Approximating Barnette's Conjecture

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Abstract

A well-known conjecture, named after David W. Barnette, asserts that every 3-regular, 3-connected, bipartite, planar graph (for short, Barnette graph) is Hamiltonian. As another step towards addressing Barnette's conjecture positively, we show that every n-vertex Barnette graph admits a subhamiltonian cycle containing $\frac{5n}{6}$ edges, improving upon the previous bound of $\frac{2n}{3}$. Equivalently, every Barnette graph admits a 2-page book embedding in which at least $\frac{5n}{6}$ consecutive vertex pairs along the spine are connected by edges. As a byproduct, we present a simple proof for a known result that guarantees the existence of Hamiltonian cycles in a certain subclass of Barnette graphs.

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1 Introduction

Barnette's conjecture [3] is an intriguing open problem in topological graph theory; it asserts that every 3-regular, 3-connected, bipartite, planar graph (for short, Barnette graph) is Hamiltonian, i.e., it contains a simple cycle visiting each vertex exactly once. The conjecture is inspired by two earlier conjectures by Tait [12] and Tutte [13], both of which were disproven.

While Barnette's conjecture [3] remains open, several related results have been established. For instance, Goodey [7] showed that Barnette graphs with faces of degree 4 and 6 are Hamiltonian. Feder and Subi [5] extended this to Barnette graphs whose faces are 3-colored such that two colors contain only degree-4 and degree-6 faces. The authors in [2] proved Hamiltonicity under certain (improper) 2-colorings of the dual faces. Alternative forms of the conjecture have also been studied. Kelmans [11] showed that Barnette's conjecture is equivalent to the existence of a Hamiltonian cycle containing exactly one of any pair of edges on a common face. Other forms of equivalency include the existence of a Hamiltonian cycle through (i) any edge [8], (ii) any path of length 3 with two consecutive edges on the boundary of a face [9], and (iii) the middle edge of any path of length 3 avoiding its outer edges [9].

Using a computer-assisted method, Brinkmann, Goedgebeur and McKay [4] verified that all Barnette graphs with up to 90 vertices are Hamiltonian. Their proof relies on the fact that all Barnette graphs can be derived from the cube graph via cube- and C_4-expansions (see Figure 1) [10]. As the former operation preserves Hamiltonicity, the main challenge in Barnette's conjecture lies in handling the latter; see a related open problem at Section 5.

Our contribution. Since Hamiltonicity in a planar graph is equivalent to admitting a 2-page book embedding with all consecutive vertices (including the first and the last) connected, studying subhamiltonian cycles (that is, linear vertex orders in such embeddings) offers a

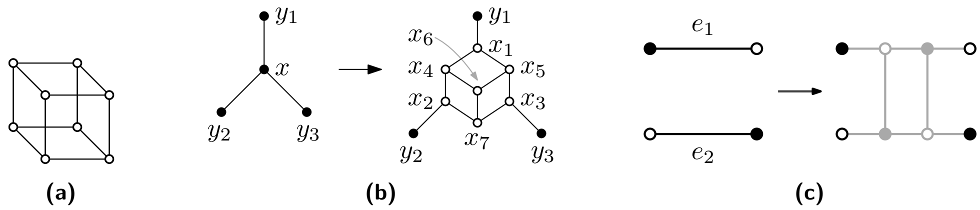

Figure 1 Starting from (a) the cube graph all Barnette graphs can be generated with either (b) cube-expansions or (c) C_4-expansions.

relaxed yet meaningful perspective. In this context, it is known that every n-vertex Barnette graph admits a subhamiltonian cycle containing at least $\frac{2n}{3}$ edges [1]. Building on this, we revisit and refine the approach of [1], improve the bound to $\frac{5n}{6}$, and discuss its potentials as a step toward resolving Barnette's conjecture. Note that a bound of n settles the conjecture.

2 Preliminaries

Basic definitions. We assume familiarity with basic concepts from Graph Theory and Graph Drawing. A k-page book embedding of a graph defines an order of its vertices and partitions its edges into k sets of non-crossing edges. A graph admitting a 2-page book embedding is called *subhamiltonian*, as it can be augmented to Hamiltonian by adding edges between consecutive vertices. The vertex order of such an embedding is called a *subhamiltonian cycle*.

Constructing Barnette graphs. As mentioned, all Barnette graphs can be constructed by iteratively applying two operations, cube-expansions and C_4-expansions, starting from the cube graph (see Figure 1). In a cube-expansion, a vertex x is replaced by seven new vertices $x_1, \dots, x_7$ and the following edges: (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_1, x_4), (x_1, x_5), (x_4, x_6), (x_5, x_6), (x_2, x_4), (x_2, x_7), (x_3, x_7), (x_6, x_7), and (x_3, x_5), where y_1, y_2, and y_3 are the neighbors of x. In a C_4-expansion, two edges e_1 and e_2 with an odd distance along the boundary of a face are each subdivided twice, and the resulting new vertices are connected with two additional edges by preserving both planarity and bipartiteness.

The algorithm by Alam et al. [1] Let G be an embedded Barnette graph with n vertices. By Lemma 2 of [1], the edges and the faces of G can be colored with three colors (red, blue and green) with the following properties (refer to Fig. 3(a) of [1] for an illustration): **(i)** every facial cycle of G is *bichromatic*, i.e., its edges alternate between two colors along its boundary, **(ii)** no two neighboring faces are of the same color, **(iii)** every face of G is colored differently from its bounding edges, **(iv)** the edge that bounds two faces of G colored x and y is of color z, where $\{x, y, z\} = \{\text{red}, \text{blue}, \text{green}\}$, **(v)** the subgraph G^*_{bg} of the dual G^* of G induced by the blue and green faces of G is connected. Let $\mathcal{T}^*_{bg}$ be a spanning tree of G^*_{bg}. This tree together with property (iv) yields a partition of the red edges of G into those that *are crossed by* $\mathcal{T}^*_{bg}$ (corresponding to those belonging to $\mathcal{T}^*_{bg}$) and the remaining ones that *are not crossed by* $\mathcal{T}^*_{bg}$. Then, the algorithm by Alam et al. [1] computes a subhamiltonian cycle C of G by a traversal of $\mathcal{T}^*_{bg}$, assuming that $\mathcal{T}^*_{bg}$ is rooted at an arbitrary blue leaf ρ as follows. For an arbitrary edge (u, v) of G that is crossed by $\mathcal{T}^*_{bg}$, let p and q be the faces that are bounded by (u, v), that is, (p, q) is an edge of $\mathcal{T}^*_{bg}$. The removal of (p, q) from $\mathcal{T}^*_{bg}$ results in two trees $\mathcal{T}^*_p$ and $\mathcal{T}^*_q$, such that w.l.o.g. ρ belongs to $\mathcal{T}^*_p$. For the recursive step, it is assumed that a *simple* and *plane* subhamiltonian cycle C_p for the subgraph G_p of G induced

(a) p is green, q blue **(b)** p is blue, q is green

Figure 2 The solid (dotted) gray edges belong to $\mathcal{T}^*_{bg}$ ($G^*_{bg} \setminus \mathcal{T}^*_{bg}$). The bold (plain) red edges are (are not, respectively) crossed by $\mathcal{T}^*_{bg}$. Cycle C_q is drawn dotted black. Figure reproduced from [1].

by the vertices of the faces of G in $\mathcal{T}^*_p$ has been computed, which does not necessarily span all vertices of G_p, but it satisfies the following invariant properties:

I.1 The edge (u, v) is on C_p.

I.2 Every red edge of G_p is in the interior of C_p or on C_p.

I.3 Every blue edge of G_p is in the exterior of C_p or on C_p.

I.4 Every green edge of G_p is on C_p.

I.5 For every red edge e of G_p that is not crossed by $\mathcal{T}^*_{bg}$ and that is adjacent to two faces h and h' of G, with $h \in \mathcal{T}^*_p$ and $h' \notin \mathcal{T}^*_p$, one of the following holds:
  - **a.** if h is a blue face, then both endpoints of e are on C_p,
  - **b.** if h is a green face, then none of the endpoints of e is on C_p.

Depending on the colors of faces p and q, the traversal of $\mathcal{T}^*_{bg}$ is continued by extending the cycle C_p to vertices of face q (thus obtaining a new cycle $\mathcal{C}_q$) and by maintaining Invariants I.1-I.5 as illustrated in Figure 2. Once the traversal of $\mathcal{T}^*_{bg}$ has been completed, a simple and plane cycle $\mathcal{C}$ has been computed, in which, by Invariants I.2-I.4, every green edge of G is on $\mathcal{C}$, while every red (blue) edge of G is in the interior (exterior) of cycle $\mathcal{C}$ or on $\mathcal{C}$. Since $\mathcal{T}^*_{bg}$ is a spanning tree of G^*_{bg}, every green edge of G bounds a face that is in $\mathcal{T}^*_{bg}$, and by Invariant I.4 both its endpoints are consecutive along $\mathcal{C}$. As every vertex is incident to a green edge, it follow that cycle $\mathcal{C}$ is indeed subhamiltonian. The following properties directly follow from Invariants I.4 and I.5, respectively (see [1]).

▶ **Property 1** ([1]). *Every green edge of G belongs to the subhamiltonian cycle $\mathcal{C}$.*

▶ **Property 2** ([1]). *Every red edge of G that is not crossed by $\mathcal{T}^*_{bg}$ belongs to the subhamiltonian cycle $\mathcal{C}$.*

The authors in [1] used Properties 1 and 2 to show that every Barnette graph admits a subhamiltonian cycle with $\frac{2n}{3}$ edges.

3 Our Main Result

To prove our main theorem, i.e., that every Barnette graph admits a subhamiltonian cycle containing at least $\frac{5n}{6}$ edges, we first establish a few more properties of the algorithm by Alam et al. [1]. To this end, let G be a Barnette graph and let f be a face of it. By Property (i) of the edge coloring of G, the edges on the boundary of f alternate between two colors. We say that two edges of the same color c are *consecutive* along f if and only if they appear consecutively in a clockwise traversal of the edges of color c in f.

▶ **Property 3.** *A blue edge appearing between two consecutive red edges of a green face of G that are crossed by $\mathcal{T}^*_{bg}$ belongs to the subhamiltonian cycle $\mathcal{C}$.*

Proof. Let f_g be a green face of G which contains a blue edge e_b appearing between two consecutive red edges e_r and e'_r of f_g that are crossed by $\mathcal{T}^*_{bg}$. Since e_r and e'_r are crossed by $\mathcal{T}^*_{bg}$, after the algorithm processes face f_g in the traversal of $\mathcal{T}^*_{bg}$, by Invariant I.1 both edges e_r and e'_r are on the subhamiltonian cycle constructed so far. By Invariant I.3, e_b is either in the exterior or on the subhamiltonian cycle constructed so far. Since e_b appears between e_r and e'_r along f_g, it follows that e_b cannot be in its exterior. Since the subhamiltonian cycle is not extended along blue edges (as those are not crossed by $\mathcal{T}^*_{bg}$), e_b will be part of $\mathcal{C}$. ◀

▶ **Property 4.** *Every green face of G that corresponds to a leaf of $\mathcal{T}^*_{bg}$ contributes a red edge (that is crossed by $\mathcal{T}^*_{bg}$) to the subhamiltonian cycle $\mathcal{C}$.*

Proof. Let f_g be a green face of G that corresponds to a leaf of $\mathcal{T}^*_{bg}$ and let f_b be its parent in $\mathcal{T}^*_{bg}$. Just before the algorithm processes face f_g in the traversal of $\mathcal{T}^*_{bg}$, the red edge of G that is crossed by the edge (f_g, f_b) of $\mathcal{T}^*_{bg}$ necessarily belongs to the subhamiltonian cycle constructed so far by Invariant I.1. Since f_g is a leaf of $\mathcal{T}^*_{bg}$ and since f_g is green, the subhamiltonian cycle will not be extended any further after the algorithm processes face f_g by Invariant I.5b. Hence, the red edge of G that is crossed by the edge (f_g, f_b) of $\mathcal{T}^*_{bg}$ will remain on the subhamiltonian cycle as desired. ◀

▶ **Property 5.** *Every green 4-face of G that does not correspond to a leaf of $\mathcal{T}^*_{bg}$ contributes two blue edges to the subhamiltonian cycle $\mathcal{C}$.*

Proof. Let f_g be a green 4-face (i.e., of degree 4) of G and assume that f_g does not correspond to a leaf of $\mathcal{T}^*_{bg}$. Since f_g is a 4-face of G, it consists of two red and two blue edges that alternate along its boundary. Since f_g does not correspond to a leaf of $\mathcal{T}^*_{bg}$, both red edges are crossed by $\mathcal{T}^*_{bg}$. Now the statement directly follows from Property 3. ◀

▶ **Property 6.** *Every green 6-face of G that does not correspond to a leaf of $\mathcal{T}^*_{bg}$ contributes at least one blue edge to the subhamiltonian cycle $\mathcal{C}$.*

Proof. Let f_g be a green 6-face of G that does not correspond to a leaf of $\mathcal{T}^*_{bg}$. Hence, its boundary contains at least two red edges that are crossed by $\mathcal{T}^*_{bg}$. These two red edges are consecutive on the boundary of f_g and thus the result follows by Property 3. ◀

So far, $\mathcal{T}^*_{bg}$ has only been assumed to be a spanning tree of G^*_{bg}. In the following, we show that it can be chosen in such a way that the preceding property generalizes to all green faces.

▶ **Property 7.** *There exists a choice of the tree $\mathcal{T}^*_{bg}$ such that every green face of G that does not correspond to a leaf of $\mathcal{T}^*_{bg}$ contributes at least one blue edge to the subhamiltonian cycle $\mathcal{C}$.*

Proof. Let f_g be a green face of G that is not a leaf in $\mathcal{T}^*_{bg}$. Since f_g is not a leaf, its boundary contains at least two red edges that are crossed by $\mathcal{T}^*_{bg}$. Our goal is to locally modify $\mathcal{T}^*_{bg}$ so that these two red edges appear consecutively along the boundary of f_g; Once this goal is achieved, we can apply Property 3 to obtain the desired result. Denote by $r_1, \ldots, r_k$ the red edges on the boundary of f_g in clockwise order. Let also $e_1, \ldots, e_k$ be the corresponding dual edges of G^*_{bg}. Finally, denote by v_g the vertex of G^*_{bg} corresponding to f_g. W.l.o.g. assume that r_1 is crossed by $\mathcal{T}^*_{bg}$, that is, e_1 belongs to $\mathcal{T}^*_{bg}$. Let r_j be the edge on the boundary of f that is crossed by $\mathcal{T}^*_{bg}$, such that $j > 1$ and j is minimum. Note that r_j exists, because f_g is not a leaf in $\mathcal{T}^*_{bg}$. If $j = 2$ or $j = k$ holds, then r_1 and r_2 are consecutive and we are done. Thus, assume $2 < j < k$. Consider the graph $\mathcal{T}^*_{bg} + e_2$. Since $\mathcal{T}^*_{bg}$ is spanning, $\mathcal{T}^*_{bg} + e_2$ contains a cycle. Let e be the edge of this cycle (distinct from e_2) that is also incident to v_g.

If $e \neq e_1$, then replacing e with e_2 in $\mathcal{T}^*_{bg}$ yields a new spanning tree in which e_1 and e_2 are consecutive, as desired. Hence, suppose $e = e_1$. In this case, we obtain a new spanning tree by replacing e_1 with e_2 in $\mathcal{T}^*_{bg}$ and we observe that the distance between the relevant pair of red edges crossed by this tree has been decreased. Repeating this argument a finite number of times will eventually yield a configuration in which the two red edges are consecutive, completing the proof. ◀

We are now ready to prove the main theorem of this section.

▶ **Theorem 1.** *Every n-vertex Barnette graph admits a subhamiltonian cycle with at least $\frac{5n}{6}$ edges.*

Proof. Let G be an n-vertex Barnette graph and let F_g, F_r and F_b be the set of green, red and blue faces of G, respectively. Since G is 3-regular, Euler's formula directly yields that G has (i) $\frac{3n}{2}$ edges out of which $\frac{n}{2}$ are green, $\frac{n}{2}$ are red and $\frac{n}{2}$ are blue, and (ii) $\frac{n}{2}+2$ faces. Hence, $|F_g|+|F_r|+|F_b| = \frac{n}{2}+2$. Assume w.l.o.g. that $|F_g| \geq |F_r| \geq |F_b|$. By this assumption, F_g contains at least $\frac{1}{3}(\frac{n}{2}+2)$ faces. Thus, for some $x > 0$, we may write:

$$|F_g| = \frac{n}{6} + x \tag{1}$$

Since $|F_g| \geq |F_r| \geq |F_b|$, it follows:

$$|F_r| \geq \frac{n}{6} - \frac{x}{2} + 1 \tag{2}$$

We next claim that the average degree of the faces in F_g is at most 6. To see this, observe that edges on the boundary of the faces of F_g are in total n, since by Properties (iii) and (iv) of the edge coloring of G, they correspond to the red and the blue edges of G. Thus, by Eq.(1) the average degree of the faces in F_g is $\frac{n}{\frac{n}{6}+x}$, which is at most 6 as claimed. It follows that for every face in F_g of degree greater than 6, there is at least one face in F_g which has degree 4. With these observations in mind, we are now ready to count the number of edges contained in the subhamiltonian cycle $\mathcal{C}$.

- By Property 1, subhamiltonian cycle $\mathcal{C}$ contains all green edges of G, which are $\frac{n}{2}$ in total.
- By Property 2, subhamiltonian cycle $\mathcal{C}$ contains all red edges of G not crossed by $\mathcal{T}^*_{bg}$. We claim that these are at least $\frac{n}{6} - \frac{x}{2}$. To see this, recall that all red edges of G are $\frac{n}{2}$ in total. Since $\mathcal{T}^*_{bg}$ is a spanning tree of $\mathcal{G}^*_{bg}$, the number of red edges crossed by $\mathcal{T}^*_{bg}$ is:

  $$|F_b| + |F_g| - 1 = \frac{n}{2} + 2 - |F_r| - 1 = \frac{n}{2} - |F_r| + 1.$$

  Therefore, the number of red edges of G which are not crossed by $\mathcal{T}^*_{bg}$ is:

  $$\frac{n}{2} - (\frac{n}{2} - |F_r| + 1) = |F_r| - 1 \overset{(2)}{\geq} \frac{n}{6} - \frac{x}{2}$$

- Each green face of G contributes an additional edge to $\mathcal{C}$. To see this, observe that a green face corresponding to leaf in $\mathcal{T}^*_{bg}$ contributes at least one (red) edge to $\mathcal{C}$ by Property 4. Each of the remaining faces contributes at least one (blue) edge by Properties 5 and 7.

From the discussion above, it follows that the number of edges belonging to the subhamiltonian cycle $\mathcal{C}$ is at least

$$\frac{n}{2} + \frac{n}{6} - \frac{x}{2} + \frac{n}{6} + x = \frac{5n}{6} + \frac{x}{2}$$

which concludes the proof. ◀

4 Implications for Restricted Subclasses

In this section, we focus on certain restricted subclasses of Barnette graphs for which the algorithm yields a Hamiltonian cycle.

▶ **Observation 2.** *Let G be a Barnette graph. If $\mathcal{T}^*_{bg}$ is such that any green face f of G is (i) a leaf in $\mathcal{T}^*_{bg}$ or (ii) its degree in $\mathcal{T}^*_{bg}$ is exactly half of the degree of f, then G is Hamiltonian.*

Proof. A green leaf f contains exactly $\frac{1}{2}|f|$ red edges, one of which is crossed by $\mathcal{T}^*_{bg}$. The crossed edge will be part of $\mathcal{C}$ due to Property 4, while the others are part of $\mathcal{C}$ by construction. A green non-leaf f has, by assumption, maximum degree in $\mathcal{T}^*_{bg}$, i.e., all its red edges are crossed by $\mathcal{T}^*_{bg}$. But then Property 3 directly implies that $\frac{1}{2}|f|$ blue edges of f are part of $\mathcal{C}$. Hence, for any green face, half of its boundary edges are part of $\mathcal{C}$. Since the faces in F_g are disjoint and cover all red and blue edges, they include exactly n edges. Summing over all green faces, this contributes a total of $n/2$ edges to $\mathcal{C}$. Together with all green edges (which are $n/2$ many), this means that $|\mathcal{C}| = n$ and thus G is Hamiltonian. ◀

The following known result from the literature due to Florek [6] can hence be obtained as a direct corollary of Observation 2.

▶ **Corollary 3.** *A Barnette graph is Hamiltonian if its faces are 3-colored, with adjacent faces having different color, such that one of the three colors contain only faces of degree 4.*

Proof. W.l.o.g. let all green faces have degree 4. Hence, a green face is either a leaf in $\mathcal{T}^*_{bg}$ or its degree is 2 in $\mathcal{T}^*_{bg}$, i.e., maximum. Thus, Observation 2 yields the desired result. ◀

5 Open Problems

In this work, we approached Barnette's conjecture from the lens of 2-page book embeddings. Our work, while not resolving Barnette's conjecture at this time, makes another step forward and raises new open problems and possible directions for Barnette's conjecture:

1. Improve the lower bound of $\frac{5n}{6}$ on the size of the subhamiltonian cycle guaranteed by Theorem 1; a more strategic choice of the underlying spanning tree may lead to a better result in this direction (although our preliminary attempts were not successful).
2. It is also worth asking whether, for a given Barnette graph with a given Hamiltonian cycle, there is always a choice of the spanning tree $\mathcal{T}^*_{bg}$ such that our algorithm yields the given Hamiltonian cycle.
3. Is it true that all Barnette graphs generated solely by C_4-expansions are Hamiltonian?
4. The previous question is interesting also when the faces have a certain maximum degree.
5. Extend the set of restricted subclasses for which Observation 2 can be shown. Potential candidates are known subclasses from the literature which admit a Hamiltonian cycle.

References

1 Muhammad Jawaherul Alam, Michael A. Bekos, Vida Dujmović, Martin Gronemann, Michael Kaufmann, and Sergey Pupyrev. On dispersable book embeddings. *Theor. Comput. Sci.*, 861:1–22, 2021. doi:10.1016/J.TCS.2021.01.035.

2 Helmut Alt, Michael S. Payne, Jens M. Schmidt, and David R. Wood. Thoughts on Barnette's conjecture. *Australas. J Comb.*, 64:354–365, 2016. URL: http://ajc.maths.uq.edu.au/pdf/64/ajc_v64_p354.pdf.

3 David W. Barnette. Conjecture 5. In W. T. Tutte, editor, *Recent Progress in Combinatorics*. Academic Press, New York, 1969. Proc. of the Third Waterloo Conference on Combinatorics.
4 Gunnar Brinkmann, Jan Goedgebeur, and Brendan D. McKay. The minimality of the Georges-Kelmans graph. *Math. Comput.*, 91(335):1483–1500, 2022. `doi:10.1090/MCOM/3701`.
5 Tomas Feder and Carlos Subi. On Barnette's conjecture. Technical report, Electronic Colloquium on Computational Complexity (ECCC), 2006.
6 Jan Florek. Remarks on Barnette's conjecture. *J. Comb. Optim.*, 39(1):149–155, 2020. `doi:10.1007/S10878-019-00460-8`.
7 P. R. Goodey. Hamiltonian circuits in polytopes with even sided faces. *Israel Journal of Mathematics*, 22(1):52–56, March 1975. `doi:10.1007/bf02757273`.
8 A. Hertel. Hamiltonian cycles in sparse graphs. Master's thesis, University of Toronto, 2004.
9 A. Hertel. A survey and strengthening of Barnette's conjecture, 2005. Undergraduate thesis.
10 Derek A. Holton, Bennet Manvel, and Brendan D. McKay. Hamiltonian cycles in cubic 3-connected bipartite planar graphs. *J. Comb. Theory B*, 38(3):279–297, 1985. `doi:10.1016/0095-8956(85)90072-3`.
11 A. K. Kelmans. Konstruktsii kubicheskih dvudolnyh 3-svyaznyh bez gamiltonovyh tsiklov. *Sbornik Trudov Vserossiiskogo NII Sistemnykh Issledovanii*, 10:64–72, 1986.
12 P. G. Tait. Listing's topologie. *Philosophical Magazine*, 17:30–46, 1884. Reprinted in *Scientific Papers*, Vol. II, pp. 85–98.
13 W. T. Tutte. On hamiltonian circuits. *Journal of the London Mathematical Society*, 21(2):98–101, 1946. `doi:10.1112/jlms/s1-21.2.98`.