Defective Linear Layouts of Graphs

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Abstract

A linear layout of a graph defines a total order of the vertices and partitions the edges into either *stacks* or *queues*, i.e., crossing-free and non-nested sets of edges along the order, respectively. In this work, we study defective linear layouts that allow forbidden patterns among edges of the same set. Our focus is on k*-defective stack layouts* and k*-defective queue layouts*, in which the *conflict graph* representing the forbidden patterns among the edges of each stack or queue has maximum degree at most k.

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Category Poster Abstract

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1 Introduction

Stack [3] and queue [5] layouts are key concepts in topological graph theory, where vertices are ordered and edges are partitioned into non-crossing (stacks) or non-nested (queues) sets. The goal is to minimize the number of stacks/queues – known as the *stack/queue number*. We introduce k-defective linear layouts, a generalization of stack/queue layouts that allows some forbidden patterns (edge crossings or nestings), controlled by a conflict graph with maximum degree k. Classical layouts correspond to the case $k = 0$. Our contributions include: (i) characterizations of k-defective layouts; (ii) bounds on edge density; and (iii) bounds or exact values of k-defective numbers for specific graph families.

2 Queue Layouts

We first bound the density of graphs admitting k-defective h-queue layouts. Let $v_0 \prec \cdots \prec v_{n-1}$ be a vertex ordering of a graph G. Partition the edges into $n-1$ *classes*, where edges (v_{i_1}, v_{j_1}) and (v_{i_2}, v_{j_2}) belong to the same class if $\lfloor \frac{i_1+j_1}{2} \rfloor = \lfloor \frac{i_2+j_2}{2} \rfloor$. We denote by C_i the class whose maximum number of edges is i ($i = 1, \ldots, n-1$). Note that if C_i has i edges, it induces a path.

▶ **Lemma 1.** *Let $\mathcal{L}$ be a k-defective h-queue layout of a graph G. Every defective queue of $\mathcal{L}$ contains at most $k+2$ edges of each class.*

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▶ **Theorem 2.** *Any n-vertex graph that admits a k-defective h-queue layout has at most $h(k+2)\left(n - \frac{h(k+2)+1}{2}\right)$ edges, for $n \geq h(k+2)+1$.*

Sketch. Let $\mathcal{L}$ be a k-defective h-queue layout of a graph G with n vertices and m edges. Classes with at most $h(k+2)$ edges are *small*, otherwise are *large*. Let n_l be the number of large classes, and m_s the number of edges in small classes. We claim $m \leq n_l h(k+2) + m_s$. Suppose, for contradiction, that $m > n_l h(k+2) + m_s$. Then some large class must have more than $h(k+2)$ edges, implying a queue with more than $k+2$ edges from that class, contradicting Lemma 1. Since each class has at most i edges, we have $n_l = n-1-h(k+2)$, and $m_s \leq \sum_{i=1}^{h(k+2)} i = \frac{h(k+2)(h(k+2)+1)}{2}$. Therefore, $m \leq (n-1-h(k+2))h(k+2) + \frac{h(k+2)(h(k+2)+1)}{2} = h(k+2)\left(n - \frac{h(k+2)+1}{2}\right)$. Finally, $h(k+2)\left(n - \frac{h(k+2)+1}{2}\right) \leq \frac{n(n-1)}{2}$ for $n \geq h(k+2)+1$. ◀

We next consider defectiveness and queue number. In this content, outerplanar graphs have queue number 2 [7], but not all admit 1-queue layouts with bounded defectiveness. Outer 1-planar graphs have queue number at most 42 [2]; we show that their 1-defective queue number is 2, so their queue number is at most 3.
For general planar graphs, we can prove an upper bound of 33 on their 1-defective queue number by adapting a well-known technique by Dujmovic et al. [4].
We next establish bounds on the k-defective queue number of K_n that are tight for $k = 1$.

▶ **Theorem 3.** *The k-defective queue number of K_n is at least $\left\lceil \frac{n-1}{k+2} \right\rceil$ and at most $\lceil \frac{n-1}{l} \rceil$, where $l = \left\lfloor \frac{3+\sqrt{8k+1}}{2} \right\rfloor$.*

Sketch. For the lower bound, consider a k-defective queue layout $\mathcal{L}$ of K_n. The edges can be partitioned into classes $C_1, \ldots, C_{n-1}$. Since C_{n-1} has $n-1$ edges and at most $k+2$ edges can share a defective queue (Lemma 1), at least $\left\lceil \frac{n-1}{k+2} \right\rceil$ queues are needed. We prove the upper bound via an explicit construction. For h to be specified, assign to each defective queue q_a (for $0 \leq a \leq h$) all edges with hop-size $al + i$ for $1 \leq i \leq l$. The edges with the most nestings are the $(al+l)$-hop edges, each nesting $\frac{1}{2}(l-2)(l-1)$ edges, ensuring the layout is k-defective. To cover all $\frac{n(n-1)}{2}$ edges of K_n, note that each q_a contains $\sum_{i=1}^{l}(n-(al+i))$ edges. For $h = \lceil \frac{n-1}{l} \rceil$, the total edges assigned satisfy $\sum_{a=0}^{h-1}\sum_{i=1}^{l}(n-(al+i)) \geq \frac{n(n-1)}{2}$. ◀

We give bounds on the k-defective queue number of $K_{n,n}$, considering both the general case and the *separated setting*, where one part precedes the other [1].

▶ **Corollary 4.** *The 1-defective queue number of $K_{n,n}$ in the separated setting is $\lceil \frac{2n-1}{3} \rceil$. For $k > 1$, it ranges between $\left\lceil \frac{2n-1}{k+2} \right\rceil$ and $\lceil \frac{2n-1}{l} \rceil$ in the separated setting, while in the non-separated setting it ranges between $\left\lceil \frac{n-1}{k+2} \right\rceil$ and $\lceil \frac{2n-1}{l} \rceil$, where $l = \left\lfloor \frac{3+\sqrt{8k+1}}{2} \right\rfloor$.*

3 Stack Layouts

A graph admits a k-defective h-stack layout if and only if its edges can be partitioned into h defective stacks, each forming an outer k-planar subgraph. This yields the following characterization.

▶ **Theorem 5.** *A graph has k-defective stack number 1 if and only if it is outer k-planar.*

We leverage the characterization given above to obtain bounds on the edge density of the graphs admitting k-defective h-stack layouts.

▶ **Theorem 6** (Ábrego et al. [8], Pach et al. [6]). *Any n-vertex graph that admits a k-defective 1-stack layout has at most:* **(i)** $\frac{5}{2}n-4$ *edges, if* $k=1$, **(ii)** $3n-5$ *edges, if* $k=2$, **(iii)** $\frac{13}{4}n-\frac{11}{2}$ *edges, if* $k=3$, **(iv)** $O(\sqrt{k}n)$, *otherwise. Also, the first three bounds are tight.*

▶ **Theorem 7.** *An n-vertex graph with a 1-defective h-stack layout has at most* $\left(\frac{3}{2}h+1\right)n-4h$ *edges.*

In the following, we present bounds on the k-defective stack numbers of K_n and $K_{n,n}$. For $k=1$, our upper bounds are tight (up to small constants).

▶ **Theorem 8.** *The k-defective stack number of K_n is at least* $\left\lceil\frac{n}{2k+2}\right\rceil$ *and at most* $\left\lceil\frac{n}{l+2}\right\rceil$, *where* $l=\left\lfloor\frac{-1+\sqrt{8k+1}}{2}\right\rfloor$.

▶ **Corollary 9.** *The 1-defective stack number of $K_{n,n}$ in the separated setting is at least* $\lceil\frac{n}{2}\rceil$ *and at most* $\lceil\frac{2n}{3}\rceil$, *while in the non-seperated setting it is at least* $\lceil\frac{n}{4}\rceil$ *and at most* $\lceil\frac{n}{2}\rceil$.

▶ **Theorem 10.** *The k-defective stack number of $K_{n,n}$ in the separated setting is at least* $\left\lceil\frac{n}{2k+2}\right\rceil$ *and at most* $\lceil\frac{n}{l}\rceil$, *where* $l=\sqrt{k}+1$.

4 Open Problems

Our work raises several new open problems, which we list below.

- Extend to other linear layouts (e.g., deques, riques) and mixed settings.
- Explore defects beyond bounded conflict graph degree, such as bounding diameter.
- Research questions from our results: (i) complexity of recognizing graphs with k-defective h-queue layouts for $k,h\geq 1$, (ii) improve bounds on k-defective stack and queue numbers of K_n for $k>1$, and (iii) study other graph classes.

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Linear Layouts of Graphs

- Given a graph, order $\prec$ its vertices
- and partition its edges into pages s.t. each page has a property.

stacks
queues

In a stack the edges are crossing-free

In a queue the edges are non-nested

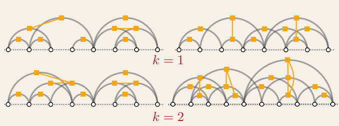
Stack/queue number = minimum required number of stacks/queues.

Conflict Graph

- Nodes $\rightarrow$ edges in the same page.
- Edges $\rightarrow$ forbidden patterns.
- Degree at most k at every page $\rightarrow$ k-defects.

k-defective stack

k-defective queue

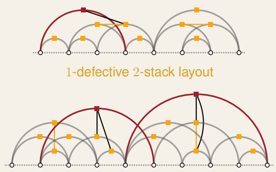


k-defective stack/queue number = minimum required number of k-defective stacks/queues.

Defective layouts and colorings

$\exists$ k-defective layout with h stacks (queues) iff $\exists \prec$ s.t. the conflict graph admits an h-coloring with monochromatic components of degree $\leq k$

1-defective 2-coloring

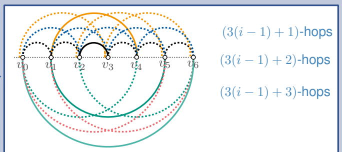


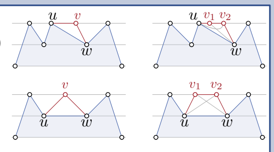
Results

k-defective queue layouts

k	h	Edge density bounds	Tight
>1	>1	$h(k+2)\left(n-\frac{h(k+2)+1}{2}\right)$	
1	>1	$3h\left(n-\frac{3h+1}{2}\right)$	✓
2	1	$\frac{10}{3}n-\frac{21+r}{3}, r = \mathrm{mod} 3$	✓
1	1	$3n-6$	✓

$h = $ #queues



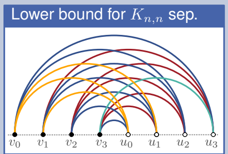


Upper Bound for $K_{n,n}$

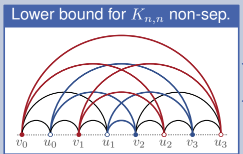
Directly from the queue number bounds of K_{2n}.

k-defective q.n.	Lower Bound	Upper Bound
Outer 1-planar $(k = 1)$	2	2
Planar $(k = 1)$	2	33
K_n	$\lceil\frac{n-1}{k+2}\rceil$	$\lceil\frac{n-1}{l_1}\rceil$ $l_1 = \left\lfloor\frac{3+\sqrt{8k+1}}{2}\right\rfloor$
$K_{n,n}$ (sep)	$\lceil\frac{2n-1}{k+2}\rceil$	$\lceil\frac{2n-1}{l_1}\rceil$
$K_{n,n}$ (non sep)	$\lceil\frac{n-1}{k+2}\rceil$	$\lceil\frac{2n-1}{l_1}\rceil$

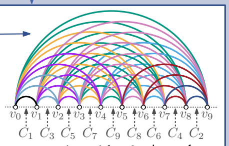
Lower bound for $K_{n,n}$ sep.

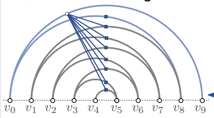


Lower bound for $K_{n,n}$ non-sep.



The edges of K_n can be partitioned into $n-1$ classes: $C_1, C_2, \ldots, C_{n-1}$. C_{n-1} has $n-1$ edges.





In a k-defective queue, at most $k+2$ edges of C_{n-1} can be in one queue. Total: $\lceil\frac{n-1}{k+2}\rceil$

k-defective stack layouts

Theorem ([1, 2])

Any n-vertex graph that admits a k-defective h-stack layout has at most

1. $(\frac{5}{2}h+1)n-4h$ edges, if $k = 1$
2. $(3h+1)n-5h$ edges, if $k = 2$
3. $(\frac{13}{4}h+1)n-\frac{11}{2}h$, if $k = 3$
4. $O(\sqrt{k}hn)$, otherwise

k	h	Edge density bounds	Tight
>1	>1	$O(\sqrt{k}hn)$	
1	>1	$\left(\frac{3}{2}h+1\right)n-4h$	
2	1	$3n-5$	✓
1	1	$\frac{5}{2}n-4$ [3, 4]	✓

$h = $ #stacks

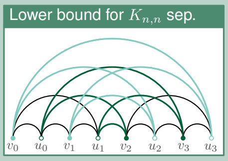
Proof

Outer 1-planar graphs have $\leq \frac{5n}{2}-4$ edges $\Rightarrow$
The 1st page has so many edges, while the rest have at most $\frac{3n}{2}-4$ (as the boundary edges are in the 1st page).
Total edges: $\frac{5n}{2}-4+(h-1)\cdot(\frac{3n}{2}-4) = (\frac{3n}{2}h+1)n-4h$

Theorem

A graph has k-defective stack number 1 iff it is outer k-planar.

Lower bound for $K_{n,n}$ sep.



Lower bound for K_n

In any vertex order of K_n, there exist $\frac{n}{2}$ pairwise crossing edges.
Thus, at most $k+1$ of them lie in the same defective stack.

k-defective s.n.	Lower Bound	Upper Bound
Outer k-planar	1	1
Planar $(k = 1)$	2	4 [5]
K_n	$\lceil\frac{n}{2k+2}\rceil$ [6]	$\lceil\frac{n}{l_2+2}\rceil$ $l_2 = \left\lfloor\frac{-1+\sqrt{8k+1}}{2}\right\rfloor$
$K_{n,n}$ (sep)	$\lceil\frac{n}{k+1}\rceil$	$\lceil\frac{2n}{l_2+2}\rceil$
$K_{n,n}$ (non sep)	$\lceil\frac{n}{2k+2}\rceil$	$\lceil\frac{n}{l_3}\rceil$ $l_3 = \sqrt{k}+1$

Problems

- Extend to other linear layouts (e.g., deques, riques) and mixed settings.
- Explore new defect types, such as bounded diameter in conflict graphs.
- Study the complexity of recognizing graphs that admit k-defective h-queue layouts.

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