

# Edge Densities of Drawings of Graphs with One Forbidden Cell

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## Abstract

A connected topological drawing of a graph divides the plane into a number of cells. The type of a cell  $c$  is the cyclic sequence of crossings and vertices along the boundary walk of  $c$ . For example, all triangular cells with three incident crossings and no incident vertex share the same cell type. When a non-homotopic drawing of an  $n$ -vertex multigraph  $G$  does not contain any such cells, Ackerman and Tardos [JCTA 2007] proved that  $G$  has at most  $8n - 20$  edges, while Kaufmann, Klemz, Knorr, Reddy, Schröder, and Ueckerdt [GD 2024] showed that this bound is tight.

In this paper, we initiate the in-depth study of non-homotopic drawings that do not contain one fixed cell type  $c$ , and investigate the edge density of the corresponding multigraphs, i.e., the maximum possible number of edges. We consider non-homotopic as well as simple drawings, multigraphs as well as simple graphs, and every possible type of cell. For every combination of drawing style, graph type, and cell type, we give upper and lower bounds on the corresponding edge density. With the exception of the cell type with four incident crossings and no incident vertex, we show for every cell type  $c$  that the edge density of  $n$ -vertex (multi)graphs with  $c$ -free drawings is either quadratic in  $n$  or linear in  $n$ . In most cases, our bounds are tight up to an additive constant.

Additionally, we improve the current lower bound on the edge density of simple graphs that admit a non-homotopic quasiplanar drawing from  $7n - 28$  to  $7.5n - 28$ .

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## 1 Introduction

Many fundamental classes of graphs are defined by admitting a drawing without a forbidden configuration or pattern. This includes planar graphs (forbidding two crossing edges), as well as many beyond-planar graph classes such as  $k$ -planar graphs (forbidding an edge with  $k + 1$  crossings), quasiplanar graphs (forbidding three pairwise crossing edges), fan planar graphs (forbidding an edge crossed by two independent edges<sup>1</sup>) and many others. Planar and

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<sup>1</sup> The actual definition is more complex but irrelevant here.



beyond-planar graphs are of eminent importance, both from a practical and a theoretical point of view. Nevertheless, many fundamental properties are still not sufficiently well understood. One of the most fundamental questions in this area concerns the *edge density* of a beyond-planar graph class  $\mathcal{G}$ , that is, the maximum possible number of edges of any  $n$ -vertex graph in  $\mathcal{G}$ . There is a long history of edge density results ranging from  $k$ -planar graphs [2, 20, 21],  $k$ -quasiplanar graphs [1, 3],  $k$ -bend RAC graphs [4, 8],  $(k, l)$ -grid free graphs [19], and more to the recently introduced Density Formula [16].

We study the *cells* of a drawing, that is, the regions of  $\mathbb{R}^2$  delimited by the vertices and edges of the drawing. The *type* of a cell  $c$  is the cyclic order of edge segments, vertices, and crossings along the boundary of  $c$ . For example, a  $\nabla$ -cell is incident to three edge segments, three crossings, and no vertex. Distinguishing cells by their type is often useful when reasoning about drawings. The recently developed Density Formula [16], whose many applications usually come down to counting and relating different cell types in drawings, has further increased the need for a good grasp on the relation between types of cells in a drawing and the structure of the underlying graph. Interestingly, Ackerman and Tardos [3] proved already in 2007 that  $n$ -vertex drawings with no  $\nabla$ -cell contain at most  $8n - 20$  edges.

**Our contribution.** In this paper, we consider for each cell type  $\mathfrak{c}$  the corresponding beyond-planar graph class  $\mathcal{G}_{\mathfrak{c}}$  consisting of all graphs that admit a drawing without  $\mathfrak{c}$ -cells. We present upper and lower bounds on the edge densities of graphs in  $\mathcal{G}_{\mathfrak{c}}$ . We determine the asymptotic edge density in the setting of arbitrary, non-homotopic, and simple drawings, except for one particular cell type. Depending on  $\mathfrak{c}$ , we obtain some linear and some quadratic edge densities. In many cases, our bounds are tight up to an additive constant. All our results and some previous related work are shown in Table 1. Our lower bounds entail various novel constructions, while our upper bounds are mostly proved via discharging. Additionally, we present a construction of simple graphs on  $7.5n - 28$  edges that admit a non-homotopic quasiplanar drawing, beating a previous lower bound by  $\frac{n}{2}$  in the highest-order term.

**Related work.** Apart from the above-mentioned work of Ackerman and Tardos [3] and Kaufmann, Klemz, Knorr, Reddy, Schröder, and Ueckerdt [16] on  $\nabla$ -free drawings, we are (as far as we know) the first to consider graph drawings with a forbidden type of cell. However, some beyond-planar drawing styles are characterized by forbidding a configuration that is somewhat similar to a particular cell type. For example, in a fan-crossing-free drawing [7] there is no edge crossed by two adjacent edges, which resembles the  $\triangleleft$ -cell. Secondly, in a quasiplanar drawing, there are no three pairwise crossing edges, which resembles the  $\nabla$ -cell. And in a  $(2, 2)$ -grid-free drawing, there are no two pairs  $e_1, e_2, f_1, f_2$  of edges such that every  $e_i$  crosses every  $f_j$ , which resembles the  $\boxplus$ -cell. For a collection of edge density results on these and other beyond-planar graph classes, we refer to [9, Chapter 4]. Let us remark that in the three cases above, every fan-crossing-free (quasiplanar,  $(2, 2)$ -grid-free) drawing is in particular a  $\triangleleft$ -free ( $\nabla$ -free,  $\boxplus$ -free) drawing. Hence, lower bound constructions carry over to our setting, while in general,  $\mathfrak{c}$ -free drawings might allow for a much higher edge density.

Let us also mention that cell types have been investigated in the context of arrangements of pseudolines. For simple arrangements of pseudolines, it is known that only  $\nabla$ -cells are forced to appear in every sufficiently large arrangement. In fact, any simple arrangement of  $n$  pseudolines in the Euclidean plane or in the projective plane contains at least  $n - 2$  or  $n$ , respectively,  $\nabla$ -cells [11, 18]. On the other hand, there are various constructions without  $\boxplus$ -cells [13, 14], and arbitrarily large simple arrangements consisting only of  $\nabla$ -cells and  $\boxplus$ -cells [17]. Further results on the number of cells of different types in pseudoline

■ **Table 1** Results on edge densities of graphs admitting a drawing without one fixed type of cell. Upper bounds hold for all  $n \geq 3$ , while lower bounds are proven for infinitely many values of  $n$ .

| Forbidden cell                                                                            | Graph type | Drawing type   | Lower Bound                      | Upper Bound            |
|-------------------------------------------------------------------------------------------|------------|----------------|----------------------------------|------------------------|
| Any                                                                                       | Simple     | No restriction | $\binom{n}{2}$<br>Construction 3 | $\binom{n}{2}$         |
|  -cell   | Multi      | Non-homotopic  | $8n - 20$<br>[16]                | $8n - 20$<br>[3]       |
|                                                                                           | Simple     | Non-homotopic  | $8n - 28$<br>Construction 8      | $8n - 20$<br>[3]       |
|                                                                                           | -          | Simple         | $7n - 30$<br>Construction 6      | $7n - 20$<br>Theorem 5 |
|  -cell   | Multi      | Non-homotopic  | $9n - 18$<br>Construction 5      | -                      |
|                                                                                           | Simple     | Non-homotopic  | $6n - 12$<br>Construction 4      | $\binom{n}{2}$         |
|                                                                                           | -          | Simple         | $6n - 12$<br>Construction 4      | $\binom{n}{2}$         |
|  -cell   | -          | Simple         | $\Omega(n^2)$<br>Construction 9  | $\binom{n}{2}$         |
|  -cell | Multi      | Non-homotopic  | $6n - 12$<br>Construction 4      | $6n - 12$<br>Theorem 4 |
|                                                                                           | -          | Simple         | $6n - 12$<br>Construction 4      | $6n - 12$<br>Theorem 4 |
| Others                                                                                    | -          | Simple         | $\binom{n}{2}$<br>Construction 1 | $\binom{n}{2}$         |

arrangements can be found in the textbook chapter [10], including upper bounds on the number of -cells and -cells in simple pseudoline arrangements from [6, 12], and a lower bound on the number of -cells and -cells from [22].

**Outline.** We start in Section 2 with relevant definitions and some statements that we will use throughout the paper. Then, in Section 3 we show that most cell types can be avoided in simple drawings of  $K_n$  and that all cell types can be avoided in arbitrary drawings of  $K_n$ . Constituting the main body of this work, we address the remaining cell types in Sections 4–7 with quasiplanarity discussed jointly with -cells in Section 6.

## 2 Preliminaries: Drawings, cells, and cell types

Throughout this paper, we consider finite graphs with no loops but possibly parallel edges. To make the (potential) presence of multiedges more explicit, we sometimes use the term *multigraph*. Similarly, when a graph has no parallel edges, it is called a *simple* graph. A drawing  $\Gamma$  of a multigraph  $G = (V, E)$  maps vertices to pairwise distinct points and edges to curves connecting the corresponding vertices. For convenience, we consider drawings on the sphere  $\mathbb{S}^2$  to avoid a special treatment of the unbounded cell. As customary, we require that no edge passes through a vertex, any two edges have only finitely many points in common, each being a common endpoint or a proper crossing, and that no three edges cross in the same point. A drawing  $\Gamma$  is *connected* if the image of the map  $\Gamma$  is connected in  $\mathbb{S}^2$ .

A drawing is *simple* if any two edges have at most one point in common, which is either a common endpoint or a crossing. As this rules out parallel edges, only simple graphs admit simple drawings. A *lens* is a region of  $\mathbb{S}^2$  bounded by exactly two parts of edges. This could be two parallel edges, two edges crossing twice, or two crossing adjacent edges. We call a lens *empty* if its interior contains neither a crossing nor a vertex. Then a drawing is *non-homotopic* if it does not contain any empty lens.

Consider any fixed connected drawing  $\Gamma$  of some multigraph  $G$ . The crossings split each edge into a number of *edge segments*. An *outer edge segment* is incident to a vertex, while an *inner edge segment* starts and ends with a crossing. The *cells* of  $\Gamma$  are the connected components of  $\mathbb{S}^2$  after the removal of all vertices and edges. For a cell  $c$ , we denote by  $v(c)$  the number of vertices, by  $e(c)$  the number of edge segments, and by  $e_{in}(c)$  the number of inner edge segments on its boundary, counted with multiplicity. The *size* of  $c$  is the sum of the former two:  $\|c\| := e(c) + v(c)$ . We denote by  $\mathcal{C}$  the set of all cells, and set  $\mathcal{C}_k = \{c \in \mathcal{C} : \|c\| = k\}$  and  $\mathcal{C}_{\geq k} = \{c \in \mathcal{C} : \|c\| \geq k\}$ .

Since  $\Gamma$  is assumed to be a connected drawing of  $G$ , the boundary of every cell is connected. The *type* of a cell  $c$  is the cyclic sequence of incidences with edge segments, crossings, and vertices along the boundary of  $c$ . We emphasize that a cell type  $\mathbf{c}$  only determines the form of incidence (edge segment, vertex, or crossing) and not the participating edge, vertex, or crossing of  $G$ . In particular, different incidences might be with the same edge segment, crossing, or vertex. We use pictograms with the size of the cell inscribed (such as  or ) to concisely denote cell types of small size. For convenience, let a cell type  $\mathbf{c}$  also denote the number of all  $\mathbf{c}$ -cells in a given drawing. If no cell in  $\Gamma$  has type  $\mathbf{c}$ , then  $\Gamma$  is called  *$\mathbf{c}$ -free*.

Let us conclude the preliminaries with an observation from [16] and two lemmas, which we will need in the course of this paper.

► **Observation 1** (Kaufmann, Klemz, Knorr, Reddy, Schröder, and Ueckerdt [16]). *Let  $\Gamma$  be any non-homotopic connected drawing of some multigraph  $G$  with at least three vertices. Then,*

- $\mathcal{C}_1 = \mathcal{C}_2 = \emptyset$ ,
- $\mathcal{C}_3$  is the set of all -cells,
- $\mathcal{C}_4$  is the set of all -cells and all -cells, and
- $\mathcal{C}_5$  is the set of all -cells, -cells, and -cells.

The next two lemmata will help prove upper bounds for -free drawings in Section 4 and -free drawings in Section 6. Lemma 2 below relates the number of vertices in a graph to the sizes of cells in a drawing of the graph. The statement already appears implicitly in various edge density proofs via discharging (e.g., [1, 2, 5]) and as a key ingredient in the proof for the Density Formula [16, Lemma 3.6]. We thus state it without proof<sup>2</sup>.

► **Lemma 2.** *For any connected drawing  $\Gamma$  of a multigraph  $G$  with at least one edge,*

$$|V(G)| - 2 = \sum_{c \in \mathcal{C}} \left( \frac{1}{4} \|c\| - 1 \right).$$

The next lemma gives a bound on the number of cells of certain types in terms of the number of inner edge segments in large cells of non-homotopic drawings. Similar statements have been implicitly used in prior work such as [2, 3, 16].

<sup>2</sup> As one of many ways to prove it, for example, subtracting the Density Formula with  $t = 0$  from the Density Formula with  $t = 1$  gives the result.

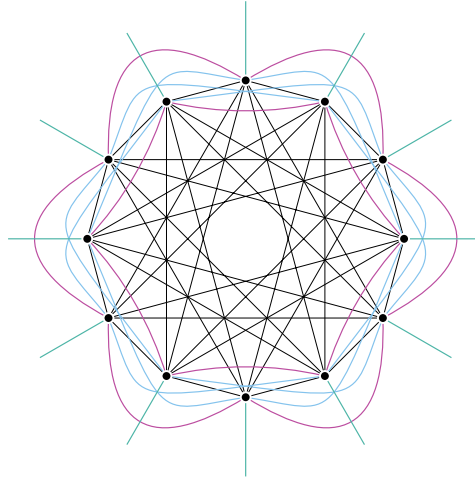


Construction 1 shows that for any cell type  $\mathfrak{c} \notin \{\nabla, \sqcup, \triangle, \diamond\}$ , every large enough complete graph admits a simple  $\mathfrak{c}$ -free drawing. In particular, the edge density of simple  $\mathfrak{c}$ -free drawings is  $\binom{n}{2}$  whenever  $\mathfrak{c}$  has size at least 6, or  $\mathfrak{c} = \text{pentagon}$ .

Next, we show that  $\triangle$ -cells can also be avoided in simple drawings of  $K_n$ . Note that any drawing without uncrossed edges will do, see for example [15]. For completeness, we present our own construction.

► **Construction 2.** For every  $n \geq 8$ , the complete graph  $K_n$  has a simple  $\triangle$ -free drawing.

**Proof.** Let  $n \geq 8$  be even and start with a straight-line drawing of  $K_n$  with vertices in convex position. Then redraw the edges of length 3 as depicted in Figure 2, so as to cross the edges of length 1 (on the convex hull). Further redraw every other edge of length 2 to be uncrossed in the outer cell. Finally, redraw all edges of length  $n/2$  to be pairwise crossing in the outer cell. For  $n$  odd, take the construction for  $n - 1$  even, add the last vertex in the central cell, and connect it to all other vertices with straight-line segments. ◀

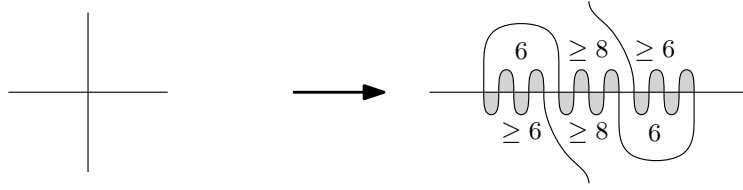


■ **Figure 2** A simple  $\triangle$ -free drawing of  $K_{12}$ . (A small perturbation eliminates the multi-crossings.) Edges of length three are drawn in blue and those of length two in red. Edges of length  $n/2$  are depicted with green edges.

With Constructions 1 and 2 at hand, only the cell types  $\nabla$ ,  $\triangle$ ,  $\sqcup$ , and  $\diamond$  remain, if we restrict to simple drawings of complete graphs. Before turning to these types, let us close the section by dropping the requirement that the drawing is simple. In fact, without any restriction, every cell type can be avoided in drawings of complete graphs.

► **Construction 3.** For any cell type  $\mathfrak{c}$  and infinitely many values of  $n$ , the complete graph  $K_n$  admits a  $\mathfrak{c}$ -free drawing.

**Proof.** For  $\mathfrak{c} \notin \{\nabla, \sqcup, \triangle, \diamond\}$ , the claim follows from Constructions 1 and 2. Every remaining cell type  $\mathfrak{c}$  has size 3, 4, or 5 and at least one crossing on its boundary. For every  $n \in \mathbb{N}$ , we now construct a drawing of  $K_n$  in which every cell with an incident crossing has size 2 or at least 6, which completes the proof. To do so, start with any drawing of  $K_n$ . Locally replace each crossing with the pattern in Figure 3. In the resulting drawing, each cell incident to a crossing is part of such a pattern and therefore of size 2 or at least 6. ◀



■ **Figure 3** A crossing can be redrawn to bound only cells of size 2 (shaded gray) or at least 6.

#### 4 Non-homotopic drawings without $\boxtimes$ -cells

In this section, we consider the  $\boxtimes$ -cell. In particular, we show that non-homotopic  $\boxtimes$ -free drawings have only linear edge density. Note that by Construction 3, the restriction to non-homotopic drawings is essential.

► **Theorem 4.** *For  $n \geq 3$ , any connected  $n$ -vertex multigraph that admits a non-homotopic  $\boxtimes$ -free drawing has at most  $6n - 12$  edges.*

**Proof.** Fix a  $\boxtimes$ -free drawing of an  $n$ -vertex multigraph with  $n \geq 3$ . First, let us compute for each cell  $c \in \mathcal{C}$  the quantity  $3e(c) + 2v(c) - 12$ , and in case  $c \in \mathcal{C}_{\geq 5}$  compare it with  $e_{in}(c)$ .

|                              |                                                                    |
|------------------------------|--------------------------------------------------------------------|
| $c$ is a $\nabla$ -cell :    | $3e(c) + 2v(c) - 12 = -3$                                          |
| $c$ is a $\triangle$ -cell : | $3e(c) + 2v(c) - 12 = -1$                                          |
| $c$ is a $\square$ -cell :   | $3e(c) + 2v(c) - 12 = 0$                                           |
| $c$ is a $\boxtimes$ -cell : | $3e(c) + 2v(c) - 12 = 2 = e_{in}(c)$                               |
| $c$ is a $\boxtimes$ -cell : | $3e(c) + 2v(c) - 12 = 1 \geq 0 = e_{in}(c)$                        |
| $\ c\  \geq 6$ :             | $3e(c) + 2v(c) - 12 = e(c) + 2\ c\  - 12 \geq e(c) \geq e_{in}(c)$ |

Note that the above case distinction is complete as  $c$  is not a  $\boxtimes$ -cell.

Summing over all cells, this gives

$$\sum_{c \in \mathcal{C}} (3e(c) + 2v(c) - 12) \geq -3 \cdot \nabla - \triangle + \sum_{c \in \mathcal{C}_{\geq 5}} e_{in}(c) \geq 0, \quad (1)$$

where the last inequality is an application of Lemma 3.

Finally, we obtain the desired result by double-counting all vertex-cell incidences as

$$2|E| = \sum_{c \in \mathcal{C}} v(c) \stackrel{(1)}{\leq} \sum_{c \in \mathcal{C}} (3e(c) + 3v(c) - 12) = 3 \sum_{c \in \mathcal{C}} (\|c\| - 4) = 12(n - 2),$$

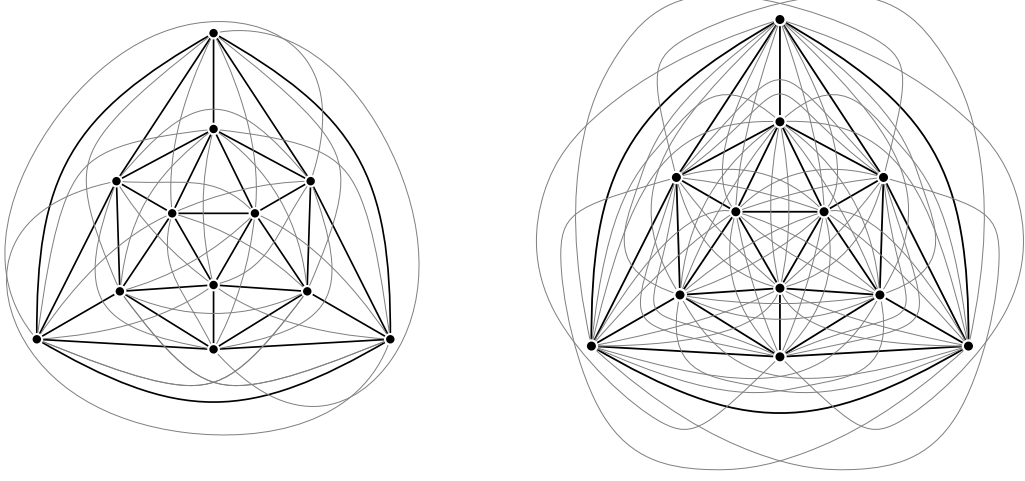
where the last equality is an application of Lemma 2. ◀

Theorem 4 gives an upper bound of  $6n - 12$  on the edge density of  $\boxtimes$ -free non-homotopic multigraphs. As the next construction shows, this bound is tight, even for simple drawings.

► **Construction 4.** *For infinitely many values of  $n$ , there is a simple  $n$ -vertex graph with  $6n - 12$  edges that admits a simple  $\square$ -free and  $\boxtimes$ -free drawing.*

**Proof.** Start with a plane drawing  $\Gamma_0$  of a 5-connected triangulation  $G_0$  on  $n$  vertices. For an edge  $e = uv$  in  $G_0$ , consider the two triangular faces  $f, f'$  in  $\Gamma_0$  incident to  $e$ , and let  $w, w'$  be the third vertex (different from  $u, v$ ) in  $f, f'$ , respectively. We insert a new edge between  $w$  and  $w'$  in the drawing by only traversing the two incident faces  $f, f'$  of  $e$ . As  $G_0$

is 5-connected,  $ww'$  is not already an edge in  $G_0$ , as otherwise there is a separating triangle in  $G_0$  with  $w, w'$  and one of  $u, v$ . Now, we do the same for all edges in  $G_0$ . Again, as  $G_0$  is 5-connected, for every edge  $\tilde{e} \neq e$  in  $G_0$  we obtain a pair different from  $w, w'$ , as otherwise there is a separating 4-cycle in  $G_0$  with  $w, w'$  and one endpoint of each of  $e, \tilde{e}$ .



**Figure 4** Left: A simple  $\square$ -free and  $\hexagon$ -free drawing with  $6n - 12$  edges. Right: A non-homotopic  $\square$ -free drawing of a multigraph with  $9n - 18$  edges.

Call the resulting graph  $G$  and the resulting drawing  $\Gamma$ . By the above reasoning, the graph  $G$  is simple (has no parallel edges). Moreover, observe that the drawing  $\Gamma$  is simple (any two edges have at most one point in common). See Figure 4 (Left) for an illustration.

The only cells in  $\Gamma$  that are not incident to any vertex are  $\nabla$ -cells. Hence,  $\Gamma$  is  $\square$ -free and  $\hexagon$ -free. Finally, since  $G_0$  is a plane triangulation, we have  $|E(G_0)| = 3n - 6$  and thus  $|E(G)| = 2|E(G_0)| = 6n - 12$ .  $\blacktriangleleft$

## 5 Non-homotopic drawings without $\square$ -cells

In this section, we briefly discuss the  $\square$ -cell. From Construction 4, we immediately obtain a  $6n - 12$  lower bound for the edge density of simple graphs that admit a  $\square$ -free drawing. A slight variation of this construction yields a better bound for non-homotopic drawings.

► **Construction 5.** *For infinitely many values of  $n$ , there is an  $n$ -vertex multigraph with  $9n - 18$  edges that admits a non-homotopic  $\square$ -free drawing.*

**Proof.** As in Construction 4, we start with any 5-connected triangulation  $G_0$  on  $n$  vertices. For each edge  $e = uv$  in  $G_0$ , we again consider the unique pair  $w, w'$  of common neighbors of  $u$  and  $v$ . But this time, we add two parallel edges between  $w$  and  $w'$ , again drawing these new edges so that they only cross the edge  $e$  of  $G_0$ . For each triangle  $uvw$  of  $G_0$ , this yields two edges per vertex that emanate from the vertices into the triangle.

Additionally, we draw the new edges in a way that in every triangular face  $uvw$  of  $G_0$ , the two parallel edges from each vertex enclose exactly two crossings of the four other edges that emanate into that face. This way,  $uvw$  is split into one cell of size 6 with no incident vertex and six  $\nabla$ -cells in the interior of  $uvw$ , as well as one  $\hexagon$ -cell at each edge of  $uvw$  and three cells at each vertex of  $uvw$  (one  $\nabla$ -cell and two  $\triangle$ -cells); see Figure 4 (right).



Thus, the obtained drawing is  $\mathbb{A}$ -free. Non-homotopicity is apparent since only the pairs of parallel edges form lenses, which are non-empty. Finally, the drawing contains  $9n - 12$  edges, since for each of the  $3n - 6$  edges of  $G_0$ , two edges are added.  $\blacktriangleleft$

By Construction 5, the edge density of  $\mathbb{A}$ -free non-homotopic multigraphs is at least  $9n - 18$ . For simple  $\mathbb{A}$ -free graphs, it is at least  $6n - 12$  by Construction 4. However, we have no non-trivial upper bound; see also Problem 1 in Section 8.

## 6 Non-homotopic $\nabla$ -free and non-homotopic quasiplanar drawings

As mentioned already,  $\nabla$ -free drawings are closely related to *quasiplanar* drawings, that is, drawings in which no three edges cross pairwise. As self-intersection of edges is forbidden, any quasiplanar drawing is  $\nabla$ -free but there exist  $\nabla$ -free drawings that are not quasiplanar.

Several results from the existing literature contain upper and lower bounds on the edge density of  $\nabla$ -free and quasiplanar drawings. See Table 2 for an overview. In fact, the edge density for both drawing styles is known to be  $8n - 20$  in the case of non-homotopic drawings of multigraphs. However, there are still some gaps in the case of non-homotopic drawings of simple graphs, and the case of simple drawings of (simple) graphs. In this section, we provide several improvements.

For quasiplanar drawings, we improve the lower bound for non-homotopic drawings of simple graphs from  $7n - 28$  [16] to  $7.5n - 28$  (Construction 7). We remark that there still remains a gap to the upper bound of  $8n - 20$  [3].

For  $\nabla$ -free drawings, we improve the lower bound for non-homotopic drawings of simple graphs from  $7.5n - \mathcal{O}(1)$  [3] to  $8n - 28$  (Construction 8), almost matching the upper bound of  $8n - 20$  [3]. For  $\nabla$ -free simple drawings of graphs, we give a lower bound of  $7n - 30$  (Construction 6) and prove a nearly tight upper bound of  $7n - 20$  (Theorem 5). The previous upper bound was  $8n - 20$  [3] in this case, too. See again Table 2.

**Table 2** Previous and new upper and lower bounds on the number of edges in  $n$ -vertex quasiplanar drawings and  $n$ -vertex  $\nabla$ -free drawings. Upper bounds hold for all  $n \geq 3$ , while lower bounds are proven for infinitely many values of  $n$ .

|                | Drawing Type               | Lower Bounds                                 |                               | Upper Bounds                            |
|----------------|----------------------------|----------------------------------------------|-------------------------------|-----------------------------------------|
| quasiplanar    | Non-homotopic multigraph   | $8n - 20$<br>[16]                            |                               | $8n - 20$<br>[3]                        |
|                | Non-homotopic simple graph | $7n - 28$<br>implicitly in [16]              | $7.5n - 28$<br>Construction 7 | $8n - 20$<br>[3]                        |
|                | Simple drawing             | $6.5n - 20$<br>[16]                          |                               | $6.5n - 20$<br>[3]                      |
| $\nabla$ -free | Non-homotopic multigraph   | $8n - 20$<br>[16]                            |                               | $8n - 20$<br>[3]                        |
|                | Non-homotopic simple graph | $7.5n - \mathcal{O}(1)$<br>implicitly in [3] | $8n - 28$<br>Construction 8   | $8n - 20$<br>[3]                        |
|                | Simple drawing             | $7n - 30$<br>Construction 6 based on [3]     |                               | $7n - 20$ $8n - 20$<br>Theorem 5    [3] |

### 6.1 Upper bound for $\nabla$ -free drawings of simple graphs

Ackerman and Tardos proved in [3] that any  $n$ -vertex quasiplanar drawing ( $n \geq 3$ ) contains at most  $8n - 20$  edges, but explicitly mention that they only use  $\nabla$ -freeness. For arbitrary (not necessarily non-homotopic) drawings, this would contradict Construction 3. However, their proof relies on the additional assumption that the drawing is crossing-minimal quasiplanar, which they use to rule out cells  $c$  with  $\|c\| \leq 2$ . In arbitrary crossing-minimal  $\nabla$ -free drawings, this cannot be concluded. But recall from Observation 1 that in any non-homotopic drawing every cell has size at least 3. Therefore, the proof of the  $8n - 20$  upper bound in [3] applies to non-homotopic  $\nabla$ -free drawings, even of multigraphs.

Ackerman and Tardos also considered the case of simple quasiplanar drawings, for which they obtained an upper bound of  $6.5n - 20$  edges via a similar proof idea. The final case distinction in this proof however, works only for quasiplanar drawings and not for  $\nabla$ -free drawings. Here, we extend their proof to simple  $\nabla$ -free drawings by adding one further case. This yields the slightly higher upper bound of  $7n - 20$ . In fact, our bound is best possible (up to an additive constant), as we shall later construct simple  $n$ -vertex graphs with  $\nabla$ -free drawings and  $7n - 30$  edges (Construction 6). In particular, the edge densities of simple quasiplanar drawings and simple  $\nabla$ -free drawings are indeed different.

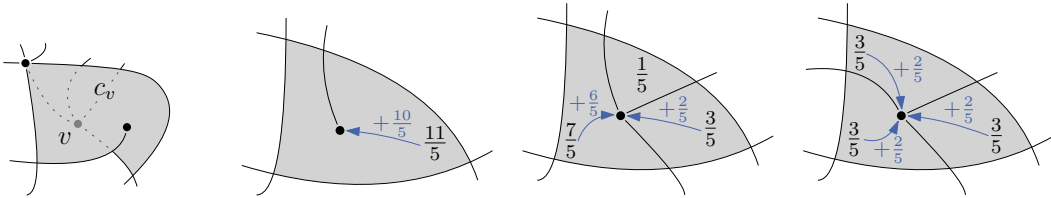
As mentioned above, we modify the proof of Ackerman and Tardos. Hence, we only sketch this proof and refer to [3] for more details.

► **Theorem 5.** *Let  $\Gamma$  be a  $\nabla$ -free drawings of an  $n$ -vertex multigraph  $G$  with  $n \geq 3$ . If  $\Gamma$  is non-homotopic, then  $G$  has at most  $8n - 20$  edges. If  $\Gamma$  is simple, then  $G$  has at most  $7n - 20$  edges.*

**Proof sketch.** Consider any  $\nabla$ -free drawing  $\Gamma$  of an  $n$ -vertex multigraph. We later distinguish whether  $\Gamma$  is non-homotopic or simple.

Initially, give each cell charge  $\text{ch}(c) := \|c\| - 4$ . By Lemma 2, the total charge is therefore  $4n - 8$ . We will redistribute the charges in two steps and, via double counting, obtain an upper bound on the number of edges. In the first discharging step, we obtain new charges  $\text{ch}'$  by setting  $\text{ch}'(c) := \text{ch}(c) + \frac{1}{5}$  for each  $\triangleleft$ -cell  $c$  and  $\text{ch}'(c) := \text{ch}(c) - \frac{e_{\text{in}}(c)}{5}$  for each  $c \in \mathcal{C}_{\geq 5}$ . By Lemma 3, the total charge does not increase in this step. Using  $\nabla$ -freeness, it can be checked via a case distinction that for every cell  $c$ , we have  $\text{ch}'(c) \geq \frac{v(c)}{5}$ .

The final charge distribution  $\text{ch}''$  is obtained by moving charge from cells to vertices. Namely, we distribute the excess charge  $\text{ch}'(c) - \frac{v(c)}{5}$  of each cell fairly among its  $v(c)$  incident vertices. We lower bound the charge  $\text{ch}''(v)$  of an arbitrary vertex  $v$  by a case distinction on the type of the cell  $c_v$  that is obtained when removing  $v$  and its incident edge segments (see Figure 5, left).



■ **Figure 5** Left: Removing a vertex  $v$  and its incident edge segments, a new cell  $c_v$  (shaded gray) is formed. Others: Case distinction in Theorem 5 where  $c_v$  is a  $\nabla$ -cell.

In the non-homotopic setting, Ackerman and Tardos prove  $\text{ch}''(v) \geq \frac{4}{5}$ . The minimum is obtained if  $c_v$  is a  $\bowtie$ -cell. When  $\Gamma$  is simple quasiplanar, they show  $\text{ch}''(v) \geq \frac{7}{5}$  and equality can be obtained when  $c_v$  is a  $\triangleleft$ -cell. To complete the case distinction in our setting of

simple  $\nabla$ -free  $\Gamma$ , it suffices to consider the additional case that  $c_v$  is a  $\nabla$ -cell: Note that  $\text{ch}''(v)$  is unchanged, whether an edge segment of  $c_v$  is crossed once or more than once, since additional crossings only increase the number of  $\triangleleft$ -cells incident to  $v$ , which cannot send any charge to  $v$ . Thus, we only have to compute  $\text{ch}''(v)$  if  $s = 1, 2, 3$  edge segments of  $c_v$  are crossed (for  $s = 0$ ,  $\Gamma$  is disconnected), see Figure 5, second-to-left to right. If  $s = 1$ ,  $v$  has final charge  $\frac{10}{5}$ , sent from a cell of size 7 with one incident vertex. If  $s = 2$ ,  $\text{ch}''(v) = \frac{6}{5} + \frac{2}{5}$  given by a cell of size 6 with one incident vertex and a  $\triangleleft$ -cell. Finally, if  $s = 3$ ,  $\text{ch}''(v) = 3 \cdot \frac{2}{5}$  from the three incident  $\triangleleft$ -cells. Thus in total,  $\text{ch}''(v) \geq \frac{6}{5}$  for every vertex  $v$ . Finally, by rearranging

$$4n - 8 = \sum_{c \in C} \text{ch}(c) \geq \sum_{c \in C} \frac{v(c)}{5} + \sum_{v \in V} \text{ch}''(v) \geq \frac{2}{5}|E| + |V| \cdot \min_v \text{ch}''(v),$$

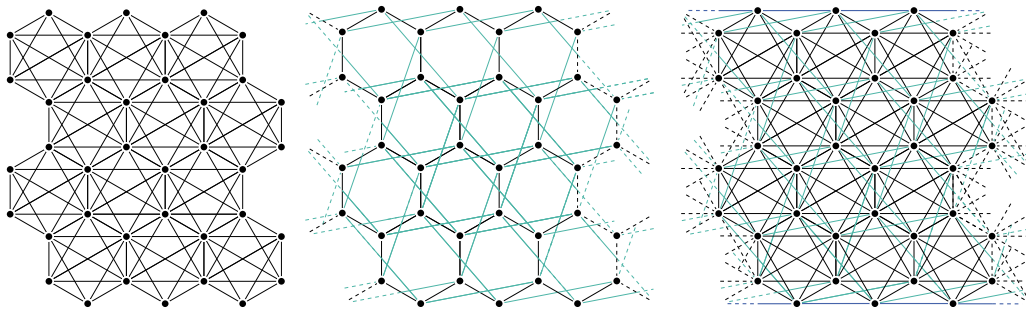
we obtain  $|E| \leq 8n - 20$  for non-homotopic and  $|E| \leq 7n - 20$  for simple  $\nabla$ -free drawings.  $\blacktriangleleft$

## 6.2 Lower bounds for $\nabla$ -free and quasiplanar drawings of simple graphs

Let us start with simple  $\nabla$ -free drawings. We give a lower bound construction that is tight up to an additive constant. In fact, in this case, it is enough to take a subdrawing of the construction in [3, Theorem 4].

► **Construction 6.** *For infinitely many values of  $n$ , there is a simple  $n$ -vertex graph with  $7n - 30$  edges that admits a  $\nabla$ -free drawing.*

**Proof.** Fix  $m \geq 1$  and start with an  $m \times 3$ -grid of hexagons on  $n = 8m + 6$  vertices, each hexagon containing a straight-line drawing of  $K_6 \setminus e$  where the vertical diagonal is missing (see Figure 6, left). Now, identify opposite vertices along the side of the grid of length  $m$  to obtain the surface of a cylinder whose top and bottom boundary contain 6 vertices. In the resulting drawing, every vertex not on the boundary has degree 11. Now, add long edges as depicted in the center of Figure 6 whenever this is possible. This increases the degree of every vertex sufficiently far away from the boundary to 14. After adding three uncrossed edges each on the top and bottom (see Figure 6, right), we arrive at the final drawing  $\Gamma$ . Counting vertex degrees near the boundary, it can be confirmed that the number of edges in  $\Gamma$  is  $7n - 30$ .



■ **Figure 6** Illustrations of Construction 6.

Before adding the long edges, the drawing is clearly  $\nabla$ -free and no  $\nabla$ -cells are created by adding the long edges and the six planar edges. Finally, the simplicity of  $\Gamma$  follows from the fact that  $\Gamma$  is locally homeomorphic to a straight-line drawing and no two edges are long enough to meet twice on the cylinder.  $\blacktriangleleft$

In the remainder of this section, we construct non-homotopic drawings of simple graphs. We start with a quasiplanar construction, followed by a construction of  $\nabla$ -free such drawings.

► **Construction 7.** *For infinitely many values of  $n$ , there is a simple  $n$ -vertex graph with  $7.5n - 28$  edges that admits a non-homotopic quasiplanar drawing.*

**Proof.** Let  $n \geq 14$  be even. We describe a drawing on  $n$  vertices with the desired properties. Precise edge paths are obtained by avoiding unnecessary crossings. Figure 7 (Top) depicts the final drawing  $\Gamma$ . We draw

1. an  $\frac{n}{2}$ -cycle  $C_{\text{in}}$  as a circle with equidistant points.
2. a concentric copy  $C_{\text{out}}$  of  $C_{\text{in}}$  with larger diameter.

We call two radially aligned vertices on  $C_{\text{in}}$  and  $C_{\text{out}}$  *partners*. For any vertex  $x$ , we denote by  $x'$  its partner, by  $x_{+k}$  the vertex reached by  $k$  clockwise steps on the cycle of  $x$ , and by  $x'_{+k}$  the partner of  $x_{+k}$ . We continue drawing

3. a straight edge between any pair of partners.
4. for any vertex  $v$ , a straight edge to  $v'_{+1}$ .
5. for any vertex  $v$  on  $C_{\text{in}}$  an edge from  $v$  to  $v_{+2}$ , drawn through the interior of  $C_{\text{in}}$ .
6. for any vertex  $v$  on  $C_{\text{out}}$  an edge from  $v$  to  $v_{+2}$ , drawn through the exterior of  $C_{\text{out}}$ .
7. for any vertex  $v$  on  $C_{\text{in}}$ , an edge to  $v'_{+2}$ . The edge passes through the space between  $C_{\text{in}}$  and  $C_{\text{out}}$ , crosses  $C_{\text{out}}$  in the edge  $v'v'_{+1}$  and continues through the exterior of  $C_{\text{out}}$ .
8. for any vertex  $v$  on  $C_{\text{out}}$ , an edge to  $v'_{+2}$ . Edges are drawn symmetrically to the last step.
9. for any vertex  $v$  on  $C_{\text{in}}$  an edge to  $v_{+3}$ . The edge first passes through the space between  $C_{\text{in}}$  and  $C_{\text{out}}$ , crosses  $C_{\text{in}}$  between  $v_{+1}$  and  $v_{+2}$  and then goes through the interior of  $C_{\text{in}}$ .
10. an edge between  $v$  and  $v_{+3}$  for any vertex  $v$  on  $C_{\text{out}}$ , symmetrically to the last step.
11. a zig-zag path inside of  $C_{\text{in}}$  whose edges connect vertices of distance at least 4 on  $C_{\text{in}}$ .
12. another different zig-zag path in the interior of  $C_{\text{in}}$  such that no parallel edges are created.
13. a zig-zag path outside of  $C_{\text{out}}$ , again connecting vertices of distance at least 4 on  $C_{\text{out}}$ .
14. another such zig-zag path in the exterior of  $C_{\text{out}}$ .

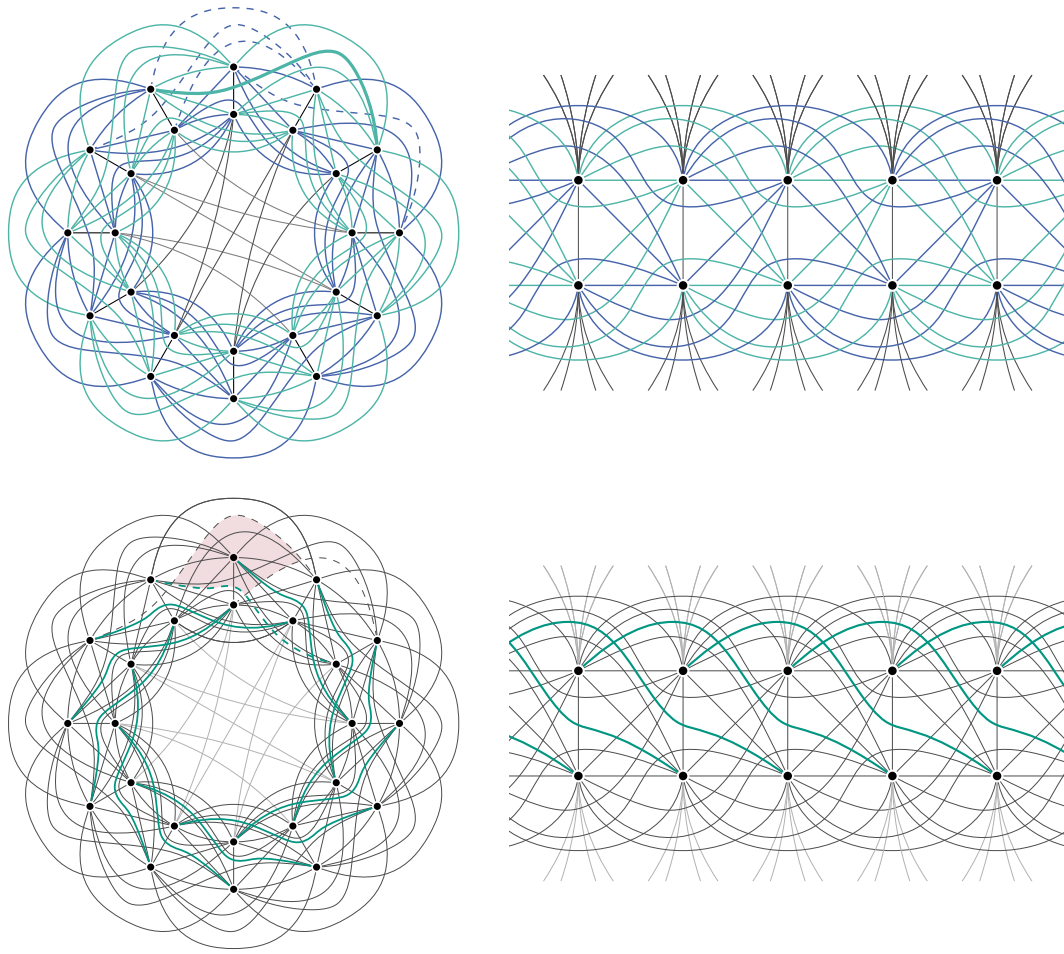
The first three steps add  $\frac{n}{2}$  edges each, step 4 adds  $n$  edges, the six steps thereafter  $\frac{n}{2}$  edges each, and the last four steps  $\frac{n}{2} - 7$  edges each, totaling the desired  $7.5n - 28$  edges. As the various steps connect vertices at different distances, the underlying graph is simple.

We now argue that  $\Gamma$  is quasiplanar. First, consider all edges except those in zig-zag paths. These can be locally 3-colored such that one color class consists only of the edges in step 3 and edges of the same color do not cross (see Figure 7). Therefore, any 3 pairwise crossing edges would contain an edge from step 3, but each such edge is only crossed by two other edges, which in turn do not cross.

Now, consider without loss of generality the zig-zag paths  $P$  and  $P'$  in the interior of  $C_{\text{in}}$  and some edge  $e = uv$  in  $P$ . Observe that  $e$  is crossed by three sets of edges: (i) Edges in  $P'$ , (ii) edges from steps 1-10 that cross  $e$  at  $u$ , and (iii) edges from steps 1-10 that cross  $e$  at  $v$ .

We argue that no edges from (i), (ii), or (iii) cross. Note that no pair of edges in the same group crosses. Next, an edge from (ii) cannot cross an edge from (iii) since  $u$  and  $v$  have distance at least 4 on  $C_{\text{in}}$ , which two edges cannot span while crossing. Further, note that no edge in (i) is incident to  $u$  or  $v$ . Since edges in (ii) and (iii) cross edges from zig-zag paths only at  $u$  and  $v$ , they cannot cross edges from (i). Thus,  $e$  is not part of three pairwise crossing edges, and in total, the drawing is quasiplanar.

Finally, we argue that  $\Gamma$  is non-homotopic. Note that in the above iterative construction, simplicity is only violated by the edges added in steps 9 and 10. Each such edge is part of four lenses (see dashed edges in Figure 7 (Top)), all of which are non-empty. ◀



■ **Figure 7** The drawings of Constructions 7 and 8 with the outer zig-zag paths omitted for visual clarity, as well as zoomed-in strips of the local edge patterns.

Top: Construction 7 where edges except on the inner zig-zag paths are partitioned by color into three plane drawings. The edges that form lenses with the thick edge are dashed.

Bottom: Construction 8 where green edges are those not present in Construction 7. The shaded area is bounded by three pairwise crossing edges.

► **Construction 8.** For infinitely many values of  $n$ , there is a simple graph on  $n$  vertices and  $8n - 28$  edges that admits a non-homotopic drawing without  $\nabla$ -cells.

**Proof.** Let  $n \geq 14$  be even and start with the  $\nabla$ -free drawing from Construction 7. We will reuse the notation established there. For any vertex  $v$  on  $C_{\text{out}}$ , we draw an edge from  $v$  to  $v'_{+3}$  that first moves through the space between  $C_{\text{in}}$  and  $C_{\text{out}}$ , crosses  $C_{\text{in}}$  in  $v'_{+1}v'_{+2}$  and then passes through the interior of  $C_{\text{in}}$ . The resulting drawing is shown in Figure 7 (Bottom).

Checking along the paths of the newly added edges, it can be verified that the drawing remains  $\nabla$ -free. Note that this process adds  $\frac{n}{2}$  edges, none of which are parallel to each other or previous edges. Therefore, we obtain the claimed density. ◀

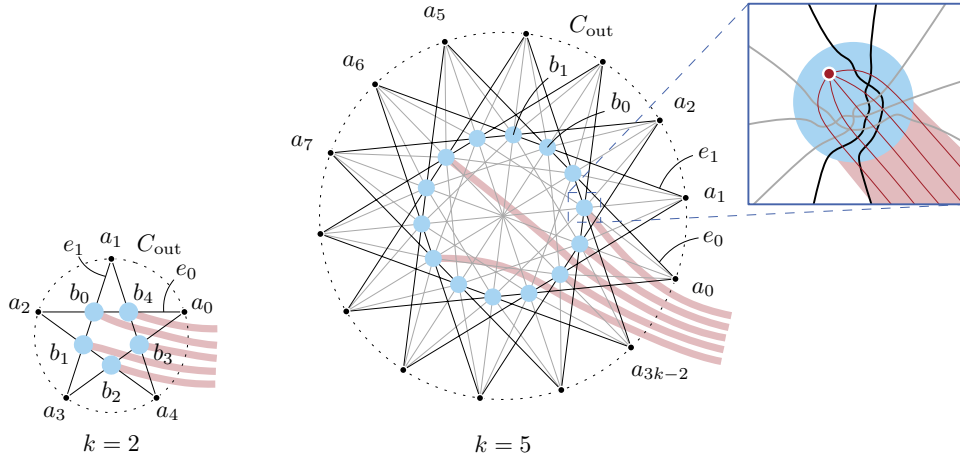
## 7 Non-homotopic drawings without $\triangleleft$ -cells

In this section, we consider the  $\triangleleft$ -cell. In particular, we show that  $n$ -vertex graphs with simple  $\triangleleft$ -free drawings can have  $\Theta(n^2)$  many edges and minimum degree  $\Omega(n)$ . The underlying graph in our construction is disconnected but can be made connected by adding a constant number of edges and vertices.

► **Construction 9.** *For infinitely many values of  $n$ , there is a simple  $n$ -vertex graph on  $\frac{n^2}{18} - \mathcal{O}(n)$  edges admitting a simple  $\triangleleft$ -free drawing.*

**Proof.** For each integer  $k \geq 2$ , we construct a simple  $k$ -regular graph  $G(k)$  on  $n = 3 \cdot (3k - 1)$  vertices together with a simple  $\triangleleft$ -free drawing; see Figures 8 and 9 for an illustration. The graph  $G(k)$  consists of three isomorphic copies of a graph  $G'(k)$  on  $3k - 1$  vertices. To describe  $G'(k)$  (and its drawing), we start with a set  $A$  of  $3k - 1$  equidistant points on a circle  $C_{\text{out}}$ . The points in  $A$  will be the vertices of  $G'(k)$ , which we denote  $a_0, \dots, a_{3k-2}$  in cyclic ordering around  $C_{\text{out}}$ . For each  $i = 0, \dots, 3k - 2$  draw a straight-line edge  $e_i$  between  $a_i$  and  $a_{i+k}$  (throughout this construction all indices are taken modulo  $3k - 1$ ). Let  $b_i$  be the crossing point of  $e_i$  and  $e_{i+1}$ , and  $B$  the set of all such  $b_i$ .

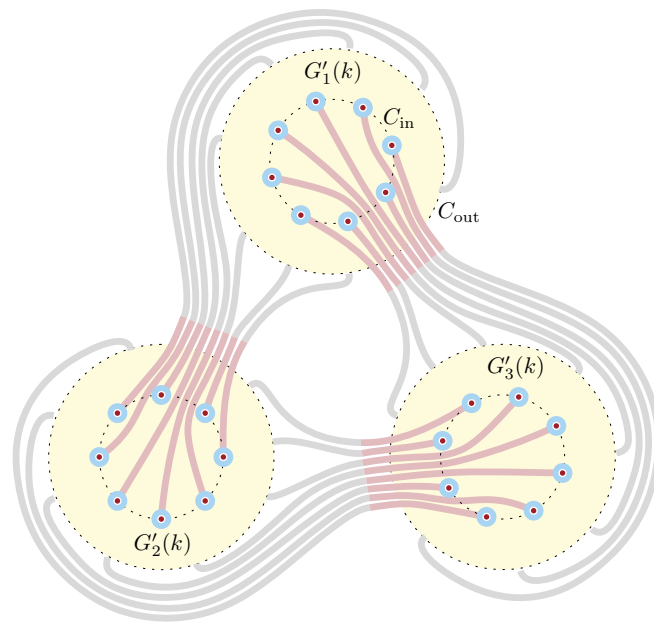
Then,  $b_0, \dots, b_{3k-2}$  are equidistant points (blue disks in Figure 8) on a smaller circle  $C_{\text{in}}$  concentric to  $C_{\text{out}}$ . Note that  $e_i$  crosses  $e_{i+1}$  at  $b_i$  and  $e_{i-1}$  at  $b_{i-1}$ . Next, we add further edges such that each  $a_i$  has exactly  $k$  incident edges and at each  $b_i$ , exactly  $k$  edges pairwise cross. To this end, we insert edges from  $a_i$  to  $a_{i+k+1}, \dots, a_{i+\lfloor 3k/2 \rfloor}$  where the edge  $a_i a_{i+k+j}$  with  $j = 1, \dots, \lfloor k/2 \rfloor$  goes through  $b_{i-j-1}$  and  $b_{i+2j}$ ; see again Figure 8 for an illustration. This completes the construction of  $G'(k)$ . Observe that the described drawing of  $G'(k)$  is simple (in fact, almost straight-line) and every vertex of  $G'(k)$  lies on the same cell.



■ **Figure 8** Illustration of the  $k$ -regular graph  $G'(k)$  on  $3k - 1$  vertices for  $k = 2$  and  $k = 5$ . At each of the  $3k - 1$  points in  $B$  (blue disks),  $k$  edges are pairwise crossing. Corridors are indicated in red. (Due to space reasons, only some corridors are shown for  $k = 5$ .)

We now take three copies of  $G'(k)$ , called  $G'_1(k), G'_2(k), G'_3(k)$ , and interleave the three drawings so as to contain no  $\triangleleft$ -cell. See Figure 9 for an illustration. We shall move the vertices in such a way that each vertex  $a$  of  $G'_1(k)$  lies close to a point  $b$  in the set  $B$  of  $G'_2(k)$ . We locally perturb the  $k$  edges of  $G'_2(k)$  that all cross in  $b$  so that each of the  $k$  edges of  $G'_1(k)$  emanating from  $a$  has its first crossing with a different edge of  $G'_2(k)$ . Loosely speaking, the edges of  $G'_2(k)$  at point  $b$  “support” the edges of  $G'_1(k)$  starting at point  $a$ . Similarly, we let the edges of  $G'_3(k)$  support the edges of  $G'_2(k)$ , and the edges of  $G'_1(k)$  support the edges of  $G'_3(k)$ .





■ **Figure 9** Connecting the drawings of  $G'_1(k)$ ,  $G'_2(k)$ , and  $G'_3(k)$  to a  $\triangleleft$ -free drawing of  $G(k)$ .

More precisely, we consider in the drawing of  $G'(k)$  for each  $b \in B$  a straight and very narrow corridor starting at  $b$  and ending on  $C_{\text{out}}$  between  $a_0$  and  $a_{3k-2}$ . Then, we perturb the  $k$  edges close to  $b$  to obtain a simple drawing in which  $b$  sees a segment of each of these  $k$  edges. Now, place the simple drawings of  $G'_1(k)$ ,  $G'_2(k)$ ,  $G'_3(k)$  with corridors disjointly next to each other. Move the vertices of  $G'_1(k)$  (without introducing new crossings) through the prepared corridors of  $G'_2(k)$  (each vertex through one corridor). Place each vertex  $a$  of  $G'_1(k)$  at the end of the corridor of  $G'_2(k)$  (at the point  $b \in B$ ) so that each edge (of  $G'_1(k)$ ) at  $a$  crosses through a different edge (of  $G'_2(k)$ ) near  $b$ . This ensures that  $a$  is not part of any  $\triangleleft$ -cell. By similarly threading the vertices of  $G'_2(k)$  through the corridors of  $G'_3(k)$  and the vertices of  $G'_3(k)$  through the corridors of  $G'_1(k)$ ,  $\triangleleft$ -freeness is ensured. The resulting drawing, as illustrated in Figure 9, is simple, and a quick calculation ensures the desired density. ◀

## 8 Concluding remarks

With this paper, we initiated the study of drawings of graphs with a forbidden type of cell. For arbitrary, non-homotopic, and simple drawings, and all cell types  $\mathfrak{c}$  but  $\triangleleft$ , we determined whether graphs that admit a  $\mathfrak{c}$ -free drawing have at most linear or quadratic edge density. In many cases we obtain near-tight bounds. Being unable to control the  $\triangleleft$ -cell with our methods and constructions, we wonder in which regime the number of edges in  $\triangleleft$ -free drawings lies.

► **Open Problem 1.** *Is the edge density of simple graphs that admit a (simple, non-homotopic)  $\triangleleft$ -free drawing quadratic in  $n$ ?*

Planar and 1-planar graphs can be characterized by forbidding not only one, but infinitely many types of cells. Perhaps there are other, less obvious, instances of beyond-planar graph classes characterized by forbidden cells. For example, in light of the density results on  $\nabla$ -free and quasiplanar drawings (Section 6), the underlying graph classes might even coincide for non-homotopic drawings.

► **Open Problem 2.** Does every graph that admits a non-homotopic  $\nabla$ -free drawing also admit a non-homotopic quasiplanar drawing?

The vast number of questions asked about other beyond planar graph classes – e.g. about complexity of recognition and stretchability – can be carried over to our setting. However, our forbidden patterns rely on the topology of concrete drawings rather than only abstract combinatorics of the graph or the drawing (such as crossing edge pairs). Hence, the resulting graph classes are not always closed under taking subgraphs. For example,  $K_5$  admits a simple  $\triangleleft$ -free drawing while  $K_3$  does not. In light of this, the results that almost all cell types can be avoided in drawings of  $K_n$  from Section 3 raise the question of which cell types can be avoided in drawings of any graph.

► **Open Problem 3.** For which cell types  $c$  does every graph admit a (simple, non-homotopic)  $c$ -free drawing?

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