and e_{j_2} inside of β_i.

curve α_j lie outside of β_i. Hence, either β_i and β_j are disjoint, or they are tangent if α_i and α_j have a common endpoint, or β_i and β_j have two points in common if an endpoint of α_i lies in the interior of α_j (or vice versa).

By construction, for each subcurve α_i, there are $t \leq k$ edges $e_{j_1}, \ldots, e_{j_t} \in E(G)$ that cross α_i. Inside of β_i, we can redraw each edge $e_{j_1}, \ldots, e_{j_t}$ so that it intersects the curve α_i at most once as follows. By starting at one of the endpoints of e_{j_r}, we trace along e_{j_r} until it intersects β_i for the first time, which is right before it would intersect α_i. We denote this subcurve as f_{j_r} (the first part of e_{j_r}). Likewise, by starting at the other endpoint of e_{j_r}, we again trace along e_{j_r} until it intersects β_i for the first time, and denote this subcurve as ℓ_{j_r} (the last part of e_{j_r}). We then delete the part of e_{j_r} between f_{j_r} and ℓ_{j_r} (the middle part of e_{j_r}), and then connect f_{j_r} and ℓ_{j_r} by drawing a curve inside of β_i from the endpoint of f_{j_r} on β_i to the endpoint of ℓ_{j_r} on β_i. Clearly, this can be done so that this new drawing of e_{j_r} crosses α_i at most once. We can perform this redrawing procedure to each $e_{j_1}, \ldots, e_{j_t}$ such that afterwards, each pair of edges $e_{j_r}, e_{j_w} \in \{e_{j_1}, \ldots, e_{j_t}\}$ will have at most one crossing point between them inside of β_i, and each such edge e_{j_r} crosses α_i at most once. See Figure 1. After applying this redrawing procedure for each α_i, let $\mathcal{D}'(G)$ denote the new drawing of G in the plane. Hence, no additional crossings were created outside of γ in $\mathcal{D}'(G)$, and the rotation system $\mathcal{R}$ in the new drawing has not changed.

Given $\mathcal{D}'(G)$, we now redraw the edges of G outside of γ as follows. First notice that for each edge $e_i \in E(G)$, e_i will cross γ at most $2k+2$ times in $\mathcal{D}'(G)$. Indeed, starting at one of the endpoints of e_i, we move along e_i and first cross γ without crossing $\mathcal{T}$. Continuing along e_i, since e_i will cross at most k subcurves α_i, we have another possible $2k$ crossing points on γ. Finally, as we approach the other endpoint of e_i, we cross γ one last time for a total of at most $2k+2$ crossing points along γ. Hence, outside of γ, each edge e_i will give rise to at most $k+1$ pairwise disjoint *outer arcs*, with end points on γ. See Figure 2. Hence, all of the edges in G will give rise to at most $(k+1)|E(G)|$ outer arcs outside of γ. Given the circular structure of the endpoints of these outer arcs by γ, we can redraw all of the outer arcs outside of γ such that any two outer arcs cross at most once, and the endpoints of each outer arc remain the same. Moreover, if edges e_i and e_j did not cross outside of γ, then they still do not cross outside of γ in the new drawing. After we redraw the outer arcs as described above, let $\mathcal{D}''(G)$ be the new and final drawing of G in the plane, and notice that the rotation system of $\mathcal{D}''(G)$ is the same as the rotation system of $\mathcal{D}(G)$. Also notice that since each edge consists of at most $k+1$ outer arcs, two edges e_i and e_j will create at most $(k+1)^2$ crossing points outside of γ in $\mathcal{D}''(G)$.

Finally, we show that each edge e_i in the final drawing $\mathcal{D}''(G)$ has at most $k(k+1)(k+2)$ crossing points. By the above, the number of crossing points on e_i outside of γ is at most $k(k+1)^2$, since e_i crosses at most k other edges, and any two will create at most $(k+1)^2$ crossing points outside of γ. In order to bound the number of crossing points on e_i inside of γ, let us consider the following two cases.

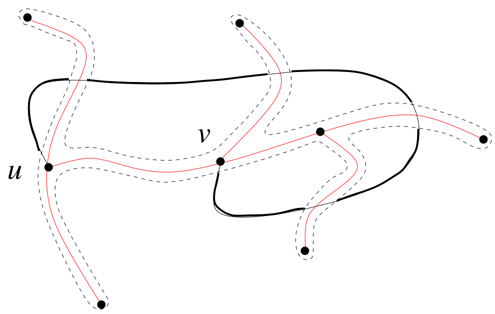


Figure 2 Example of an edge e with endpoints u and v, consisting of 5 outer arcs in bold.

Case 1. Suppose that $e_i \notin T$. Recall that there are at most k subcurves α_j that cross e_i, and for each subcurve α_j, there are at most $k-1$ other edges that cross α_j. Hence, e_i will have at most k crossing points inside of β_j, giving us a total of at most k^2 crossing points on e_i inside of γ.

Case 2. Suppose $e_i \in T$. Again, there are at most k subcurves α_j that cross e_i, and for each subcurve α_j with $j \neq i$, there are at most $k-1$ other edges that cross α_j. Moreover, e_i could have an additional k crossing points along α_i inside of β_i. Hence, e_i will have at most k^2+k crossing points inside of γ.

Putting everything together, each edge in $\mathcal{D}''(G)$ has at most $k(k+1)^2 + k^2 + k = k(k+1)(k+2)$ crossings. ◀

3 Concluding remarks

1. Suppose G has local pair-crossing number at most k. If there is an edge with more than 2^k crossings on it, then by applying Lemma 9.2 in [17], we can simplify the drawing without creating any new pair of crossing edges (that is, the local pair-crossing number remains at most k), and without increasing the number of crossings along any edge. Repeating this step for each edge of G, we end up with a drawing of G having local crossing number at most 2^k.

2. In the forthcoming journal version of this note, we show that it is possible to improve the bound in Theorem 2 to $\text{loc-cr}(G) \leq \text{loc-pair-cr}(G)^{2.5+o(1)}$ by a more involved argument.

3. Using the redrawing technique used in the proof of Theorem 2, one can show that for any multigraph G, $\text{cr}(G) = O(\text{pair-cr}(G)^{5/3})$. Although this is weaker than the result stated in the introduction due to Solé Pi, we plan to include the proof in the forthcoming journal version for the interested reader.

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