# Graph Tiles

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## Abstract

We define a graph tile to be a unit square (or more generally, a polygon) on which a piece of a graph has been drawn/embedded; in particular, it may have vertices in its interior, edges connecting those vertices, or half-edges that extend to the boundary of the tile. In a graph tiling problem, we are given as input a set of graph tiles, with multiplicities, and the output is an arrangement of those tiles forming a graph of larger area. We focus on a simple tile set: unit square tiles with a central vertex and either a half-edge or no half-edge on each side. Up to symmetry this gives us six different types. We characterize which multiplicities are compatible for sets of at most three different tiles.

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## 1 Introduction

**Motivation from material engineering.** Irregular architected materials have outstanding mechanical properties, like high stiffness-to-weight ratios and energy absorption, but their rational design is still an open challenge. The recent development of computer-aided virtual growth algorithms (VGAs) has emerged as a new promising venue that allows the generation of architected materials with tailorable mechanical properties [3]. VGAs rely on a specific set of simplified tiles that are assembled on a predefined grid, following a precise set of connectivity rules. Depending on the relative concentration of different tiles, different materials architectures and patterns can be generated, leading in turn to completely different mechanical (and functional) material properties [2, 4].

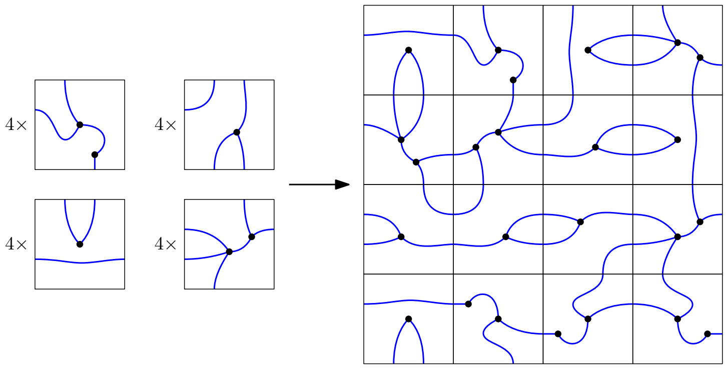


Figure 1 A graph tiling problem and possible solution.

**The graph tiling problem.** Underlying this engineering challenge lies a mathematical problem. We define a *graph tile* to be a unit square (or more generally, a polygon) on which a piece of a graph has been drawn/embedded; in particular, it may have vertices in its interior, edges connecting those vertices, or half-edges that extend to the boundary of the tile. In a *graph tiling* problem, we are given as input a set of graph tiles, with *multiplicities*, and the output is an arrangement of those tiles into a larger area. Figure 1 shows an example. We can then ask for the output graph to satisfy certain properties, or to optimize over certain criteria, e.g. connectedness, shortest paths, stretch factor, tortuosity, etc. In order to effectively inform VGAs, it is necessary to understand the relation between tile sets and the achievable graph properties in a resulting graph tiling.

**Contribution.** In this work, we initiate the fundamental study into this area by considering one of the simplest possible sets of tiles: square tiles, with a single vertex in the center, and which do or do not have a half-edge extending to each of the four sides. Taking symmetries into account, this gives us six distinct tiles, see Figure 2. These tiles have been previously identified as the simplest set which is rich enough to encode meaningful material design [4].

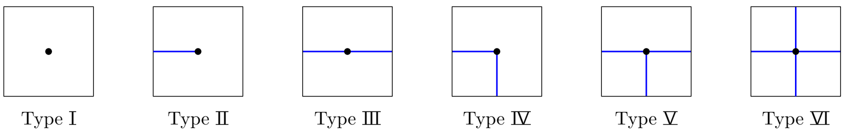


Figure 2 The six different types of tiles considered in this work.

An instance can then be characterized by six integers, the multiplicities of each tile, and a desired output region. Already determining whether the tiles can be arranged with their sides matching is a non-trivial problem, and is related to well-studied topics such as Wang tiles [5], tile self-assembly [1], and more. However, those problems typically do not specify multiplicities of the available tiles, and understanding which multiplicities are compatible is a necessary first step towards solving graph tiling problems. In this work, we characterize which multiplicities are compatible for all sets of at most three different tiles. We restrict our attention to instances in which the total number of tiles is a square number $n^2$, that need to be assembled into an $n \times n$ grid, and assume input integers are encoded in binary.

**Table 1** Results for all sets of at most three distinct tiles. The class with the value "?" contains problems that are compatible and those that are not compatible and we conjecture that there exists a polynomial-time algorithm to distinguish them, but this is still open.

| | | | | | |
|---|---|---|---|---|---|
| | | | | ⟨I, II, III⟩≙⟨III, V, VI⟩ | YES |
| | | ⟨I, II⟩≙⟨V, VI⟩ | YES | ⟨I, II, IV⟩≙⟨IV, V, VI⟩ | YES |
| | | ⟨I, III⟩≙⟨III, VI⟩ | P | ⟨I, II, V⟩≙⟨II, V, VI⟩ | YES |
| | | ⟨I, IV⟩≙⟨IV, VI⟩ | YES | ⟨I, II, VI⟩≙⟨I, V, VI⟩ | ? |
| ⟨I⟩≙⟨VI⟩ | YES | ⟨I, V⟩≙⟨II, VI⟩ | P | ⟨I, III, IV⟩≙⟨III, IV, VI⟩ | YES |
| ⟨II⟩≙⟨V⟩ | YES | ⟨I, VI⟩ | NO | ⟨I, III, V⟩≙⟨II, III, VI⟩ | P |
| ⟨III⟩ | YES | ⟨II, III⟩≙⟨III, V⟩ | YES | ⟨I, III, VI⟩ | P |
| ⟨IV⟩ | YES | ⟨II, IV⟩≙⟨IV, V⟩ | YES | ⟨I, IV, V⟩≙⟨II, IV, VI⟩ | YES |
| | | ⟨II, V⟩ | YES | ⟨I, IV, VI⟩ | P |
| | | ⟨III, IV⟩ | YES | ⟨II, III, IV⟩≙⟨III, IV, V⟩ | YES |
| | | | | ⟨II, III, V⟩ | YES |
| | | | | ⟨II, IV, V⟩ | YES |

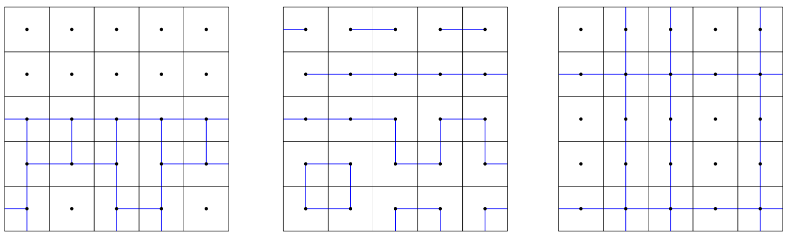

**Figure 3** Example tilings for classes ⟨I, V⟩, ⟨II, III, IV⟩, and ⟨I, III, VI⟩.

## 2 Results

We consider the setting where we need to fill an $n \times n$ grid with pre-specified numbers of tiles of each type. We will use the notation I, II, . . . , VI to refer to the tiles in Figure 2. Note that, apart from rotational symmetry, the tiles also have another type of symmetry: if we replace all sides with half-edges by sides without half-edges and vice versa, we arrive at an equivalent *dual* problem (for the purpose of testing compatibility). We will write I≙VI, II≙V, III≙III, and IV≙IV to denote this duality.

We will use the notation $\langle x_1, x_2, \ldots, x_k\rangle$ for the problem class of determining whether a nonzero number of tiles of each type $x_1$, $x_2$, to $x_k$ (for $1 \leq k \leq 6$) can be assembled into an $n \times n$ square grid. Each problem class may have value YES (if all problems in the class are compatible), NO (if none of the problems in the class are compatible), P (if compatible and incompatible problems both exist in the class, but it is possible to distinguish them in polynomial time), or NP (if it is NP-hard to distinguish them). Our results are summarized in Table 1. Some interesting cases can be seen in Figure 3.

## 3 Future Work

We have characterized which multiplicities of at most three tile types are compatible. The clear next step is to extend this classification to classes with up to six different tile types.

Once this is understood, we can begin to investigate graph properties that may be achieved for specific instances, such as obtaining a connected graph, or a tree, or minimizing the area of the largest face / maximizing the area of the smallest face, etc.

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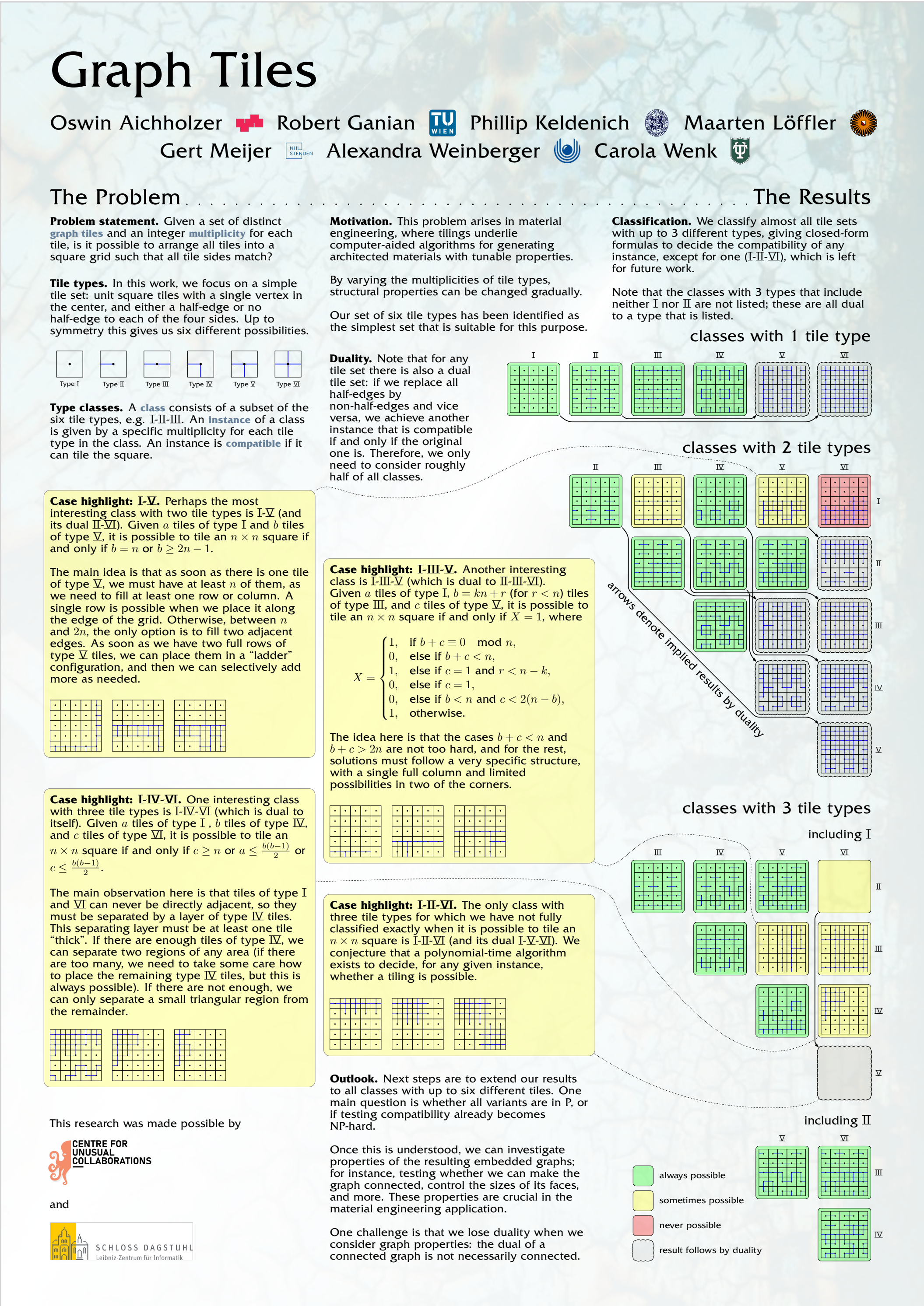
Graph Tiles
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The Problem
The Results
Problem statement. Given a set of distinct graph tiles and an integer multiplicity for each tile, is it possible to arrange all tiles into a square grid such that all tile sides match?
Tile types. In this work, we focus on a simple tile set: unit square tiles with a single vertex in the center, and either a half-edge or no half-edge to each of the four sides. Up to symmetry this gives us six different possibilities.
Type I
Type II
Type III
Type IV
Type V
Type VI
Type classes. A class consists of a subset of the six tile types, e.g. I-II-III. An instance of a class is given by a specific multiplicity for each tile type in the class. An instance is compatible if it can tile the square.
Motivation. This problem arises in material engineering, where tilings underlie computer-aided algorithms for generating architected materials with tunable properties.
By varying the multiplicities of tile types, structural properties can be changed gradually.
Our set of six tile types has been identified as the simplest set that is suitable for this purpose.
Duality. Note that for any tile set there is also a dual tile set: if we replace all half-edges by non-half-edges and vice versa, we achieve another instance that is compatible if and only if the original one is. Therefore, we only need to consider roughly half of all classes.
Classification. We classify almost all tile sets with up to 3 different types, giving closed-form formulas to decide the compatibility of any instance, except for one (I-II-VI), which is left for future work.
Note that the classes with 3 types that include neither I nor II are not listed; these are all dual to a type that is listed.
classes with 1 tile type
I II III IV V VI
classes with 2 tile types
II III IV V VI
I
II
III
IV
V
arrows denote implied results by duality
Case highlight: I-V. Perhaps the most interesting class with two tile types is I-V (and its dual II-VI). Given $a$ tiles of type I and $b$ tiles of type V, it is possible to tile an $n \times n$ square if and only if $b = n$ or $b \geq 2n - 1$.
The main idea is that as soon as there is one tile of type V, we must have at least $n$ of them, as we need to fill at least one row or column. A single row is possible when we place it along the edge of the grid. Otherwise, between $n$ and $2n$, the only option is to fill two adjacent edges. As soon as we have two full rows of type V tiles, we can place them in a "ladder" configuration, and then we can selectively add more as needed.
Case highlight: I-III-V. Another interesting class is I-III-V (which is dual to II-III-VI). Given $a$ tiles of type I, $b = kn + r$ (for $r < n$) tiles of type III, and $c$ tiles of type V, it is possible to tile an $n \times n$ square if and only if $X = 1$, where
$$X = \begin{cases} 1, & \text{if } b + c \equiv 0 \mod n, \\ 0, & \text{else if } b + c < n, \\ 1, & \text{else if } c = 1 \text{ and } r < n - k, \\ 0, & \text{else if } c = 1, \\ 0, & \text{else if } b < n \text{ and } c < 2(n - b), \\ 1, & \text{otherwise.} \end{cases}$$
The idea here is that the cases $b + c < n$ and $b + c > 2n$ are not too hard, and for the rest, solutions must follow a very specific structure, with a single full column and limited possibilities in two of the corners.
Case highlight: I-IV-VI. One interesting class with three tile types is I-IV-VI (which is dual to itself). Given $a$ tiles of type I , $b$ tiles of type IV, and $c$ tiles of type VI, it is possible to tile an $n \times n$ square if and only if $c \geq n$ or $a \leq \frac{b(b-1)}{2}$ or $c \leq \frac{b(b-1)}{2}$.
The main observation here is that tiles of type I and VI can never be directly adjacent, so they must be separated by a layer of type IV tiles. This separating layer must be at least one tile "thick". If there are enough tiles of type IV, we can separate two regions of any area (if there are too many, we need to take some care how to place the remaining type IV tiles, but this is always possible). If there are not enough, we can only separate a small triangular region from the remainder.
Case highlight: I-II-VI. The only class with three tile types for which we have not fully classified exactly when it is possible to tile an $n \times n$ square is I-II-VI (and its dual I-V-VI). We conjecture that a polynomial-time algorithm exists to decide, for any given instance, whether a tiling is possible.
classes with 3 tile types
including I
III IV V VI
II
III
IV
V
including II
V VI
III
IV
Outlook. Next steps are to extend our results to all classes with up to six different tiles. One main question is whether all variants are in P, or if testing compatibility already becomes NP-hard.
Once this is understood, we can investigate properties of the resulting embedded graphs; for instance, testing whether we can make the graph connected, control the sizes of its faces, and more. These properties are crucial in the material engineering application.
One challenge is that we lose duality when we consider graph properties: the dual of a connected graph is not necessarily connected.
always possible
sometimes possible
never possible
result follows by duality
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