Reconfigurations of Plane Caterpillars and Paths

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Abstract

Let S be a point set in the plane, and let $\mathcal{P}(S)$ and $\mathcal{C}(S)$ be the sets of all plane spanning paths and caterpillars on S. We study reconfiguration operations on $\mathcal{P}(S)$ and $\mathcal{C}(S)$. In particular, we prove that all of the commonly studied reconfigurations on plane spanning trees still yield connected reconfiguration graphs for caterpillars when S is in convex position. If S is in general position, we show that the rotation, compatible flip and flip graphs of $\mathcal{C}(S)$ are connected while the slide graph is sometimes disconnected, but always has a component of size $\frac{1}{4}(3^n - 1)$. We then study sizes of connected components in reconfiguration graphs of plane spanning paths. In this direction, we show that no component of size at most 7 can exist in the flip graph on $\mathcal{P}(S)$.

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Supplementary Material
Software (Source code): https://github.com/jelena-glisic/Caterpillar-Slides [6]
archived at swh:1:dir:552a6b334147879c561c75db5d9b586bf6c23aa2

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1 Introduction

Given a set of structures C, and a reconfiguration operation that transforms one object in C to another, the *reconfiguration graph* is a graph with vertex set C in which two vertices form an edge if one can be transformed into the other using a single reconfiguration operation. Often, in computer science, objects are solutions to a problem and reconfigurations are local

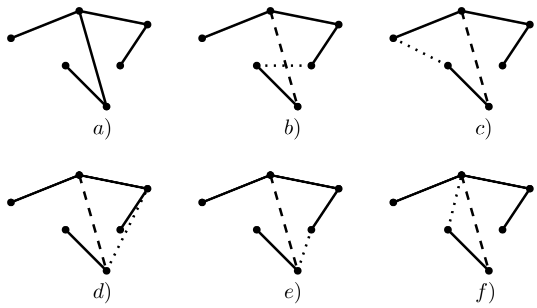

Figure 1 a) A plane spanning tree. Replacing the dashed line with the dotted line corresponds to: b) a flip, c) a compatible flip, d) a rotation, e) an empty triangle rotation and f) a slide.

changes that transform one solution into another. Then, to understand the solution space, it is important to study the reconfiguration graph. For an introduction to the topic, see [14].

Given a point set S in the plane, a *plane spanning tree* on S is a spanning tree of S whose edges are straight line segments that do not cross. Let $\mathcal{T}(S)$ be the set of all plane spanning trees on S. We consider five reconfigurations on $\mathcal{T}(S)$ (see Figure 1). For the following, we are given $T_1 = (S, E_1), T_2 = (S, E_2) \in \mathcal{T}(S)$, and we say that:

1. T_1 and T_2 are connected by a *flip* if $E_2 = E_1 \setminus \{e\} \cup \{f\}$ for some edges e, f.
2. T_1 and T_2 are connected by a *compatible flip* if $E_2 = E_1 \setminus \{e\} \cup \{f\}$ for some edges e, f which do not cross.
3. T_1 and T_2 are connected by a *rotation* if $E_2 = E_1 \setminus \{e\} \cup \{f\}$ for some edges e, f which share an endpoint.
4. T_1 and T_2 are connected by an *empty triangle rotation* if $E_2 = E_1 \setminus \{e\} \cup \{f\}$ for some edges e, f which share an endpoint and the triangle spanned by their endpoints is empty.
5. T_1 and T_2 are connected by a *slide* if $E_2 = E_1 \setminus \{e\} \cup \{f\}$ for some edges e, f which are as in 4. and if $e = ab$ and $f = ac$ then $bc \in E_1 \cap E_2$.

Note that every slide is an empty triangle rotation, every empty triangle rotation is a rotation, and so on. This hierarchy will be useful when studying the structural properties of the corresponding reconfiguration graphs.

2 Background

The reconfiguration graphs associated with the operations described above have been a topic of interest for a long time with many results appearing through the years. These results have concerned connectivity [1, 7, 13], lower and upper bounds on the diameter [2, 9, 10], etc.

It is known that the reconfiguration graph of plane spanning trees is connected even in the most restrictive case when the reconfigurations are slides [1]. In the case of slides, a tight $\Theta(n^2)$ bound on the diameter is shown in [4]. For the empty triangle rotations, an upper bound of $O(n \log n)$ on the diameter was shown in [13], while for the remaining reconfigurations, a linear upper bound is known [7]. In [2], an upper bound of $2n - 3$ is shown for flip graphs. The best known lower bound on the diameter of the reconfiguration graphs is $\frac{14}{9}n - O(1)$, as per [8].

However, there has been little research on induced subgraphs of these reconfiguration graphs. The only such subgraph that has been explored is the one induced by plane spanning paths. For a set of n points in convex position, it is known that the flip graph is connected [5] and that it has diameter $2n - 5$ for $n \in \{3, 4\}$ and $2n - 6$ for $n \geq 5$ [11]. The flip graph of point sets with at most two convex layers is connected [12], as is the flip graph of generalized double circles [3]. For point sets in general position, connectivity was first conjectured by Akl, Islam and Meijer [5] 18 years ago but still remains unsolved.

3 Our contribution

We further the study of induced subgraphs of reconfiguration graphs of plane spanning trees by exploring reconfigurations of plane spanning caterpillars, and by expanding on the topic of reconfigurations of plane spanning paths. A *caterpillar* is a tree in which all non-leaf vertices form a path. We call this path the *spine* of the caterpillar. A *plane spanning caterpillar* of a point set S is a plane spanning tree of S which is a caterpillar. For a set S, we will denote by $\mathcal{C}(S)$ the set of all plane spanning caterpillars on S. We will denote the reconfiguration graphs on $\mathcal{C}(S)$ by $G_{\mathcal{C}}^{\text{flip}}(S)$, $G_{\mathcal{C}}^{\text{comp-flip}}(S)$, $G_{\mathcal{C}}^{\text{rot}}(S)$, $G_{\mathcal{C}}^{\text{emp-rot}}(S)$, and $G_{\mathcal{C}}^{\text{slide}}(S)$. First, we focus on the case when S is in convex position. We show that the *slide graph* $G_{\mathcal{C}}^{\text{slide}}$ is connected, which implies that all of the reconfiguration graphs are connected in this case. For compatible flips, we prove a stronger bound for the diameter.

▶ **Theorem 1.** *Let S be a set of $n \geq 3$ points in convex position in the plane. Then, the graph $G_{\mathcal{C}}^{slide}(S)$ is connected with diameter at most $3n - 8$.*

▶ **Theorem 2.** *Let S be a set of $n \geq 3$ points in convex position in the plane. Then, the graph $G_{\mathcal{C}}^{comp\text{-}flip}(S)$ is connected with diameter at most $2n - 5$.*

Next, we consider the case when S is a point set in general position. For a caterpillar $C \in \mathcal{C}(S)$, and consecutive spine vertices $v_i, \ldots, v_j$ of C, we write $S_{i,j}$ for the point set consisting of the spine vertices and all of the leaves attached to them. We call $C \in \mathcal{C}(S)$ with spine $v_1, v_2, \ldots, v_k$ a *well-separated* caterpillar if for each $i \geq 1$, the convex hull of $S_{1,i}$ is disjoint from the rest of S. We can prove that all caterpillars in this class are mutually connected in $G_{\mathcal{C}}^{\text{slide}}(S)$. This large connected component is significant, as we can also show that $G_{\mathcal{C}}^{\text{slide}}(S)$ is disconnected. On the other hand, we can prove that the rotation graph $G_{\mathcal{C}}^{\text{rot}}(S)$ is connected. We leave open the question of the connectivity of $G_{\mathcal{C}}^{\text{emp-rot}}$.

▶ **Theorem 3.** *Any two well-separated caterpillars are connected in $G_{\mathcal{C}}^{slide}(S)$.*

▶ **Theorem 4.** *The graph $G_{\mathcal{C}}^{slide}(S)$ is connected for every set S of n points in the plane if $n \leq 7$. If $n \geq 8$, there exists a set S of n points such that $G_{\mathcal{C}}^{slide}(S)$ has isolated vertices.*

▶ **Proposition 5.** *The graph $G_{\mathcal{C}}^{rot}(S)$ is connected.*

Lastly, we focus on connected components of the reconfiguration graph of plane spanning paths. Given a set of points in general position S, we will call the corresponding *flip graph of plane spanning paths* $G_{\mathcal{P}}(S)$. Currently, the main open problem is deciding whether $G_{\mathcal{P}}(S)$ is connected. Here we find a large connected component consisting of what we call *generalized peeling paths*. We note that Theorem 6 was independently discovered by Kleist, Kramer, and Rieck [12]. We use the number of these paths to prove that there are at least $\frac{1}{4}(3^n - 1)$ well-separated caterpillars on S. Finally, we investigate the minimal size of components in $G_{\mathcal{P}}(S)$.

▶ **Theorem 6.** *Let S be a set of n points in general position. Then $G_{\mathcal{P}}(S)$ contains a connected component of size $2^{\Omega(n)}$.*

▶ **Theorem 7.** *Let S be a point set of $n \geq 5$ points in general position. Then, $G_{\mathcal{P}}(S)$ contains no connected component of size at most 7.*

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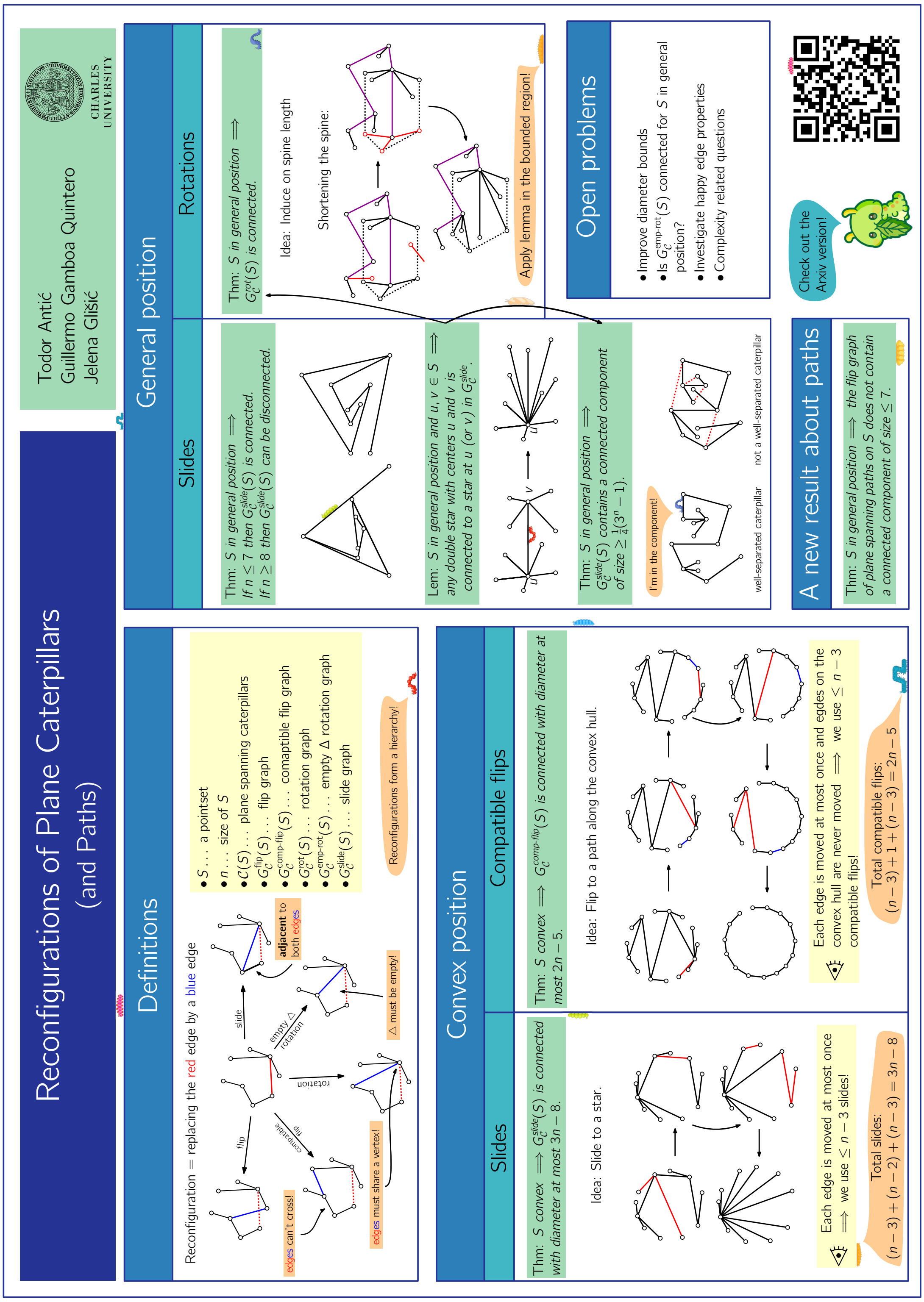
Reconfigurations of Plane Caterpillars
(and Paths)
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Definitions
Reconfiguration = replacing the red edge by a blue edge
flip
compatible flip
rotation
slide
empty Δ rotation
edges can't cross!
adjacent to both edges
edges must share a vertex!
Δ must be empty!
• S . . . a pointset
• n . . . size of S
• $\mathcal{C}(S)$ . . . plane spanning caterpillars
• $G^{\text{flip}}_{\mathcal{C}}(S)$ . . . flip graph
• $G^{\text{comp-flip}}_{\mathcal{C}}(S)$ . . . compatible flip graph
• $G^{\text{rot}}_{\mathcal{C}}(S)$ . . . rotation graph
• $G^{\text{emp-rot}}_{\mathcal{C}}(S)$ . . . empty Δ rotation graph
• $G^{\text{slide}}_{\mathcal{C}}(S)$ . . . slide graph
Reconfigurations form a hierarchy!
Convex position
Slides
Thm: S convex $\implies G^{\text{slide}}_{\mathcal{C}}(S)$ is connected with diameter at most $3n-8$.
Idea: Slide to a star.
Each edge is moved at most once $\implies$ we use $\leq n-3$ slides!
Total slides: $(n-3)+(n-2)+(n-3)=3n-8$
Compatible flips
Thm: S convex $\implies G^{\text{comp-flip}}_{\mathcal{C}}(S)$ is connected with diameter at most $2n-5$.
Idea: Flip to a path along the convex hull.
Each edge is moved at most once and egdes on the convex hull are never moved $\implies$ we use $\leq n-3$ compatible flips!
Total compatible flips: $(n-3)+1+(n-3)=2n-5$
General position
Slides
Thm: S in general position $\implies$ If $n\leq 7$ then $G^{\text{slide}}_{\mathcal{C}}(S)$ is connected. If $n\geq 8$ then $G^{\text{slide}}_{\mathcal{C}}(S)$ can be disconnected.
Lem: S in general position and $u,v\in S\implies$ any double star with centers u and v is connected to a star at u (or v) in $G^{\text{slide}}_{\mathcal{C}}$.
Thm: S in general position $\implies G^{\text{slide}}_{\mathcal{C}}(S)$ contains a connected component of size $\geq \frac{1}{4}(3^n-1)$.
I'm in the component!
well-separated caterpillar
not a well-separated caterpillar
Rotations
Thm: S in general position $\implies G^{\text{rot}}_{\mathcal{C}}(S)$ is connected.
Idea: Induce on spine length
Shortening the spine:
Apply lemma in the bounded region!
Open problems
• Improve diameter bounds
• Is $G^{\text{emp-rot}}_{\mathcal{C}}(S)$ connected for S in general position?
• Investigate happy edge properties
• Complexity related questions
Check out the Arxiv version!
A new result about paths
Thm: S in general position $\implies$ the flip graph of plane spanning paths on S does not contain a connected component of size ≤ 7.