



Hitting Geodesic Intervals in Structurally Restricted Graphs

Tatsuya Gima  

Hokkaido University, Sapporo, Japan

Yasuaki Kobayashi  

Hokkaido University, Sapporo, Japan

Yuto Okada  

Nagoya University, Japan

Yota Otachi  

Nagoya University, Japan

Hayato Takaike 

Nagoya University, Japan

Abstract

Given a graph $G = (V, E)$, a set T of vertex pairs, and an integer k , HITTING GEODESIC INTERVALS asks whether there is a set $S \subseteq V$ of size at most k such that for each terminal pair $\{u, v\} \in T$, the set S intersects at least one shortest u – v path. Aravind and Saxena [WALCOM 2024] introduced this problem and showed several parameterized complexity results. In this paper, we extend the known results in both negative and positive directions and present sharp complexity contrasts with respect to structural graph parameters.

We first show that the problem is NP-complete even on graphs with highly restricted shortest-path structures. More precisely, we show the NP-completeness on graphs obtained by adding a single vertex to a disjoint union of 5-vertex paths. By modifying the proof of this result, we also show the NP-completeness on graphs obtained from a path by adding one vertex and on graphs obtained from a disjoint union of triangles by adding one universal vertex. Furthermore, we show the NP-completeness on graphs of bandwidth 4 and maximum degree 5 by replacing the universal vertex in the last case with a long path. Under standard complexity assumptions, these negative results rule out fixed-parameter algorithms for most of the structural parameters studied in the literature (if the solution size k is not part of the parameter).

We next present fixed-parameter algorithms parameterized by k plus modular-width and by k plus vertex integrity. The algorithm for the latter case does indeed solve a more general setting that includes the parameterization by the minimum vertex multiway-cut size of the terminal vertices. We show that this is tight in the sense that the problem parameterized by the minimum vertex multicut size of the terminal pairs is W[2]-complete. We then modify the proof of this intractability result and show that the problem is W[2]-complete parameterized by k even in the setting where $T = \binom{Q}{2}$ for some $Q \subseteq V$.

2012 ACM Subject Classification Mathematics of computing → Graph algorithms; Theory of computation → Parameterized complexity and exact algorithms

Keywords and phrases Terminal monitoring set, Structural graph parameter, Geodesic interval

Digital Object Identifier 10.4230/LIPIcs.IPEC.2025.29

Related Version Full Version: <https://arxiv.org/abs/2509.01413>

Funding Tatsuya Gima: JSPS KAKENHI Grant Numbers JP24K23847, JP25K03077

Yasuaki Kobayashi: JSPS KAKENHI Grant Numbers JP23K28034, JP24H00686, JP24H00697

Yuto Okada: Supported by JST SPRING, Grant Number JPMJSP2125

Yota Otachi: JSPS KAKENHI Grant Numbers JP22H00513, JP24H00697, JP25K03076, JP25K03077



© Tatsuya Gima, Yasuaki Kobayashi, Yuto Okada, Yota Otachi, and Hayato Takaike; licensed under Creative Commons License CC-BY 4.0

20th International Symposium on Parameterized and Exact Computation (IPEC 2025).

Editors: Akanksha Agrawal and Erik Jan van Leeuwen; Article No. 29; pp. 29:1–29:16

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

1 Introduction

For vertices u and v in a graph G , the *geodesic interval* $I_G[u, v]$ is the set of vertices that belong to at least one shortest u - v path in G . In this paper, we study the problem HITTING GEODESIC INTERVALS that asks to find a minimum hitting set for a given family of geodesic intervals, which is formalized as follows.

Problem: HITTING GEODESIC INTERVALS (HGI)

Input: A graph $G = (V, E)$, a set T of vertex pairs, and an integer k .

Question: Is there $S \subseteq V$ with $|S| \leq k$ such that $S \cap I_G[u, v] \neq \emptyset$ for each $\{u, v\} \in T$?

We also study the same problem on edge-weighted graphs, which we call WEIGHTED HITTING GEODESIC INTERVALS (WHGI). We call a vertex belonging to $V(T) = \bigcup_{\{u, v\} \in T} \{u, v\}$ a *terminal vertex*. We say that a vertex w *interrupts* a pair $\{u, v\}$ if $w \in I_G[u, v]$. Also, we say that a vertex set S *interrupts* $\{u, v\}$ if one of the members of S interrupts $\{u, v\}$.

The setting of HGI may have many practical applications. For example, it can be seen as a facility location problem in a network G with a set T of departure-destination point pairs of transportation demands. The goal here is to locate as small number of facilities (e.g., gas stations) as possible in such a way that, for each demand $\{u, v\} \in T$, one can take a shortest u - v path in G that visits at least one facility.

HGI has been introduced recently by Aravind and Saxena [2], who motivated the setting as a communication network monitoring problem (and called it TERMINAL MONITORING SET). They initiated the study of the parameterized complexity of HGI and showed several positive and negative results (see Section 1.1). Our contribution in this paper is to considerably tighten the known tractability/intractability border of HGI with respect to structural graph parameters, as we describe in Section 1.2.

1.1 Known results

Observations. To illustrate connections to other problems, we first observe that some known facts imply (in)tractability of some cases of HGI. (Some of them will be useful later.)

1. WHGI parameterized by $|T|$ is fixed-parameter tractable.
2. WHGI on trees is polynomial-time solvable.
3. If VERTEX COVER is NP-complete on a graph class \mathcal{C} , then so is HGI.
4. HGI on complete graphs is NP-complete.
5. HGI on $r \times r$ grids is NP-complete.

The first one is an easy observation that this is a special case of HITTING SET parameterized by the number of subsets to be hit, which is fixed-parameter tractable (see [8, Theorem 6.1]). The second follows from the fact that the following more general problem is polynomial-time solvable (see, e.g., the discussion by Jansen [19]): given a set of subtrees of a tree, find a minimum vertex set that intersects all subtrees in the set.

The third and fourth observations come from the equivalence of an instance $\langle (V, E), k \rangle$ of VERTEX COVER and the instances $\langle (V, E), \binom{V}{2}, k \rangle$ and $\langle (V, \binom{V}{2}), E, k \rangle$ of HGI, where $\binom{V}{2} = \{\{u, v\} \mid u, v \in V, u \neq v\}$. The equivalence holds since the only way to interrupt an adjacent pair is to pick one of them.

The fifth fact can be shown by observing an interesting connection to a geometric problem. PIERCING RECTANGLES asks, given axis-parallel rectangles R_1, \dots, R_r in the plane and an integer k , whether there is a set of k points that intersects all rectangles R_i (see [1]). It

is known that even if all coordinates are integers, each rectangle is a 2×2 square, and all rectangles are placed in a region of $cr \times cr$ for some constant c , this problem is NP-complete [16].¹ Such an instance of **PIERCING RECTANGLES** can be transformed to an equivalent instance $\langle G, T, k \rangle$ of HGI by setting G to the $cr \times cr$ grid and adding the pair of two opposite corners of each rectangle R_i to T . Observe that a geodesic interval $I_G[u, v]$ for u and v is the set of vertices in the axis-parallel rectangle having u and v at opposite corners.

Results of Aravind and Saxena [2]. Their main results can be summarized as follows.

1. HGI is W[2]-complete parameterized by the solution size k .
2. HGI is fixed-parameter tractable parameterized by k plus cluster vertex deletion number.
3. HGI is fixed-parameter tractable parameterized by k plus neighborhood diversity.
4. WHGI is fixed-parameter tractable parameterized by feedback edge set number.
5. WHGI is fixed-parameter tractable parameterized by vertex cover number.

Their first result complements the fixed-parameter tractability with respect to $|T|$ as $k < |T|$ in nontrivial instances. Note that this result implies the W[2]-completeness of WHGI parameterized by k even on complete graphs (consider the reduction from HGI that replaces each nonedge with a huge-weight edge). In their second and third results, having k in the parameter is necessary because of the NP-completeness of HGI on complete graphs. Their fourth result generalizes polynomial-time solvability on trees. Note that they did not explicitly mention the weighted case for feedback edge set number, but their algorithm works without any modifications for the weighted case as well.

Since vertex cover number is an upper bound on the solution size k , the parameterization solely by vertex cover number is equivalent to the one by k plus vertex cover number. Thus, most of the known positive results (except for the one parameterized by feedback edge set number) have “plus k ” in their parameters.

As open questions, Aravind and Saxena [2] asked for the complexity of HGI parameterized by treewidth, pathwidth, feedback vertex set number, and distance to path forest.

1.2 Our results

NP-completeness of HGI on highly restricted graphs. Our first results demonstrate that HGI is NP-complete even if the target graph classes are highly restricted, and thus answer the open questions of Aravind and Saxena [2] in a negative way. More precisely, we show that the problem is NP-complete on the following classes:

1. graphs of vertex-deletion distance 1 to the class of disjoint unions of 5-vertex paths;
2. graphs of vertex-deletion distance 1 to the class of paths;
3. graphs of vertex-deletion distance 1 to the class of disjoint unions of triangles;
4. graphs of bandwidth 4 and maximum degree 5.

Our main technical contribution here is the first one, which already implies that HGI is paraNP-complete for most of the structural parameters studied in the literature (see Figure 1 (left)) as the graphs there have distance to path forest 1 and vertex integrity 6. The second one is an easy corollary of the first. The third one, which improves the vertex integrity bound to 4, is based on the same idea as the first. The fourth one can be obtained from the third by a minor modification.

¹ In [16], the problem is described as a point-set covering problem by 2×2 squares, which is equivalent to the one described here because in each problem one object hits or covers another object if the L_∞ distance between their centers is at most 1.

FPT with k plus structural parameters. To cope with the rather hopeless situation discussed above, we consider the setting where the solution size k is also part of the parameter. In this setting, we present the following positive results.

1. HGI is fixed-parameter tractable parameterized by k plus modular-width.
2. WHGI is fixed-parameter tractable parameterized by k plus vertex integrity.
3. WHGI is fixed-parameter tractable parameterized by the minimum vertex multiway-cut size of the set of terminal vertices $V(T)$.

The first result for k plus modular-width generalizes the one for k plus neighborhood diversity [2]. The second and third results are achieved by a single algorithm that solves WHGI when a small separator of a special kind is given as part of the input. After presenting the algorithm, we show that the two cases can be handled by this algorithm without the assumption that the separator is given as part of the input (see Section 4.2.2 for details). Observe that the second result generalizes the one parameterized solely by vertex cover number [2] since vertex cover number is an upper bound of both k and vertex integrity. The third result generalizes the one parameterized by $|T|$, which can be seen as an example of the conceptual framework of Jansen and Swennenhuis [20] for parameterizing terminal-involved problems with a parameter structurally smaller than the number of terminals. Although the third result does not explicitly have k in the parameter, we can see that its parameter is an upper bound of k .

Figure 1 (right) shows the complexity of HGI parameterized by solution size k plus a structural parameter. As we can see, it is not well understood yet, and many cases remain unsettled.

Parameterized intractability beyond k . As mentioned above, there are many unsettled cases in the “plus k ” setting. Actually, we do not know any intractability result like “HGI is W-hard parameterized by k plus X” for any structural graph parameter X.

As intermediate steps toward this direction, we present two intractability results for the following cases, where k is part of the parameter, implicitly or explicitly.

1. HGI is W[2]-complete parameterized by the minimum vertex multicut size of the set of terminal pairs T .
2. HGI is W[2]-complete parameterized by k even if $T = \binom{Q}{2}$ for some $Q \subseteq V$.

The first one implicitly has k in the parameter since such a cut size is an upper bound of k . Also, it is a lower bound of the minimum vertex multiway-cut size of $V(T)$. In this sense, this intractability result tightens the gap between the W[2]-completeness parameterized solely by k [2] and our fixed-parameter tractability result parameterized by the minimum vertex multiway-cut size of $V(T)$.

The second result shows that a natural restriction of HGI is still W[2]-complete parameterized solely by k , where we want to hit pairwise geodesic intervals among a given vertex subset $Q \subseteq V$. This result is shown by a modification of the proof of the first result.

1.3 Related work

In the fields of graph theory and graph algorithms, the concept of geodesic intervals (also known as *geodetic intervals* or just as *intervals*) is well studied, especially in the context of the problem GEODETIC SET [18] (see the survey on GEODETIC SET [5] and the references therein). In a graph $G = (V, E)$, a set $S \subseteq V$ is a *geodetic set* if $\bigcup_{\{u,v\} \in \binom{S}{2}} I_G[u, v] = V$; that is, the union of pairwise geodesic intervals among the vertices in S covers all vertices of G .

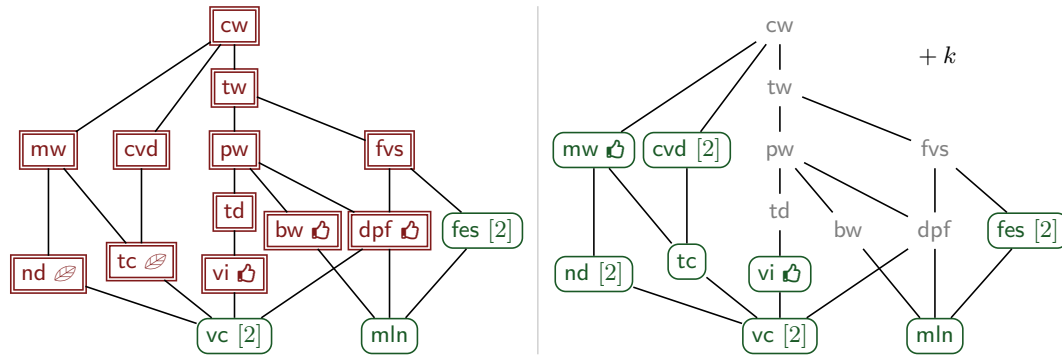


Figure 1 The figure on the left shows the complexity of HITTING GEODESIC INTERVALS parameterized solely by each parameter, and the one on the right shows the complexity parameterized by each parameter plus solution size k . (See Section 2 for the non-abbreviated names of the parameters.) The results marked with 👍 are shown in this paper and the ones with 🍃 are easy observations. The double rectangles and the rounded rectangles indicate paraNP-complete and fixed-parameter tractable cases, respectively. Ones without frames remain unsettled. A connection between two parameters means that if the one below (say β) is bounded, then so is the one above (say α); that is, there is a function f such that $\alpha(G) \leq f(\beta(G))$ for every graph G . Thus, a positive result propagates downward, while a negative result propagates upward.

Given a graph G and an integer k , GEODETIC SET asks whether G has a geodetic set of size at most k . GEODETIC SET is NP-complete [3, 10] and several parameterized complexity results have been shown recently [21, 28]. Although HITTING GEODESIC INTERVALS and GEODETIC SET share some concepts in common, there seem to be no direct connections.

2 Preliminaries

Throughout the paper, we assume that the reader is familiar with the concepts in parameterized complexity and fixed-parameter tractability (see standard textbooks, e.g., [8, 11, 12, 15, 25]).

All graphs considered in this paper are finite, simple, and undirected. In Section 4.2, we consider edge-weighted graphs, where the weight of each edge represents its length. To guarantee the existence of shortest paths, we assume that the input graph does not contain a cycle of a negative total weight. It is well known that even in the edge-weighted setting, we can compute all-pairs shortest paths (or find a negative cycle) in polynomial time using standard algorithms (see e.g., [7]).

As mentioned in the introduction, WHGI is a special case of HITTING SET. Given a finite set U , a family $\mathcal{F} \subseteq 2^U$, and an integer k , HITTING SET asks whether there is $S \subseteq U$ with $|S| \leq k$ such that $S \cap F \neq \emptyset$ for each $F \in \mathcal{F}$. We can see that an instance $\langle G = (V, E), T, k \rangle$ of WHGI is equivalent to the instance $\langle V, \{I_G[u, v] \mid \{u, v\} \in T\}, k \rangle$ of HITTING SET, where the family $\{I_G[u, v] \mid \{u, v\} \in T\}$ can be computed in polynomial time. HITTING SET is known to be W[2]-complete parameterized by k , while it is fixed-parameter tractable parameterized by $|U|$ and by $|\mathcal{F}|$ (see [8]).

Although our results involve many structural graph parameters as shown in Figure 1, not all of the parameters need their formal definitions for the presentations in this paper. Thus, we only define some of them when it is necessary. For the definitions and relationships of the parameters, we refer the reader to the survey on structural graph parameters [27] (see also [26, 29]).

In Figure 1, we abbreviated the names of graph parameters as follows: *cw* is cliquewidth; *tw* is treewidth; *pw* is pathwidth; *td* is treedepth; *vi* is vertex integrity; *vc* is vertex cover number; *mw* is modular-width; *nd* is neighborhood diversity; *cvd* is cluster vertex deletion number; *tc* is twin cover number; *bw* is bandwidth; *mln* is max-leaf number; *fvs* is feedback vertex set number; *fes* is feedback edge set number; and *dpf* is distance to path forest.

3 NP-completeness on highly restricted graphs

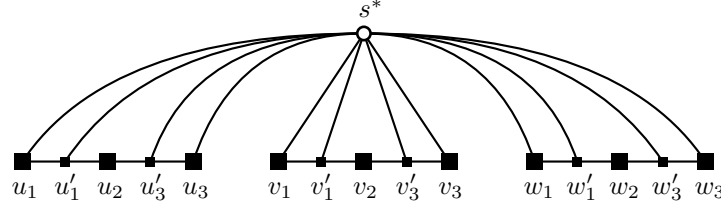
Since HGI clearly belongs to NP, we only prove the NP-hardness in the following proofs.

► **Theorem 3.1.** *HITTING GEODESIC INTERVALS is NP-complete on graphs of vertex-deletion distance 1 to the class of disjoint unions of 5-vertex paths.*

Proof. We present a polynomial-time reduction from 3-COLORING, which is NP-complete [17, GT4]. Given a graph $G = (V, E)$, 3-COLORING asks whether there is a *proper 3-coloring* of G , i.e., a mapping $c: V \rightarrow \{1, 2, 3\}$ such that $c(u) \neq c(v)$ for every $\{u, v\} \in E$.

Let $G = (V, E)$ be an instance of 3-COLORING. In the following, we construct an equivalent instance $\langle H, T, k \rangle$ of HITTING GEODESIC INTERVALS.

Construction of $\langle H, T, k \rangle$. We set $k = 2|V|$. For each vertex $v \in V$, we construct a path P_v of five vertices $v_1, v'_1, v_2, v'_3, v_3$ that appear in this order along the path. To complete the construction of H , we add one more vertex, s^* , that is adjacent to almost all other vertices except for the center vertex v_2 in each path P_v for each $v \in V$. See Figure 2. Clearly, $H - s^*$ is a disjoint union of 5-vertex paths. For each $v \in V$, we add pairs $\{v_1, v'_1\}$ and $\{v_3, v'_3\}$ into T . For each $\{u, v\} \in E$, we add pairs $\{u_1, v_1\}$, $\{u_2, v_2\}$, and $\{u_3, v_3\}$ into T .



■ **Figure 2** The construction of H in Theorem 3.1 (when G has only three vertices u, v, w). We represent terminal vertices by squares. Some terminal vertices are depicted as small squares since they appear only in “local” terminal pairs.

If G is a yes instance. Let $c: V \rightarrow \{1, 2, 3\}$ be a proper 3-coloring of G . For each $v \in V$, let $S_v \subseteq \{v_1, v'_1, v_2, v'_3, v_3\}$ be the following set depending on $c(v)$:

$$S_v = \begin{cases} \{v'_1, v_3\} & \text{if } c(v) = 1, \\ \{v_1, v_3\} & \text{if } c(v) = 2, \\ \{v_1, v'_3\} & \text{if } c(v) = 3. \end{cases}$$

We set $S = \bigcup_{v \in V} S_v$. Since $|S| = 2|V| = k$, it suffices to show that S interrupts all pairs in T . For each $v \in V$, it is easy to see that S_v interrupts the pairs $\{v_1, v'_1\}$ and $\{v_3, v'_3\}$.

For each edge $\{u, v\} \in E$, we show that $S_u \cup S_v$ interrupts the pairs $\{u_1, v_1\}$, $\{u_2, v_2\}$, and $\{u_3, v_3\}$. Since c is a proper 3-coloring, we have $c(u) \neq c(v)$, and thus $\{c(u), c(v)\} \cap \{1, 3\} \neq \emptyset$. By symmetry, we may assume that $c(u) \in \{1, 3\}$. Assume that $c(u) = 1$. Then, $S_u = \{u'_1, u_3\}$

interrupts $\{u_3, v_3\}$ and also $\{u_2, v_2\}$ as $u'_1 \in I_H[u_2, v_2]$. Since $c(v) \in \{2, 3\}$, S_v includes v_1 , and thus it interrupts $\{u_1, v_1\}$. The other case of $c(u) = 3$ can be handled in the same way by swapping 1 and 3 in the discussion above.

If $\langle H, T, k \rangle$ is a yes instance. Let $S \subseteq V(H)$ be a set with $|S| \leq k$ that interrupts all pairs in T . Since $\{v_1, v'_1\}, \{v_3, v'_3\} \in E(H) \cap T$ for every $v \in V$ and $k = 2|V|$, we have $|S \cap \{v_1, v'_1\}| = |S \cap \{v_3, v'_3\}| = 1$ for each $v \in V$ and $S \subseteq \bigcup_{v \in V} \{v_1, v'_1, v_3, v'_3\}$. Let $S_v = S \cap \{v_1, v'_1, v_3, v'_3\}$. We define $c(v) \in \{1, 2, 3\}$ for each $v \in V$ as follows:

$$c(v) = \begin{cases} 1 & \text{if } v_1 \notin S_v, \\ 3 & \text{else if } v_3 \notin S_v, \\ 2 & \text{otherwise } (S_v = \{v_1, v_3\}). \end{cases}$$

We show that c is a proper 3-coloring of G . Suppose to the contrary that there is an edge $\{u, v\} \in E$ with $c(u) = c(v) = x$. If $x \in \{1, 3\}$, then $u_x, v_x \notin S_u \cup S_v$. Since $I_H[u_x, v_x] = \{u_x, s^*, v_x\}$ and $s^* \notin S$, S does not interrupt the pair $\{u_x, v_x\} \in T$. If $x = 2$, then $S_u \cup S_v = \{u_1, u_3, v_1, v_3\}$ does not intersect $I_H[u_2, v_2] = \{u_2, u'_1, u'_3, s^*, v'_1, v'_3, v_2\}$, and thus $S (\not\supseteq s^*)$ does not interrupt the pair $\{u_2, v_2\} \in T$. \blacktriangleleft

To the graph H constructed in the proof of Theorem 3.1, we may add some vertices and edges without invalidating the reduction as long as the geodesic intervals to be hit do not change. In particular, we can pick arbitrary two vertices u and v of G and add a long path between u_3 and v_1 to H .² Repeating this in an appropriate way, we can make $H - s^*$ a path and obtain the following corollary.

► **Corollary 3.2.** *HITTING GEODESIC INTERVALS is NP-complete on graphs of vertex-deletion distance 1 to the class of paths.*

The next theorem can be shown by replacing 5-vertex paths in the proof of Theorem 3.1 with triangles.

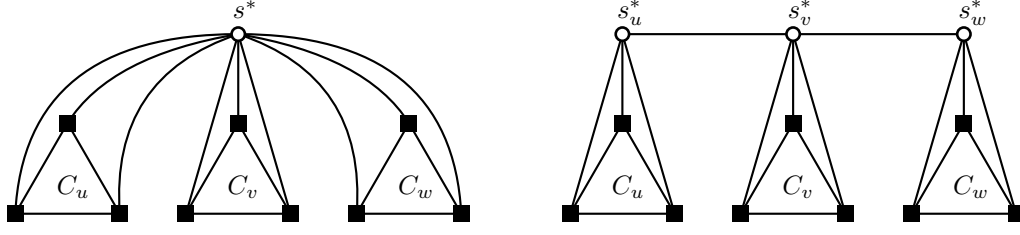
► **Theorem 3.3.** *HITTING GEODESIC INTERVALS is NP-complete on graphs of vertex-deletion distance 1 to the disjoint unions of triangles.*

Proof. Let $G = (V, E)$ be an instance of 3-COLORING. We set $k = 2|V|$. For each $v \in V$, the graph H contains a triangle C_v with vertices v_1, v_2 , and v_3 . Additionally, H has vertex s^* adjacent to all other vertices. See Figure 3 (left). For each $v \in V$, we add pairs $\{v_1, v_2\}$, $\{v_2, v_3\}$, and $\{v_3, v_1\}$ into T . For each $\{u, v\} \in E$, we add pairs $\{u_1, v_1\}$, $\{u_2, v_2\}$, and $\{u_3, v_3\}$ into T .

Assume that G admits a proper 3-coloring c . For each $v \in V$, let $S_v = \{v_1, v_2, v_3\} \setminus \{v_{c(v)}\}$. For each $v \in V$, S_v interrupts the pairs $\{v_1, v_2\}$, $\{v_2, v_3\}$, and $\{v_3, v_1\}$. For each edge $\{u, v\} \in E$, since $c(u) \neq c(v)$, $S_u \cup S_v$ includes at least one of u_i and v_i for each $i \in \{1, 2, 3\}$, and thus, $S_u \cup S_v$ interrupts the pairs $\{u_1, v_1\}$, $\{u_2, v_2\}$, and $\{u_3, v_3\}$. Since $\bigcup_{v \in V} S_v$ has size $2|V| = k$, $\langle H, T, k \rangle$ is a yes instance of HGI.

Next assume that $\langle H, T, k \rangle$ is a yes instance of HGI, and that $S \subseteq V(H)$ interrupts all pairs in T and has size at most k . Since $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_1\} \in E(H) \cap T$ and $k = 2|V|$, we have $s^* \notin S$ and $|S \cap \{v_1, v_2, v_3\}| = 2$ for each $v \in V$. We define $c(v) \in \{1, 2, 3\}$ for each $v \in V$ to be the unique index $i \in \{1, 2, 3\}$ such that $v_i \notin S$. Suppose that there is an edge $\{u, v\} \in E$ with $c(u) = c(v) = x$. The definition of c implies that $u_x, v_x \notin S \cap \{u_1, u_2, u_3, v_1, v_2, v_3\}$. Since $s^* \notin S$ and $I_H[u_x, v_x] = \{u_x, s^*, v_x\}$, S does not interrupt the pair $\{u_x, v_x\} \in T$. \blacktriangleleft

² Actually, we can show that adding an edge between u_3 and v_1 is good enough.

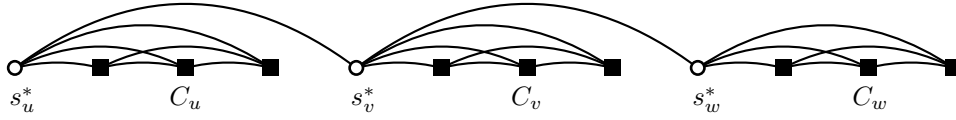


■ **Figure 3** The construction of H and H' in Theorem 3.3 and Corollary 3.4.

To show the hardness on graphs of bandwidth 4 and maximum degree 5, we only need to slightly modify the proof of Theorem 3.3. For an n -vertex graph $G = (V, E)$, its bandwidth $\text{bw}(G)$ is defined as $\text{bw}(G) = \min_{\sigma} \max_{\{u,v\} \in E} |\sigma(u) - \sigma(v)|$, where the minimum is taken overall bijections $\sigma: V \rightarrow [n]$, where $[n] = \{i \in \mathbb{Z} \mid 1 \leq i \leq n\}$.

► **Corollary 3.4.** *HITTING GEODESIC INTERVALS is NP-complete on graphs of bandwidth 4 and maximum degree 5.*

Proof. Let H be the graph constructed in the proof of Theorem 3.3. We first remove s^* from H . Next, for each $v \in V$, we add a vertex s_v^* adjacent to all vertices in the triangle C_v . Finally, we add $|V| - 1$ edges to make a path that goes through the new vertices $\{s_v^* \mid v \in V\}$. See Figure 3 (right). Let us call the modified graph H' . We can see that $\langle H, T, k (= 2|V|) \rangle$ and $\langle H', T, k \rangle$ are equivalent instances of HGI since a solution S of $\langle H', T, k \rangle$ cannot include any vertex in $\{s_v^* \mid v \in V\}$. Clearly, H' has maximum degree 5. To see that H' has bandwidth 4, consider the ordering that places the vertices along the path on $\{s_v^* \mid v \in V\}$, while the vertices of C_v are placed right after s_v^* . This ordering has bandwidth 4. See Figure 4. ◀



■ **Figure 4** The embedding of H' in Corollary 3.4.

4 Fixed-parameter tractability with k plus structural parameters

4.1 Modular-width

Let $G = (V, E)$ be a graph. A set $M \subseteq V$ is a *module* of G if $N_G(u) \setminus M = N_G(v) \setminus M$ holds for all $u, v \in M$. In other words, each vertex $w \notin M$ is adjacent to all or no vertices of M . Note that if M and M' are two disjoint modules of G , then either there are all or no possible edges between them. Two modules M and M' are *adjacent* if there are all possible edges between them, and *nonadjacent* otherwise.

The *modular-width* of $G = (V, E)$, denoted $\text{mw}(G)$, is defined as the minimum integer μ that satisfies the following recursive condition:

- $|V| \leq \mu$; or
- V can be partitioned into modules $M_1, \dots, M_{\mu'}$ of G such that $2 \leq \mu' \leq \mu$ and $\text{mw}(G[M_i]) \leq \mu$ for each $i \in [\mu']$.

► **Theorem 4.1.** *HITTING GEODESIC INTERVALS is fixed-parameter tractable parameterized by solution size k plus modular-width.*

Proof. Let $\langle G = (V, E), T, k \rangle$ be an instance of HGI. Observe that the vertex set of each connected component C of G is a module of G , and thus $\text{mw}(C) \leq \text{mw}(G)$. Hence, we may assume that G is connected; otherwise we solve each connected component of G independently for all $k' \leq k$ and combine the answers for them.

Let M_1, \dots, M_p be a partition of V into modules of G with $2 \leq p \leq \text{mw}(G)$. It is known that such a partition can be computed in linear time [24]. Note that, although the definition of modular-width requires a complete recursive structure, our algorithm described below only needs a partition at the very first level.

For each $\{u, v\} \in T$, we take a set $J(u, v)$ and initialize it by $I_G[u, v]$. Now we set $\mathcal{I} = \{J(u, v) \mid \{u, v\} \in T\}$. Our goal is to find a hitting set S of \mathcal{I} with size at most k . To this end, we make two phases of branching that make the rest of the problem almost trivially polynomial-time solvable. Although we do not explicitly state the trivial termination conditions in each step, we stop and reject a branch if it results in $k < 0$ or $\emptyset \in \mathcal{I}$.

The first branching. We first branch on the 2^p candidates of the set of modules that S intersects, which we call \mathcal{M}_S . We assume that S is chosen in such a way that $|\mathcal{M}_S|$ is maximized. According to \mathcal{M}_S , we update \mathcal{I} as follows.

- Add each $M_i \in \mathcal{M}_S$ into \mathcal{I} .
- For each $\{u, v\} \in T$, if there is $M_i \in \mathcal{M}_S$ with $M_i \subseteq J(u, v)$, then remove $J(u, v)$ from \mathcal{I} .
- For each $\{u, v\} \in T$ and $M_i \notin \mathcal{M}_S$, update $J(u, v)$ as $J(u, v) := J(u, v) \setminus M_i$.

We now see that, for every $\{u, v\} \in T$, if u and v belong to different modules, say M_i and M_j , then $J(u, v) \subseteq \{u, v\}$ after the update above. Observe that $I_G[u, v] = \{u, v\} \cup \bigcup_{M \in \mathcal{M}} M$ for some $\mathcal{M} \subseteq \{M_1, \dots, M_p\} \setminus \{M_i, M_j\}$. Since $J(u, v) \in \mathcal{I}$, we have $\mathcal{M} \cap \mathcal{M}_S = \emptyset$, and thus $J(u, v) \subseteq \{u, v\}$.

The second branching. Let \mathcal{I}' be the subset of \mathcal{I} consisting of the sets $J(u, v)$ such that $\{u, v\} \in E$ or u and v belong to different modules. Observe that every hitting set S of \mathcal{I} with size at most k contains a *minimal* hitting set S' of \mathcal{I}' with size at most k . We list all possible candidates for S' and branch on them. Since each member of \mathcal{I}' has size at most 2, we can use the algorithm of Damaschke [9] for enumerating all minimal vertex covers of size at most k in $O(|\mathcal{I}'| + k^2 2^k)$ time. It gives us all (at most 2^k) minimal hitting sets of \mathcal{I}' with size at most k (or answers that there is no such a set). After we pick one candidate S' , we decrease k by $|S'|$ and update \mathcal{I} by removing all elements hit by S' . In particular, we remove all $J(u, v) \in \mathcal{I}'$ from \mathcal{I} .

Solving the reduced instance in polynomial time. We first show that, at this point, each member of \mathcal{I} is a subset of some module M_i . This is trivial for the members of \mathcal{M}_S . For $J(u, v) \in \mathcal{I}$, we have $\{u, v\} \notin E$ and there is a module M_i such that $u, v \in M_i$. This implies that $\text{dist}_G(u, v) = 2$ since M_i has an adjacent module by the connectivity of G . Thus, $I_G[u, v]$ consists of the modules adjacent to M_i and a subset of M_i . Since $J(u, v) \in \mathcal{I}$ now, the adjacent modules do not belong to \mathcal{M}_S . This implies that $J(u, v) \subseteq M_i$.

For $i \in [p]$, let \mathcal{I}_i be the (possibly empty) subset of \mathcal{I} consisting of the members that are subsets of M_i . Since $\mathcal{I}_1, \dots, \mathcal{I}_p$ form a partition of \mathcal{I} with disjoint universes, we can handle them separately. That is, if we know a minimum hitting set S_i of \mathcal{I}_i for each $i \in [p]$, then we can set $S = S' \cup \bigcup_{i \in [p]} S_i$.

We now claim that, under the assumption that \mathcal{M}_S and S' are correct ones, each nonempty \mathcal{I}_i admits a hitting set of size 1. Suppose to the contrary that a minimum hitting set S'_i of \mathcal{I}_i has size at least 2 for some $i \in [p]$. Let M_j be a module adjacent to M_i , and let $v_i \in S'_i$ and $v_j \in M_j$. Let $S'_i = (S_i \setminus \{v_i\}) \cup \{v_j\}$ and $R = (S \setminus S_i) \cup S'_i$. Observe that S'_i hits \mathcal{I}_i : it contains at least one vertex of M_i , which hits M_i ; and $v_j \in M_j$ hits all $J(u, v) \in \mathcal{I}_i$ since $\text{dist}_G(u, v) = 2$. This implies that R interrupts all pairs in T as S does. However, this contradicts the assumption on S since R intersects one more module (M_j) than S . Thus, we can conclude that at least one of \mathcal{M}_S and S' is not correct.

Given the observation above, we can solve the rest of the current branch in polynomial time as follow. If there are at most k indices $i \in [p]$ such that \mathcal{I}_i is nonempty and each of them admits a hitting set of size 1, then return yes. Otherwise, we reject the current branch.

Total running time. We first branch on 2^p cases. In each case, we take $O^*(2^k)$ time to enumerate at most 2^k candidates for the next branch. We branch on these candidates and take polynomial time in each of them. In total, the running time is $O^*(2^{p+k})$. ◀

4.2 Vertex integrity and vertex multiway cut

To show that WHGI is fixed-parameter tractable parameterized by k plus vertex integrity and by the minimum vertex multiway-cut size of terminal vertices $V(T)$, we first give a single algorithm that applies to a more general setting that includes both cases under the assumption that a certain kind of a separator is given as part of the input. We then present simple observations of how the requirement that the separator is given as part of the input can be dropped for our cases.

4.2.1 The main algorithm

Let us first give an overview of the main algorithm. It can be seen as a generalization of the one parameterized by vertex cover number due to Aravind and Saxena [2]. Instead of a vertex cover, we use a small set $Z \subseteq V$ such that each connected component of $G - Z$ contains a small number of terminal vertices. Given Z , we guess how our solution S interrupts pairs in $\binom{Z}{2}$. To reflect the guess, we then update the family \mathcal{I} of vertex sets to be hit by the solution (which was originally the family of geodesic intervals for T), in an appropriate way. We show that, after this update, each member of \mathcal{I} is either a geodesic interval between two vertices in Z or a vertex set intersecting at most two connected components of $G - Z$. We then show that although each connected component of $G - Z$ may have unbounded size, we can focus on a bounded number of “representatives” in each connected component. This allows us to shrink the members of \mathcal{I} and then apply the sunflower lemma to a subfamily of \mathcal{I} . As a result, we get an upper bound of $|\mathcal{I}|$, and thus the rest becomes easy to solve.

► **Theorem 4.2.** *Consider a special case of WHGI where, in addition to $\langle G = (V, E), T, k \rangle$, we are given a set $Z \subseteq V$ such that $|Z| \leq p$ and each connected component of $G - Z$ contains at most q terminal vertices. There is a fixed-parameter algorithm for this problem parameterized by $k + p + q$.*

Proof. For simplicity, we assume that Z does not include any terminal vertex. This assumption can be justified by the following simple modification: if a vertex $z \in Z$ is involved in terminal pairs, then add a new vertex z' adjacent only to z and replace all terminal pairs $\{z, x\}$ with $\{z', x\}$; this modification is safe as any path to z' must go through z . We also

assume that our solution does not use any vertex in Z . For this assumption, we guess from at most 2^p candidates the subset Z' of Z that actually is contained in the solution, remove the terminal pairs interrupted by Z' , and reduce k by $|Z'|$.

Let $u, v \in V \setminus Z$ and C_u and C_v be the connected components of $G - Z$ that contain u and v , respectively. Note that C_u and C_v are not necessarily different. We now show that

$$I_G[u, v] \setminus \left(Z \cup \bigcup_{\{z, z'\} \in \binom{Z}{2}} I_G[z, z'] \right) \subseteq V(C_u) \cup V(C_v). \quad (1)$$

Let P be a shortest u - v path. To show (1), it suffices to show that either $V(P) \subseteq V(C_u) \cup V(C_v)$ or $V(P) \setminus V(P_Z) \subseteq V(C_u) \cup V(C_v)$ for some shortest path P' connecting (possibly the same) vertices $z, z' \in Z$. If P does not visit any vertex in Z , then $V(P) \subseteq V(C_u)$ holds. Assume that P visits some vertices in Z . Let P_Z be the maximal subpath of P such that the first and last vertices belong to Z . Note that P_Z is a shortest path. Let P_u and P_v be the paths containing u and v , respectively, obtained from P by removing $V(P_Z)$. Since Z separates C_u from the rest of the graph, $V(P_u) \subseteq V(C_u)$ holds. By the same argument, we can see that $V(P_v) \subseteq V(C_v)$. Hence, $V(P) \setminus V(P_Z) \subseteq V(C_u) \cup V(C_v)$ holds. This implies (1).

Branching. For each $\{u, v\} \in T$, we take a set $J(u, v)$ and initialize it by $I_G[u, v] \setminus Z$. (Recall that we already guessed and preprocessed the intersection of Z and the solution.)

Now we set $\mathcal{I} = \{J(u, v) \mid \{u, v\} \in T\}$. We branch on the $2^{\binom{|Z|}{2}}$ candidates of the subset of $\binom{Z}{2}$ that S interrupts, which we call Y . That is, we assume that we are looking for a solution S such that $S \cap I_G[u, v] \neq \emptyset$ for each $\{u, v\} \in Y$, and $S \cap I_G[u, v] = \emptyset$ for each $\{u, v\} \in \binom{Z}{2} \setminus Y$. Thus, we update \mathcal{I} as follows.

- For each $\{u, v\} \in Y$, let $J(u, v) = I_G[u, v] \setminus Z$ and add $J(u, v)$ into \mathcal{I} .
- For each $\{u, v\} \in T$ and $\{u', v'\} \in Y$, if $J(u', v') \subseteq J(u, v)$, then remove $J(u, v)$ from \mathcal{I} .
- For each $\{u, v\} \in T$ and $\{u', v'\} \in \binom{Z}{2} \setminus Y$, if $J(u', v') \subseteq J(u, v)$, then update $J(u, v)$ as $J(u, v) := J(u, v) \setminus J(u', v')$.

Let $\mathcal{I}_T = \{J(u, v) \in \mathcal{I} \mid \{u, v\} \in T\}$ and $\mathcal{I}_Y = \{J(u, v) \in \mathcal{I} \mid \{u, v\} \in Y\}$. The condition (1) implies that, after the update above, $J(u, v) \subseteq V(C_u) \cup V(C_v)$ holds for each $J(u, v) \in \mathcal{I}_T$.

Reducing the size of each set in \mathcal{I}_T . Two vertices u and v are T -equivalent if u and v interrupt the same set of pairs in T . In other words, u and v are T -equivalent if, for each $\{x, y\} \in T$, either $\{u, v\} \subseteq I_G[x, y]$ or $\{u, v\} \cap I_G[x, y] = \emptyset$ holds. Observe that the T -equivalence among vertices is an equivalence relation. Every inclusion-wise minimal solution for $\langle G, T, k \rangle$ includes at most one vertex of a T -equivalence class, and if it includes one vertex of a T -equivalence class, then taking any of them is equivalent. The partition of the vertices to the T -equivalence classes can be computed in polynomial time by first computing all-pairs shortest distance in G , and then iteratively refining the partition of the vertices with respect to the interruption relation to each pair $\{u, v\} \in T$.

Let V' be a subset of V constructed by picking one vertex from each T -equivalence class. Using V' , we further update \mathcal{I} by compressing each $J(u, v) \in \mathcal{I}$ as $J(u, v) := J(u, v) \cap V'$. We now show that this update allows us to upper-bound the size of each $J(u, v) \in \mathcal{I}_T$ by a function depending only on $p + q$. To this end, we prove the following claim.

▷ **Claim 4.3.** Let C be a connected component of $G - Z$. If two vertices $x, y \in V(C)$ are $\binom{Z \cup (V(T) \cap V(C))}{2}$ -equivalent, then they are T -equivalent.

Proof. It suffices to show that if one of x and y interrupts a pair $\{a, b\} \in T$, then the other also interrupts $\{a, b\}$. By symmetry, we may assume that x interrupts $\{a, b\}$.

Let P_{ax} be a shortest a - x path and P_{xb} a shortest x - b path. Since x interrupts $\{a, b\}$, the concatenation $P_{ab}^{(x)} := P_{ax}P_{xb}$ is a shortest a - b path. Now we pick two vertices $a' \in V(P_{ax})$ and $b' \in V(P_{xb})$ as follows: we set $a' = a$ if $a \in V(C)$; otherwise, we set a' to an arbitrary vertex in $V(P_{ax}) \cap Z$ (such a vertex exists as Z separates C from the rest of the graph); analogously, we set $b' = b$ if $b \in V(C)$; otherwise, we set b' to an arbitrary vertex in $V(P_{xb}) \cap Z$. By the definition, $\{a', b'\} \subseteq Z \cup (V(T) \cap V(C))$ holds.

Let $P_{a'b'}^{(x)}$ be the subpath of $P_{ab}^{(x)}$ from a' to b' . Since $P_{a'b'}^{(x)}$ is a shortest a' - b' path containing x , x interrupts $\{a', b'\}$, and thus so does y by their $(Z \cup (V(T) \cap V(C)))$ -equivalence. Thus, there is a shortest a' - b' path $P_{a'b'}^{(y)}$ that passes through y . This implies that we can obtain a shortest a - b path that visits y by replacing $P_{a'b'}^{(x)}$ in $P_{ab}^{(x)}$ with $P_{a'b'}^{(y)}$. Hence, y interrupts $\{a, b\}$. \triangleleft

For every connected component C of $G - Z$, the assumption on Z implies that $|Z \cup (V(T) \cap V(C))| \leq p + q$. Thus, Claim 4.3 implies that C intersects at most $2^{\binom{|Z \cup (V(T) \cap V(C))|}{2}} \leq 2^{\binom{p+q}{2}}$ T -equivalence classes, and thus, $|V' \cap V(C)| \leq 2^{\binom{p+q}{2}}$. This implies that $|J(u, v)| \leq 2^{\binom{p+q}{2}+1}$ for each $J(u, v) \in \mathcal{I}_T$ since $J(u, v) \subseteq V' \cap (V(C_u) \cup V(C_v))$.

Reducing the size of \mathcal{I} . Now we shrink the size of \mathcal{I}_T by a well-known application of the sunflower lemma [14] to d -HITTING SET, a variant of HITTING SET with each set having size at most d . There is a polynomial-time algorithm based on the sunflower lemma that, given an instance \mathcal{H} of d -HITTING SET, either

- answers that \mathcal{H} admits no hitting set of size at most k , or
- returns an instance \mathcal{H}' of d -HITTING SET with at most $k^d \cdot d!$ sets such that \mathcal{H}' and \mathcal{H} admit exactly the same (possibly empty) family of hitting sets of size at most k .

See [15, Section 9.1] or [8, Theorem 2.26]. By setting $d = 2^{\binom{p+q}{2}+1}$, we apply this algorithm to \mathcal{I}_T and obtain \mathcal{I}'_T with at most $k^d \cdot d!$ sets (or find that it is a no-instance). Finally, we set $\mathcal{I}' = \mathcal{I}'_T \cup \mathcal{I}_Y$. The discussion so far guarantees that \mathcal{I}' and \mathcal{I} are equivalent. Since $|\mathcal{I}'| (= |\mathcal{I}'_T| + |\mathcal{I}_Y| \leq k^d \cdot d! + p^2)$ depends only on $k + p + q$, we can find a minimum hitting set of \mathcal{I}' by applying the standard fixed-parameter algorithm for HITTING SET parameterized by the number of sets (see Section 1.1). \triangleleft

4.2.2 Applications of the main algorithm

Theorem 4.2 requires a separator Z with special properties as part of the input. A natural question would be whether the original problem WHGI is fixed-parameter tractable parameterized by $k + p + q$. To this end, it suffices to show that finding Z is fixed-parameter tractable parameterized by $p + q$. Although we leave this general problem unsettled, we now observe that for our special cases finding Z is fixed-parameter tractable.

The *vertex integrity* of a graph $G = (V, E)$, denoted $\text{vi}(G)$, is the minimum integer ι such that there is a set $X \subseteq V$ satisfying that $\iota = |X| + \max_{C \in \text{cc}(G-X)} |V(C)|$, where $\text{cc}(G - X)$ is the set of connected components of $G - X$. For an n -vertex graph G with $\text{vi}(G) = \iota$, finding a set $X \subseteq V$ satisfying $\iota = |X| + \max_{C \in \text{cc}(G-X)} |V(C)|$ can be done in time $O(\iota^{\iota+1}n)$ [13]. Observe that such a set X satisfies the conditions of Z in Theorem 4.2 with $p = q = \iota$. Thus we get the following result as a direct corollary.

► **Corollary 4.4.** *WEIGHTED HITTING GEODESIC INTERVALS is fixed-parameter tractable parameterized by solution size k plus vertex integrity.*

For a graph $G = (V, E)$ and $U \subseteq V$, a set $X \subseteq V$ is a *vertex multiway cut* for U if each connected component of $G - X$ contains at most one vertex of U . It is known that the problem of finding a vertex multiway cut of size r is fixed-parameter tractable parameterized by r [6, 22]. A multiway cut X for $V(T)$ with size r satisfies the conditions of Z in Theorem 4.2 with $p = r$ and $q = 1$. Furthermore, X interrupts all pairs in T as no path can connect a pair in T without passing through X . Thus, we can assume that $k \leq r$ as otherwise we already have a desired solution (i.e., X). Now we apply Theorem 4.2 and get the following result.

► **Corollary 4.5.** *WEIGHTED HITTING GEODESIC INTERVALS is fixed-parameter tractable parameterized by the minimum vertex multiway-cut size of the set of terminal vertices $V(T)$.*

As a final remark to this section, we note that the problem of finding Z in Theorem 4.2 is *nonuniformly* fixed-parameter tractable parameterized by $p + q$. This follows from the discussion by Marx [22, p. 396] on the case of $q = 1$ (i.e., the vertex multiway cut problem), which directly applies to general q as well. Given $G = (V, E)$ and $R \subseteq V$, we construct an edge-colored graph $G' = (V', E')$ as follows: for each $v \in R$, add a vertex v' and an edge $\{v, v'\}$; color E in black and $\{\{v, v'\} \mid v \in R\}$ in red. Now, there is $Z \subseteq V$ such that $|Z| \leq p$ and each connected component of $G - Z$ contains at most q terminal vertices if and only if there is $Z' \subseteq V'$ such that $|Z'| \leq p$ and each connected component of $G' - Z'$ contains at most q red edges. Observe that the latter is a minor-closed property. More precisely, let $\mathcal{G}_{p,q}$ be the family of graphs with black and red edges such that each graph $H \in \mathcal{G}_{p,q}$ admits a separator $X \subseteq V(H)$ such that $|X| \leq p$ and each connected component of $H - X$ contains at most q red edges. Then, $\mathcal{G}_{p,q}$ is closed under the operation of taking minors. Thus, the edge-colored version of the graph minor theorem (see [12, Exercise 19.5.6]) implies that testing the membership to $\mathcal{G}_{p,q}$ is fixed-parameter tractable parameterized by $p + q$.

5 Parameterized intractability beyond k

Since HGI parameterized by k belongs to $W[2]$, both subcases handled in this section also belong to $W[2]$. Hence, we only prove the $W[2]$ -hardness in each case.

For a graph $G = (V, E)$ and a set T of vertex pairs, a set $X \subseteq V$ is a *vertex multicut* for T if no connected component of $G - X$ contains a pair in T . It is known that the problem of finding a vertex multicut is fixed-parameter tractable parameterized by its size [4, 23].

Observe that a vertex multiway cut for $V(T)$ is a vertex multicut for T , but not vice versa. Observe also that a vertex multicut for T interrupts all pairs in T . Therefore, the minimum vertex *multicut* size is upper-bounded by the minimum vertex *multiway-cut* size, for which WHGI is fixed-parameter tractable (Corollary 4.5), and lower-bounded by the solution size k , for which HGI is $W[2]$ -complete [2]. Thus, the following result sharpens the border of (in)tractability in this parameter hierarchy.

► **Theorem 5.1.** *HITTING GEODESIC INTERVALS is $W[2]$ -complete parameterized by the minimum vertex multicut size of the terminal pairs T .*

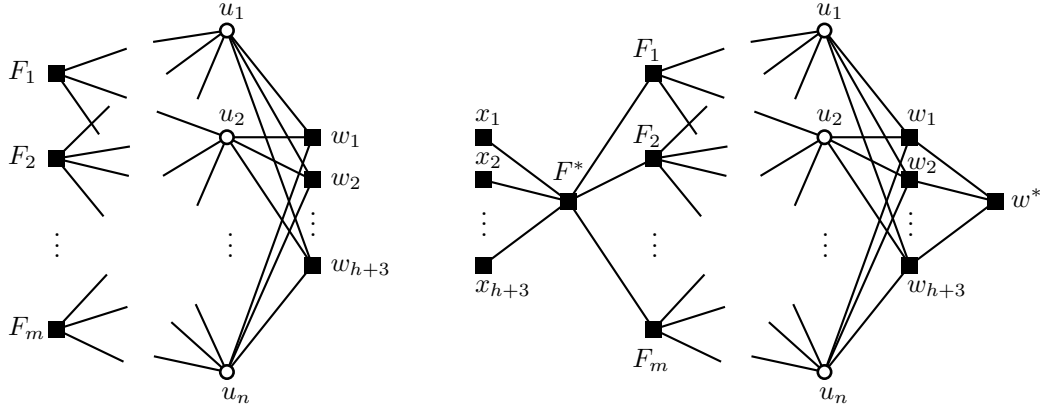
Proof. Let $\langle U, \mathcal{F}, h \rangle$ be an instance of HITTING SET, where $U = \{u_1, \dots, u_n\}$ and $\mathcal{F} = \{F_1, \dots, F_m\} \subseteq 2^U$. From this instance, we construct an equivalent instance $\langle G, T, h \rangle$ of HGI with $G = (V, E)$ as follows (see Figure 5 (left)). Let $W = \{w_i \mid i \in [h + 3]\}$.³ We set

³ For proving Theorem 5.1, it suffices to set $|W| = h + 1$ instead of $h + 3$. We use $h + 3$ to make the proof of Theorem 5.2 simpler.

$V = \mathcal{F} \cup U \cup W$. For $u_i \in U$ and $F_j \in \mathcal{F}$, we add $\{u_i, F_j\}$ into E if and only if $u_i \in F_j$. We then add all possible edges between U and W . We set $T = \{\{F_i, w_j\} \mid i \in [m], j \in [h+3]\}$. We can see that W is a vertex multicut of T with size $h+3$.

Assume that $\langle U, \mathcal{F}, h \rangle$ is a yes-instance of HITTING SET. Let $S \subseteq U$ be a hitting set of \mathcal{F} with $|S| \leq h$. Since S is a hitting set of \mathcal{F} , each $F_i \in \mathcal{F}$ includes a member of S , say u_j , and that member u_j interrupts the pair $\{F_i, w_p\}$ for each $p \in [h+3]$.

Assume that $\langle G, T, h \rangle$ is a yes-instance of HGI. We first claim that there is a solution for $\langle G, T, h \rangle$ that does not include any vertex in \mathcal{F} . To see this, observe that F_i interrupts all pairs $\{F_i, w_p\}$ for $p \in [h+3]$ but no others, while each $u_j \in F_i$ also interrupts them. Let S be a solution for $\langle G, T, h \rangle$ with $S \cap \mathcal{F} = \emptyset$. Since $|S| \leq h < |W|$, there is a vertex $w_p \in W \setminus S$. Since no other vertex $w_q \in W \setminus \{w_p\}$ belongs to the geodesic interval $I_G[w_p, F_i]$ for every $i \in [m]$, the set $S \setminus W (= S \cap U)$ has to interrupt $\{w_p, F_i\}$ for every $i \in [m]$. This implies that, each F_i has a neighbor u_j in $S \cap U$, and thus, $S \cap U$ is a hitting set of \mathcal{F} . ◀



■ **Figure 5** The graphs G (left) and H (right) in Theorems 5.1 and 5.2, respectively.

We now modify the proof of Theorem 5.1 and show that HGI is $W[2]$ -complete parameterized by k even if the structure formed by the terminal pairs is quite limited, i.e., $T = \binom{Q}{2}$ for some $Q \subseteq V$. In other words, we consider the setting where we want to hit pairwise geodesic intervals in a given vertex set Q . As this variant could be of independent interest, we define it as a separate problem as follows.

Problem: HITTING PAIRWISE GEODESIC INTERVALS (HPGI)

Input: A graph $G = (V, E)$, a set $Q \subseteq V$, and an integer k .

Question: Is there $S \subseteq V$ with $|S| \leq k$ such that $S \cap I_G[u, v] \neq \emptyset$ for each $\{u, v\} \in \binom{Q}{2}$?

► **Theorem 5.2.** *HITTING PAIRWISE GEODESIC INTERVALS is $W[2]$ -complete parameterized by solution size k .*

Proof. We first repeat the construction in the proof of Theorem 5.1 to obtain an instance $\langle G = (V, E), T, h \rangle$ of HGI from an instance $\langle U, \mathcal{F}, h \rangle$ of HITTING SET. From $\langle G, T, h \rangle$, we construct an equivalent instance $\langle H, Q, h+2 \rangle$ of HPGI (see Figure 5 (right)): to obtain H from G , we add vertices F^* and w^* and a set $X = \{x_i \mid i \in [h+3]\}$ of vertices, and then add all possible edges between F^* and $X \cup \mathcal{F}$ and between w^* and W ; we set $Q = V(H) \setminus U$.

Assume that $\langle G, T, h \rangle$ is a yes-instance of HGI. Let $S \subseteq U$ be a solution for $\langle G, T, h \rangle$. We set $S' = S \cup \{F^*, w^*\}$. We can see that S' interrupts all pairs in $\binom{Q}{2}$: S interrupts all pairs in $T = \{\{F_i, w_j\} \mid i \in [m], j \in [h+3]\}$; F^* interrupts all pairs involving a vertex in X and all pairs $\{F_i, F_j\}$; and w^* interrupts all pairs $\{w_i, w_j\}$.

Assume that $\langle H, Q, h+2 \rangle$ is a yes-instance of HPGL. Let S be a solution for $\langle G, Q, h \rangle$. Observe that $F^* \in S$ since otherwise we have to take all $h+3$ vertices of X into S . Similarly, we can see that $w^* \in S$. Since neither F^* nor w^* can interrupt pairs $\{F_i, w_p\} \in T$ and $I_G[F_i, w_p] = I_H[F_i, w_p]$ for all $i \in [m]$ and $j \in [h+3]$, we can conclude that $S \setminus \{F^*, w^*\}$ interrupts all pairs in T . ◀

6 Concluding remarks

In this paper, we showed that HITTING GEODESIC INTERVALS is intractable even on graphs with highly restricted structures, while combinations of the solution size k and some structural graph parameters make the problem fixed-parameter tractable (see Figure 1). As Figure 1 (right) shows, when k is part of the parameter, many cases remain unsettled. It would be interesting to further investigate these cases.

In one of our fixed-parameter algorithms (Theorem 4.2), we have a restriction that a certain kind of a separator is given as part of the input, and then we discussed that for our target cases it is actually not a restriction (Corollaries 4.4 and 4.5). We wonder if requiring such a separator in the input is not a restriction in general. More precisely, we would like to ask whether the following problem is fixed-parameter tractable parameterized by $p+q$.

Input: A graph $G = (V, E)$, a set of terminal vertices $R \subseteq V$, and integers p and q .
Question: Is there $Z \subseteq V$ with $|Z| \leq p$ such that each connected component C of $G - Z$ contains at most q terminal vertices (i.e., $|V(C) \cap R| \leq q$)?

Note that, as we discussed in the last paragraph of Section 4.2.2, the graph minor theorem gives a nonuniform fixed-parameter algorithm parameterized by $p+q$ for this problem, while we are looking for a uniform one.

References

- 1 Pankaj K. Agarwal, Sarel Har-Peled, Rahul Raychaudhury, and Stavros Sintos. Fast approximation algorithms for piercing boxes by points. In *SODA 2024*, pages 4892–4908, 2024. doi:10.1137/1.9781611977912.174.
- 2 N. R. Aravind and Roopam Saxena. The parameterized complexity of terminal monitoring set. In *WALCOM 2024*, volume 14549 of *Lecture Notes in Computer Science*, pages 76–90, 2024. doi:10.1007/978-981-97-0566-5_7.
- 3 Mustafa Atici. Computational complexity of geodetic set. *Int. J. Comput. Math.*, 79(5):587–591, 2002. doi:10.1080/00207160210954.
- 4 Nicolas Bousquet, Jean Daligault, and Stéphan Thomassé. Multicut is FPT. *SIAM J. Comput.*, 47(1):166–207, 2018. doi:10.1137/140961808.
- 5 Boštjan Brešar, Matjaž Kovše, and Aleksandra Tepeh. Geodetic sets in graphs. In *Structural Analysis of Complex Networks*, pages 197–218. Birkhäuser, 2011. doi:10.1007/978-0-8176-4789-6_8.
- 6 Jianer Chen, Yang Liu, and Songjian Lu. An improved parameterized algorithm for the minimum node multiway cut problem. *Algorithmica*, 55(1):1–13, 2009. doi:10.1007/S00453-007-9130-6.

- 7 Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Introduction to Algorithms*. MIT Press, 4th edition, 2022.
- 8 Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshantov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015. doi:10.1007/978-3-319-21275-3.
- 9 Peter Damaschke. Parameterized enumeration, transversals, and imperfect phylogeny reconstruction. *Theor. Comput. Sci.*, 351(3):337–350, 2006. doi:10.1016/J.TCS.2005.10.004.
- 10 Aaron L. Douthett and Man C. Kong. Computing geodesic bases of chordal and split graphs. *J. Comb. Math. Comb. Comput.*, 22:67–77, 1996.
- 11 Rodney G. Downey and Michael R. Fellows. *Parameterized Complexity*. Springer, 1999. doi:10.1007/978-1-4612-0515-9.
- 12 Rodney G. Downey and Michael R. Fellows. *Fundamentals of Parameterized Complexity*. Springer, 2013. doi:10.1007/978-1-4471-5559-1.
- 13 Pål Grønås Drange, Markus S. Dregi, and Pim van ’t Hof. On the computational complexity of vertex integrity and component order connectivity. *Algorithmica*, 76(4):1181–1202, 2016. doi:10.1007/s00453-016-0127-x.
- 14 Paul Erdős and Richard Rado. Intersection theorems for systems of sets. *J. London Math. Soc.*, 35(1):85–90, 1960. doi:10.1112/jlms/s1-35.1.85.
- 15 Jörg Flum and Martin Grohe. *Parameterized Complexity Theory*. Springer, 2006. doi:10.1007/3-540-29953-X.
- 16 Robert J. Fowler, Mike Paterson, and Steven L. Tanimoto. Optimal packing and covering in the plane are NP-complete. *Inf. Process. Lett.*, 12(3):133–137, 1981. doi:10.1016/0020-0190(81)90111-3.
- 17 Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.
- 18 Frank Harary, Emmanuel Loukakis, and Constantine Tsouros. The geodesic number of a graph. *Math. Comput. Model.*, 17(11):89–95, 1993. doi:10.1016/0895-7177(93)90259-2.
- 19 Bart M. P. Jansen. On structural parameterizations of hitting set: Hitting paths in graphs using 2-SAT. *J. Graph Algorithms Appl.*, 21(2):219–243, 2017. doi:10.7155/JGAA.00413.
- 20 Bart M. P. Jansen and Céline M. F. Swennenhuis. Steiner tree parameterized by multiway cut and even less. In *ESA 2024*, volume 308 of *LIPICs*, pages 76:1–76:16, 2024. doi:10.4230/LIPICs.ESA.2024.76.
- 21 Leon Kellerhals and Tomohiro Koana. Parameterized complexity of geodesic set. *J. Graph Algorithms Appl.*, 26(4):401–419, 2022. doi:10.7155/JGAA.00601.
- 22 Dániel Marx. Parameterized graph separation problems. *Theor. Comput. Sci.*, 351(3):394–406, 2006. doi:10.1016/J.TCS.2005.10.007.
- 23 Dániel Marx and Igor Razgon. Fixed-parameter tractability of multicut parameterized by the size of the cutset. *SIAM J. Comput.*, 43(2):355–388, 2014. doi:10.1137/110855247.
- 24 Ross M. McConnell and Jeremy P. Spinrad. Modular decomposition and transitive orientation. *Discrete Mathematics*, 201(1-3):189–241, 1999. doi:10.1016/S0012-365X(98)00319-7.
- 25 Rolf Niedermeier. *Invitation to Fixed-Parameter Algorithms*. Oxford University Press, 2006. doi:10.1093/ACPROF:OSO/9780198566076.001.0001.
- 26 Johannes C. B. Schröder. Comparing graph parameters. Bachelor thesis, Technische Universität Berlin, 2019. URL: <https://ftp.akt.tu-berlin.de/publications/theses/BA-Schr%C3%B6der.pdf>.
- 27 Manuel Sorge and Mathias Weller. The graph parameter hierarchy, 2019. URL: <https://manyu.pro/assets/parameter-hierarchy.pdf>.
- 28 Prafullkumar Tale. Geodesic set on graphs of constant pathwidth and feedback vertex set number. *CoRR*, abs/2504.17862, 2025. doi:10.48550/arXiv.2504.17862.
- 29 Duc Long Tran. Expanding the graph parameter hierarchy. Bachelor thesis, Technische Universität Berlin, 2022. URL: <https://ftp.akt.tu-berlin.de/publications/theses/BA-Duc-Long-Tran.pdf>.