Reeb Lobsters Are 1-Planar

Maarten Löffler
Department of Information and Computing Sciences, Utrecht University, The Netherlands
Miriam Münch
Faculty of Computer Science and Mathematics, University of Passau, Germany
Ignaz Rutter
Faculty of Computer Science and Mathematics, University of Passau, Germany

Abstract

Very recently, Chambers, Fasy, Hosseini Sereshgi and Löffler [3] showed that every Reeb caterpillar admits a crossing-free drawing. It turns out that this does not hold for Reeb lobsters but we show that these graphs admit drawings with at most one crossing per edge.

2012 ACM Subject Classification Mathematics of computing → Graphs and surfaces; Mathematics of computing → Topology

Keywords and phrases Reeb graphs, layered drawings, local crossing number

Digital Object Identifier 10.4230/LIPIcs.GD.2025.50

Category Poster Abstract

Funding Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 541433306.

1 Introduction

A *layered* graph (G, h) is a graph together with pairwise-distinct preassigned heights of its vertices. In a drawing of a layered graph each vertex v is mapped to a unique point $(x, h(v)) \in \mathbb{R}^2$ and every edge is mapped to a continuous y-monotone curve [1, 7]. For a vertex v in a layered graph we denote the horizontal line through v by ℓ_v. Let $\deg_\downarrow(v)$ denote the number of neighbors w of v with $h(w) < h(v)$. Similarly we let $\deg_\uparrow(v)$ denote the number of neihgbors w of v with $h(w) > h(v)$. A *Reeb graph* (G, h), introduced in [6], is a layered graph such that for every vertex v we have $\deg(v) \in \{1, 3\}$, $\deg_\downarrow(v) \leq 2$ and $\deg_\uparrow(v) \leq 2$. Reeb graphs can be computed effieciently [4, 5] and capture the evolution of level sets of functions from a topological space to the real numbers. They have many applications in topological data analysis and visualization [8, 2].

A *caterpillar* is a tree C such that removing all leaves from C yields a path, called the *spine* of C. A tree L such that removing all leaves yields a caterpillar is a *lobster*. The spine of a lobster L is the spine of the caterpillar we obtain by removing all leaves from L. The *local crossing number* $\mathrm{lcr}(G)$ of a (layered) graph G is the smallest integer k such that G admits a drawing with at most k crossings per edge. A graph G is *k-planar* if $\mathrm{lcr}(G) \leq k$.

2 Our Results

First we show that crossings are needed to draw Reeb lobsters.

▶ **Theorem 1.** *Not every Reeb lobster admits a crossing-free drawing.*

Proof. Consider the lobster G shown in Figure 1(a). Assume that G admits a crossing-free drawing Γ. Let G' be the graph we obtain from G by adding two new independent vertices x, y with $h(x) > h(v)$ and $h(y) < h(v)$ for every $v \in V$ such that x is adjacent exactly to the

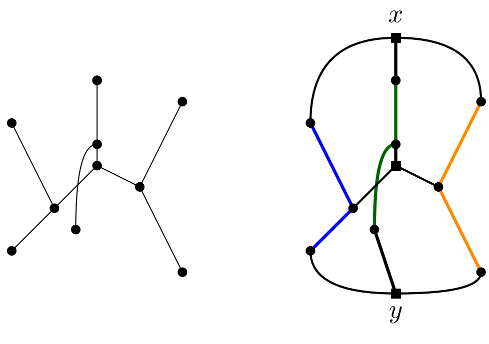


Figure 1 (a) A Reeb lobster and (b) its extension to a $K_{3,3}$-subdivision.

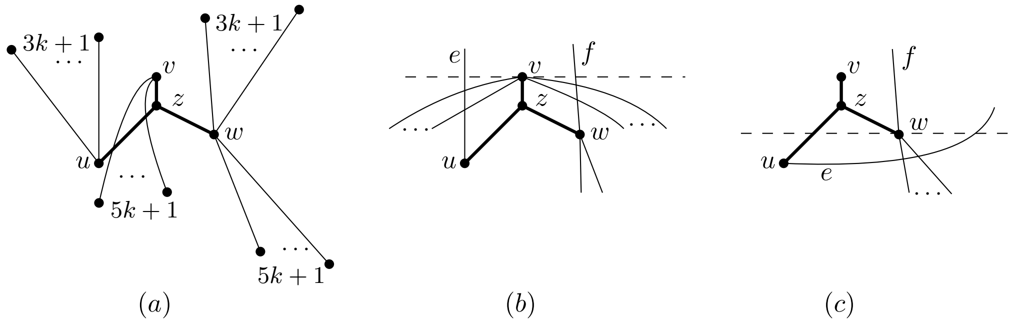


Figure 2 Illustration of the proof of Theorem 2.

three vertices with greatest heights and y is adjacent exactly to the three vertices with lowest heights; see Figure 1(b). Then it is easy to extend Γ to a crossing-free drawing of G' but G' is a $K_{3,3}$-subdivision and thus non-planar; a contradiction. ◀

Next we show that the local crossing-number of a layered lobster may be arbitrarily large.

▶ **Theorem 2.** *For every $k \in \mathbb{N}$ there is a layered lobster L such that $\mathrm{lcr}(L) > k$.*

Proof. For a fixed $k \in \mathbb{N}$ consider the lobster L sketched in Figure 2(a). We call a set F of l edges an *l-fan* if all edges in F share a common endpoint c, the remaining endpoints have degree 1 and the heights of these degree-1 endpoints lie either all above $h(c)$ or all below $h(c)$. For each of the four fans in L the heights of the corresponding degree-1 vertices are chosen such that each other vertex either lies below all of them or above all of them. Assume for the sake of contradiction that there exists a drawing Γ of L with at most k crossings per edge. Let u' be a degree-1 vertex adjacent to u, whose incident edge does not cross an edge incident to z; such a vertex exists as there are $3k+1$ degree-1 vertices adjacent to u and the edges incident to z have at most $3k$ crossings. Analogously, there exists a degree-1 vertex w' adjacent to w with $h(w') > h(w)$, whose incident edge does not cross an edge incident to z. Similarly, we can find sets V'' and W'' each consisting of $2k+1$ degree-1 vertices adjacent to v and w, respectively, that lie below w and whose incident edges do not cross an edge incident to z. Let Γ' denote the drawing induced by $\{u, u', v, w, w', z\} \cup V'' \cup W''$, where we possibly reroute zu and zw so that they do not cross each other. Note that all edges incident to z are crossing-free in Γ'. Consider the edges $e = uu'$ and $f = ww'$. Without loss of generality assume that zu crosses ℓ_w left of w. Since the path uzw is crossing-free, f crosses ℓ_z to the right of z and since zv is crossing-free, f crosses ℓ_v to the right of v. We distinguish two cases based on whether e crosses ℓ_v on the left or on the right side of v.

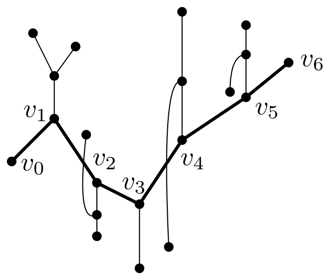


Figure 3 Illustration of the construction of a 1-planar drawing of a lobster that is a Reeb graph.

Assume that e crosses ℓ_v on the left side of v; see Figure 2(b). As v lies between e and f on ℓ_v and the edges incident to z are crossing-free in Γ', each edge connecting v to V'' crosses e or f. Since $|V''| = 2k + 1$, e or f has more than k crossings; a contradiction.

It remains to consider the case where e crosses ℓ_v on the right side of v; see Figure 2(c). Note that in this case e crosses the horizontal line ℓ_w to the right of w since the path wzv is uncrossed. Since the path zuu' starts and ends above w, crosses ℓ_w once to the left of w and once to the right of w, and zu is crossing-free, all edges connecting w to W'' cross e. Since $|W''| = 2k + 1$ it follows that e has more than k crossings; a contradiction. ◀

Finally we show that Reeb lobsters admit a drawing with at most one crossing per edge.

▶ **Theorem 3.** *Every Reeb lobster is 1-planar.*

Proof. Let L be a Reeb graph. We consider an *extended spine* of L; i.e, a path between degree-1 vertices of L that contains the spine. Let $\{v_0, \dots, v_k\}$ be an extended spine of L such that for every $0 < i < k$ the two neighbors of v_i are v_{i-1} and v_{i+1}. We start by drawing the extended spine x-monotone; see Figure 3 for an illustration. Next we insert all vertices of L that are adjacent to the extended spine crossing-free by assigning each such vertex u an x-coordinate close to the x-coordinate of its neighbor v on the extended spine. It remains to add the missing leaves of L that have a neighbor that is not adjacent to a vertex on the extended spine. Let w be such a leaf of L with neighbor u. Let v_i be the neighbor of u on the extended spine. If $h(v_i)$ does not lie between $h(w)$ and $h(u)$, then we can easily insert w crossing-free by assigning it an x-coordinate close to or equal to the x-coordinate of u. This does not result in a crossing. Hence assume that $h(w) < h(v_i) < h(u)$ or $h(w) > h(v_i) > h(u)$. Now we assign w an x-coordinate between the x-coordinates of v_i and v_{i-1}. Then wu crosses the edge v_iv_{i-1}. Hence by construction for every such leaf w, the edge wu receives at most one crossing. Recall that $\deg_\downarrow(u) \le 2$, $\deg_\uparrow(u) \le 2$. Thus every u that is neither a leaf of L nor part of the extended spine has at most one neighbor w with $h(w) < h(v_i) < h(u)$ or $h(w) > h(v_i) > h(u)$. Hence for every $0 < i < k$ also v_iv_{i-1} receives at most one crossing. Thus the construction described above yields a 1-planar drawing of L. ◀

3 Conclusion

Our results show that the property to be a Reeb graph yields an upper bound of 1 for the otherwise unbounded local crossing number of layered lobsters. An interesting question is how this generalizes to more general Reeb trees.

References

1. Oliver Bastert and Christian Matuszewski. *Layered Drawings of Digraphs*, pages 87–120. Springer Berlin Heidelberg, Berlin, Heidelberg, 2001. doi:10.1007/3-540-44969-8_5.
2. Silvia Biasotti, Daniela Giorgi, Michela Spagnuolo, and Bianca Falcidieno. Reeb graphs for shape analysis and applications. *Theoretical computer science*, 392(1-3):5–22, 2008. doi:10.1016/j.tcs.2007.10.018.
3. Erin W. Chambers, Brittany Terese Fasy, Erfan Hosseini Sereshgi, and Maarten Löffler. Drawing reeb graphs. In *Combinatorial Algorithms - 36th International Workshop, IWOCA 2025*, volume 15885 of *Lecture Notes in Computer Science*, pages 58–71. Springer, 2025. doi:10.1007/978-3-031-98740-3_5.
4. William Harvey, Yusu Wang, and Rephael Wenger. A randomized $O(m \log m)$ time algorithm for computing Reeb graphs of arbitrary simplicial complexes. In *Proceedings of the 2010 annual symposium on Computational geometry*, SoCG '10, pages 267–276. ACM, 2010. doi:10.1145/1810959.1811005.
5. Salman Parsa. A deterministic O (m log m) time algorithm for the reeb graph. *Discret. Comput. Geom.*, 49(4):864–878, 2013. doi:10.1007/s00454-013-9511-3.
6. Georges Reeb. Sur les points singuliers d'une forme de pfaff completement integrable ou d'une fonction numerique [on the singular points of a completely integrable pfaff form or of a numerical function]. *Comptes Rendus Acad. Sciences Paris*, 222:847–849, 1946. URL: https://cir.nii.ac.jp/crid/1571417125676878592.
7. Kozo Sugiyama, Shôjirô Tagawa, and Mitsuhiko Toda. Methods for visual understanding of hierarchical system structures. *IEEE Transactions on Systems, Man, and Cybernetics, SMC*, 11(2):109–125, 1981. doi:10.1109/TSMC.1981.4308636.
8. Lin Yan, Talha Bin Masood, Raghavendra Sridharamurthy, Farhan Rasheed, Vijay Natarajan, Ingrid Hotz, and Bei Wang. Scalar field comparison with topological descriptors: Properties and applications for scientific visualization. In *Computer Graphics Forum*, volume 40, pages 599–633. Wiley Online Library, 2021. doi:10.1111/cgf.14331.

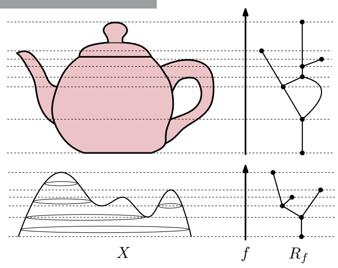
Reeb Lobsters are 1-Planar

Maarten Löffler[1] • Miriam Münch[2] • Ignaz Rutter[2]

[1] Utrecht University [2] Universität Passau, Fakultät für Informatik und Mathematik

Reeb Graphs

- Given: topological space X, function $f : X \to \mathbb{R}$
- for two points $x, y \in X$ we have $x \sim y$ iff
  - $f(x) = f(y) = a$ and
  - x, y are in the same path-connected component of $f^{-1}(a)$
- Reeb graph R_f is the quotient space $X/\sim$



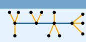
- capture evolution of level sets of functions from a topological space to the real numbers
- applications in topological data analysis and visualization (e.g. [2, 3])

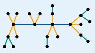
Properties of Reeb Graphs

Layered graph with height function $h : V \to \mathbb{R}$ s.t. for each $v \in V$

- $h(v) \neq h(w)$ for any $w \neq v$
- $\deg(v) \in \{1, 3\}$
- downward degree $\deg_\downarrow(v) \leq 2$, upward degree $\deg_\uparrow(v) \leq 2$

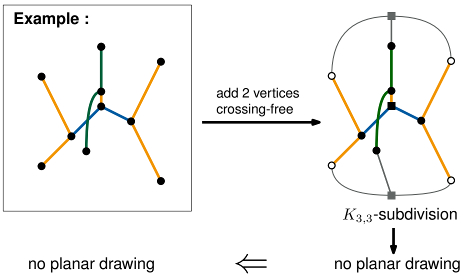
Question: How to draw Reeb graphs with few crossings?

Known: Reeb caterpillars can be drawn crossing-free (Chambers, Fasy, Sereshgi, Löffler '25 [1]) 

What about lobsters? 

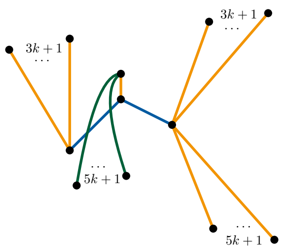
Theorem 1

Not every Reeb lobster admits a planar drawing.



Theorem 2

For every $k \in \mathbb{N}$ there is a layered lobster that is not k-planar.



Theorem 3

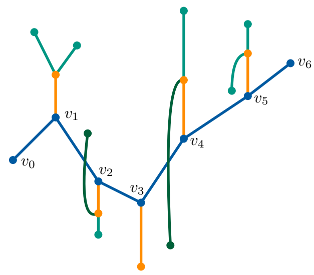
Every Reeb lobster is 1-planar.

Construction :

- Draw **extended spine** x-monotone (path between degree-1 vertices that contains the spine)
- Add **caterpillar leaves** crossing-free
- Add **lobster leaves** as follows:
  Let w be a lobster leaf with neighbor u and let v_i be the spine-neighbor of u.
  - **Case 1:** $h(v_i) = \max(h(u), h(w), h(v_i))$ or $h(v_i) = \min(h(u), h(w), h(v_i))$
    $\to$ add w crossing-free
  - **Case 2:** $h(u) < h(v_i) < h(u)$ or $h(u) < h(v_i) < h(w)$
    $\to$ add w s.t. wu crosses $v_{i-1}v_i$

By construction: wu crossed at most once

$\deg_\downarrow(u) \leq 2, \deg_\uparrow(u) \leq 2 \Rightarrow v_{i-1}v_i$ crossed at most once



Conclusion

First step towards efficient crossing minimization for Reeb graphs $\to$ How do our results generalize to more general Reeb trees?

1. E. W. Chambers, B. T. Fasy, E. Hosseini Sereshgi, and M. Löffler. Drawing Reeb Graphs. *CoRR*, abs/2504.21329, 2025.
2. S. Biasotti, D. Giorgi, M. Spagnuolo, and B. Falcidieno. Reeb graphs for shape analysis and applications. *Theoretical computer science*, 392(1-3):5-22, 2008.
3. L. Yan, T. B. Masood, R. Sridharamurthy, F. Rasheed, V. Natarajan, I. Hotz, and B. Wang. Scalar field comparison with topological descriptors: Properties and applications for scientific visualization. *Computer Graphics Forum* , 40(3):599-633, 2021.