Counting Triangulations of Fixed Cardinal Degrees

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Abstract

A fixed set of vertices in the plane may have multiple planar straight-line triangulations in which the degree of each vertex is the same. As such, the degree information does not completely determine the triangulation. We show that even if we know, for each vertex, the number of neighbors in each of the four cardinal directions, the triangulation is not completely determined. We show that counting such triangulations is #P-hard via a reduction from #3-regular bipartite planar vertex cover.

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Category Poster Abstract

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Paper Description

We consider *plane straight-line (PSL) graphs*: graphs whose vertices are points in the plane, and whose edges are interior-disjoint segments connecting the vertices at their endpoints. A PSL graph is *maximal* if adding any edge would result in a graph that is not a PSL graph. Given a PSL graph $G = (V, E)$ and $v \in V$, we write $\deg_{\mathcal{N}}(v)$ to mean the number of edges incident to v whose other endpoint lies above v in the y-direction ("to the north"). The *south-, east-, and west-degrees of* v, written $\deg_{\mathcal{S}}, \deg_{\mathcal{E}}, \deg_{\mathcal{W}} \colon V \to \mathbb{N}$ respectively, are defined analogously, and the four are collectively called the *cardinal degrees* of G. We consider only graphs with vertices at distinct x- and y-coordinates and in general position. We call $\sigma = (V, \deg_{\mathcal{N}}, \deg_{\mathcal{S}}, \deg_{\mathcal{E}}, \deg_{\mathcal{W}})$ a *cardinal signature* and call G a *realization* of σ if and

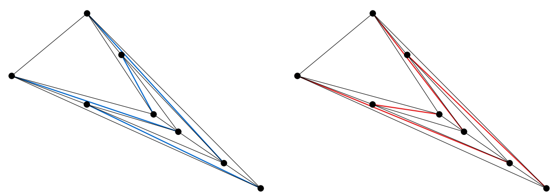

Figure 1 Maximal PSL graphs with identical cardinal signatures. Common edges are black.

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only if G is a maximal PSL graph with cardinal signature σ. Although each graph has a unique cardinal signature, distinct graphs can have the same cardinal signature, even in the restricted setting of maximal PSL graphs; see Figure 1.

We therefore consider the problem of *counting* realizations of a cardinal degree signature, called #cardinal signature realization, and, in particular, we show this problem is #P-hard. We do so by a reduction from #3-regular bipartite planar vertex cover. However, we first reduce to an intermediary problem, namely, the #tiled noncrossing cycle-set problem, whose input is a *tiling*: a grid of *tiles*, where each tile has one of the *tile types* as in Figure 2, and are arranged to form a collection of red and blue cycles. A solution to this problem is a selection of noncrossing cycles. Figure 3 illustrates the connection between #tiled noncrossing cycle-set and #3-regular bipartite planar vertex cover.

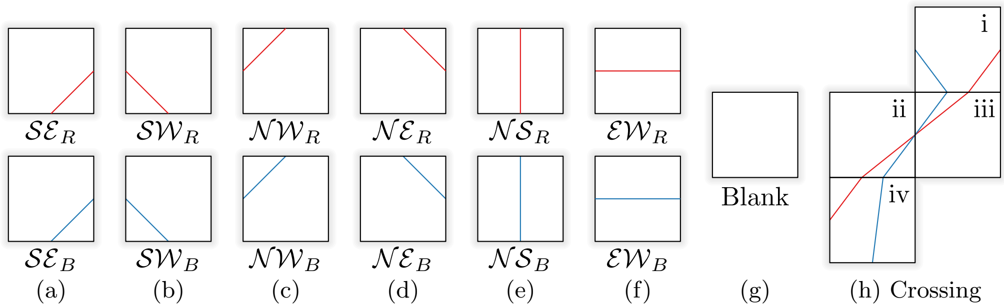

Figure 2 The 17 different tile types in the #tiled noncrossing cycle-set problem. In any tiling, the four tile types in (h) always appear together as pictured.

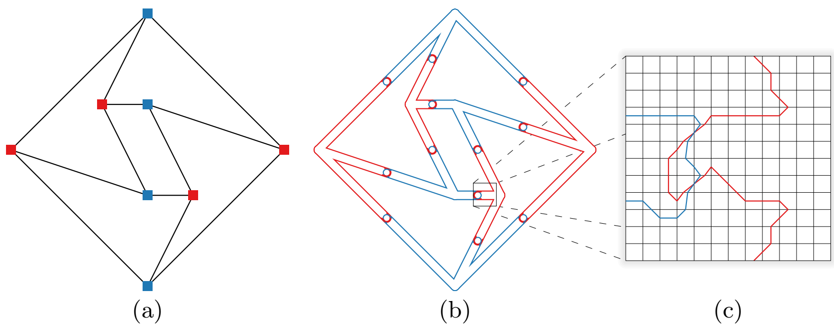

Figure 3 (a) An instance of #3-regular bipartite planar independent set. (b) A corresponding set of cycles in the plane. (c) Part of a corresponding tiling.

To connect our central problem to #tiled noncrossing cycle-set, we introduce *frame graphs*. A frame graph F is a grid-like pattern of convex chains of edges, forming roughly square faces called *frame cells*, corresponding to the tile structure of #tiled noncrossing cycle-set; see the black edges of Figure 4(c). Suppose a roughly horizontal path of F lies just below height h. This path is part of the boundary of the subgraph of F whose vertices lie below h. The sum of south degrees of all vertices below h tells us how many edges lie entirely in this halfspace. Using Euler's formula, we confirm that this lower subgraph is maximal PSL, i.e., the edges in this path must be present in every cardinal signature of F. Using a left-to-right argument for the nearly vertical paths tells us the realization of the cardinal signature of F is unique. By extension, any graph made by adding edges to F has a cardinal signature whose realizations all have the edges of F, i.e., these edges are *forced*. Thus, we include extra edges to a frame graph to form *gadgets*, corresponding to tile types; see Figure 4.

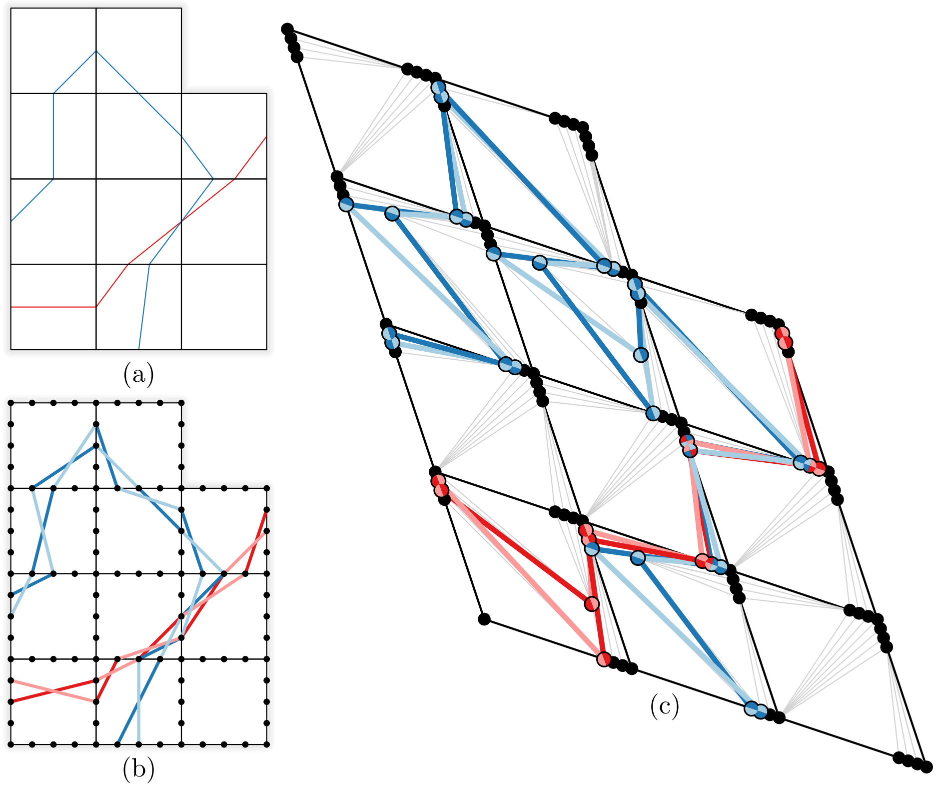

Figure 4 (a) Tile types for #tiled noncrossing cycle-set (up to exchanging colors). (b) Schematic representation of the gadgets of (c). (c) Corresponding gadgets in the same relative positions for #cardinal signature realization. Forced edges are grey and black.

We show that a frame graph triangulated by edges chosen from gadgets has a cardinal signature whose realizations are only the ones shown. We first establish our inductive step.

▶ **Lemma 1** (informal). *Let F be a frame graph triangulated by edges of gadgets. Let $\mathcal{G}$ be a gadget and suppose that the triangulation in F of the frame cell to the south of $\mathcal{G}$ is made by choosing light (dark) colored edges. Then the same choice of light (dark) colored edges triangulates $\mathcal{G}$ so that vertices on the southern boundary of $\mathcal{G}$ have the same cardinal degrees as they do in F. A similar statement applies to the west of $\mathcal{G}$.*

Since having a fixed triangulation to the south and west of a frame cell establishes the triangulation of the frame cell itself, and a consistent choice of light/dark leads to a consistent choice within the cell, we induct on the row and column of a frame cell to show the following.

▶ **Lemma 2** (informal). *All realizations of the cardinal signature of F correspond to triangulating frame cells using a subset of edges of the kind shown in Figure 4(c).*

We bijectively match each solution of a noncrossing tile selection to a realization of a cardinal signature σ; choosing (not choosing) a tile cycle corresponds to that portion of the triangulation made using light (dark) colored edges. We conclude that the number of cardinal signature realizations of σ is the same as the number of noncrossing tile selections of the corresponding cycle-set.

▶ **Theorem 3.** *#cardinal signature realization is #P-hard.*

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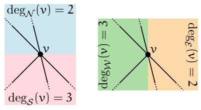
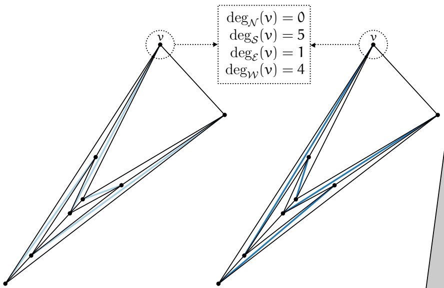
1

Cardinal degree of a vertex:
counts of degree to the north, south, east, and west



Cardinal degree signature of a plane graph:
position of vertices, and their cardinal degrees

Multiple graphs can have same cardinal signature!

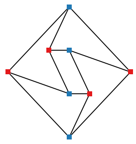


Light or dark blue edges → different triangulations, but every vertex keeps its cardinal degrees

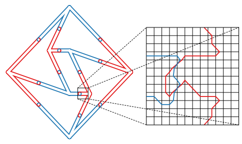
Computational question
How many realizations does a given cardinal degree signature have?

2

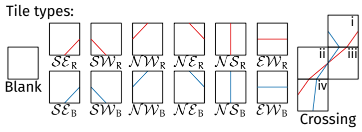
#P-hardness We reduce from the #P-hard problem of counting independent vertex sets in 3-regular planar bipartite graphs.



Reduce this to counting non-crossing subsets of cycles, where cycles are drawn on *tiles* of specific types.

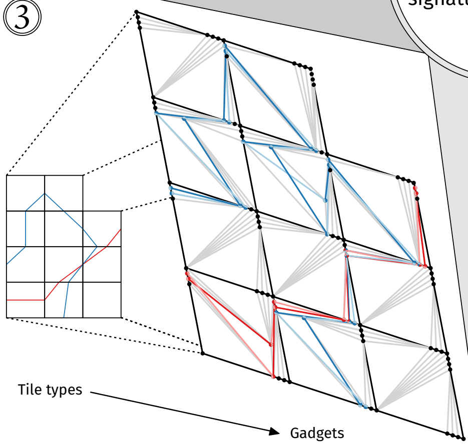


Tile types:



Counting graphs with a given cardinal degree signature is #P-hard.

3



Including a blue/red cycle in the set → analogous to choosing a triangulation using light blue/red in the corresponding places.

Not including a cycle → triangulations that use *dark* blue/red.

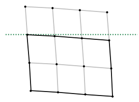
Any non-crossing cycle subset corresponds to a triangulation.

4

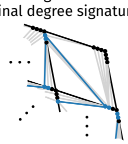
Why tiles and grid-like subgraphs? Why "tilted?"

- We know #edges below (left of) a horizontal line by summing degrees. By Euler's formula, subgraph below (to the left) must be convex
- Boundary of convex subgraph below/to the left is present in every realization

⇒ Grid edges must be present in any graph with the same cardinal degree signature.



Choosing light/dark red/blue to the south and west forces a cell to be triangulated with the same choices. (Otherwise cardinal degree signature changes).



Using gadgets to turn a tiled cycle set into a triangulation, non-crossing subsets of cycles are in bijection with triangulations with this cardinal signature!