Geodetic Set on Graphs of Constant Pathwidth and Feedback Vertex Set Number

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Abstract

In the Geodetic Set problem, the input consists of a graph G and a positive integer k. The goal is to determine whether there exists a subset S of vertices of size k such that every vertex in the graph is included in a shortest path between two vertices in S. Kellerhals and Koana [IPEC 2020; J. Graph Algorithms Appl 2022] proved that the problem is W[1]-hard when parameterized by the pathwidth or the feedback vertex set number of the input graph. They posed the question of whether the problem admits an XP-algorithm when parameterized by the combination of these two parameters. We answer this in the negative by proving that the problem remains NP-hard even on graphs of constant pathwidth and feedback vertex set number.

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1 Introduction

Consider a simple undirected graph G with vertex set $V(G)$ and edge set $E(G)$. For two vertices $u, v \in V(G)$, let $I(u, v)$ be the set of all vertices in G that are part of some shortest path between u and v. For a subset S of vertices, we generalize this definition to $I(S)$ by taking the union of $I(u, v)$ for all pairs of vertices u, v in S. A subset of vertices T is said to be *covered* by S if $T \subseteq I(S)$. A set of vertices S is a *geodetic set* if $V(G)$ is covered by S. In the Geodetic Set problem, the input is a graph G and a positive integer k, and the objective is to determine whether there is a geodetic set of size k. This problem was introduced in 1993 by Harary, Loukakis, and Tsouros [27]. We refer readers to [9, 21] for application of the problem. See [22, 37] for a broader discussion on geodesic convexity in graphs.

The Geodetic Set problem falls under the broad category of *metric graph problems*. These problems are defined using a metric on the graph, with the shortest distance between two vertices being the most commonly used metric. Metric graph problems include many important NP-complete graph problems such as Distance d-Dominating Set (also called (k, d)-Center), Distance d-Independent Set (also called d-Scattered Set), and Metric Dimension to name a few. These problems have played a central role in the development of both classical and parameterized algorithms and complexity theory [32, 33, 3, 35, 4, 10, 23, 29, 11], as they behave quite differently from their more "local" (neighborhood-based) counterparts such as Dominating Set or Independent Set.

Due to the non-local nature of metric graph problems, most algorithmic techniques fail to yield positive results. Consider the case of the Metric Dimension problem in which an input is a graph G and an integer k, and the objective is to determine whether there

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is a set S of at most k vertices such that for every pair of distinct vertices u and v, there exists a vertex in S that has different distances to u and v. In a seminal paper, Hartung and Nichterlein [28] proved that the problem is W[2]-hard when parameterized by the solution size k. This motivated the structural parameterization of the problem. See, for example, Galby et al. [25] for an overview. The complexity of Metric Dimension parameterized by treewidth remained an open problem for a long time, until Bonnet and Purohit [5] made the first breakthrough by proving that the problem is W[1]-hard when parameterized by pathwidth. This result was strengthened in two directions: Li and Pilipczuk [35] proved that it is para-NP-hard when parameterized by pathwidth, whereas Galby et al. [25] showed that the problem is W[1]-hard when parameterized by pathwidth plus the feedback vertex set number of the graph.

Continuing the exploration of hardness results, we mention some results that hold for both Metric Dimension and Geodetic Set. Foucaud et al. [23] demonstrated that these two problems on graphs of bounded diameter admit double exponential conditional lower bounds when parameterized by treewidth, along with a matching algorithm. The same set of authors also proved that both problems admit "exotic" conditional lower bounds when parameterized by the vertex cover number [24]. Bergougnoux et al. [4] showed that enumerating minimal solution sets for both problems in split graphs is equivalent to enumerating minimal transversals of hypergraphs, arguably the most important open problem in algorithmic enumeration.

As illustrated in the examples above, Metric Dimension and Geodetic Set behave similarly concerning hardness results. Kellerhals and Koana [34] proved that Geodetic Set is W[1]-hard when parameterized by the pathwidth and feedback vertex set number (and the solution size) combined, a result similar to that of Galby et al. [25] regarding Metric Dimension. However, a counterpart to the result of Li and Pilipczuk [35] for Geodetic Set is not yet known. More formally, we do not know the complexity of the Geodetic Set problem on graphs of constant treewidth. Chakraborty et al. [8] mentioned the possibility of obtaining XP-algorithm parameterized by the treewidth. Kellerhals and Koana [34] posed an even stronger question: Does Geodetic Set admit an XP-algorithm when parameterized by the pathwidth and feedback vertex set number? We answer these questions in the negative.

▶ **Theorem 1.** *Geodetic Set is* NP*-complete even on graphs of pathwidth at most* 17 *and feedback vertex set number at most* 13*.*

This result is particularly surprising in light of the closely related problem Geodetic Hull. In Geodetic Hull, the input is a graph G and an integer k, and the objective is to determine whether there is a vertex set S of size k such that $I^j[S] = V(G)$ for some $j > 0$, where $I^0[S] = S$ and $I^j[S] = I[I^{j-1}[S]]$ for $j > 0$. It is easy to see that both these problems are trivial on trees since selecting all leaves is both necessary and sufficient for a solution. Kante et al. [30] proved, among other things, that the problem admits an XP-algorithm parameterized by treewidth. Hence, while Geodetic Hull is polynomial-time solvable on graphs of constant treewidth, Geodetic Set remains NP-hard.

Related Work. As is often the case with metric-based problems, Geodetic Set is computationally hard – even on highly structured graph classes. See [2, 7, 8, 9, 12, 13, 18, 19, 20, 21, 27] for various earlier hardness results. Geodetic Set can be solved in polynomial time on split graphs [19, 20], well-partitioned chordal graphs [1], outerplanar graphs [36], ptolemaic graphs [22], cographs [19], distance-hereditary graphs [31], block-cactus graphs [21], solid grid graphs [8, 12], and proper interval graphs [21]. A two-player game variant of Geodetic Set was introduced in [6].

As mentioned before, Kellerhals and Koana in [34] studied the parameterized complexity of Geodetic Set. They observed that a reduction from [19] implies that the problem is W[2]-hard when parameterized by the solution size – even for chordal bipartite graphs. They proved the problem to be W[1]-hard for the parameters solution size, feedback vertex set number, and pathwidth, combined [34]. On the positive side, they showed that Geodetic Set is FPT for the parameters treedepth, modular-width (more generally, clique-width plus diameter), and feedback edge set number [34]. Chakraborty et al. [8] proved that the problem is FPT on chordal graphs when parameterized by the treewidth. On the approximation side, it is known that the minimization variant of the problem is NP-hard to approximate within a factor of $o(\log n)$, even on graphs of diameter 2 [9] and subcubic bipartite graphs of arbitrarily large girth [15].

2 Preliminaries

For an integer n, we let $[n] = \{1, \ldots, n\}$. We use standard graph-theoretic notation and refer the reader to [16] for any undefined notation. For an undirected graph G, the sets $V(G)$ and $E(G)$ denote its set of vertices and edges, respectively. Two vertices $u, v \in V(G)$ are *adjacent* or *neighbors* if $(u, v) \in E(G)$. We say a vertex v is a *pendant vertex* if its degree is one. A vertex v is a *branching vertex* if its degree is at least three. A path is a collection of vertices $\{v_1, v_2, \ldots, v_\ell\}$ such that $(v_i, v_{i+1}) \in E(G)$ for every i in $[\ell - 1]$. In this case, v_1 and v_ℓ are two endpoints, whereas all the other vertices are called *internal vertices*. We say a path is simple if all its internal vertices have degree precisely two. The length of a path is the number of edges in it. The *distance* between two vertices u, v in G, denoted by $\text{dist}(u, v)$, is the length of a shortest path connecting u to v. For a subset S of $V(G)$, we denote the graph obtained by deleting S from G by $G - S$. Recall that a subset $S \subseteq V(G)$ is a *geodetic set* if for every $u \in V(G)$ there exist $s_1, s_2 \in S$ such that u lies on a shortest path from s_1 to s_2. Consider a pendant vertex v in G. Note that v does not belong to any shortest path between any pair x, y of vertices which are distinct from v. Hence, v belongs to every geodetic set of G. This observation was also made in [13].

We refer readers to Cygan et al. [14] for unspecified terms related to parameterized complexity. We define some of the parameters mentioned in this article. For a graph G, a set $X \subseteq V(G)$ is a *feedback vertex set* of G if $G - X$ is an acyclic graph. The *feedback vertex set number* of graph G, denoted by $\texttt{fvs}(G)$, is the size of a minimum feedback vertex set of G. We denote by $\texttt{tw}(G)$ and $\texttt{pw}(G)$ as the treewidth and pathwidth of graph G, respectively. It is easy to verify that $\texttt{tw}(G) \leq \texttt{fvs}(G) + 1$ and $\texttt{tw}(G) \leq \texttt{pw}(G)$ whereas $\texttt{pw}(G)$ and $\texttt{fvs}(G)$ are incomparable.

We define an equivalent definition of pathwidth via mixed search games. In a mixed search game, a graph G is considered a system of tunnels. Initially, all edges are contaminated by a gas. An edge is cleared by placing searchers at both its endpoints simultaneously or by sliding a searcher along the edge. A cleared edge is re-contaminated if there is a path from an uncleared edge to the cleared edge without any searchers on its vertices or edges. A search is a sequence of operations that can be of the following types: (*i*) placement of a new searcher on a vertex; (*ii*) removal of a searcher from a vertex; (*iii*) sliding a searcher on a vertex along an incident edge to its other endpoint and placing the searcher there. A search strategy is winning if, after its termination, all edges are cleared. The mixed search number of a graph G, denoted by $\texttt{ms}(G)$, is the minimum number of searchers required for a winning strategy of mixed searching on G. Takahashi et al. [38] proved that $\texttt{pw}(G) \leq \texttt{ms}(G) \leq \texttt{pw}(G) + 1$, which we use to bound the pathwidth of the graphs obtained in the reduction.

3 NP-Hardness

In this section, we prove that the problem is para-NP-hard when parameterized by pathwidth and the feedback vertex set number of the input graph. We present a polynomial-time reduction from the 3-Dimensional Matching problem, which is known to be NP-hard [26, SP 1]. For notational convenience, we work with the following definition of the problem. The input consists of a universe $\mathcal{U} = \{\alpha, \beta, \gamma\} \times [n]$, a family $\mathcal{S}$ of m subsets of $\mathcal{U}$ such that for every set $S \in \mathcal{S}$, $S = \{(\alpha, a), (\beta, b), (\gamma, c)\}$ for some $a, b, c \in [n]$. The goal is to find a subset $\mathcal{S}' \subseteq \mathcal{S}$ of size n that partitions $\mathcal{U}$.

Reduction

The reduction takes as input an instance $(\mathcal{U}, \mathcal{S})$ of 3-Dimensional Matching and returns an instance (G, k) of Geodetic Set in polynomial time where pathwidth plus feedback vertex set number of G is a constant. Recall that a branching vertex is a vertex of degree at least three. We start by specifying the common branching vertices in G and the lengths of simple paths connecting them.

Common Branching Vertices. The reduction starts constructing G by adding the following vertices: $\{g_1, g_2, g_3, g_4\}$, $\{p^\alpha, q^\alpha, r^\alpha\}$, $\{p^\beta, q^\beta, r^\beta\}$, and $\{p^\gamma, q^\gamma, r^\gamma\}$. All set-encoding gadgets and element-encoding gadgets, which we define below, are connected via these common branching vertices. These common branching vertices are denoted using grey shaded region in all the figures. Apart from the vertices in $\{q^\alpha, q^\beta, q^\gamma\}$, the reduction adds pendant vertices adjacent to each common branching vertex. In all the figures, vertices that are adjacent to a pendant vertex are highlighted by enclosed squares around them. Define an integer M such that $M \in \mathcal{O}(n^2)$ and for any $a \in [n]$, we have $0.99 \cdot M^2 \leq M^2 - 2aM, M^2 + 2aM \leq 1.01 \cdot M^2$. The reduction connects g_1 to g_2 and g_3 to g_4 using a path of length M^2 for each connection. See Figure 1.

Set-encoding Gadget. The reduction adds n identical set-encoding gadgets denoted by $\mathcal{S}^1, \mathcal{S}^2, \ldots, \mathcal{S}^n$. For any $i \in [n]$, set-encoding gadget $\mathcal{S}^i$ contains sets of vertices X^i, Y^i, and two auxiliary vertices z_1^i and z_2^i. Sets X^i and Y^i contain m vertices each. The reduction adds matching edges across X^i and Y^i. Each of these matching edges and their endpoints corresponds to a unique set in $\mathcal{S}$. Then, the reduction replaces each of these matching edges with a simple path of length M^2. For each vertex in Y^i, the reduction adds a pendant vertex adjacent to it. It connects z_1^i with every vertex in $\{z_2^i\} \cup X^i$ using a simple path of length M^2 for each connection.

We now specify how the vertices mentioned above are connected to common branching vertices. The reduction connects (i) g_1 to z_1^i and z_2^i, (ii) g_2 to z_1^i, and (iii) g_4 to every vertex in Y^i, using paths of length M^2 for each connection. It connects g_2 to every vertex in Y^i using a path of length $M^2 - 1$. We remark that these are the only paths that are of length $M^2 - 1$. To specify the connection to the remaining common branching vertices, consider a set $S = \{(\alpha, a'), (\beta, b'), (\gamma, c')\}$ in $\mathcal{S}$ and the vertex x corresponding to it in X^i. The reduction connects x to all the remaining common branching vertices such that

- $\text{dist}(x, p^\alpha) = M^2 + 2a'M$, $\text{dist}(x, q^\alpha) = M^2 - a'M$, and $\text{dist}(x, r^\alpha) = M^2 - 2a'M$;
- $\text{dist}(x, p^\beta) = M^2 + 2b'M$, $\text{dist}(x, q^\beta) = M^2 - b'M$, and $\text{dist}(x, r^\beta) = M^2 - 2b'M$; and
- $\text{dist}(x, p^\gamma) = M^2 + 2c'M$, $\text{dist}(x, q^\gamma) = M^2 - c'M$, and $\text{dist}(x, r^\gamma) = M^2 - 2c'M$.

See Figure 2 for an illustration.

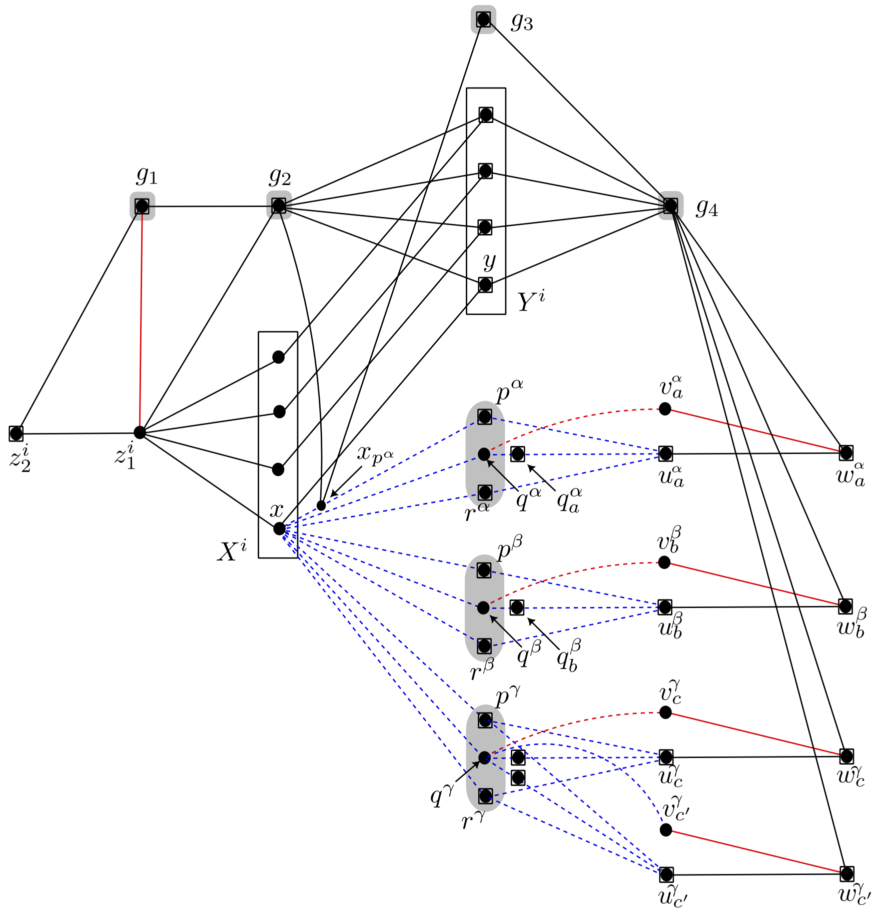

Figure 1 The figure shows the i^{th} copy of set-encoding gadget $\{z_2^i, z_1^i\} \cup X^i \cup Y^i$, element-encoding gadgets encoding elements of the form (α, a), (β, b), (γ, c) and (γ, c'), for some $a, b, c, c' \in [n]$, and the common branching vertices. Each solid (black or red) edge represents a path of fixed length which is either M^2 or $M^2 - 1$. Each dotted (blue or red) edge represents a path whose length depends on the set or element corresponding to one of its endpoints. Apart from the internal vertices in the red edges, all the other vertices are covered by pendant vertices in the graph. For clarity, we show connecting paths starting at only one vertex in X^i. Moreover, it only shows one of the nine vertices, viz x_{p^α}, that are adjacent to x and are connected to g_2 and g_3.

Now, consider the vertex on the path connecting x to p^α which is adjacent to x. We denote this vertex by x_{p^α}. The reduction connects x_{p^α} with g_2 and g_3 using the paths of length M^2 each. It repeats this process to add vertices x_{q^α}, x_{r^α}, x_{p^β}, x_{q^β}, x_{r^β}, x_{p^γ}, x_{q^γ}, and x_{r^γ}, and their connection to g_2 and g_3.

Element-Encoding Gadget. Consider an element (α, a) in $\mathcal{U}$. The reduction adds three vertices u_a^α, v_a^α and w_a^α. It adds a pendant vertex adjacent to u_a^α and another pendant vertex adjacent to w_a^α. It connects w_a^α with u_a^α and v_a^α using a simple path of length M^2 for each connection.

The reduction connects u_a^α with $\{p^\alpha, q^\alpha, r^\alpha\}$ using simple paths such that

- $\text{dist}(u_a^\alpha, p^\alpha) = M^2 - 2aM$, $\text{dist}(u_a^\alpha, q^\alpha) = M^2 + aM$, and $\text{dist}(u_a^\alpha, r^\alpha) = M^2 + 2aM$.

It connects v_a^α (only) to q^α such that $\text{dist}(v_a^\alpha, q^\alpha) = M^2 + aM$. See Figure 2 for an illustration. Consider the vertex in the simple path connecting q^α and u_a^α which is adjacent with q^α. We denote the vertex by q_a^α. The reduction adds a pendant vertex and makes it adjacent with q_a^α.

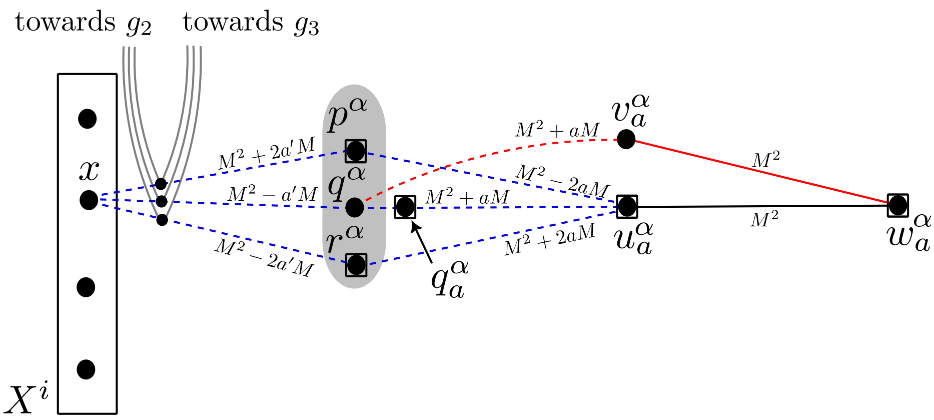

Figure 2 On the left of the grey (shaded) area: Connection between some common branching vertices and a vertex x in X^i which corresponds to set $S = \{(\alpha, a'), (\beta, b'), (\gamma, c')\}$, for some $a', b', c' \in [n]$. The figure also shows vertices x_{p^α}, x_{q^α}, and x_{r^α}. On the right of the grey (shaded) area: Connection between some common vertices and element-encoding gadget encoding (α, a) for some $a \in [n]$.

The reduction repeats the constructions for elements (β, b) and (γ, c) in $\mathcal{U}$. In the first case, the vertices in the gadgets are connected to $\{p^\beta, q^\beta, r^\beta\}$ whereas in the latter case they are connected to $\{p^\gamma, q^\gamma, r^\gamma\}$.

Finally, for any vertices of the type w_a^α, w_b^β, and w_c^γ added in the above steps, the reduction connects each of them to g_4 using simple paths of length M^2 for each connection. See Figure 1.

This completes the construction. Suppose P is the collection of all the pendant vertices in G. The reduction returns $(G, k = n + |P|)$ as an instance of Geodetic Set.

Note that each edge in Figure 1 corresponds to a simple path whose length is at least $0.99 \cdot M^2$ and at most $1.01 \cdot M^2$. Hence, the distance between any two vertices in Figure 1, where we use an edge to denote a simple path connecting two of its endpoints, is proportional to their distance in G.

Define P to be the collection of all the pendants vertices in G. To prove the correctness of the reduction, we first argue that vertices in P covers most of the vertices in G. We argue that the only vertices they do not cover are internal vertices in the red edges of Figure 1. We define these uncovered vertices as follows. Define V^{g_1} as the collection of all the internal vertices in the simple path connecting g_1 to z_1^i for any $i \in [n]$. For every $a \in [n]$, define V_a^α as the collection of internal vertices in the simple path connecting q^α to u_a^α and simple path connecting u_a^α to w_a^α. Define $V^\alpha = \bigcup_{a \in [n]} V_a^\alpha$. Similarly, define V^β and V^γ.

Recall that $I(u, v)$ denotes the set of vertices in G that are part of some shortest path between u and v. Also, we generalize this definition to set S by taking the union of $I(u, v)$ for all pairs of vertices u, v in S.

▶ **Lemma 2.** *Let P be the collection of pendant vertices in G. Then, $V(G) \setminus (V^{g_1} \cup V^\alpha \cup V^\beta \cup V^\gamma) = I(P)$.*

Proof. We first prove that $V(G) \setminus (V^{g_1} \cup V^\alpha \cup V^\beta \cup V^\gamma) \subseteq I(P)$. By the construction, any vertex u in G is adjacent with at most one pendant vertex. We use $\mathtt{pndt}(u)$ to denote the pendant vertex adjacent with u, if it exists. We first consider the partition of vertices in G based on the simple paths they are part of.

Consider the simple paths ending at g_1. From the construction, $I(\mathsf{pndt}(g_1), \mathsf{pndt}(g_2))$ contains all the vertices in the simple path connecting g_1 and g_2. Similarly, $I(\mathsf{pndt}(z_2^i), \mathsf{pndt}(g_1))$ covers all the vertices in the simple path connecting z_2^i and g_1 for every $i \in [n]$. Note that the statement of the lemma excludes the vertices in simple paths connecting g_1 and z_1^i.

Consider the simple paths ending at g_2. From the construction, $I(\mathsf{pndt}(z_2^i), \mathsf{pndt}(g_2))$ covers all the vertices in the simple path connecting z_1^i to g_2 for every $i \in [n]$. Consider vertices x in X^i and y in Y^i which are the part of i^{th} set-encoding gadget $\mathcal{S}^i$ for some i in $[n]$. $I(\mathsf{pndt}(y), \mathsf{pndt}(g_2))$ covers all the vertices in the simple path connecting y to g_2. Also, $I(\mathsf{pndt}(g_2), \mathsf{pndt}(g_3))$ covers all the vertices in the path connecting g_2 with x_{p^α}. A similar statement is true for the other branching vertices adjacent to x.

Consider the simple paths ending at g_3. From the construction, $I(\mathsf{pndt}(g_3), \mathsf{pndt}(g_4))$ contains all the vertices in the simple path connecting g_3 and g_4. As argued in the previous paragraph, $I(\mathsf{pndt}(g_2), \mathsf{pndt}(g_3))$ covers all the vertices in the path connecting g_3 with all the branching vertices like x_{p^α} that are adjacent with x.

Consider the simple paths ending at g_4. For all such paths, there other endpoints are adjacent to some pendant vertex. By the construction, the shortest path between the pendant vertices adjacent to the endpoints covers all the vertices in the simple path.

Consider the simple paths ending at p^α, p^β or p^γ. All the simple paths ending at p^α can be partitioned into two parts: ones that have the other endpoint in a set-encoding gadget (i.e. x_{p^α} for some x in X^i for some $i \in [n]$) or ones that have the other endpoint in an element-encoding gadget (i.e. u_a^α for some $a \in [n]$). In the second case, both the endpoints of the simple path are adjacent to pendant vertices. Hence, by the construction, the shortest path between the pendant vertices adjacent to the endpoints cover all the vertices in the simple path. In the first case, suppose the other endpoint of the simple path is x_{p^α} for some x in X^i. Consider vertex x and the unique vertex y in Y^i which is at the distant M^2 from x. By the construction, there is a shortest path from p^α to y that contains x. Hence, $I(\mathsf{pndt}(p^\alpha), \mathsf{pndt}(y))$ covers all the vertices in the simple path connecting p^α to x. Since x is an arbitrary point in X^i, the statement is true for all the vertices in the first type of simple path. These vertices are also covered by $I(\mathsf{pndt}(p^\alpha), \mathsf{pndt}(g_3))$. A similar set of arguments implies that all the vertices in the simple path ending at p^β and p^γ are covered by vertices in P.

Consider the simple paths ending at q^α, q^β or q^γ. Once again, all the simple paths ending at q^α can be partitioned into two parts: ones that have the other endpoint in a set-encoding gadget (i.e. x_{q^α} for some x in X^i for some $i \in [n]$) or ones that have the other endpoint in an element-encoding gadget (i.e. u_a^α or v_a^α for some $a \in [n]$). Fix an integer $a \in [n]$ and recall that q_a^α is adajcent with q^α. Using the same arguments as in the above paragraph, $I(\mathsf{pndt}(q_a^\alpha), \mathsf{pndt}(y))$ covers all the vertices in the simple path connecting q^α and x. This proves the claim for the vertices in the first type of path. $I(\mathsf{pndt}(q_a^\alpha), \mathsf{pndt}(u_a^\alpha))$ covers all the vertices in the path connecting q_a^α and u_a. Note that the vertices in the simple path connecting q_a^α and v_a are in V^α, which are excluded in the statement of the lemma. A similar set of arguments implies that all the vertices in the simple path ending at q^β and q^γ are covered by vertices in P.

Consider the simple paths ending at r^α, r^β or r^γ. A similar set of arguments regarding paths ending at p^α, p^β or p^γ proves that all the vertices in simple paths ending at r^α, r^β or r^γ are covered by vertices in P.

Consider the simple paths whose endpoints are contained in a set-encoding gadget. Consider a set-encoding gadget $\mathcal{S}^i$ for a fixed i in $[n]$. The shortest paths connecting z_2^i and g_3 covers all vertices in simple paths connecting z_2^i to z_1^i and z_1^i to x for every x in X^i. As argued before, there is a shortest path from p^α to y, for any y in Y^i, that contains the corresponding x in X^i. Hence, $I(\mathsf{pndt}(p^\alpha), \mathsf{pndt}(y))$ covers all the vertices in the simple path connecting x to y. This implies that all the vertices in $\mathcal{S}^i$, except the vertices in V^{g_1} are covered. Recall that V^{g_1} contains the internal vertices of the path connecting g_1 with z_1^i.

Consider the simple paths whose both endpoints are contained in an element-encoding gadget. Consider an element-encoding gadget encoding (α, a) for some a in $[n]$. By the construction, $I(\mathsf{pndt}(u_a^\alpha), \mathsf{pndt}(w_a^\alpha))$ covers all the vertices in the simple path connecting u_a^α and w_a^α. We remark that the vertices in the simple path connecting v_a^α and w_a^α are part of V^α and hence excluded from the statement of the lemma. Using similar arguments for elements of the form (β, b) and (γ, c) for some b, c in $[n]$.

The arguments above imply that $V(G) \setminus (V^{g_1} \cup V^\alpha \cup V^\beta \cup V^\gamma) \subseteq I(P)$. It remains to prove that no vertex in $(V^{g_1} \cup V^\alpha \cup V^\beta \cup V^\gamma)$ is covered by the vertices in P.

Consider the vertices in V^{g_1} which are in the i^{th} copy of set-encoding gadget for some i in $[n]$. It is easy to see that vertices in $P \setminus \{\mathsf{pndt}(z_2^i), \mathsf{pndt}(g_1)\}$ can not cover the vertices mentioned in the previous sentence. Any shortest path whose one endpoint is $\mathsf{pndt}(z_2^i)$ and other endpoint is in $P \setminus \{\mathsf{pndt}(g_1)\}$ contains either $\{g_1, g_2\}$ or $\{z_1^i, g_2\}$. Hence, none of such paths can cover vertices in the path connecting g_1 and z_1^i. Similarly, any shortest path whose one endpoint is $\mathsf{pndt}(g_1)$ and other endpoint is in $P \setminus \{\mathsf{pndt}(z_2^i)\}$ contains g_2 and hence can not contain z_1^i. This implies that P can not cover any vertex in V^{g_1}.

Consider the vertices in V^α. It is easy to verify that vertices in V^α are not covered by the shortest path between any two common branching vertices. Also, the shortest paths between the pendant vertices in the element-encoding gadget do not cover vertices in V^α. Consider vertices in V^α, which are part of the element-encoding gadget corresponding to element (α, a) for some a in $[n]$. Any shortest path between the vertices in P that can cover these vertices has $\mathsf{pndt}(w_a^\alpha)$ as one of its endpoint and should contain q^α. By the construction, no path connecting $\mathsf{pndt}(w_a^\alpha)$ to any of the pendant vertex adjacent with common branching vertices contains q^α. It is easy to see for p^α, r^α, g_3 and g_4. For the remaining common vertices, it is sufficient to prove that the shortest path from $\mathsf{pndt}(w_a^\alpha)$ to g_2 do not contain q^α. Note that the path from w_a^α to g_2 that contains q^α and x_{q^α} for some x in X^i, is of length close to $4M^2$. However, the shortest path from $\mathsf{pndt}(w_a^\alpha)$ to g_2 is of length $3M^2$ and contains y in some Y^i and g_4. This implies that P does not cover any vertex in V^α. Using similar arguments, the statement holds for vertices in V^β and V^γ which concludes the proof of the lemma. ◀

In the following two lemmas, we prove that the reduction is safe.

▶ **Lemma 3.** *If $(\mathcal{U}, \mathcal{S})$ is a* Yes*-instance of* 3-Dimensional Matching *then (G, k) is a* Yes*-instance of* Geodetic Set*.*

Proof. Without loss of generality, suppose $\{S_1, S_2, \dots, S_n\}$ is a collection of n sets in collection of $\mathcal{S}$ (which contains m sets) that partitions all the vertices in $\mathcal{U}$. For every i in $[n]$, consider a vertex in X^i in the set-encoding gadget $\mathcal{S}^i$ corresponding to set S_i. Define set $Q = \{x_1^1, x_2^2, \dots, x_n^n\}$, where x_i^i is in X^i. Recall that P is the collection of all the pendant vertices in G. It is easy to see that $|P \cup Q| = k$. Due to Lemma 2, to prove that $P \cup Q$ is a geodetic set of G, it is sufficient to prove that Q covers vertices in $V^{g_1} \cup V^\alpha \cup V^\beta \cup V^\gamma$. See the paragraph above Lemma 2 for the definition of these sets. Also, recall that for any vertex u, $\mathsf{pndt}(u)$ denotes the unique pendant vertex adjacent to it, if one exists.

For a fixed i in $[n]$, $I(x_i^i, \mathsf{pndt}(g_1))$ covers all the vertices in the simple path connecting z_1^i to g_1. As this is true for any i in $[n]$, Q covers vertices in V^{g_1}. Suppose $\mathcal{S}^i = \{(\alpha, a), (\beta, b), (\gamma, c)\}$ for some a, b, c in $[n]$. Consider vertex x_i^i in X^i. We argue that the shortest paths between x_i^i and w_a^α covers the vertices in V^α that are part of the element-encoding gadget that is encoding element (α, a). Consider Figure 2 with $a' = a$. In this case, all three paths from x_i^i to u_a^α via p^α, q^α, and q^α as the same length of $2M^2$. This implies the shortest distance between x_i^i and w_a^α is $3M^2$. Moreover, the path connecting x_i^i to w_a^α that contains q^α and v_a^α is the shortest path between these two vertices. Hence, the shortest path between x_i^i and $\mathsf{pndt}(w_a^\alpha)$ covers all the vertices in V^α that are part of element-encoding gadget. As $\{S_1, S_2, \ldots, S_n\}$ partitions $\mathcal{U}$, (exactly) one of these n set contains element (α, a) for every a in $[n]$, all the vertices in V^α are covered by a vertex in Q and a vertex in P. Using the similar arguments, all the vertices in V^β and V^γ are covered by vertices in $P \cup Q$. This concludes the proof of the lemma. ◀

▶ **Lemma 4.** *If (G, k) is a* Yes*-instance of* Geodetic Set *then $(\mathcal{U}, \mathcal{S})$ is a* Yes*-instance of 3-*Dimensional Matching*.*

Proof. Recall that P is the collection of all the pendant vertices in G, and P is a subset of any geodetic set of G. Suppose $P \cup Q$ is the geodetic set of G of size of k, and hence $|Q| \leq n$. By Lemma 2, vertices in Q covers all the vertices in $V^{g_1} \cup V^\alpha \cup V^\beta \cup V^\gamma$. See the paragraph before Lemma 2 for the definition of these sets.

We first prove that Q contains at least one vertex in the i^{th} copy of set-encoding gadget $\mathcal{S}^i$ for every i in $[n]$. Assume, for the sake of contradiction, that this is not the case for an index i. Recall that V^{g_1} is the union of internal vertices connecting paths g_1 and z_1^i for i in $[n]$. Define $V_i^{g_1}$ as the set of vertices in V^{g_1} which are in $\mathcal{S}^i$. Consider a vertex, say h, in $V_i^{g_1}$. As $P \cup Q$ is a geodetic set of G, there are two vertices in it, say s_1 and s_2, that covers the vertex h. By Lemma 2, s_1 and s_2 are *not* in P. By the above assumption, s_1 and s_2 are *not* in $\mathcal{S}^i$. Consider the shortest path from s_1 to s_2 that contains the vertex h. Suppose that the paths enter and exit $\mathcal{S}^i$ via common branching vertices c_1 and c_2. Consider the case when $\{c_1, c_2\} \cap \{q^\alpha, q^\beta, q^\gamma\} = \emptyset$. In this case, the shortest path between $\mathsf{pndt}(c_1)$ and $\mathsf{pndt}(c_2)$ also contains the vertex h. However, this contradicts Lemma 2, which states vertices in P can not cover any vertex in V^{g_1}. Consider the other case when the shortest path enters $\mathcal{S}^i$ via, say, q^α. In this case, the above arguments can be extended to the shortest path from $\mathsf{pndt}(q_a^\alpha)$ for some a in $[n]$. This also leads to contradiction to Lemma 2. Hence, our assumption is wrong and Q contains at least one vertex in $\mathcal{S}^i$.

As the cardinality of Q is at most n, this implies that Q contains exactly one vertex in $\mathcal{S}^i$ for every i in $[n]$. Let q be the unique vertex in Q which is in $\mathcal{S}^i$. We determine the position of Q in $\mathcal{S}^i$ using the fact that the shortest paths from q to some other vertices in $(P \cup Q) \setminus \{q\}$ covers all the uncovered vertices in the set-encoding gadget $\mathcal{S}^i$. We now restrict the possible cases for this other vertex in $P \cup Q$. Let h be the other vertex in $(P \cup Q) \setminus \{q\}$, so the shortest path from q to h covers some vertices in $V_i^{g_1}$. By the arguments above, h is not in $\mathcal{S}^i$. As argued in the previous paragraph, the vertices covered by the shortest path from q to h are also covered by the shortest path from q to $\mathsf{pndt}(c)$ where c is a common branching vertex or the vertex with pendant neighbor which is adjacent to either q^α, q^β, or q^γ. Hence, while considering the cases of shortest path from q to h, it is sufficient to check the cases when h is one of the vertex mentioned in the previous sentence. Note that as no vertex in Q is present in any element-encoding gadget, vertices in Q, which are in set-encoding gadgets, are also used to cover the uncovered vertices in the element-encoding gadget. We use this property to narrow the location of q in $\mathcal{S}^i$.

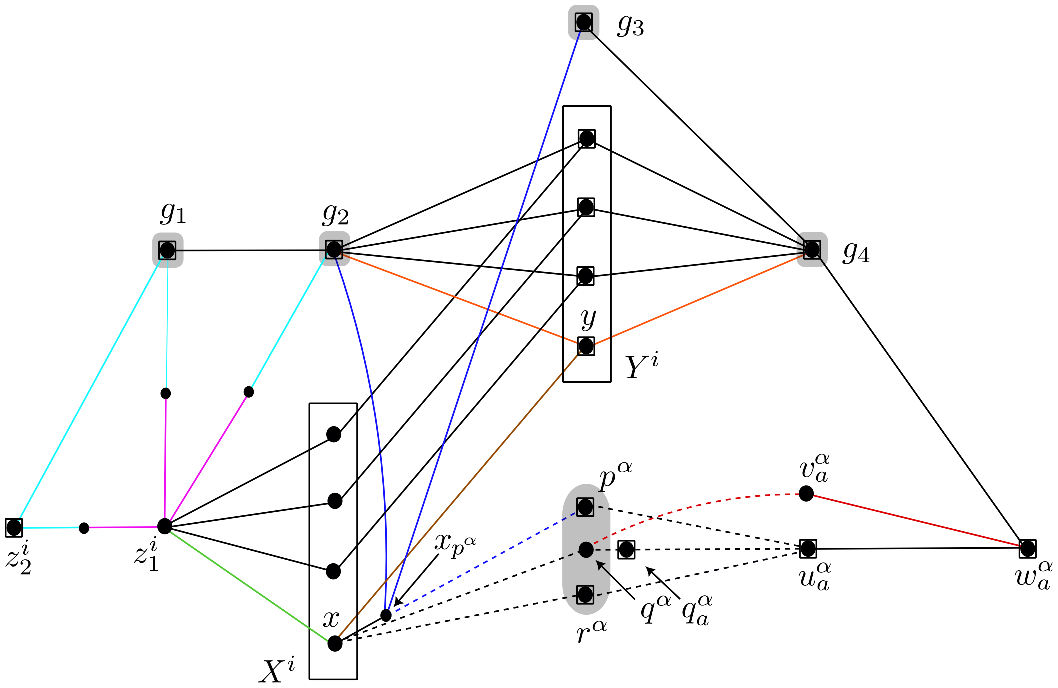

Figure 3 Types of partition of vertices in $\mathcal{S}^i$. Different colors correspond to different types of parts in the set-encoding gadget. For clarity, the figure does not show all the vertices in $\mathcal{S}^i$.

We partition $\mathcal{S}^i$ into the following parts and prove that q can be located in particular types of parts. By the construction, deleting all the common branching vertices results in a collection of trees. Consider $\mathcal{S}^i$ and root it at z_1^i.

- For a vertex x in X^i, consider the vertex x_{p^α} adjacent with it and on the path connecting p^α. Define x_{p^α}*-part* as the subtree rooted at x_{p^α}. Similarly, define x_{p^β}*-part* and x_{p^γ}*-part*. In Figure 3, x_{p^α}-part is highlighted in blue.
- For every y in Y^i, define y-part as subtree rooted at y, which is highlighted using orange color in Figure 3.
- For every x in X^i and the corresponding y in Y^i, i.e. the unique vertex in Y^i which is at distance M^2 from x, define (x, y)*-part* as the collection of internal vertices of the path from x to y. See the brown edge in Figure 3.
- For every x in X^i, define $(z_1^i, x]$*-part* as the vertex x and the internal vertices of the path from x to z_1^i. See the green part in Figure 3. Note that the part contains x but not z_1^i.
- Let h, h_1 and h_2 be the middle points of the paths from z_1^i to z_2^i, from z_1^i to g_1, and from z_2^i to g_2, respectively. Define z_1^i*-part* as the tree rooted at z_1^i with h, h_1 and h_2 as its leaves. This is highlighted by the purple color in Figure 3.
- All the remaining vertices are considered as *left-over part*, highlighted using cyan color in Figure 3. Note that this is the only disconnected part in the partition.

Using the fact that q is used to cover the vertices in $V_i^{g_1}$, we first argue it is not in the part of the first-type or of the second type mentioned above, nor can it be in the leftover part. Assume q is in x_{p^α}-part for some x in X^i. In this case, the shortest path from q to g_1 is via g_2 and hence does not contain vertices in $V_i^{g_1}$. Similarly, the shortest path from q to z_2^i is via g_1 or via z_1^i and can not contain vertices in $V_i^{g_1}$. It is easy to see that in this case, there are no paths from q to any other vertex in P can cover vertices in $V_i^{g_1}$. Hence, q cannot be in x_{p^α}-part. Now, assume q is in y-part for some y in Y^i. The shortest path from q to g_1 is via g_2 and to z_2^i is via g_1 or z_1^i; hence, no such short paths can cover vertices in $V_i^{g_1}$. Finally, it is easy to see that there is no vertex in the leftover part can cover any vertex in $V_i^{g_1}$.

We now turn our attention to the remaining parts, viz (x, y)-part, $(z_1^i, x]$-part, for some x in X^i, or z_1^i-part. Note that the shortest path from any vertex in $(z_1^i, x]$-part or z_1^i-part to g_1 covers all the vertices in $V_i^{g_1}$ whereas the same is true for half the vertices in (x, y)-part which are closer to x than they are to y. In the next few paragraphs, we argue that the only vertices in the $(z_1^i, x]$-part can be used to cover the uncovered vertices in the element encoding gadget. Recall that for every $a \in [n]$, define V_a^α as the collection of internal vertices in the simple path connecting q^α to u_a^α and the simple path connecting u_a^α to w_a^α. We also defined $V^\alpha = \bigcup_{a\in[n]} V_a^\alpha$ and defined V^β and V^γ in a similar way. Note that V^α, V^β, and V^γ are the collection of uncovered vertices in the element-encoding gadget.

Suppose vertex x in X^i corresponds to set $S = \{(\alpha, a'), (\beta, b'), (\gamma, c')\}$ and h is a vertex in $(z_1^i, x]$-part. We prove that $I(h, w_a^\alpha)$ covers V_a^α if and only if $a' = a$. For a vertex h in $(z_1^i, x]$-part, suppose $\text{dist}(h, x) = d_h$. Consider the four paths from h to w_a^α that are of length at most $d_h + 3M^2$ and contains exactly one common branching point amongst $\{g_4, p^\alpha, q^\alpha, r^\alpha\}$. Consider the path from h to w_a^α that contains x, the corresponding y, and g_4. By the construction, the length of this path is $d_h + 3M^2$. For the remaining three paths, consider Figure 2. The lengths of these three paths is $d_h + 3M^2 + 2(a' - a)M$, $d_h + 3M^2 + (a - a')M$, and $d_h + 3M^2 + 2(a - a')M$ when it contains p^α, q^α, and r^α, respectively. Note that $I(h, w_a^\alpha)$ covers V_a^α if and only the shortest path from h to w_a^α contains q^α. From the distances above, it is easy to see that this condition is true only when $a = a'$. Following the identical arguments, we can prove that $I(h, w_b^\beta)$ covers V_b^β if and only if $b' = b$, and $I(h, w_c^\gamma)$ covers V_a^γ if and only if $c' = c$.

We next prove that no vertex from (x, y)-part or z_1^i-part can cover any uncovered vertex in the element-encoding gadget. Consider a vertex h in (x, y)-part. As argued in the previous paragraph, $I(h, w_a^\alpha)$ can cover the vertices in V^α if and only if the shortest path from h to w_a^α contains q^α. However, the shortest path from h to w_a^α contains g_4 and hence cannot cover the vertices in V^α. Now consider the vertices in z_1^i-part. For any vertex h in z_1^i-part which is an internal vertex of path from z_1^i to g_2, the shortest path h to w_a^α contains g_4 and hence can not cover vertices in V^α. Note that the shortest path from z_1^i to w_a^α also contains g_4 in it. This is implied by the fact that the distance between g_2 and every vertex in Y^i is $M^2 - 1$ (instead of M^2 like most of the other paths). Hence, $I(z_1^i, w_a^\alpha)$ does not cover any vertex in V^α. For any vertex in the remaining z_1^i-part, the shortest path from it to w_a^α is via z^i, which contains g_4. These arguments imply that q can not be in (x, y)-part or z_1^i- part for any x in X^i.

We are in a position to conclude the proof of the lemma. Recall that $P \cup Q$ is a geodetic set of G. Considering the position of vertices in Q, we construct a collection of n subsets $\mathcal{S}'$ of $\mathcal{S}$ as follows: For every i in $[n]$, let x be the unique vertex in X^i such that the vertex in Q is in $(z_1^i, x]$-part of $\mathcal{S}^i$. If x corresponds to set S, then include S in $\mathcal{S}'$. It is easy to see that the cardinality of $\mathcal{S}'$ is at most n. We argue that $\mathcal{S}'$ covers all the vertices in $\mathcal{U}$. Consider an arbitrary element, say (α, a), in $\mathcal{U}$. The vertices in V^α are part of the element-encoding gadget that encodes (α, a). By above arguments, for some i in $[n]$, there is x in X^i such that $(z_1^i, x_1]$-part of $\mathcal{S}^i$ contains vertex q in Q with the property that $I(q, w_a^\alpha)$ covers vertices in V^α in the element-encoding gadget. Suppose x corresponds to the vertex $S = \{(\alpha, a'), (\beta, b'), (\gamma, c')\}$ for some $a', b', c' \in [n]$. From the arguments above, $I(q, w_a^\alpha)$ covers vertices in V^α if and only if $a' = a$. This implies that (α, a) is covered by some set in $\mathcal{S}'$. Since this is an arbitrary element in $\mathcal{U}$, one can conclude that $\mathcal{S}'$ covers all the vertices in $\mathcal{U}$. This implies that $(\mathcal{U}, \mathcal{S})$ is a YES-instance of 3-DIMENSIONAL MATCHING. ◀

▶ **Lemma 5.** *Pathwidth and feedback vertex set the number of G are at most* 17 *and* 13*, respectively.*

Proof. We present a mixed search strategy to clean G using 17 searchers to bound the pathwidth of G. Place 13 searchers on common branching vertices. These searchers will never move from their places. (This is equivalent to deleting common branching vertices and presenting a mixed search strategy using four searchers for the resulting graph.) We search the graph in $3n + n$ rounds, one round for each element-encoding gadget and one round for each set of set-encoding gadget, respectively.

Consider an element-encoding gadget encoding an element, say (α, a) for some a in $[n]$. We can place three searchers on u^{α}_a, v^{α}_a, and w^{α}_a, and the fourth searcher can move along the simple path connecting these vertices and with common branching vertices to clean all the edges in the connecting paths.

By the construction, deleting all the common branching vertices results in a collection of trees. Consider the set-encoding gadget $\mathcal{S}^i$ for some i in $[n]$ and root it at z^i_1. In each round, we fix a searcher on z^i_1. We need one searcher to clean all the edges between the simple path connecting z^i_1 to g_1 via z^i_2, path of length M^2 connecting z^i_1 to g_1, and finally path connecting z^i_1 to g_2. Once these paths are cleaned, the searcher is free and can be used to clean the remaining part of the set-encoding gadget. We divide the remaining part of this round into m sub-rounds corresponding to each vertex in X^i. For a sub-round, fix two searchers on a vertex x in X^i and another searcher on the corresponding vertex y in Y^i which is at distance M^2 from x. The remaining searcher can clean all the edges in the simple paths connecting x to y, y to common vertices, and x to z^i_1. Next, the searcher placed on y can be relocated to each of the branching vertices adjacent to x, i.e. vertices of the form $x_{p^{\alpha}}$. Now, the other searcher can be used to clean the subtree rooted at the branching vertex. This completes one sub-round. Completing such m sub-rounds clears all the paths in $\mathcal{S}^i$.

As we need 17 searchers to clean the graph, result of Takahashi et al. [38] implies that $\texttt{pw}(G) \leq 17$.

As deleting all the common branching vertices results in a collection of trees, feedback vertex set number of the graph is 13. ◀

4 Conclusion

In this article, we proved that Geodetic Set is NP-complete even on graphs with constant pathwidth and feedback vertex set number answering the open question by Kellerhals and Koana [34]. In the same paper, the authors provided an FPT algorithm (with impractical running time) when parameterized by treedepth (denoted by $\texttt{td}$) of the input graph. Recent work by Foucaud et al. [23] showed that Geodetic Set admits an algorithm running in time $2^{\texttt{diam}^{\mathcal{O}(\texttt{tw})}} \cdot poly(|V(G)|)$, where $\texttt{diam}$ denotes the diameter of the graph. This result, along with the fact that $\texttt{diam}$ and $\texttt{tw}$ are upper bounded by $2^{\texttt{td}}$ and $\texttt{td} + 1$, respectively, imply that Geodetic Set admits an algorithm running in time $2^{2^{\mathcal{O}(\texttt{td}^2)}} \cdot poly(|V(G)|)$. It would be interesting to obtain an FPT algorithm parameterized by treedepth with improved running time. We remark that Foucaud et al. [23] proved that the problem does not admit an algorithm running in time $2^{2^{o(\texttt{td})}} \cdot poly(|V(G)|)$, unless the ETH fails.

As highlighted in the introduction of this article and also in [8, 34], Metric Dimension and Geodetic Set share many hardness properties. We mention one notable exception: Metric Dimension is FPT when parameterized by the solution size plus the treelength [3], however, Geodetic Set is W[2]-hard when parameterized by the solution size even when treelength is a constant. See [17, Theorem 2]. Regarding the parameterized complexity of

METRIC DIMENSION when parameterized by pathwidth and feedback vertex set number of the graph, Galby et al. [25] showed that the problem is W[1]-hard. Can we improve this result to obtain a result for METRIC DIMENSION as we did for GEODETIC SET in this article?

References

1. J. Ahn, L. Jaffke, O. Kwon, and P. T. Lima. Well-partitioned chordal graphs. *Discrete Mathematics*, 345(10):112985, 2022. doi:10.1016/J.DISC.2022.112985.
2. M. Atici. Computational complexity of geodetic set. *International journal of computer mathematics*, 79(5):587–591, 2002. doi:10.1080/00207160210954.
3. R. Belmonte, F. V. Fomin, P. A. Golovach, and M. S. Ramanujan. Metric dimension of bounded tree-length graphs. *SIAM J. Discrete Math.*, 31(2):1217–1243, 2017. doi:10.1137/16M1057383.
4. B. Bergougnoux, O. Defrain, and F. Mc Inerney. Enumerating minimal solution sets for metric graph problems. In *Graph-Theoretic Concepts in Computer Science - 50th International Workshop, WG*, 2024. doi:10.1007/978-3-031-75409-8_4.
5. E. Bonnet and N. Purohit. Metric dimension parameterized by treewidth. *Algorithmica*, 83:2606–2633, 2021. doi:10.1007/S00453-021-00808-9.
6. F. Buckley and F. Harary. Geodetic games for graphs. *Quaestiones Mathematicae*, 8(4):321–334, 1985. doi:10.1080/16073606.1985.9631921.
7. L. R. Bueno, L. D. Penso, F. Protti, V. R. Ramos, D. Rautenbach, and U. S. Souza. On the hardness of finding the geodetic number of a subcubic graph. *Inf. Process. Lett.*, 135:22–27, 2018. doi:10.1016/J.IPL.2018.02.012.
8. D. Chakraborty, S. Das, F. Foucaud, H. Gahlawat, D. Lajou, and B. Roy. Algorithms and complexity for geodetic sets on planar and chordal graphs. In *31st International Symposium on Algorithms and Computation (ISAAC)*, 2020. doi:10.4230/LIPIcs.ISAAC.2020.7.
9. D. Chakraborty, F. Foucaud, H. Gahlawat, S. K. Ghosh, and B. Roy. Hardness and approximation for the geodetic set problem in some graph classes. In *Proc. of the 6th International Conference on Algorithms and Discrete Applied Mathematics (CALDAM)*, 2020.
10. D. Chakraborty, F. Foucaud, and A. Hakanen. Distance-based covering problems for graphs of given cyclomatic number. In *Fundamentals of Computation Theory - 24th International Symposium, FCT*, 2023.
11. D. Chakraborty, F. Foucaud, D. Majumdar, and P. Tale. Tight (double) exponential bounds for identification problems: Locating-dominating set and test cover. In *35th International Symposium on Algorithms and Computation, ISAAC*, 2024.
12. D. Chakraborty, H. Gahlawat, and B. Roy. Algorithms and complexity for geodetic sets on partial grids. *Theoretical Computer Science*, 979:114217, 2023. doi:10.1016/J.TCS.2023.114217.
13. G. Chartrand, F. Harary, and P. Zhang. On the geodetic number of a graph. *Networks*, 39(1):1–6, 2002. doi:10.1002/NET.10007.
14. M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Springer, 2015.
15. T. Davot, L. Isenmann, and J. Thiebaut. On the approximation hardness of geodetic set and its variants. In *Proc. of the 27th International Computing and Combinatorics Conference, COCOON*, 2021.
16. R. Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012.
17. M. Dourado, F. Protti, D. Rautenbach, and J. Szwarcfiter. Some remarks on the geodetic number of a graph. *Discret. Math.*, 310(4):832–837, 2010. doi:10.1016/J.DISC.2009.09.018.
18. M. C. Dourado, F. Protti, D. Rautenbach, and J. L. Szwarcfiter. On the complexity of the geodetic and convexity numbers of a graph. In *Proc. of the International Conference on Discrete Mathematics (ICDM)*, volume 7 of *RMS Lecture Notes Series*, pages 101–108. Ramanujan Mathematical Society, 2008.

19 M. C. Dourado, F. Protti, D. Rautenbach, and J. L. Szwarcfiter. Some remarks on the geodetic number of a graph. *Discrete Mathematics*, 310(4):832–837, 2010. doi:10.1016/J.DISC.2009.09.018.

20 A. L. Douthat and M. C. Kong. Computing geodetic bases of chordal and split graph. *Journal of Combinatorial Mathematics and Combinatorial Computing*, pages 67–77, 1996.

21 T. Ekim, A. Erey, P. Heggernes, P. van't Hof, and D. Meister. Computing minimum geodetic sets of proper interval graphs. In *Proc. of the 10th Latin American Symposium on Theoretical Informatics (LATIN 2012)*, 2012.

22 M. Farber and R. E. Jamison. Convexity in graphs and hypergraphs. *SIAM Journal on Algebraic Discrete Methods*, 7(3):433–444, 1986.

23 F. Foucaud, E. Galby, L. Khazaliya, S. Li, F. Mc Inerney, R. Sharma, and P. Tale. Problems in NP can admit double-exponential lower bounds when parameterized by treewidth or vertex cover. In *51st International Colloquium on Automata, Languages, and Programming, ICALP*, 2024. doi:10.4230/LIPIcs.ICALP.2024.66.

24 F. Foucaud, E. Galby, L. Khazaliya, S. Li, F. Mc Inerney, R. Sharma, and P. Tale. Metric dimension and geodetic set parameterized by vertex cover. In *42nd International Symposium on Theoretical Aspects of Computer Science, STACS*, 2025. doi:10.4230/LIPIcs.STACS.2025.33.

25 E. Galby, L. Khazaliya, F. Mc Inerney, R. Sharma, and P. Tale. Metric dimension parameterized by feedback vertex set and other structural parameters. *SIAM J. Discrete Math.*, 37(4):2241–2264, 2023. doi:10.1137/22M1510911.

26 M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W. H. Freeman, 1979.

27 F. Harary, E. Loukakis, and C. Tsouros. The geodetic number of a graph. *Mathematical and Computer Modelling*, 17(11):89–95, 1993.

28 S. Hartung and A. Nichterlein. On the parameterized and approximation hardness of metric dimension. In *Proc. of the 28th Conference on Computational Complexity, CCC 2013*, pages 266–276. IEEE Computer Society, 2013. doi:10.1109/CCC.2013.36.

29 L. Jaffke, O j. Kwon, T. J. F. Strømme, and J. A. Telle. Mim-width III. Graph powers and generalized distance domination problems. *Theoretical Computer Science*, 796:216–236, 2019. doi:10.1016/J.TCS.2019.09.012.

30 M. Kanté, T. Marcilon, and R. M. Sampaio. On the parameterized complexity of the geodesic hull number. *Theor. Comput. Sci.*, 791:10–27, 2019. doi:10.1016/J.TCS.2019.05.005.

31 M. M. Kanté and L. Nourine. Polynomial time algorithms for computing a minimum hull set in distance-hereditary and chordal graphs. *SIAM J. Discrete Math.*, 30(1):311–326, 2016. doi:10.1137/15M1013389.

32 I. Katsikarelis, M. Lampis, and V. Th. Paschos. Structural parameters, tight bounds, and approximation for (k, r)-center. *Discret. Appl. Math.*, 264:90–117, 2019. doi:10.1016/J.DAM.2018.11.002.

33 I. Katsikarelis, M. Lampis, and V. Th. Paschos. Structurally parameterized d-scattered set. *Discrete Applied Mathematics*, 308:168–186, 2022. doi:10.1016/J.DAM.2020.03.052.

34 L. Kellerhals and T. Koana. Parameterized complexity of geodetic set. *Journal of Graph Algorithms and Applications*, 26(4):401–419, 2022. doi:10.7155/JGAA.00601.

35 S. Li and M. Pilipczuk. Hardness of metric dimension in graphs of constant treewidth. *Algorithmica*, 84(11):3110–3155, 2022. doi:10.1007/S00453-022-01005-Y.

36 M. Mezzini. Polynomial time algorithm for computing a minimum geodetic set in outerplanar graphs. *Theoretical Computer Science*, 745:63–74, 2018. doi:10.1016/J.TCS.2018.05.032.

37 I. M. Pelayo. *Geodesic Convexity in Graphs.* Springer, 2013.

38 A. Takahashi, S. Ueno, and Y. Kajitani. Mixed searching and proper-path-width. *Theor. Comput. Sci.*, 137(2):253–268, 1995. doi:10.1016/0304-3975(94)00160-K.