Directed Disjoint Paths Remains W[1]-Hard on Acyclic Digraphs Without Large Grid Minors

Ken-ichi Kawarabayashi
National Institute of Informatics, Tokyo, Japan

Nicola Lorenz
Universität Hamburg, Germany

Marcelo Garlet Milani
National Institute of Informatics, Tokyo, Japan

Jacob Stegemann
Universität Hamburg, Germany

Abstract

In the Vertex-Disjoint-Paths-With-Congestion problem, the input consists of a digraph D, an integer c and k pairs of vertices (s_i, t_i), and the task is to find a set of paths connecting each s_i to its corresponding t_i, whereas each vertex of D appears in at most c many paths. The case where $c = 1$ is known to be NP-complete even if $k = 2$ [Fortune, Hopcroft and Wyllie, 1980] on general digraphs and is W[1]-hard with respect to k (excluding the possibility of an $f(k)n^{O(1)}$-time algorithm under standard assumptions) on acyclic digraphs [Slivkins, 2010]. The proof of [Slivkins, 2010] can also be adapted to show W[1]-hardness with respect to k for every congestion $c \geq 1$.

We strengthen the existing hardness result by showing that the problem remains W[1]-hard for every congestion $c \geq 1$ even if: **(1)** the input digraph D is acyclic, **(2)** D does not contain an acyclic $(5, 5)$-grid as a butterfly minor, **(3)** D does not contain an acyclic tournament on 9 vertices as a butterfly minor, and **(4)** D has ear-anonymity at most 5.

Further, we also show that the edge-congestion variant of the problem remains W[1]-hard for every congestion $c \geq 1$ even if: **(1)** the input digraph D is acyclic, **(2)** D has maximum undirected degree 3, **(3)** D does not contain an acyclic $(7, 7)$-wall as a weak immersion and **(4)** D has ear-anonymity at most 5.

2012 ACM Subject Classification Theory of computation → Problems, reductions and completeness; Mathematics of computing → Paths and connectivity problems

Keywords and phrases digraphs, parameterized complexity, disjoint paths, butterfly minors, immersions, ear anonymity

Digital Object Identifier 10.4230/LIPIcs.IPEC.2025.2

Related Version *Full Version*: `https://doi.org/10.48550/arXiv.2507.09868` [20]

Funding Research of Ken-ichi Kawarabayashi and Marcelo Garlet Milani supported by JSPS KAKENHI JP20A402 and 22H05001, and by JST ASPIRE JPMJAP2302.

1 Introduction

The Vertex-Disjoint-Paths (VDP) problem is a classic problem for directed graphs (digraphs) where the input consists of a digraph D and k vertex pairs (called terminal pairs) of the form (s_i, t_i), and the task is to decide whether k pairwise disjoint paths $P_1, P_2, \ldots, P_k$ exist in D such that each P_i is a path from s_i to t_i.

VDP is NP-complete even if $k = 2$ [15]. On directed acyclic graphs (DAGs), the problem can be solved in $n^{O(k)}$ time [15] and is W[1]-hard with respect to k (that is, there is no $f(k)n^{O(1)}$-time algorithm under standard assumptions) [27]. Under the assumption of the Exponential Time Hypothesis (ETH), no $n^{o(k)}$-time algorithm exists for VDP on DAGs [8], implying that the $n^{O(k)}$-time algorithm is essentially optimal.

20th International Symposium on Parameterized and Exact Computation (IPEC 2025).
Editors: Akanksha Agrawal and Erik Jan van Leeuwen; Article No. 2; pp. 2:1–2:17
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

We investigate the impact on VDP on DAGs of excluding minors and imposing further structural restrictions on the input digraph. There are several notions of minors for digraphs, including topological minors, butterfly minors and strong minors. Strong minors are not meaningful on acyclic digraphs, and if a digraph D contains H as a topological minor, then it also contains H as a butterfly minor. Hence, out of the three relations above, butterfly minors are the most adequate for obtaining our results in the most general form.

Directed treewidth was introduced by [18], who also proved that VDP can be solved in $n^{f(k,\mathrm{dtw}(D))}$ time, where $\mathrm{dtw}(D)$ is the *directed treewidth* of the input digraph D. As the directed treewidth of DAGs is 0, this roughly generalizes the $n^{O(k)}$-time algorithm for DAGs mentioned above.

The edge-disjoint variant of the problem is called Edge-Disjoint-Paths (EDP). [3] proved that EDP can be solved in $f(k)n^{O(1)}$-time on Eulerian digraphs. EDP is W[1]-hard with respect to k even if the input DAG is planar [8]. We observe that the reduction from the edge-disjoint to the vertex-disjoint variant of EDP mentioned above does not preserve planarity, and so this hardness result does not immediately transfer to the vertex-disjoint variant. In fact, vertex-disjoint EDP is solvable in $f(k)n^{O(1)}$-time on planar digraphs [12].

Relaxations of VDP were considered in order to find tractable settings. One such relaxation is Vertex-Disjoint-Paths-With-Congestion (VDPwC), where each vertex/edge can be used by at most g many paths in the solution instead of requiring disjointness. The value g is referred to as the maximum *congestion* at a vertex/edge.

The hardness reduction due to [27] can be adapted to VDPwC, also yielding W[1]-hardness with respect to k on DAGs [1]. This was further strengthened by [1], who showed that, assuming the *Exponential Time Hypothesis* (ETH), for every computable function f and every constant $g \geq 1$, there is no $f(k)n^{o(k/\log k)}$-time algorithm for VDPwC on DAGs.

[28] studied an approximation variant of VDP where the goal is to either find a solution for the given instance or decide that no solution joining at least k/q terminal pairs exist, for some constant q. They proved that such a variant is W[1]-hard on DAGs for all constants q.

A maximization variant of VDP was also studied. Approximation properties of this variant have been considered by [6, 28, 4, 7, 5, 9, 22, 10, 14, 21].

The undirected analog of VDP is computationally much easier and can be solved in $f(k)n^3$ time for some function f [26]. The exponent of n was later improved to 2 [19] and recently to $1 + o(1)$ [23].

In rough terms, the algorithms of [26, 19, 23] above work by applying the *irrelevant vertex technique* in order to repeatedly remove vertices from the graph until the *treewidth* is sufficiently small so that the problem can be solved efficiently. Towards this end, the algorithms rely on the *Flat Wall Theorem* [26], which roughly states that, for every graph G, we have three cases: G contains a large clique as a minor, G contains a large *flat wall*, or G has bounded treewidth.

On a somewhat different direction, [25] introduced the parameter *ear anonymity*, a parameter which tries to capture the “structure” of maximal paths in a digraph, and asked if VDP can be solved in $f(k)n^{h(\mathrm{ea}(D))}$ time on DAGs, where $\mathrm{ea}(D)$ is the *ear anonymity* of the input digraph D.

We prove that VDPwC and its edge-disjoint variant remain computationally hard even if we impose structural restrictions as described above. We obtain additional lower bounds when assuming the Exponential Time Hypothesis (ETH). Below, $\overrightarrow{TT}_9$ is the unique acyclic orientation of the undirected clique on 9 vertices, and the acyclic $(5,5)$-grid is the orientation of the undirected $(5,5)$-grid where each edge is oriented from “left” to “right” and from “top” to “bottom”.

▶ **Theorem 13.** *For every congestion $g \geq 1$, Vertex-Disjoint-Paths-With-Congestion is W[1]-hard with respect to the number k of terminal pairs, even if*

- *The input digraph D is acyclic,*
- *D contains no $\overrightarrow{TT}_9$ as a butterfly minor,*
- *D contains no acyclic $(5,5)$-grid as butterfly minor, and*
- $\mathrm{ea}(D) \leq 5$.

Furthermore, assuming the ETH, no $f(k)n^{o(\sqrt{k/g})}$ time algorithm exists for VDPwC, even under the above conditions.

Our reduction may create large complete bipartite digraphs where all edges are oriented from one partition to the other. Hence, it would be interesting to determine whether VDP remains W[1]-hard even on DAGs without large acyclic grids and without such complete bipartite digraphs as butterfly minors.

We also consider excluding *weak immersions* instead of butterfly minors, as immersions are more closely related to edge-disjoint than to vertex-disjoint paths. Since the maximum degree is closed under immersions, restricting the maximum degree of a digraph D also exclude the possibility of D having any digraph with larger degree as a weak immersion. For this reason, we consider here the acyclic wall instead of the acyclic grid, as the grid contains vertices of degree 4. We present the result above for the edge-disjoint variant of the problem, but we note that both reductions can be adapted to both variants of the problem.

▶ **Theorem 18** ($\star$). *For every congestion $g \geq 1$, Edge-Disjoint-Paths-With-Congestion is W[1]-hard with respect to the number k of terminal pairs, even if*

- *The input digraph D is acyclic,*
- *D has maximum undirected degree 3,*
- *D contains no acyclic $(7,7)$-wall as a weak immersion, and*
- $\mathrm{ea}(D) \leq 5$.

Furthermore, assuming the ETH, no $f(k)n^{o(\sqrt{k/g})}$ time algorithm exists for Edge-Disjoint-Paths-With-Congestion, even under the above conditions.

Observe that the hardness reductions of [27, 1] do not exclude the existence of a $\overrightarrow{TT}_k$ in the reduced instance, and the reduction due to [8] for the planar case does not exclude the existence of acyclic grids of size n.

The paper is organized as follows. Section 2 contains definitions and notation. In Section 3 we prove Theorem 13. Section 3.1 contains our hardness reduction, Sections 3.2–3.4 contain the structural analysis and Section 3.5 completes the proof. In Section 4 we prove Theorem 18. Due to space constraints, proofs marked with ($\star$) are deferred to full version of this manuscript.

2 Preliminaries

Sets. For $n \in \mathbb{N}$ we define $[n] \coloneqq \{1, \dots, n\}$. A *multiset over a set X* is a function $f\colon X \to \mathbb{N}$. For every $x \in X$, the value $f(x)$ is the *multiplicity* of that element in the multiset. We write multisets as $\{x_1 \colon f(x_1), x_2 \colon f(x_2), \dots, x_n \colon f(x_n)\}$, since this uniquely defines f. When explicitly stated to be a multiset, we may also write multisets like $\{x_1, x_2, \dots, x_n\}$. In this case, the underlying set X is $\{x_1, x_2, \dots, x_n\}$ (understood as a usual set) and the multiplicity of an element $x \in X$ is given by $f(x) \coloneqq |\{i \in [n] \mid x_i = x\}|$.

Digraphs. We refer the reader to [2] for an introduction to digraph theory. A directed graph (digraph) D is a tuple (V, E) where $E \subseteq V \times V$ and E does not contain *loops*, that is, edges of the form (v, v). We write $V(D) \coloneqq V$ and $E(D) \coloneqq E$.

Given a set $X \subseteq V(D)$, we write $D - X$ for the digraph $(Y \coloneqq V(D) \setminus X,\ E(D) \cap (Y \times Y))$. Similarly, given a set $F \subseteq E(D)$, we write $D - F$ for the digraph $(V(D), E(D) \setminus F)$.

If D is a digraph and $v \in V(D)$, then $\mathsf{in}_D(v) \coloneqq \{u \in V \mid (u, v) \in E(D)\}$ is the set of *in-neighbours* and $\mathsf{out}_D(v) \coloneqq \{u \in V \mid (v, u) \in E(D)\}$ the set of *out-neighbours* of v. By $\mathsf{indeg}_D(v) \coloneqq |\mathsf{in}(v)|$ we denote the *in-degree* of v and by $\mathsf{outdeg}_D(v) \coloneqq |\mathsf{out}(v)|$ its out-degree.

Given a vertex v in a digraph D, the *split* of v is the operation which consists of replacing v with two vertices $v_{\text{in}}, v_{\text{out}}$, adding the edge $(v_{\text{in}}, v_{\text{out}})$ and the edge $(u, v_{\text{in}}), (v_{\text{out}}, w)$ for each $u \in \mathsf{in}(v)$ and each $w \in \mathsf{out}(v)$.

Paths and walks. A *walk* of length ℓ in a digraph D is a sequence of vertices $W \coloneqq (v_0, v_1, \ldots, v_\ell)$ such that $(v_i, v_{i+1}) \subseteq E(D)$, for all $0 \leq i < \ell$. We say that W is a v_0-v_ℓ-walk.

A walk $W \coloneqq (v_0, v_1, \ldots, v_\ell)$ is called a *path* if no vertex appears twice in it and it is called a *cycle* if $v_0 = v_\ell$ and $v_i \neq v_j$ for all $0 \leq i < j < \ell$. We often identify a walk W in D with the corresponding subgraph and write $V(W)$ and $E(W)$ for the set of vertices and arcs appearing on it.

Given two walks $W_1 \coloneqq (x_1, x_2, \ldots, x_j)$ and $W_2 \coloneqq (y_1, y_2, \ldots, y_k)$, we write $W_1 \cdot W_2$ for the *concatenation* of W_1 and W_2, defined as follows. If $x_j = y_1$, $W_1 \cdot W_2 \coloneqq (x_1,\ x_2,\ \ldots,\ x_j,\ y_2,\ y_3,\ \ldots,\ y_k)$. If W_1 or W_2 is an empty sequence, then $W_1 \cdot W_2$ is the other walk (or the empty sequence if both walks are empty). Finally, if $x_j \neq y_1$, $W_1 \cdot W_2 \coloneqq W_1 \cdot (x_j, y_1) \cdot W_2$.

Given a walk $W \coloneqq (x_1, x_2, \ldots, x_n)$, we write Wx_i for the walk $(x_1, \ldots, x_i)$ and similarly x_iW for the walk $(x_i, \ldots, x_n)$.

A digraph D without any cycles is called a *directed acyclic graph* (DAG). A vertex v in D is called a *source* if $\mathsf{indeg}_D(v) = 0$ and a *sink* if $\mathsf{outdeg}_D(v) = 0$.

Let $x_1, x_2, \ldots, x_n$ be an ordering of the vertices of a digraph D. We say that this ordering is a *topological ordering* of D if for every edge $(x_i, x_j) \in E(D)$ we have $i < j$. We note that a digraph admits a topological ordering if, and only if, it is a DAG.

Digraph classes. An *in-tree (out-tree) with root* $r \in V(D)$ is a DAG where for all $v \in V(D)$ there exists exactly one directed path from v to r (from r to v for out-trees).

The *transitive tournament on k vertices* is the digraph $\overrightarrow{TT}_k$ with vertex set $v_1, v_2, \ldots, v_k$ and edge set $\{(v_i, v_j) \mid 1 \leq i < j \leq k\}$.

We consider two types of directed grids (see Figure 1). An *acyclic (p, q)-grid* is the digraph with vertex set $\{(r, c) \mid 1 \leq r \leq p \text{ and } 1 \leq c \leq q\}$ and edge set $\{((r, c), (r+1, c)) \mid 1 \leq r \leq p-1 \text{ and } 1 \leq c \leq q\} \cup \{((r, c), (r, c+1)) \mid 1 \leq r \leq p \text{ and } 1 \leq c \leq q-1\}$.

An *acyclic (p, q)-wall* is the digraph obtained from an acyclic (p, q)-grid by applying the split operation to every vertex of indegree and outdegree exactly 2.

Immersions and minors. Let D and H be two digraphs. A weak immersion of H in D is a function μ with domain $V(H) \cup E(H)$ satisfying the following:

1. μ maps $V(H)$ injectively into $V(D)$,
2. $\mu((u, v))$ is a directed path from $\mu(u)$ to $\mu(v)$ in D for every $(u, v) \in E(H)$ and
3. the directed paths $\mu(e)$ and $\mu(f)$ are edge disjoint for every pair of distinct edges $e, f \in E(H)$.

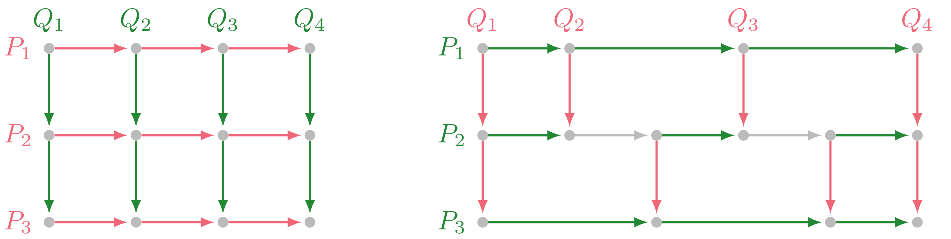

Figure 1 An acyclic $(3,4)$-grid on the left and an acyclic $(3,4)$-wall on the right.

Let H and D be directed graphs. A **butterfly-model** of H in D is a function μ which assigns to every $x \in V(H) \cup E(H)$ a subdigraph of D such that:

1. for every pair of distinct vertices $u, v \in V(H)$, $\mu(u)$ and $\mu(v)$ are vertex-disjoint,
2. for a vertex $v \in V(H)$ and a non-incident edge $e \in E(H)$, $\mu(v)$ and $\mu(e)$ are vertex-disjoint,
3. for every $v \in V(H)$, $\mu(v)$ is the union of an in-tree and an out-tree intersecting exactly on their common root, and
4. for every $(u,v) \in E(H)$, $\mu((u,v))$ is a directed path starting at a vertex of the out-tree of $\mu(u)$ and ending at a vertex of the in-tree of $\mu(v)$.

Additionally we define a *center* function $c_\mu : V(H) \to V(D)$ as follows. Let $v \in V(H)$. As $\mu(v)$ is an acyclic graph, it therefore admits a topological ordering. We let $c_\mu(v) \in V(\mu(v))$ be minimal in this ordering with the property that it can be reached by all sources of $\mu(v)$ (note that this does not depend on the choice of the topological ordering).

Note that $\mu(v)$ being a union of an in-tree and an out-tree intersecting exactly on their common root implies that $c_\mu(v)$ can also reach all sinks of $\mu(v)$.

Ear anonymity. An *ear* in a digraph D is a subgraph of D which is either a path or a cycle. Let P be an ear. A sequence $(a_1, a_2, \ldots, a_k)$ of arcs of P is an *identifying sequence for P* if $k \geq 1$ and every ear Q containing $(a_1, a_2, \ldots, a_k)$ in this order is a subgraph of P.

Let $P = (a_1, a_2, \ldots, a_k)$ be a maximal ear in a digraph D, given by its arc-sequence (in the case of a cycle, any arc of P can be chosen as a_1). The *ear anonymity* of P in D, denoted by $\mathrm{ea_D}(P)$, is the length of the shortest identifying sequence for P. If $k = 0$, we say that $\mathrm{ea_D}(P) = 0$. The ear anonymity of a digraph D, denoted by $\mathrm{ea}(D)$, is the maximum ear anonymity of the maximal ears of D.

Complexity theory. We refer the reader to [11, 13] for formal definitions of parameterized complexity concepts.

We skip a formal definition of W[1]-*hardness* and recall only the properties that we require for our results. If a parameterized problem L_1 is W[1]-hard with respect to k_1 and there is a reduction from L_1 to L_2 parameterized by k_2 with running time at most $f(k_1)n^{O(1)}$ such that $k_2 = h(k_1)$, then L_2 is W[1]-hard with respect to k_2. Under standard assumptions, if a problem is W[1]-hard with respect to some parameter k, then no algorithm with running time $f(k)n^{O(1)}$ exists for said problem.

The *Exponential Time Hypothesis* (ETH) states that 3-SAT on n variables and m clauses cannot be solved in $2^{o(n)} \cdot (n+m)^{O(1)}$ time [16, 17].

Computational problems. We recall the definitions of the following decision problems.

VERTEX-DISJOINT-PATHS-WITH-CONGESTION (VDPWC)

Input A digraph D, two integers g and k and a multiset $T = \{(s_1,t_1),(s_2,t_2),\ldots,(s_k,t_k)\}$ of vertex pairs.

Question Is there a multiset $\mathcal{L}$ of paths in D such that for each $(s_i,t_i) \in S$ there are $\mathrm{mult}_T((s_i,t_i))$ many s_i-t_i paths in $\mathcal{L}$ and for each $v \in V(D)$ there are at most g paths in $\mathcal{L}$ which contain v?

EDGE-DISJOINT-PATHS-WITH-CONGESTION (EDPWC)

Input A digraph D, two integers g and k and a multiset $T = \{(s_1,t_1),(s_2,t_2),\ldots,(s_k,t_k)\}$ of vertex pairs.

Question Is there a multiset $\mathcal{L}$ of paths in D such that for each $(s_i,t_i) \in T$ there are $\mathrm{mult}_T((s_i,t_i))$ many s_i-t_i paths in $\mathcal{L}$ and for each $e \in E(D)$ there are at most g paths in $\mathcal{L}$ which contain e?

The special cases where the congestion g above is 1 are called VERTEX-DISJOINT-PATHS and EDGE-DISJOINT-PATHS. Our main reduction is from the following W[1]-hard problem.

GRIDTILING

Input Numbers $n,k \in \mathbb{N}$ and sets $S_{i,j} \subseteq [n] \times [n]$ for every $1 \le i,j \le k$.

Question Are there $a_1,a_2,\ldots,a_k,b_1,b_2,\ldots,b_k \in [n]$ such that $(a_i,b_j) \in S_{i,j}$ holds for all $1 \le i,j \le k$?

▶ **Theorem 1** ([24, Lemma 1]). *GRIDTILING is W[1]-hard with respect to k. Furthermore, unless the ETH fails, there is no $f(k)n^{o(k)}$ time algorithm for GRIDTILING.*

3 Vertex disjoint paths

We provide a reduction from GRIDTILING to VDPWC. Our gadgets are inspired by the reduction due to [27].

3.1 Hardness reduction

▶ **Construction 2.** Let $(n,\{S_{i,j}\}_{i,j=1}^{k})$ be an instance of GRIDTILING. Let $g \ge 1$ be an integer. We construct a VERTEX-DISJOINT-PATHS-WITH-CONGESTION instance (D,T,g) as follows.

We use two types of *gadgets* in our construction: selector and verifier gadgets. For each row and each column, we have one selector gadget $\mathrm{SEL}^{r,i}$ and $\mathrm{SEL}^{c,i}$, respectively. For each $\circ \in \{r,c\}$ and each $i \in [k]$, we construct $\mathrm{SEL}^{\circ,i}$ as follows (see Figure 2).

For each $j \in [k]$, add the source $s^{\circ,i,j}$ if $\circ = r$ and $s^{\circ,j,i}$ if $\circ = c$ instead.

Define two paths $P^{\circ,i}$ and $Q^{\circ,i}$ by their subpaths as follows. For each $1 \le \ell \le n$, define

$$P_\ell^{\circ,i} = p_{\ell,0}^{\circ,i},\ldots,p_{\ell,k}^{\circ,i} \qquad \text{and} \qquad Q_\ell^{\circ,i} = q_{\ell,1}^{\circ,i},\ldots,q_{\ell,k+1}^{\circ,i}.$$

Then,

$$P^{\circ,i} = P_1^{\circ,i} \cdot \ldots \cdot P_n^{\circ,i} \qquad \text{and} \qquad Q^{\circ,i} = Q_1^{\circ,i} \cdot \ldots \cdot Q_n^{\circ,i}.$$

Connect the paths above as follows. For each $\ell \in [n]$ and each $j \in [k]$, add the edges $(p_{\ell,0}^{\circ,i}, q_{\ell,k+1}^{\circ,i})$ and $(p_{\ell,j}^{\circ,i}, q_{\ell,j}^{\circ,i})$.

Add the terminal pairs $(p_{1,0}^{\circ,i}, p_{n,k}^{\circ,i})$ and $(q_{1,1}^{\circ,i}, q_{n,k+1}^{\circ,i})$ with multiplicity $g-1$ to T. Add the terminal pair $(p_{1,0}^{\circ,i}, q_{n,k+1}^{\circ,i})$ with multiplicity 1 to T.

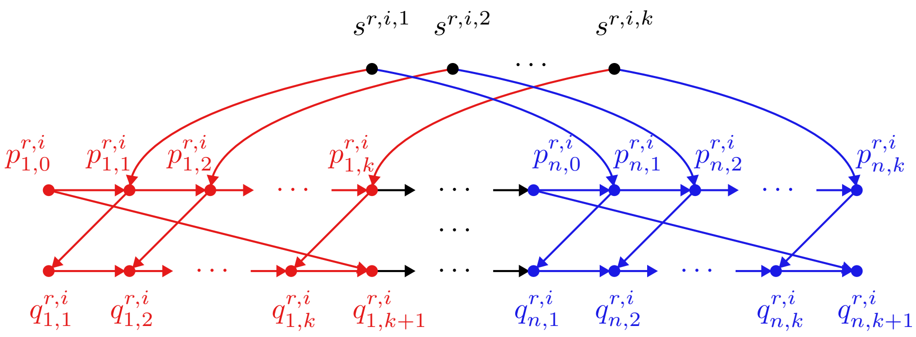

Figure 2 The selector gadget $\mathrm{SEL}^{r,i}$. The colored subgraphs are repeating building blocks.

This concludes the construction of the selector gadget. Vertices and edges of the selector can be summarized as follows.

$$
\begin{aligned}
V(\mathrm{SEL}^{\circ,i}) &= V(P^{\circ,i}) \cup V(Q^{\circ,i}) \cup \left\{ s^{\circ,i,j} \;\middle|\; j \in [k] \right\}, \\
E(\mathrm{SEL}^{\circ,i}) &= E(P^{\circ,i}) \cup E(Q^{\circ,i}) \\
&\quad \cup \left\{ (s^{\circ,i,j}, p^{\circ,i}_{\ell,j}), (p^{\circ,i}_{\ell,j}, q^{\circ,i}_{\ell,j}) \;\middle|\; j \in [k], \ell \in [n] \right\} \\
&\quad \cup \left\{ (p^{\circ,i}_{\ell,0}, q^{\circ,i}_{\ell,k+1}) \;\middle|\; \ell \in [n] \right\}.
\end{aligned}
$$

Next, we construct the verifier gadgets $\mathrm{VER}^{i,j}$ for $1 \le i, j \le k$ (see Figure 3).

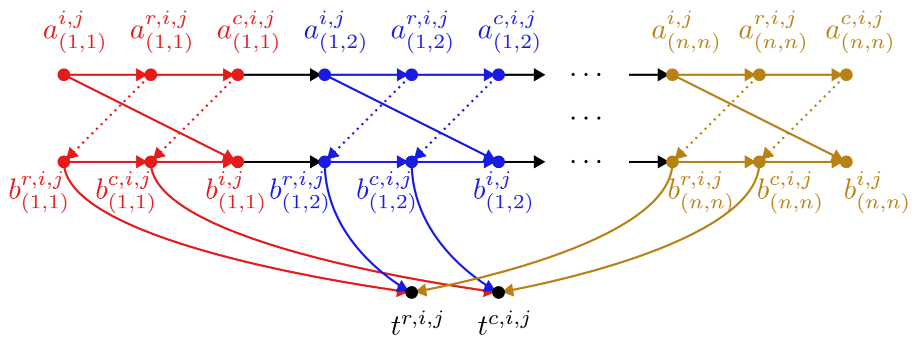

Figure 3 The verifier gadget $\mathrm{VER}^{i,j}$. The colored subgraphs are repeating building blocks, where the existence of the dashed arrows depends on the set $S_{i,j}$.

First, add the targets $t^{r,i,j}, t^{c,i,j}$.

Define two paths $A^{i,j}$ and $B^{i,j}$ by their subpaths as follows. For $d \in [n] \times [n]$, let

$$A^{i,j}_d = a^{i,j}_d \cdot a^{r,i,j}_d \cdot a^{c,i,j}_d \qquad \text{and} \qquad B^{i,j}_d = b^{r,i,j}_d \cdot b^{c,i,j}_d \cdot b^{i,j}_d.$$

For $1 \le x \le n$, let

$$A^{i,j}_x = A^{i,j}_{(x,1)} \cdot \ldots \cdot A^{i,j}_{(x,n)} \qquad \text{and} \qquad B^{i,j}_x = B^{i,j}_{(x,1)} \cdot \ldots \cdot B^{i,j}_{(x,n)}.$$

Now define

$$A^{i,j} = A^{i,j}_1 \cdot \ldots \cdot A^{i,j}_n \qquad \text{and} \qquad B^{i,j} = B^{i,j}_1 \cdot \ldots \cdot B^{i,j}_n.$$

Connect $A^{i,j}$ to $B^{i,j}$ and $B^{i,j}$ to $t^{r,i,j}, t^{c,i,j}$ by adding, for each $d \in S_{i,j}$, the edges $(a_d^{i,j}, b_d^{i,j})$, $(a_d^{r,i,j}, b_d^{r,i,j})$, $(a_d^{c,i,j}, b_d^{c,i,j})$, $(b_d^{r,i,j}, t^{r,i,j})$ and $(b_d^{c,i,j}, t^{c,i,j})$.

Add the terminal pairs $(a_{(1,1)}^{i,j}, a_{(n,n)}^{c,i,j})$ and $(b_{(1,1)}^{r,i,j}, b_{(n,n)}^{i,j})$ with multiplicity $g-1$ to T. Add the terminal pair $(a_{(1,1)}^{i,j}, b_{(n,n)}^{i,j})$ with multiplicity 1 to T.

This concludes the construction of the verifier gadgets. We can summarize the vertices and edges of the verifier as follows.

$$V(\mathrm{VER}^{i,j}) = V(A^{i,j}) \cup V(B^{i,j}) \cup \{t^{r,i,j}, t^{c,i,j}\}.$$

$$E(\mathrm{VER}^{i,j}) = E(A^{i,j}) \cup E(B^{i,j}) \cup \left\{ \begin{array}{lll} (a_d^{i,j}, b_d^{i,j}), & (a_d^{r,i,j}, b_d^{r,i,j}), & (a_d^{c,i,j}, b_d^{c,i,j}), \\ & (b_d^{r,i,j}, t^{r,i,j}), & (b_d^{c,i,j}, t^{c,i,j}) \end{array} \;\middle|\; d \in S_{i,j} \right\}.$$

Finally, connect the selectors and verifiers via additional edges as follows.

$$E' = \left\{ (q_{\ell,j}^{r,i}, a_{(\ell,x)}^{r,i,j}), (q_{\ell,i}^{c,j}, a_{(x,\ell)}^{c,i,j}) \;\middle|\; i,j \in [k] \text{ and } \ell, x \in [n] \right\}$$

This completes the construction of D. The multiset of terminal pairs is given by

$$\begin{aligned} T = & \left\{ (s^{r,i,j}, t^{r,i,j}) : 1, (s^{c,i,j}, t^{c,i,j}) : 1, (a_{(1,1)}^{i,j}, b_{n,n}^{i,j}) : 1 \;\middle|\; i,j \in [k] \right\} \\ & \cup \left\{ (p_{1,0}^{r,i}, q_{n,k+1}^{r,i}) : 1, (p_{1,0}^{c,i}, q_{n,k+1}^{c,i}) : 1 \;\middle|\; i \in [k] \right\} \\ & \cup \left\{ \begin{array}{ll} (p_{1,0}^{r,i}, p_{n,k}^{r,i}) : g-1, & (q_{1,1}^{r,i}, q_{n,k+1}^{r,i}) : g-1 \\ (p_{1,0}^{c,i}, p_{n,k}^{c,i}) : g-1, & (q_{1,1}^{c,i}, q_{n,k+1}^{c,i}) : g-1 \end{array} \;\middle|\; i \in [k] \right\} \\ & \cup \left\{ (a_{(1,1)}^{i,j}, a_{(n,n)}^{c,i,j}) : g-1, (b_{(1,1)}^{r,i,j}, b_{(n,n)}^{i,j}) : g-1 \;\middle|\; i,j \in [k] \right\} \end{aligned}$$

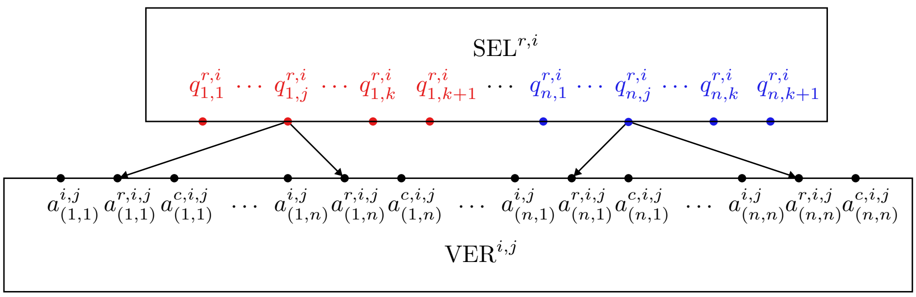

Figure 4 The edges between SEL^{r,i} and VER^{i,j}. The colors in the selector again mark the repeating building blocks.

In total, D contains k selectors SEL^{r,i} and k selectors SEL^{c,i}. Each selector contains $k + 2g - 1$ sources. There are k^2 verifier gadgets, each containing $2g - 1$ sources. Hence, we have in total $2k(k + 2g - 1) + k^2(2g - 1) = 2g(k^2 + k) + k^2 - 2k$ terminal pairs in the constructed instance.

Before analyzing the structural properties of the digraph constructed above, we show that our reduction is correct.

▶ **Lemma 3.** *Let $I_G \coloneqq (n, \{S_{i,j}\}_{i,j=1}^k)$ be an instance of* GridTiling*. Let $g \geq 1$ be an integer. Let $I_L \coloneqq (D, T, g)$ be the VDPwC instance obtained from Construction 2. Then I_G is a* yes *instance if, and only if, I_L is a* yes *instance.*

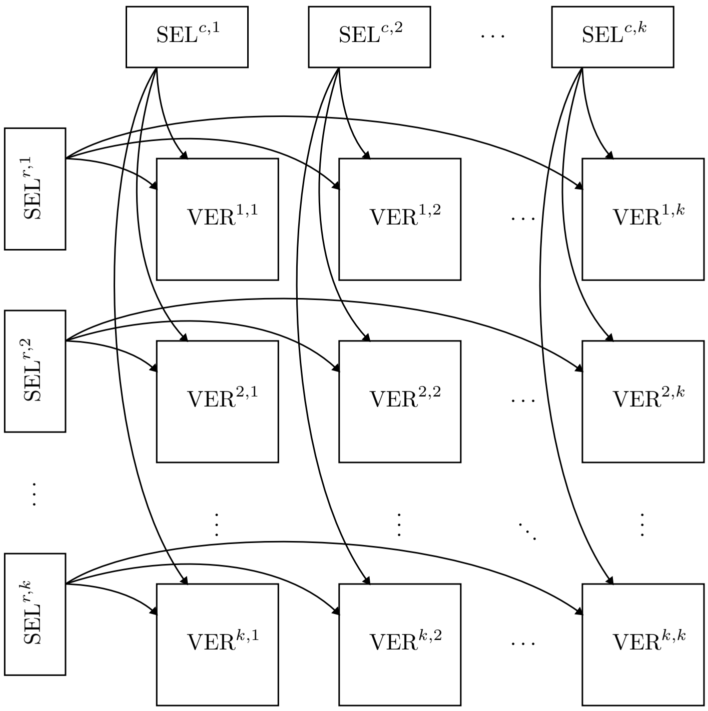

Figure 5 A simplified representation of all edges between selectors and verifiers. Each edge represents in fact multiple edges as shown in Figure 4.

Proof.

First direction. If I_G is a *yes* instance, then I_L is a *yes* instance.

Let $a_1, a_2, \ldots a_k, b_1, b_2, \ldots, b_k \in [n]$ be a solution for I_G. We construct a solution for I_L as follows. Recall from Construction 2 that the set of terminal pairs is given by

$$\begin{aligned} T^* &= \left\{ \begin{array}{ll} (p^{r,i}_{1,0}, p^{r,i}_{n,k}) : g-1, & (q^{r,i}_{1,1}, q^{r,i}_{n,k+1}) : g-1 \\ (p^{c,i}_{1,0}, p^{c,i}_{n,k}) : g-1, & (q^{c,i}_{1,1}, q^{c,i}_{n,k+1}) : g-1 \end{array} \middle| \; i \in [k] \right\} \\ &\quad \cup \left\{ (a^{i,j}_{(1,1)}, a^{c,i,j}_{(n,n)}) : g-1, (b^{r,i,j}_{(1,1)}, b^{i,j}_{(n,n)}) : g-1 \;\middle|\; i,j \in [k] \right\}, \\ T &= \left\{ (s^{r,i,j}, t^{r,i,j}) : 1, (s^{c,i,j}, t^{c,i,j}) : 1, (a^{i,j}_{(1,1)}, b^{i,j}_{n,n}) : 1 \;\middle|\; i,j \in [k] \right\} \\ &\quad \cup \left\{ (p^{r,i}_{1,0}, q^{r,i}_{n,k+1}) : 1, (p^{c,i}_{1,0}, q^{c,i}_{n,k+1}) : 1 \;\middle|\; i \in [k] \right\} \cup T^*. \end{aligned}$$

For each $i, j \in [k]$ and each $h \in [g-1]$, we define paths connecting the terminal pairs.

First, we consider the terminal pairs whose corresponding path does not depend on the solution for I_G.

For each $\circ \in \{r, c\}$, the pair $(p^{\circ,i}_{(0,1)}, p^{\circ,i}_{(n,k)})$ gets connected via $g-1$ copies of the path $(p^{\circ,i}_{(1,0)}, p^{\circ,i}_{(1,1)}) \cdot P^{\circ,i} \cdot (p^{\circ,i}_{(n,k-1)}, p^{\circ,i}_{(n,k)})$ and the pair $(q^{\circ,i}_{(1,1)}, q^{\circ,i}_{(n,k+1)})$ gets connected via $g-1$ copies of the path $(q^{\circ,i}_{(1,1)}, q^{\circ,i}_{(1,2)}) \cdot Q^{\circ,i} \cdot (q^{\circ,i}_{(n,k)}, q^{\circ,i}_{(n,k+1)})$.

The pair $(a^{i,j}_{(1,1)}, a^{i,j,c}_{(n,n)})$ gets connected via $g-1$ copies of the path $(a^{i,j}_{(1,1)}, a^{r,i,j}_{(1,1)}) \cdot A^{i,j} \cdot (a^{r,i,j}_{(n,n)}, a^{c,i,j}_{(n,n)})$ and the pair $(b^{i,j}_{(1,1)}, b^{i,j}_{(n,n)})$ gets connected via $g-1$ copies of the path $(b^{r,i,j}_{(1,1)}, b^{c,i,j}_{(1,1)}) \cdot B^{i,j} \cdot (b^{c,i,j}_{(n,n)}, b^{i,j}_{(n,n)})$.

Let $\mathcal{L}^*$ be the linkage connecting the terminal pairs above. Observe that $\mathcal{L}^*$ provokes a congestion of $g-1$ on every vertex in $P^{r,i}, P^{c,i}, Q^{r,i}, Q^{c,i}, A^{i,j}$ and $B^{i,j}$.

The solution for the remaining pairs will depend on the values a_i and b_j above. Recall that the remaining terminal pairs all have multiplicity 1 in T.

The terminal pair $(s^{r,i,j}, t^{r,i,j})$ gets connected via the path

$$L^{r,i,j} \coloneqq (s^{r,i,j}, p^{r,i}_{a_i,j}, q^{r,i}_{a_i,j}, a^{r,i,j}_{(a_i,b_j)}, b^{r,i,j}_{(a_i,b_j)}, t^{r,i,j}).$$

The terminal pair $(s^{c,i,j}, t^{c,i,j})$ gets connected via the path

$$L^{c,i,j} \coloneqq (s^{c,i,j}, p^{c,j}_{b_j,i}, q^{c,j}_{b_j,i}, a^{c,i,j}_{(a_i,b_j)}, b^{c,i,j}_{(a_i,b_j)}, t^{c,i,j}).$$

Observe that, because $(a_i, b_j) \in S_{i,j}$, we know that the edges $(a^{r,i,j}_{(a_i,b_j)}, b^{r,i,j}_{(a_i,b_j)})$ and $(a^{c,i,j}_{(a_i,b_j)}, b^{c,i,j}_{(a_i,b_j)})$ do in fact exist.

The terminal pair $(a^{i,j}, b^{i,j})$ gets connected via the path

$$L^{i,j} \coloneqq (a^{i,j}_{(1,1)}, a^{r,i,j}_{(1,1)}) \cdot A^{i,j} \cdot (a^{i,j}_{(a_i,b_j)}, b^{i,j}_{(a_i,b_j)}) \cdot B^{i,j} \cdot (b^{c,i,j}_{(n,n)}, b^{i,j}_{(n,n)}).$$

The terminal pair $(p^{r,i}, q^{r,i})$ gets connected via the path

$$L^{r,i} \coloneqq (p^{r,i}, p^{r,i}_{1,0}) \cdot P^{r,i} \cdot (p^{r,i}_{a_i,0}, q^{r,i}_{a_i,k+1}) \cdot Q^{r,i} \cdot (q^{r,i}_{n,k+1}, q^{r,i}).$$

The terminal pair $(p^{c,j}, q^{c,j})$ gets connected via the path

$$L^{c,j} \coloneqq (p^{c,j}, p^{c,j}_{1,0}) \cdot P^{c,j} \cdot (p^{c,j}_{b_j,0}, q^{c,j}_{b_j,k+1}) \cdot Q^{c,j} \cdot (q^{c,j}_{n,k+1}, q^{c,j}).$$

These paths are pairwise vertex disjoint. Further, each path visits at most one vertex in each $P^{r,i}, P^{c,i}, Q^{r,i}, Q^{c,i}, A^{i,j}$ and $B^{i,j}$. Hence, together with the linkage $\mathcal{L}^*$, we obtain congestion at most g in order to connect all terminal pairs of T.

Second direction. If I_L is a *yes* instance, then I_G is a *yes* instance.

Let $\mathcal{L}$ be a solution for I_L. For each $i, j \in [k]$, let $L^{r,i,j}$, $L^{c,i,j}$, $L^{r,i}$, $L^{c,j}$ and $L^{i,j}$ be the paths connecting the pairs $(s^{r,i,j}, t^{r,i,j})$, $(s^{c,i,j}, t^{c,i,j})$, $(p^{r,i}_{0,1}, q^{r,i}_{n,k+1})$, $(p^{c,j}_{0,1}, q^{c,j}_{n,k+1})$ and $(a^{i,j}_{(1,1)}, b^{i,j}_{(1,1)})$, respectively, and let $\mathcal{L}' \subseteq \mathcal{L}$ be the set containing these paths. Let $\mathcal{L}^* \subseteq \mathcal{L}$ be the multiset of paths connecting the pairs in T^*.

Observe that, for each $L \in \mathcal{L}^*$, the path L is the unique path in D from $\mathsf{start}(L)$ to $\mathsf{end}(L)$, as there are no paths from $Q^{r,i}$ to $P^{r,i}$ in the selector gadgets and also no path from $B^{i,j}$ to $A^{i,j}$ in the verifier gadgets. Hence, $\mathcal{L}^*$ provokes a congestion of $g-1$ on every vertex in $P^{r,i}, P^{c,i}, Q^{r,i}, Q^{c,i}, A^{i,j}$ and $B^{i,j}$.

As every non-terminal vertex inside $\mathcal{L}'$ lies in $P^{r,i}, P^{c,i}, Q^{r,i}, Q^{c,i}, A^{i,j}$ or $B^{i,j}$, we deduce that the paths in $\mathcal{L}'$ are pairwise vertex disjoint, as otherwise we would have congestion greater than g at some vertex.

Let $i \in [k]$ and consider the path $L^{r,i} \in \mathcal{L}'$ connecting $(p^{r,i}_{0,1}, q^{r,i}_{n,k+1})$. Note that the set of vertices which can reach $q^{r,i}_{n,k+1}$ and are reachable from $p^{r,i}_{0,1}$ is given by $\{p^{r,i}_{0,1}, q^{r,i}_{n,k+1}\} \cup V(P^{r,i}) \cup V(Q^{r,i})$. Hence, $L^{r,i}$ lies inside $\mathrm{SEL}^{r,i}$. Further, it must contain exactly one edge from $P^{r,i}$ to $Q^{r,i}$, as there are no edges from $Q^{r,i}$ back to $P^{r,i}$.

Let v_p be the last vertex along $L^{r,i}$ which lies on $P^{r,i}$ and let v_q be the first vertex along $L^{r,i}$ which lies on $Q^{r,i}$. Observe that there is exactly one path from $p^{r,i}_{0,1}$ to v_p, which we call P', and exactly one path from v_q to $q^{r,i}_{n,k+1}$, which we call Q'. Because of the linkage $\mathcal{L}^*$, the congestion at P' and at Q' is equal to g, and so no other path of $\mathcal{L}'$ can intersect P' or Q'.

Let $j \in [k]$ and let v_j be the first vertex of $P^{r,i}$ along $L^{r,i,j}$. By construction, v_j is necessarily of the form $p^{r,i}_{a_j,j}$, for some $a_j \in [n]$. As $L^{r,i,j}$ is disjoint from P', the last vertex of P' must occur before $p^{r,i}_{a_j,j}$ along $P^{r,i}$.

Similarly, let u_j be the first vertex of $Q^{r,i}$ along $L^{r,i,j}$. As $L^{r,i,j}$ is disjoint from Q', the first vertex of Q' must occur after u_j along $Q^{r,i}$. Additionally, u_j must occur at or after $q^{r,i}_{a_j,j}$ along $Q^{r,i}$, as earlier vertices are not reachable from v_j.

This implies that $L^{r,i}$ must contain the edge $(p^{r,i}_{a_j,0}, q^{r,i}_{a_j,k+1})$. As this holds for every $j \in [k]$ and every $L^{r,i,j}$, we have that $a_j = a_\ell$ for all $j, \ell \in [k]$. Let $a_i \in [n]$ be the value equal to all a_j. Also, for each $j \in [k]$, the path $L^{r,i,j}$ leaves the gadget $\mathrm{SEL}^{r,i}$ at the vertex $q^{r,i}_{a_i,j}$.

By an analogous argument, we obtain a value b_j from the selector $\mathrm{SEL}^{c,j}$ and conclude that, for every $i \in [k]$, the path $L^{c,i,j}$ leaves $\mathrm{SEL}^{c,j}$ at the vertex $q^{c,i,j}_{b_j,i}$. We claim that $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$ is a solution for I_G.

Let $i, j \in [k]$. We show that $(a_i, b_j) \in S_{i,j}$. As shown above, the path $L^{r,i,j}$ enters $\mathrm{VER}^{i,j}$ by an edge of the form $(q^{r,i}_{a_i,j}, a^{r,i,j}_{a_i,x})$, for some $x \in [n]$. Analogously, the path $L^{c,i,j}$ enters $\mathrm{VER}^{i,j}$ by an edge of the form $(q^{c,j}_{b_j,i}, a^{c,i,j}_{\ell,b_j})$, for some $\ell \in [n]$.

By an analogous argument as before, we conclude that the path $L^{i,j}$ must contain exactly one edge of the form $(a^{i,j}_{\ell',x'}, b^{i,j}_{\ell',x'})$, as otherwise it would intersect both $L^{r,i,j}$ and $L^{c,i,j}$, causing congestion greater than g at some vertex. In particular, this implies that $\ell' = a_i = \ell$ and $x' = b_j = x$. Hence, $L^{r,i,j}$ must contain the edge $(a^{r,i,j}_{a_i,b_j}, b^{r,i,j}_{a_i,b_j})$ and $L^{c,i,j}$ must contain the edge $(a^{c,i,j}_{a_i,b_j}, b^{c,i,j}_{a_i,b_j})$. This is only possible if $(a_i, b_j) \in S_{i,j}$, as otherwise these edges do not exist in D.

Hence, for all $i, j \in [k]$, we have $(a_i, b_j) \in S_{i,j}$, and so I_G is a *yes* instance. ◀

3.2 Excluding an acyclic $(5, 5)$-grid

▶ **Lemma 4** ($\star$). *For each $1 \leq i, j \leq k$, none of the gadgets $\mathrm{VER}^{i,j}$, $\mathrm{SEL}^{r,i}$ and $\mathrm{SEL}^{c,i}$ contain any acyclic $(3, 3)$-grid as a butterfly minor.*

The following observation can easily be verified from the construction and will be useful in subsequent proofs.

▶ **Observation 5.** *Let $(n, \{S_{i,j}\}^k_{i,j=1})$ be a* GridTiling *instance. Let (D, T) be the instance given by Construction 2. For all $i, i', j, j' \in [k]$ and all $\circ \in \{r, c\}$, the following holds*
1. *there are no edges leaving $V(\mathrm{VER}^{i,j})$,*
2. *there are no edges coming into $V(\mathrm{SEL}^{\circ,i})$,*
3. *if there is an edge from $V(\mathrm{SEL}^{r,i'})$ to $V(\mathrm{VER}^{i,j})$, then $i' = i$, and*
4. *if there is an edge from $V(\mathrm{SEL}^{c,j'})$ to $V(\mathrm{VER}^{i,j})$, then $j' = j$.*

▶ **Corollary 6.** *Let $(n, \{S_{i,j}\}^k_{i,j=1})$ be an instance of* GridTiling. *Let (D, T) be the reduced instance constructed by Construction 2. Let H be an acyclic digraph with exactly one source and exactly one sink. If H is a butterfly minor of D with model μ, then there exist $i, j \in [n]$ such that $\mathrm{Im}(\mu) \subseteq D[V(\mathrm{SEL}^{r,i}) \cup V(\mathrm{VER}^{i,j})]$ or $\mathrm{Im}(\mu) \subseteq D[V(\mathrm{SEL}^{c,j}) \cup V(\mathrm{VER}^{i,j})]$.*

We can now exclude an acyclic $(5, 5)$-grid by arguing that such a grid in the constructed digraph would imply the existence of an $(3, 3)$-grid in one of the selectors or one of the verifiers.

▶ **Lemma 7.** *Let $(n, \{S_{i,j}\}^k_{i,j=1})$ be an instance of* GridTiling. *Let (D, T) be the reduced instance constructed by Construction 2. Then D does not contain any acyclic $(5, 5)$-grid as a butterfly minor.*

Proof. We assume towards a contradiction that D contains an acyclic $(5,5)$-grid G as a butterfly minor. Let μ be the model of G in D.

G has exactly one source and one sink. By Corollary 6, there exist $i,j \in [n]$ such that $\text{Im}(\mu) \subseteq D[V(\text{SEL}^{r,i}) \cup V(\text{VER}^{i,j})]$ or $\text{Im}(\mu) \subseteq D[V(\text{SEL}^{c,j}) \cup V(\text{VER}^{i,j})]$. Without loss of generality we assume that $\text{Im}(\mu) \subseteq D[V(\text{SEL}^{r,i}) \cup V(\text{VER}^{i,j})]$.

Define a partition of the vertices of G as follows

$$A \coloneqq \{v \in V(G) \mid c_\mu(v) \in V(\text{SEL}^{r,i})\},$$
$$B \coloneqq \{v \in V(G) \mid c_\mu(v) \in V(\text{VER}^{i,j})\}.$$

By Observation 5, there is no edge from B to A, as otherwise the model would give a path from $c_\mu(b) \in V(\text{VER}^{i,j})$ to $c_\mu(a) \in V(\text{SEL}^{r,i})$ for some $b \in B$ and $a \in A$.

Let v be the vertex in the center of the acyclic $(5,5)$-grid G.

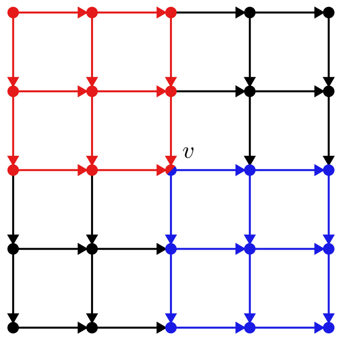

Figure 6 The acyclic $(5,5)$-grid G, $v \in V(G)$ and two acyclic $(3,3)$-subgrids (red and blue).

If $v \in A$, then the top left acyclic $(3,3)$-subgrid (see the red colored subgraph in Figure 6) is also in A as there is no edge from B to A. Since there is no edge from $\text{VER}^{i,j}$ to $\text{SEL}^{r,i}$, restricting the domain of μ to this acyclic $(3,3)$-grid and its codomain to $\text{SEL}^{r,i}$ would give a model for the acyclic $(3,3)$-grid as a butterfly minor in $\text{SEL}^{r,i}$. But by Lemma 4, $\text{SEL}^{r,i}$ does not contain any acyclic $(3,3)$-grid as a butterfly minor. Similarly, if $v \in B$, then the bottom right acyclic $(3,3)$-subgrid (see the blue colored subgraph in Figure 6) is also in B. We would get an acyclic $(3,3)$-grid as a butterfly minor in $\text{VER}^{i,j}$. But by Lemma 4, $\text{VER}^{i,j}$ does not contain any acyclic $(3,3)$-grid as a butterfly minor. Hence, we obtain a contradiction. ◀

3.3 Excluding a $\overrightarrow{TT}_9$

▶ **Lemma 8.** *For each $1 \leq i,j \leq k$, none of the gadgets $\text{VER}^{i,j}$, $\text{SEL}^{r,i}$ and $\text{SEL}^{c,i}$ contain any $\overrightarrow{TT}_6$ as a butterfly minor.*

Proof. We prove the statement for the selector $\text{SEL}^{r,i}$. The proof for the remaining gadgets follow analogously. We assume towards a contradiction that $\text{SEL}^{r,i}$ contains a $G \coloneqq \overrightarrow{TT}_6$ as a butterfly minor. Let μ be the model of G in $\text{SEL}^{r,i}$.

Let s be the vertex of G with in-degree 0 and $U \coloneqq V(G - s)$. Then, the indegree of all vertices $u \in U$ is at least 1. Therefore, $c_\mu(u) \neq s^{r,i,j}$ for all $1 \leq j \leq k$.

Define $A \coloneqq V(P^{r,i}), B \coloneqq V(Q^{r,i})$. The five vertices of U lie in $A \cup B$. So $|U \cap A| \geq 3$ or $|U \cap B| \geq 3$.

Without loss of generality, $\{u, w, v\} \in U \cap A$ and $c_\mu(u), c_\mu(w)$ and $c_\mu(v)$ appear in this order on $P^{r,i}$. Then every path from $c_\mu(u)$ to $c_\mu(v)$ contains $c_\mu(w)$. Thus, $\mu(w)$ is not disjoint from $\mu(u) \cup \mu((u, v)) \cup \mu(v)$. A contradiction. The case where $|U \cap B| \geq 3$ follows analogously. ◀

▶ **Lemma 9** ($\star$). *Let $(n, \{S_{i,j}\}_{i,j=1}^k)$ be an instance of* GridTiling*. Let (D, T) be the reduced instance constructed by Construction 2. Then D contains no $\overrightarrow{TT}_9$ as a butterfly minor.*

The analysis of Lemma 9 is essentially tight, as shown by the observation below.

▶ **Observation 10.** *Let $n \geq 5$ and k be a natural number and define $S_{i,j} \coloneqq [n] \times [n]$ for all $i, j \in [k]$ as an instance of* GridTiling*. Let (D, T) be the reduced instance constructed by Construction 2. Then D contains a $\overrightarrow{TT}_8$ as a butterfly minor as shown in Figure 7.*

3.4 Bounded ear anonymity

▶ **Lemma 11.** *Let $(n, \{S_{i,j}\}_{i,j=1}^k)$ be an instance of* GridTiling*. Let (D, T) be the reduced instance constructed by Construction 2. Then $\mathrm{ea}(D) \leq 5$.*

Proof. Let P be a maximal path in D. Let s be the starting point and t the endpoint of P. We construct an ear identify sequence $\bar{a}$ for P as follows.

Let $\mathcal{H}$ be the set of paths of the form $P^{\circ,i}, Q^{\circ,i}, A^{i,j}$ or $B^{i,j}$ in D. Let F be the set of edges leaving some path in $\mathcal{H}$ and entering a different path in $\mathcal{H}$. The next claim follows easily from the construction of D.

▷ Claim 12. Let $(v, u) \in F$ be an edge from some path R_1 to another path R_2, where R_1 is some path of the form $P^{\circ,i}$, $Q^{\circ,i}$ or $A^{i,j}$, and R_2 is some path of the form $Q^{\circ,i}$, $A^{i,j}$ or $B^{i,j}$. Let $v_1 \in V(R_1)$ be a vertex which can reach v and let $u_2 \in V(R_2)$ be a vertex which can be reached by u. Then there is exactly one v_1-u_2 path in D using (v, u).

Let e_s be the outgoing edge of s in P and let e_t be the incoming edge of t in P. Add e_s to $\bar{a}$. Add every arc in $F \cap E(P)$ to $\bar{a}$. Add e_t to $\bar{a}$. As $|F \cap E(P)| \leq 3$, we have that $|\bar{a}| \leq 5$.

We show that P is the unique maximal path of D visiting $\bar{a}$ in order. Observe that every edge leaving some source in D must end in some path in $\mathcal{H}$. Similarly, every edge entering some sink in D must start at some path in $\mathcal{H}$. Let $R_1 \in \mathcal{H}$ be the path in which e_s ends and let $R_2 \in \mathcal{H}$ be the path in which e_t starts.

If $R_1 = R_2$, then there is exactly one path in D from the head of e_s to the tail of e_t. Otherwise, there is an edge in $F \cap E(P)$. Let P' be the shortest subpath of P containing all edges in $F \cap E(P)$. By Claim 12, P' is the unique minimal path in D visiting those edges Further, by construction there is a unique path from the head of e_s to the start of P' and a unique path from the end of P' to the tail of e_t. Hence, P is the unique path in D visiting the edges of $\bar{a}$ in order. As $|\bar{a}| \leq 5$, we conclude that $\mathrm{ea}(D) \leq 5$. ◀

3.5 W[1]-hardness

▶ **Theorem 13.** *For every congestion $g \geq 1$,* Vertex-Disjoint-Paths-With-Congestion *is W[1]-hard with respect to the number k of terminal pairs, even if*

- *The input digraph D is acyclic,*
- *D contains no $\overrightarrow{TT}_9$ as a butterfly minor,*
- *D contains no acyclic $(5, 5)$-grid as butterfly minor, and*
- *$\mathrm{ea}(D) \leq 5$.*

Furthermore, assuming the ETH, no $f(k)n^{o(\sqrt{k/g})}$ time algorithm exists for VDPwC*, even under the above conditions.*

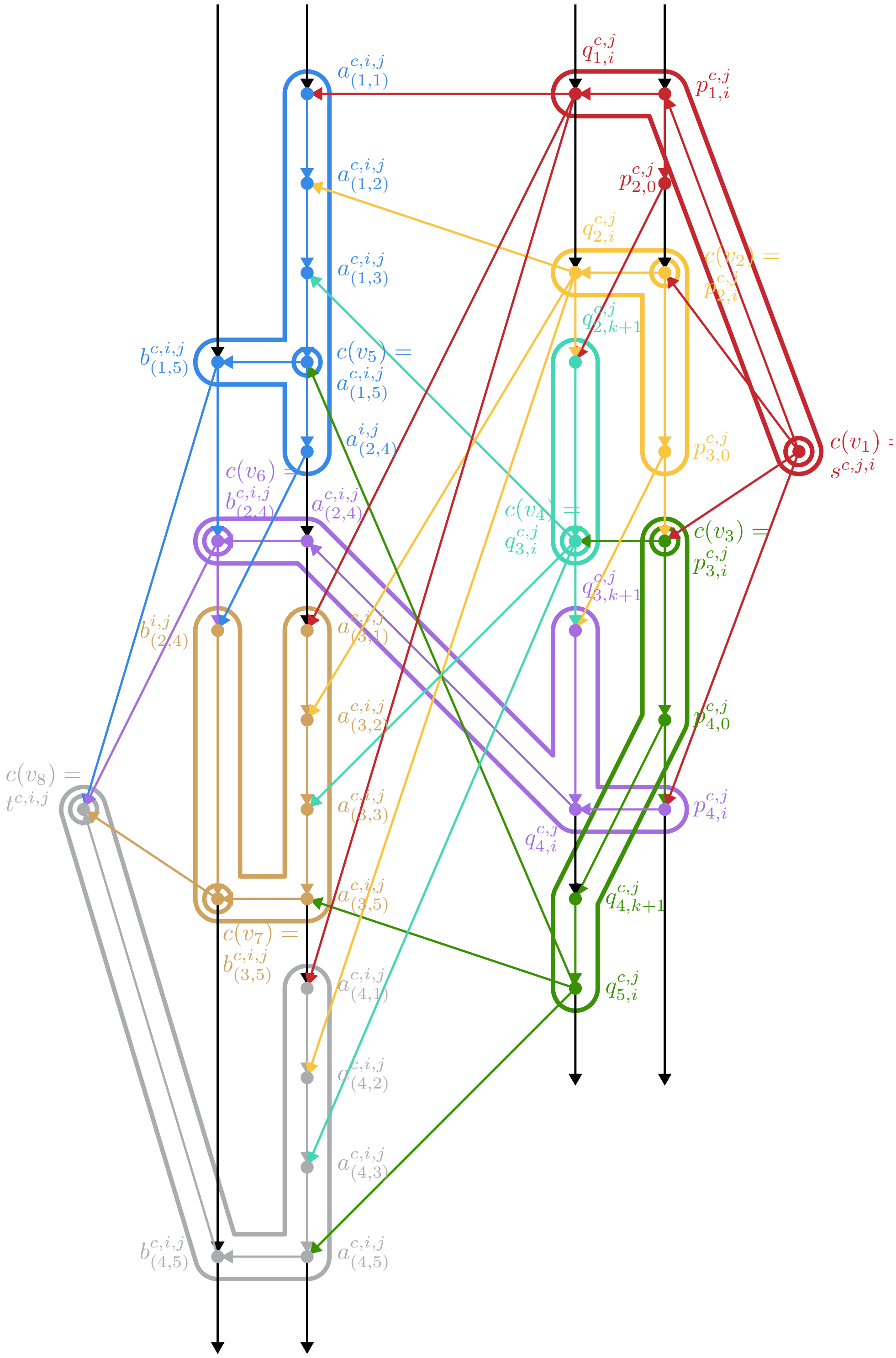

Figure 7 A butterfly model μ of the $\overrightarrow{TT}_8$ in D. Subdivided edges are still drawn as straight lines. The left part is $\mathrm{VER}^{i,j}$ and the right part is $\mathrm{SEL}^{c,j}$. The subgraph $\mu(v_i)$ is indicated by the colored region containing the center $c_\mu(v_i)$ and the path $\mu((v_i, v_j))$ is indicated by the paths leaving and entering the regions and is colored as $\mu(v_i)$.

Proof. We provide a reduction from GridTiling, which is W[1]-hard with respect to k by Theorem 1. Let $I_G = (n, \{S_{i,j}\}_{i,j=1}^k)$ be a GridTiling instance and let $g \geq 1$ be an integer. We construct a VDPwC instance $I_L = (D, T, g)$ by using Construction 2. By Lemma 3, I_L is a *yes* instance if, and only if, I_G is a *yes* instance. By Lemmas 7 and 9, D contains no $\overrightarrow{TT}_9$ and no acyclic $(5,5)$-grid as a butterfly minor. Further, D is acyclic. Additionally, by Lemma 11, $\mathrm{ea}(D) \leq 5$. Finally, $|T| \in O(gk^2)$, and so the existence of an $f(|T|)n^{o(\sqrt{|T|/g})}$-time algorithm for VDPwC would imply the existence of an $f(k)n^{o(k)}$ algorithm for GridTiling, contradicting Theorem 1. ◀

We observe that VDPwC can be solved in polynomial time if $\mathrm{ea}(D) = 1$, as in this case every s_i-t_i path is disjoint from every s_j-t_j path if $i \neq j$, and so taking the shortest s_i-t_i path for every terminal pair (s_i, t_i) yields a solution if any exist.

Further, $\mathrm{ea}(\overrightarrow{TT}_k) = \lceil \frac{k-2}{2} \rceil$ and the acyclic (p,p)-grid has ear anonymity $p - 1$. Since ear anonymity is closed under taking butterfly minors on DAGs [25], bounding $\mathrm{ea}(D)$ also excludes the presence of large acyclic grids and $\overrightarrow{TT}_k$ as butterfly minors.

4 Edge disjoint paths

4.1 Hardness reduction

▶ **Construction 14.** Let (D', T', g) be an instance of VDPwC. We construct an EDPwC instance (D, T, g) iteratively as follows. Apply the split operation to every vertex $v \in V(D')$, obtaining two vertices v_{in} and v_{out} in D.

If $\mathsf{indeg}_{D'}(v) \geq 3$, add a minimal in-tree T_v^{in} with $\mathsf{indeg}_{D'}(v)$ leaves where each vertex of the in-tree has indegree at most 2. Identify each vertex in $\mathsf{in}_D(v_{\mathrm{in}})$ with a distinct leaf of T_v^{in} and remove the edges from $\mathsf{in}_D(v_{\mathrm{in}})$ to v_{in}. Finally, add an edge from the root of T_v^{in} to v_{in}.

If $\mathsf{outdeg}_{D'}(v) \geq 3$, we proceed in an analogous fashion as above, adding an out-tree T_v^{out} instead. This completes the construction of D. Note that $\mathsf{outdeg}_D(v) + \mathsf{indeg}_D(v) \leq 3$ holds for all $v \in V(D)$.

The multiset of terminal pairs T is obtained from T' by replacing every occurrence of $(s,t) \in T'$ with $(s_{\mathrm{in}}, t_{\mathrm{out}})$.

4.2 Excluding a $(7,7)$-wall

It is not too difficult to see that the digraph D given by Construction 14 contains an acyclic $(5,5)$-wall as an immersion. To exclude the acyclic $(7,7)$-wall, we use the fact that the original instance D' does not contain an acyclic $(5,5)$-grid.

▶ **Lemma 15** ($\star$)**.** *Let (D', T', g) be a VDPwC instance. Let (D, T, g) be the EDPwC instance given by Construction 14. If D contains an acyclic $(7,7)$-wall as a weak immersion, then D' contains an acyclic $(5,5)$-grid as a butterfly minor.*

4.3 Bounded ear anonymity

▶ **Observation 16** ($\star$)**.** *Let (D', T', g) be a VDPwC instance where D' is a DAG. Let (D, T, g) be the EDPwC instance given by Construction 14. Let P be a maximal path in D and let $P_1 \cdot P_2 \cdot \ldots \cdot P_n = P$ be a partition of P into subpaths such that P_i is a maximal subpath of P contained inside $T_{v_i}^{in}, (v_{i,in}, v_{i,out})$ and $T_{v_i}^{out}$ for some $v_i \in V(D')$. Let P' be the path obtained by replacing every P_i in P by v_i. Then P' is a path in D'.*

▶ **Lemma 17** ($\star$). *Let (D', T', g) be a VDPwC instance. Let (D, T, g) be the EDPwC instance given by Construction 14. Then $\mathrm{ea}(D) \leq \mathrm{ea}(D')$.*

4.4 W[1]-hardness

The last remaining step for our hardness result is to show that the reduction given by Construction 14 is correct.

▶ **Theorem 18** ($\star$). *For every congestion $g \geq 1$, Edge-Disjoint-Paths-With-Congestion is W[1]-hard with respect to the number k of terminal pairs, even if*

- *The input digraph D is acyclic,*
- *D has maximum undirected degree 3,*
- *D contains no acyclic $(7,7)$-wall as a weak immersion, and*
- $\mathrm{ea}(D) \leq 5$.

Furthermore, assuming the ETH, no $f(k)n^{o(\sqrt{k/g})}$ time algorithm exists for Edge-Disjoint-Paths-With-Congestion, even under the above conditions.

References

1. Saeed Akhoondian Amiri, Stephan Kreutzer, Dániel Marx, and Roman Rabinovich. Routing with congestion in acyclic digraphs. *Inf. Process. Lett.*, 151, 2019. doi:10.1016/J.IPL.2019.105836.
2. Jørgen Bang-Jensen and Gregory Z. Gutin. *Digraphs - Theory, Algorithms and Applications, Second Edition.* Springer Monographs in Mathematics. Springer, 2009.
3. Dario Giuliano Cavallaro, Ken-ichi Kawarabayashi, and Stephan Kreutzer. Edge-disjoint paths in eulerian digraphs. In Bojan Mohar, Igor Shinkar, and Ryan O'Donnell, editors, *Proceedings of the 56th Annual ACM Symposium on Theory of Computing, STOC 2024, Vancouver, BC, Canada, June 24-28, 2024*, pages 704–715. ACM, 2024. doi:10.1145/3618260.3649758.
4. Parinya Chalermsook, Bundit Laekhanukit, and Danupon Nanongkai. Pre-reduction graph products: Hardnesses of properly learning dfas and approximating EDP on dags. In *55th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2014, Philadelphia, PA, USA, October 18-21, 2014*, pages 444–453. IEEE Computer Society, 2014. doi:10.1109/FOCS.2014.54.
5. Chandra Chekuri, Sanjeev Khanna, and F. Bruce Shepherd. Edge-disjoint paths in planar graphs. In *45th Symposium on Foundations of Computer Science (FOCS 2004), 17-19 October 2004, Rome, Italy, Proceedings*, pages 71–80. IEEE Computer Society, 2004. doi:10.1109/FOCS.2004.27.
6. Chandra Chekuri, Sanjeev Khanna, and F. Bruce Shepherd. An O(sqrt(n)) approximation and integrality gap for disjoint paths and unsplittable flow. *Theory Comput.*, 2(7):137–146, 2006. doi:10.4086/TOC.2006.V002A007.
7. Chandra Chekuri, Sanjeev Khanna, and F. Bruce Shepherd. Edge-disjoint paths in planar graphs with constant congestion. *SIAM J. Comput.*, 39(1):281–301, 2009. doi:10.1137/060674442.
8. Rajesh Chitnis. A tight lower bound for edge-disjoint paths on planar dags. *SIAM J. Discret. Math.*, 37(2):556–572, 2023. doi:10.1137/21M1395089.
9. Julia Chuzhoy, David H. K. Kim, and Shi Li. Improved approximation for node-disjoint paths in planar graphs. In Daniel Wichs and Yishay Mansour, editors, *Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2016, Cambridge, MA, USA, June 18-21, 2016*, pages 556–569. ACM, 2016. doi:10.1145/2897518.2897538.
10. Julia Chuzhoy, David H. K. Kim, and Rachit Nimavat. New hardness results for routing on disjoint paths. *SIAM J. Comput.*, 51(2):17–189, 2022. doi:10.1137/17M1146580.
11. Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms.* Springer, 2015. doi:10.1007/978-3-319-21275-3.

12 Marek Cygan, Dániel Marx, Marcin Pilipczuk, and Michal Pilipczuk. The planar directed k-vertex-disjoint paths problem is fixed-parameter tractable. In *54th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2013, 26-29 October, 2013, Berkeley, CA, USA*, pages 197–206. IEEE Computer Society, 2013. doi:10.1109/FOCS.2013.29.

13 Rodney G. Downey and Michael R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013. doi:10.1007/978-1-4471-5559-1.

14 Alina Ene, Matthias Mnich, Marcin Pilipczuk, and Andrej Risteski. On routing disjoint paths in bounded treewidth graphs. In Rasmus Pagh, editor, *15th Scandinavian Symposium and Workshops on Algorithm Theory, SWAT 2016, June 22-24, 2016, Reykjavik, Iceland*, volume 53 of *LIPIcs*, pages 15:1–15:15. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2016. doi:10.4230/LIPICS.SWAT.2016.15.

15 Steven Fortune, John Hopcroft, and James Wyllie. The directed subgraph homeomorphism problem. *Theoretical Computer Science*, 10(2):111–121, 1980. doi:10.1016/0304-3975(80)90009-2.

16 Russell Impagliazzo and Ramamohan Paturi. On the complexity of k-sat. *J. Comput. Syst. Sci.*, 62(2):367–375, 2001. doi:10.1006/JCSS.2000.1727.

17 Russell Impagliazzo, Ramamohan Paturi, and Francis Zane. Which problems have strongly exponential complexity? *J. Comput. Syst. Sci.*, 63(4):512–530, 2001. doi:10.1006/JCSS.2001.1774.

18 Thor Johnson, Neil Robertson, Paul Seymour, and Robin Thomas. Directed tree-width. *Journal of Combinatorial Theory, Series B*, 82(1):138–154, 2001. doi:10.1006/JCTB.2000.2031.

19 Ken-ichi Kawarabayashi, Yusuke Kobayashi, and Bruce A. Reed. The disjoint paths problem in quadratic time. *J. Comb. Theory B*, 102(2):424–435, 2012. doi:10.1016/J.JCTB.2011.07.004.

20 Ken-ichi Kawarabayashi, Nicola Lorenz, Marcelo Garlet Milani, and Jacob Stegemann. Directed disjoint paths remains w[1]-hard on acyclic digraphs without large grid minors. *CoRR*, abs/2507.09868, 2025. doi:10.48550/arXiv.2507.09868.

21 Jon M. Kleinberg and Éva Tardos. Approximations for the disjoint paths problem in high-diameter planar networks. *J. Comput. Syst. Sci.*, 57(1):61–73, 1998. doi:10.1006/JCSS.1998.1579.

22 Stavros G. Kolliopoulos and Clifford Stein. Approximating disjoint-path problems using packing integer programs. *Math. Program.*, 99(1):63–87, 2004. doi:10.1007/S10107-002-0370-6.

23 Tuukka Korhonen, Michal Pilipczuk, and Giannos Stamoulis. Minor containment and disjoint paths in almost-linear time. In *65th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2024, Chicago, IL, USA, October 27-30, 2024*, pages 53–61. IEEE, 2024. doi:10.1109/FOCS61266.2024.00014.

24 Dániel Marx. A tight lower bound for planar multiway cut with fixed number of terminals. In Artur Czumaj, Kurt Mehlhorn, Andrew M. Pitts, and Roger Wattenhofer, editors, *Automata, Languages, and Programming - 39th International Colloquium, ICALP 2012, Warwick, UK, July 9-13, 2012, Proceedings, Part I*, volume 7391 of *Lecture Notes in Computer Science*, pages 677–688. Springer, 2012. doi:10.1007/978-3-642-31594-7_57.

25 Marcelo Garlet Milani. Directed ear anonymity. In José A. Soto and Andreas Wiese, editors, *LATIN 2024: Theoretical Informatics - 16th Latin American Symposium, Puerto Varas, Chile, March 18-22, 2024, Proceedings, Part II*, volume 14579 of *Lecture Notes in Computer Science*, pages 77–97. Springer, 2024. doi:10.1007/978-3-031-55601-2_6.

26 Neil Robertson and Paul D. Seymour. Graph minors .xiii. the disjoint paths problem. *J. Comb. Theory B*, 63(1):65–110, 1995. doi:10.1006/JCTB.1995.1006.

27 Aleksandrs Slivkins. Parameterized tractability of edge-disjoint paths on directed acyclic graphs. *SIAM Journal on Discrete Mathematics*, 24(1):146–157, 2010. doi:10.1137/070697781.

28 Michal Wlodarczyk. Constant approximating disjoint paths on acyclic digraphs is w[1]-hard. In Julián Mestre and Anthony Wirth, editors, *35th International Symposium on Algorithms and Computation, ISAAC 2024, December 8-11, 2024, Sydney, Australia*, volume 322 of *LIPIcs*, pages 57:1–57:16. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2024. doi:10.4230/LIPICS.ISAAC.2024.57.