

Realization of Temporally Connected Graphs Based on Degree Sequences

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Abstract

Given an undirected graph G , the problem of deciding whether G admits a simple and proper time-labeling that makes it temporally connected is known to be NP-hard (Göbel *et al.*, 1991). In this article, we relax this problem and ask whether a given degree sequence can be realized as a temporally connected graph. Our main results are a complete characterization of the feasible cases, and a recognition algorithm that runs in $\mathcal{O}(n)$ time for graphical degree sequences (realized as simple temporal graphs) and in $\mathcal{O}(n + m)$ time for multigraphical degree sequences (realized as non-simple temporal graphs, where the number of time labels on an edge corresponds to the multiplicity of the edge in the multigraph). In fact, these algorithms can be made constructive at essentially no cost. Namely, we give a constructive $\mathcal{O}(n + m)$ time algorithm that outputs, for a given (multi)graphical degree sequence $\mathbf{d}$, a temporally connected graph whose underlying (multi)graph is a realization of $\mathbf{d}$, if one exists.

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Proof of statements marked with ($\star$) are deferred to the full version.

1 Introduction

The problem of assigning time labels to the edges of a graph in such a way that every vertex can reach every other vertex by a non-decreasing path (temporal path) is known as the gossiping problem. Problems of this type were extensively studied in the late 70s - early 90s (see, e.g. [21, 22]), initially motivated by optimal scheduling of phone calls, where the agents are represented by vertices and the calls by time-labeled edges (e.g. [17, 6]). Most of the research in this line consider interactions that are mutually exclusive (an agent cannot have multiple phone calls at the same time) and non-repetitive (two agents cannot call each other twice). One of the landmark results in this area was obtained in 1991 by Göbel *et al.* [16], who showed that deciding whether a given graph admits such a labeling is NP-complete.



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The past decade have seen a renewed interest in this type of problems, motivated by the modeling of various types of dynamic networks, such as wireless networks, social networks, autonomous vehicles, and transportation scheduling. In today's terminology, the gossip problem can naturally be rephrased in terms of temporal graphs. Formally, a temporal graph is a pair $\mathcal{G} = (G, \lambda)$ where $G = (V, E)$ is the underlying graph (in this work, undirected) and λ is a mapping that assigns time labels to the edges. Such a labeling is *simple* if every edge of G has a single time label, and it is *proper* if adjacent edges do not share any time label. The graph $\mathcal{G}$ is *temporally connected* (i.e. in class TC) if there exists a path traversing edges with non-decreasing times (temporal path) between each ordered pair of nodes. In this terminology, the gossip problem corresponds to deciding, given a graph G , if there exists a simple and proper labeling λ such that $\mathcal{G} = (G, \lambda) \in \text{TC}$ (and finding such a labeling).

A number of similar realizability questions have been considered recently in the temporal graph literature, focusing on labelings that satisfy additional prescribed properties. For example, several works investigate whether the labeling can ensure that the fastest temporal paths between vertices match (or is upper-bounded by) a given matrix of pairwise durations [24, 12, 27, 26, 28]. Another problem asks that the temporal paths realize a target reachability relation [11], under various restrictions of the labeling function (see [7, 8] for discussions on the resulting expressivity). Further studies consider minimizing the number of labels [1], minimizing reachability times [10], maximizing overall reachability sets [5], or avoiding temporal cycles [2], to name a few.

Broadly speaking, all these problems, and their non-temporal versions before them, are part of a long series of *realizability questions* in graph theory, having their root in the seminal work of Erdős and Gallai [14] and Havel and Hakimi [20, 18] on whether (and how, respectively) a given degree sequence can be realized as a (static) graph.

In this work, we return to the roots of realizability questions, studying the temporal analog of Erdős and Gallai's problem, namely, whether (and how) a given degree sequence can be realized as a temporal graph in TC, for both the simple and non-simple cases. This question is partly motivated by the hardness results of Göbel et al. [16], a natural question being how to relax the input specification in a way that offers more flexibility (and tractability) for realization. Another motivation is to investigate whether this format allows for an elegant and concise characterization of all the feasible cases, a situation that is pretty rare in the literature on temporal graphs.

Main results. Before stating the results, we recall that a *degree sequence* $\mathbf{d}$ is a non-increasing list of integers. A degree sequence of length n is *graphical* if there exist an n -vertex (static) graph whose vertices have exactly these values as their degrees. Moreover, $\mathbf{d}$ is *multigraphical* if a multigraph with this degree distribution exists.

Our main result is a complete characterization of the graphical sequences that admit a realization as a simple and proper temporal graph in TC. We refer to these sequences as *TC-realizable graphical sequences*. The characterization, obtained in Section 5, is the following:

► **Theorem 5.1.** *A graphical sequence $\mathbf{d} = (d_1, \dots, d_n)$ with $m := \frac{1}{2} \cdot \sum_{i=1}^n d_i$ edges is TC-realizable if and only if one of the following holds:*

- $m = 2n - 4$, $d_1 < n - 1$, and $d_n \geq 2$
- $m \geq 2n - 3$ and either (i) $n \leq 2$ or (ii) $n > 2$, $d_{n-1} \geq 2$, and $d_n \geq 1$.

We also characterize the degree sequences that can be realized as a non-simple and proper temporal graph in TC, where *non-simple* means that a same edge can receive multiple

time labels. Conveniently, non-simple temporal graphs are equivalent to simple temporal graphs whose underlying graph is a multigraph and where each edge of the multigraph only receives one label. Thus, the problem amounts to characterizing *TC-realizable multigraphical sequences*. The characterization is:

► **Theorem 5.2.** *A multigraphical sequence $\mathbf{d} = (d_1, \dots, d_n)$ with $m := \frac{1}{2} \cdot \sum_{i=1}^n d_i$ edges is TC-realizable if and only if one of the following holds:*

- $m = 2n - 4$ and $d_n \geq 2$
- $m \geq 2n - 3$ and either (i) $n \leq 2$ or (ii) $n > 2$, $d_{n-1} \geq 2$, and $d_n \geq 1$.

As one can see, both characterizations are close to each other, the set of TC-realizable multigraphical sequences being of course more general (indeed, TC-realizable graphical sequences are special cases of these). However, the difference is more important than one may think by looking only at these theorems, because multigraphical sequences already differ from graphical sequences. (Both standard characterizations will be recalled in the paper.)

On the algorithmic side, we show that TC-realizable graphical sequences can be recognized in $\mathcal{O}(n)$ time and TC-realizable multigraphical sequences in $\mathcal{O}(n + m)$ time. We also give a constructive $\mathcal{O}(n + m)$ time algorithm that outputs a temporal graph satisfying the desired constraints (if one exists), this algorithm being obviously asymptotically optimal.

The fact that we consider proper labelings is not a limitation. Precisely, if non-proper labelings are considered, then one may either consider strict (i.e. increasing) or non-strict (i.e. non-decreasing) temporal paths. In the case of strict temporal paths, we know that non-proper labelings can always be turned into proper labelings without reducing the reachability relation [7]. Together with the fact that proper labelings are particular cases of non-proper labelings, this implies that the feasible cases are the same as in the proper setting. If one considers non-strict temporal paths instead, then the problem becomes trivial to solve: a single spanning tree suffices (with the same time label on every edge). Thus, in this case, the problem reduces to testing if the sequence can be realized as a connected graph, which from earlier works [25, 23, 15] is the case if and only if $m \geq n - 1$ and $d_n \geq 1$.

Technical overview. The characterization of TC-realizable graphs in terms of input graphs is complicated. Some necessary and sufficient conditions are known from the work of Göbel et al. [16] (itself based on earlier works in gossip theory [19, 17, 6]), the remaining cases being NP-hard to decide. A necessary condition for these graphs is to contain two spanning trees that share at most two edges. If the graphs admit two spanning trees that share at most one edge, then this is sufficient: the graph is TC-realizable. The case with two shared edges is further constrained. If the number of edges is exactly $2n - 4$, then it is necessary and sufficient that the two shared edges belong to a central cycle of length 4. Göbel et al. [16] called such graphs “ C_4 -graphs” (although they do not consist only of a cycle of length 4. To avoid this confusion, we use the term C_4 -pivotal graphs instead. A formal definition is given in Section 2). Otherwise, there may exist feasible cases which are difficult to characterize, and these cases are precisely the ones making the problem NP-hard. Interestingly, our results imply that the framework of degree sequences allows one to circumvent these hard cases entirely.

Our characterization proceeds as follows: First, we note that the characterization of degree sequences which are realizable as a graph admitting two edge-disjoint spanning trees is known [25]. Thus, we focus on characterizing the degree sequences that can be realized as a graph admitting two spanning trees sharing exactly one edge, and, for degree sequences corresponding to exactly $2n - 4$ edges, we characterize the ones admitting a realization with the above C_4 -pivotal graph property. Surprisingly, these cases already cover all the

TC-realizable graphical sequences. Namely, if a graph has strictly more than $2n - 4$ edges and is TC-realizable, then its degree sequence could *also* be realized as a graph with two spanning trees sharing at most one edge, falling back on the previous cases. In other words, a significant by-product of our analysis is that the difficult graphs can safely be ignored in the framework of degree sequences.

In light of these explanations, the feasible cases stated in Theorem 5.1 correspond respectively to the following theorems in the paper: A degree sequence is TC-realizable if and only if it admits (i) a realization which is a C_4 -pivotal graph (Theorem 4.1) or (ii) a realization with two spanning trees that share at most one edge (Theorem 3.1). For multigraphical sequences, we prove the characterization similarly.

All of our proofs are constructive and rely on deconstructing recursively the degree sequence to a smaller degree sequence, using gadgets that preserve two edge-disjoint spanning trees in the constructed graph (up to a central component). These constructive proofs can be implemented efficiently by using a suitable data structure that stores the degree sequence and the respective graph efficiently. The labeling is then handled separately by a dedicated procedure that achieves the claimed time using our data structure and extra features offered by the above structural algorithms.

Temporal graphs are notoriously intractable objects. Most of the results in this young field are negative and most of the problems turn out to be computationally hard, often due to the non-symmetric and non-transitive nature of reachability. In this respect, the fact the feasible cases for TC-realizability admit a characterization that is at the same time compact, purely structural, and easy to recognize appears to be significant. This situation is clearly an exception in the landscape of temporal graphs studies.

2 Definitions and Important Existing Results

A temporal graph is a pair $\mathcal{G} = (G, \lambda)$, where $G = (V, E)$ is a standard (in this work, *undirected*) graph called the *underlying graph* of $\mathcal{G}$, and $\lambda : E \rightarrow 2^{\mathbb{N}}$ is a labeling function that assigns a non-empty set of time labels to every edge of E , interpreted as availability times. The labeling function λ can be restricted in various ways. It is called *proper* if adjacent edges cannot share a common time label (λ is locally injective), and it is called *simple* if every edge has exactly one time label (λ is single-valued). The typical setting of gossiping, including that of Göbel et al. [16], requires that the labeling is both proper and simple.

A pair (e, t) such that $t \in \lambda(e)$ is a *temporal edge* of $\mathcal{G}$. A *temporal path* in $\mathcal{G}$ is a sequence of temporal edges $\langle (e_i, t_i) \rangle$ such that $\langle e_i \rangle$ is a path in the underlying graph and $\langle t_i \rangle$ is non-decreasing (such a path is *strict* if $\langle t_i \rangle$ is increasing). A temporal graph $\mathcal{G}$ is *temporally connected* (in class TC) if temporal paths exist between all ordered pairs of nodes. Observe that, beyond modeling mutually exclusive interaction, the proper setting has the technical advantage of removing the distinction between strict and non-strict temporal paths (indeed, all the temporal paths in this case are *de facto* strict), which allows us to rely on a single definition of TC throughout the paper.

We can now state the gossiping problem as follows: We are given an undirected (multi)graph G and the question is whether there exists a simple and proper labeling of G such that the resulting temporal graph is temporally connected (i.e., in TC).

The original realizability question of Erdős and Gallai asks the following question:

DEGREE SEQUENCE REALIZATION

Input: A degree sequence $\mathbf{d} = (d_1, \dots, d_n)$.

Question: Does there exist a graph $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$ such that v_i has degree d_i ?

When the answer is yes, the degree sequence $\mathbf{d}$ is called *graphical* and G is called a *realization* of $\mathbf{d}$. If multiedges are permitted in G and if there is a multigraph that realizes $\mathbf{d}$, then the sequence is instead called *multigraphical* [9]. In this work, we study the following temporal version of the problem:

DEGREE SEQUENCE TC REALIZATION

Input: A degree sequence $\mathbf{d} = (d_1, \dots, d_n)$.

Question: Does there exist a realization $G = (V, E)$ of $\mathbf{d}$ and a labeling λ such that $(G, \lambda) \in \text{TC}$?

Next, we state important existing results on (multi)graphical degree sequences and TC-realizable graphs and derive immediate consequences that we use throughout the paper.

Graphical Degree Sequences.

► **Theorem 2.1** (Graphical Characterization [14, 29]). *A degree sequence $\mathbf{d} = (d_1, \dots, d_n)$ is graphical if and only if*

$$(i) \sum_{i=1}^n d_i \text{ is even, and } (ii) \text{ for each } r \in [1, n-1]: \sum_{i=1}^r d_i \leq r \cdot (r-1) + \sum_{i=r+1}^n \min(r, d_i).$$

Based on this characterization, we obtain the following sufficient condition under which degree sequences with few edges are graphical.

► **Corollary 2.2** ($\star$). *Let $\mathbf{d} = (d_1, \dots, d_n)$ be a degree sequence with $\sum_{i=1}^n d_i$ being even, $\sum_{i=1}^n d_i \leq 4(n-1)$, $n > 4$, $d_1 \leq n-1$, and $d_n \geq 2$. Then $\mathbf{d}$ is graphical if $d_4 \geq 3$.*

We will make use of this corollary several times in this work to easily argue that specific degree sequences are graphical. Similarly, we also obtain the following sufficient condition.

► **Corollary 2.3.** *Let $\mathbf{d} = (d_1, \dots, d_n)$ be a degree sequence with $\sum_{i=1}^n d_i = 4(n-1) - 2$, $n > 4$, $d_1 \leq n-1$, $d_{n-1} \geq 2$, and $d_n \geq 1$. Then $\mathbf{d}$ is graphical if $d_4 \geq 3$.*

The following algorithm constructs a graph from a graphical degree sequence.

► **Definition 2.4** (Graphical Laying Off Process [18]). *Let $\mathbf{d} = (d_1, \dots, d_n)$ be a graphical sequence. The laying off procedure consists of connecting the vertex v_i to the first d_i vertices, excluding v_i . The resulting residual sequence is given by:*

$$\begin{aligned} (d_1 - 1, \dots, d_{d_i} - 1, d_{d_i+1}, \dots, d_{i-1}, d_{i+1}, \dots, d_n) & \quad \text{if } d_i < i, \\ (d_1 - 1, \dots, d_{i-1} - 1, d_{i+1} - 1, \dots, d_{d_i+1} - 1, d_{d_i+2}, \dots, d_n) & \quad \text{if } d_i \geq i. \end{aligned}$$

The next lemma implies that the laying off process can be used iteratively to build a realization for a graphical degree sequence.

► **Lemma 2.5** ([13, 18]). *If $\mathbf{d} = (d_1, \dots, d_n)$ is a graphical sequence, then the residual sequence after laying off any entry is also graphical.*

Multigraphical degree sequences.

► **Theorem 2.6** (Multigraphical Characterization [18]). *A degree sequence $\mathbf{d} = (d_1, \dots, d_n)$ is multigraphical if and only if*

$$(i) \sum_{i=1}^n d_i \text{ is even, and } (ii) d_1 \leq \sum_{i=2}^n d_i.$$

For multigraphical sequences there also exists a “laying off” process which is similar to Definition 2.4, but more flexible. Contrary to the graphical case, where a whole degree is laid off, in the multigraphical case, only a single edge is laid off.

► **Theorem 2.7** (Multigraphical Laying Off Process [4]). *Let $\mathbf{d} = (d_1, \dots, d_n)$ be a multigraphical degree sequence with $d_n > 0$. Then, for each $j, 2 \leq j \leq n$, the degree sequence $(d_1 - 1, d_2, \dots, d_{j-1}, d_j - 1, d_{j+1}, \dots, d_n)$ is multigraphical (after reordering).*

(Multi)graphical sequences admitting two edge-disjoint spanning trees. The characterization of sequences admitting a realization with k edge-disjoint spanning trees is identical for graphical and multigraphical sequences. For graphical sequences, it was proven constructively by Kundu for $k = 2$ [25] and extended to general k by Kleitman and Wang [23]. The multigraphical case was shown non-constructively by Gu, Hai, and Liang [15]. While we focus on the case $k = 2$, we state the full characterization below.

► **Theorem 2.8** ([25, 23, 15]). *A (multi)graphical sequence $\mathbf{d} = (d_1, \dots, d_n)$ with $n \geq 2$ admits a (multigraphical) realization with $k \in \mathbb{N}$ edge-disjoint spanning trees if and only if*

$$(i) \ d_n \geq k, \text{ and } (ii) \ \sum_{i=1}^n d_i \geq 2 \cdot k(n-1).$$

TC-realizable graphs. Finally, we provide an overview on some known necessary and sufficient conditions of TC-realizable (multi)graphs.

► **Lemma 2.9** ([17]). *Let G be a TC-realizable (multi)graph on n vertices. Then G has at least $2n - 4$ edges.*

► **Lemma 2.10** ([3]). *Let G be a (multi)graph with two spanning trees that share at most one edge. Then G is TC-realizable.*

► **Definition 2.11** (Reformulation of Göbel et al. [16]). *A graph G on n vertices and $2n - 4$ edges is a C_4 -pivotal graph if G contains an induced cycle C of length 4 (called a central cycle) and two spanning trees that share exactly two edges, where both shared edges are from C .*

This definition is equivalent to the one provided by Göbel et al. [16], since no tree can contain more than three edges of the central cycle, and due to the number of edges, each edge of the cycle is contained in at least one of the two spanning trees.

► **Lemma 2.12** ([16]). *Let G be a graph with n vertices and $2n - 4$ edges. Then G is TC-realizable if and only if G is a C_4 -pivotal graph.*

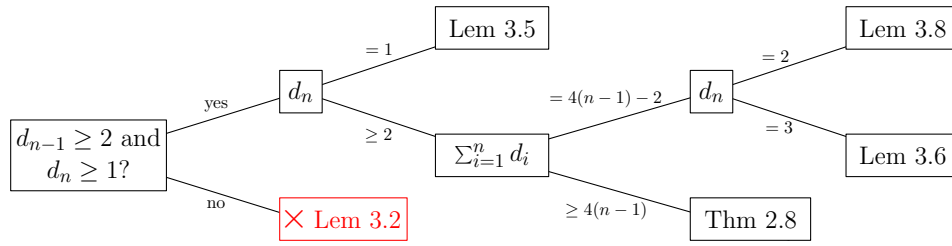
Even though it is not explicitly stated [16], the following generalization of C_4 -pivotal graphs to multigraphs also is TC-realizable by the same labeling procedure used for C_4 -pivotal graphs.¹

► **Definition 2.13.** *A multigraph G on n vertices and $2n - 4$ edges is a C_4 -pivotal multigraph if G contains an induced cycle C of length 4 (called a central cycle) and two spanning trees that share exactly two edges, where both shared edges are from C .*

Here, an induced cycle in a multigraph G has the additional property that each edge of C exists exactly once in G . Note that a C_4 -pivotal graph is also a C_4 -pivotal multigraph.

► **Corollary 2.14.** *Let G be a C_4 -pivotal multigraph. Then G is TC-realizable.*

¹ We will recall this labeling procedure in the full version.



■ **Figure 1** A guideline for the individual steps for the proof of Theorem 3.1.

3 Realizations with Two Spanning Trees Sharing at Most one Edge

In this section, we establish the following characterization of sequences that allow for a realization with two spanning trees that share (at most) one edge.

► **Theorem 3.1.** *Let $\mathbf{d} = (d_1, \dots, d_n)$ be a graphical sequence. Then, $\mathbf{d}$ admits a realization with two spanning trees that share at most one edge if and only if $\sum_{i=1}^n d_i \geq 4(n-1) - 2$ and (i) $n \leq 2$ or (ii) $n > 2$, $d_{n-1} \geq 2$, and $d_n \geq 1$.*

First, we consider the special cases of $n \leq 2$. To this end, note that $\{(0), (0, 0), (1, 1)\}$ are exactly the graphical sequences of length at most 2. Each of these sequences has a unique realization and $(0, 0)$ is the only sequence for which the unique realization is not connected. Hence for $\mathbf{d} = (0, 0)$, there is no realization with two spanning trees that share at most one edge and $\sum \mathbf{d} = 0 < 2 = 4(n-1) - 2$. For $\mathbf{d} \in \{(0), (1, 1)\}$, $\sum \mathbf{d} \geq 4(n-1) - 2$ and there is no realization with two spanning trees that share at most one edge. This proves the equivalence for part (i) of Theorem 3.1.

By the above, we may assume $n > 2$ and proceed to show the forward direction for part (ii) of Theorem 3.1. Figure 1 provides a guideline for the steps of the proof of this case.

► **Lemma 3.2** ($\star$). *Let $\mathbf{d} = (d_1, \dots, d_n)$ be a graphical sequence with $n > 2$. If $\mathbf{d}$ admits a realization with two spanning trees that share at most one edge, then $\sum_{i=1}^n d_i \geq 4(n-1) - 2$, $d_{n-1} \geq 2$, and $d_n \geq 1$.*

Realizability. Now, we consider the backward direction for part (ii) of Theorem 3.1. That is, we will show the following.

► **Lemma 3.3.** *Let $\mathbf{d} = (d_1, \dots, d_n)$ be a graphical sequence with $\sum_{i=1}^n d_i \geq 4(n-1) - 2$, $n > 2$, $d_{n-1} \geq 2$, and $d_n \geq 1$. Then $\mathbf{d}$ admits a realization with two spanning trees that share at most one edge.*

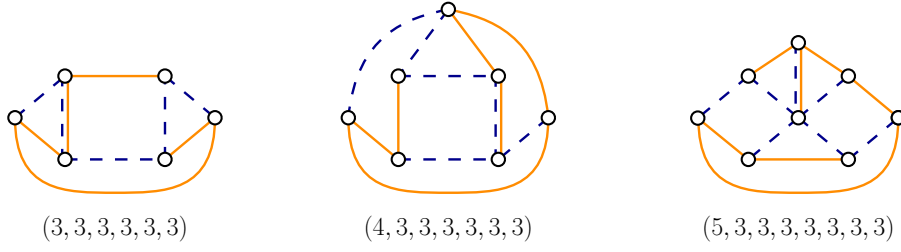
First, recall Theorem 2.8, for which the following is a corollary.

► **Corollary 3.4** ([23]). *Let $\mathbf{d} = (d_1, \dots, d_n)$ be a graphical sequence with $\sum_{i=1}^n d_i \geq 4(n-1)$, $n > 2$, and $d_n \geq 2$. Then $\mathbf{d}$ admits a realization that has two edge-disjoint spanning trees.*

Hence, it remains to consider (i) $\sum \mathbf{d} \geq 4(n-1) - 2$, $d_{n-1} \geq 2$, and $d_n = 1$, and (ii) $\sum \mathbf{d} = 4(n-1) - 2$ and $d_n \geq 2$. We first consider the former.

► **Lemma 3.5** ($\star$). *Let $\mathbf{d}$ be a graphical sequence with $\sum_{i=1}^n d_i \geq 4(n-1) - 2$, $n > 2$, $d_{n-1} \geq 2$, and $d_n = 1$. Then $\mathbf{d}$ has a realization with two spanning trees that share one edge.*

Now, we consider $\sum \mathbf{d} = 4(n-1) - 2$ and $d_n \geq 2$. Note that these sequences are guaranteed to fulfill $d_n \in \{2, 3\}$, as otherwise, $\sum \mathbf{d} \geq 4n > 4(n-1) - 2$. We first prove the case of $d_n = 3$ in Lemma 3.6, which is needed for the subsequent case $d_n = 2$ in Lemma 3.8.



■ **Figure 2** Realizations of the degree sequences $\mathbf{d}^6 = (3, 3, 3, 3, 3, 3)$, $\mathbf{d}^7 = (4, 3, 3, 3, 3, 3)$, and $\mathbf{d}^8 = (5, 3, 3, 3, 3, 3)$ with two spanning trees T_1 (solid orange edges) and T_2 (dashed blue edges) that share exactly one edge, shown from left to right.

► **Lemma 3.6.** *Let $\mathbf{d} = (d_1, \dots, d_n)$ be a graphical sequence with $\sum_{i=1}^n d_i = 4(n-1) - 2$ and $d_n = 3$. Then $\mathbf{d}$ admits a realization with two spanning trees that share one edge.*

Proof. We will show that $n \geq 6$ in this case. To this end, we provide a lower bound on the number of entries of value 3 in $\mathbf{d}$.

▷ **Claim 3.7 (★).** Let ℓ be the smallest index of $[1, n]$ for which $d_\ell = 3$. Then, the number of entries of value 3 in $\mathbf{d}$ is $6 + \sum_{i=1}^{\ell-1} (d_i - 4)$.

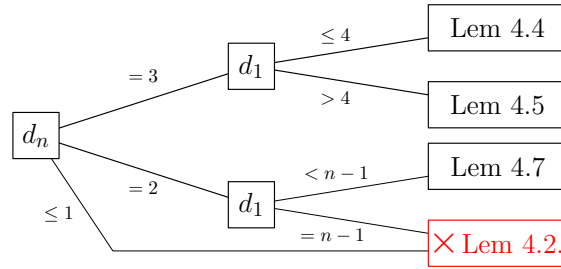
Note that this implies that $d_1 < n-1$, as otherwise, $\mathbf{d}$ would contain at least $n-1-4+6 > n$ entries of value 3. We distinguish between $d_2 = 3$ and $d_2 \geq 4$.

Firstly, consider the case that $d_2 = 3$. Note that this implies by Claim 3.7 that $d_1 = n-3$, since $\sum \mathbf{d} = 4(n-1) - 2$. In other words, for each $n \geq 6$, there is exactly one degree sequence $\mathbf{d}^n := (d_1^n, \dots, d_n^n)$ of length n with $d_2^n = d_n^n = 3$ fulfilling $\sum \mathbf{d}^n = 4(n-1) - 2$. Realizations with two spanning trees that share one edge for the three sequences $\mathbf{d}^6$, $\mathbf{d}^7$, and $\mathbf{d}^8$ are depicted in Figure 2. Now, consider $n \geq 9$. Recall that $d_1 = n-3$ and $d_i = 3$ for each $i \in [2, n]$. Moreover, recall that $\mathbf{d}^6 = (3, 3, 3, 3, 3, 3)$ has a realization G' with two spanning trees T'_1 and T'_2 that share exactly one edge (see Figure 2). Let v_1 be an arbitrary vertex of degree 3 in G' . We obtain a graph G by adding a cycle C of length $d_1 - 3 = n-6$ to G' and adding the edge v_1x to G' for each vertex x of the cycle C . Note that G is a realization of $\mathbf{d}$, since G contains $5 + d_1 - 3 = n-1$ vertices of degree 3 and the degree of v_1 was increased by $d_1 - 3$ to d_1 . Since $G^c := G[\{v_1\} \cup C]$ consists of a universal vertex attached to a cycle of length at least 3 (since $d_1 - 3 = n-6 \geq 3$ by $n \geq 9$), G^c contains two edge-disjoint spanning trees T_1^c and T_2^c . Hence, $T_1 := T'_1 \cup T_1^c$ and $T_2 := T'_2 \cup T_2^c$ are spanning trees of G , and T_1 and T_2 share only one edge, namely the edge shared by T'_1 and T'_2 . Hence, $\mathbf{d}$ admits a realization with two spanning trees that share one edge.

Secondly, consider the case that $d_2 \geq 4$. Note that Claim 3.7 thus implies that $n \geq d_1 + 4$, since $d_2 \geq 4$ and $\mathbf{d}$ contains at least $d_1 - 4 + 6 = d_1 + 2$ entries of value 3. Let $\mathbf{d}'$ be the sequence obtained from $\mathbf{d}$ by removing the entry d_1 , removing $d_1 - 1$ entries of value 3, and adding a 1. We first show that $\mathbf{d}'$ admits a realization with two spanning trees that share one edge, and then describe how to construct the realization for $\mathbf{d}$.

The remainder of this case is deferred to the full version. Intuitively, to then obtain a desired realization, we attach, similar to the first case, a large cycle to a degree-1 vertex v_1 . This time, the cycle has length $d_1 - 1$ instead of $d_1 - 3$ as in the first case. ◀

Finally, we consider the case where $\sum \mathbf{d} = 4(n-1) - 2$ and $d_n = 2$. Essentially, we perform an induction over the length of the sequence and lay off the smallest degree $d_n = 2$, until we reach a sequence of constant length or d_n has value unequal to 2. In the latter case, the previous lemma acts as a base case.



■ **Figure 3** A guideline for the individual steps for the proof of Theorem 4.1.

► **Lemma 3.8** (*). Let $\mathbf{d} = (d_1, \dots, d_n)$ be a graphical sequence with $\sum_{i=1}^n d_i = 4(n-1) - 2$ and $d_n = 2$. Then $\mathbf{d}$ admits a realization with two spanning trees that share one edge.

4 Realizations of C_4 -pivotal Graphs

In this section, we establish the following characterization of graphical sequences that allow for a C_4 -pivotal graph realization.

► **Theorem 4.1.** Let $\mathbf{d} = (d_1, \dots, d_n)$ be a graphical sequence. Then, $\mathbf{d}$ admits a realization which is a C_4 -pivotal graph if and only if $\sum_{i=1}^n d_i = 4(n-1) - 4$, $d_1 < n-1$, and $d_n \geq 2$.

Figure 3 provides a guideline on the individual steps for proving Theorem 4.1. First, we show the forward direction of Theorem 4.1. To this end, we analyze the minimum and maximum degree of C_4 -pivotal graphs. This then directly proves the forward direction of Theorem 4.1.

► **Lemma 4.2** (*). Let G be a C_4 -pivotal graph on n vertices. Then G has a minimum degree of at least 2 and a maximum degree of at most $n-2$.

Realizability. Now, we show the backward direction. That is, we show the following.

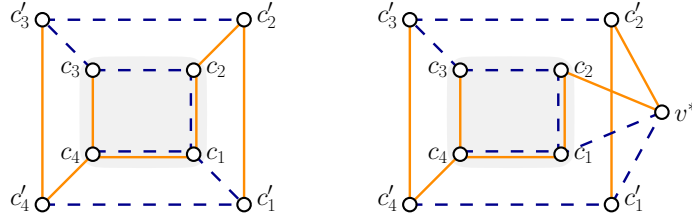
► **Lemma 4.3.** Let $\mathbf{d} = (d_1, \dots, d_n)$ be a graphical sequence with $\sum_{i=1}^n d_i = 4(n-1) - 4$, $d_1 < n-1$, and $d_n \geq 2$. Then, $\mathbf{d}$ admits a realization which is a C_4 -pivotal graph.

To prove this statement, we assume that we are given a graphical sequence $\mathbf{d}$ with $\sum \mathbf{d} = 4(n-1) - 4$, $d_1 < n-1$, and $d_n \geq 2$. Note that these sequences are guaranteed to fulfill $d_n \in \{2, 3\}$, as otherwise, $\sum \mathbf{d} \geq 4n > 4(n-1) - 4$. We split the analysis in three cases. First, we consider the case of $d_n = 3$ and $d_1 \leq 4$ in Lemma 4.4, then $d_n = 3$ and $5 \leq d_1 < n-1$ in Lemma 4.5, and lastly $d_n = 2$ and $d_1 < n-1$ in Lemma 4.6.

► **Lemma 4.4.** Let $\mathbf{d} = (d_1, \dots, d_n)$ be a graphical sequence with $\sum_{i=1}^n d_i = 4(n-1) - 4$, $d_1 \leq 4$ and $d_n = 3$. Then $\mathbf{d}$ admits a realization which is a C_4 -pivotal graph.

Proof. Since $\sum \mathbf{d} = 4(n-1) - 4$, $d_1 \leq 4$, and $d_n = 3$, we get that $n \geq 8$, $d_{n-7} = 3$, and (if $n > 8$), $d_{n-8} = 4$. In other words, for each $n \geq 8$, there is exactly one degree sequence $\mathbf{d}^n$ of length n with the required properties.

We show that $\mathbf{d}^n$ admits a realization which is a C_4 -pivotal graph via induction over n . In fact, we show a stronger result, namely, that for each $n \geq 8$, $\mathbf{d}^n$ has a realization G which is a C_4 -pivotal graph with central cycle C , such that (i) there are two spanning trees T_1 and T_2 of G that share exactly two edges and both belong to C , and (ii) for each $i \in [1, 2]$,



■ **Figure 4** Realization of the degree sequence $\mathbf{d}^8 = (3, 3, 3, 3, 3, 3, 3, 3)$ from the base case in the proof of Lemma 4.4, featuring the central 4-cycle C highlighted in gray, and two spanning trees T_1 (solid orange edges) and T_2 (dashed blue edges) that share exactly two edges, both contained in C . The right side illustrates the inductive step.

there are two distinct edges e_i, f_i from T_i and outside of C for which $\{e_1, e_2\}$ and $\{f_1, f_2\}$ are matchings in G . We call edge pairs (e_1, e_2) and (f_1, f_2) fulfilling these conditions *matching edge pairs*.

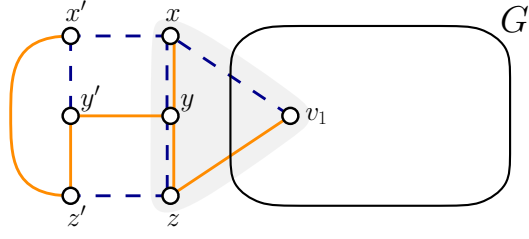
For the base case of $n = 8$, consider the graph given in Figure 4 which is a realization of $\mathbf{d}^8 = (3, 3, 3, 3, 3, 3, 3, 3)$. As highlighted in the figure, there is a central cycle C on the vertices c_1, c_2, c_3, c_4 . Moreover, $(c_1c'_1, c_2c'_2)$ and $(c_3c'_3, c_4c'_4)$ are matching edge pairs. Hence, the statement holds for the base case of $n = 8$.

Now let $n > 8$ and assume that the statement holds for $n - 1$. We show that the statement also holds for n . Let G' be a realization of $\mathbf{d}^{n-1}$ according to the properties of the induction hypothesis. That is, G' is a C_4 -pivotal graph with central cycle C , two spanning trees T'_1 and T'_2 that share exactly two edges, both of which belong to C , and matching edge pairs (e'_1, e'_2) and (f'_1, f'_2) . Recall that $\mathbf{d}^n$ can be obtained from $\mathbf{d}^{n-1}$ by adding a single entry of value 4. The idea to obtain a realization for $\mathbf{d}^n$ with these properties is by essentially subdividing both edges $e'_1 = pq$ and $e'_2 = xy$ and identifying both newly added degree-2 vertices to obtain a new vertex v^* (which then has degree 4). Formally, we obtain a graph G by removing the edges e'_1 and e'_2 from G' and afterwards adding a new vertex v^* together with the edges pv^*, qv^*, xv^*, yv^* .

Note that each vertex of $V(G) \setminus \{v^*\}$ has the same degree in both G and G' and v^* has degree 4. Hence, G is a realization of $\mathbf{d}^n$. Moreover, since e'_1 and e'_2 are not part of the central cycle C of G' , C is still a central cycle where $T_1 := (T'_1 - e'_1) \cup \{pv^*, qv^*\}$ and $T_2 := (T'_2 - e'_2) \cup \{xv^*, yv^*\}$ being spanning trees of G that each share exactly two edges and both of which belong to C . This implies that G is a C_4 -pivotal graph. It thus remains to show that there are matching edge pairs (e_1, e_2) and (f_1, f_2) in G . To define these pairs, we use the so far undiscussed pair (f'_1, f'_2) . Since e'_1, e'_2, f'_1, f'_2 are pairwise distinct edges in G' , f'_1 and f'_2 are still edges of G and moreover, f'_1 is an edge of T_1 and f'_2 is an edge of T_2 . Recall that xv^* and yv^* are edges of T_2 , and note that f'_1 contains at most one of x and y as an endpoint, as otherwise f'_1 would be identical to e'_2 . Hence, there is an edge $e_2 \in \{xv^*, yv^*\}$ which is not adjacent to f'_1 , that is, $\{f'_1, e_2\}$ is a matching in G . With the same arguments, there is an edge $e_1 \in \{pv^*, qv^*\}$ which is not adjacent to f'_2 , that is, $\{e_1, f'_2\}$ is a matching in G . This implies that (f'_1, e_2) and (e_1, f'_2) are matching edge pairs. Consequently, the statement holds for n .

Concluding, for each $n \geq 8$, the degree sequence $\mathbf{d}^n$ admits a C_4 -pivotal graph realization. ◀

Next, we consider the case that $5 \leq d_1 < n - 1$ and $d_n = 3$.



■ **Figure 5** Construction of the graph G described in the proof of Lemma 4.5.

► **Lemma 4.5** ($\star$). *Let $\mathbf{d} = (d_1, \dots, d_n)$ be a graphical sequence with $\sum_{i=1}^n d_i = 4(n-1) - 4$, $5 \leq d_1 < n-1$, and $d_n = 3$. Then $\mathbf{d}$ admits a realization which is a C_4 -pivotal graph.*

Sketch. Essentially, we can show that the degree sequence $\mathbf{d}'$ obtained by reducing v_1 by 2 and removing six entries of value 3, has a realization G' with two edge-disjoint spanning trees. To this graph, we then attach the six vertices as depicted in Figure 5 to a vertex v_1 of degree $d_1 - 2$ in G' . The resulting graph is then a C_4 -pivotal graph realization of $\mathbf{d}$. ◀

Finally, we consider the case where $d_1 < n-1$ and $d_n = 2$. This proof is similar to the proof of Lemma 3.8. Essentially, we perform an induction over the length of the sequence and lay off the smallest degree $d_n = 2$, until we reach a sequence of constant length or d_n has value 3. In the latter case, one of the previous two lemmas acts as a base case.

► **Lemma 4.6** ($\star$). *Let $\mathbf{d} = (d_1, \dots, d_n)$ be a graphical sequence with $\sum_{i=1}^n d_i = 4(n-1) - 4$, $d_1 < n-1$, and $d_n = 2$. Then $\mathbf{d}$ admits a realization which is a C_4 -pivotal graph.*

5 The Characterizations for TC-realizable Sequences

We now prove our characterization for graphical and multigraphical TC-realizable sequences.

5.1 Realizations of TC-realizable Graphs

We now prove our characterization of TC-realizable graphical sequences restated as follows.

► **Theorem 5.1.** *A graphical sequence $\mathbf{d} = (d_1, \dots, d_n)$ with $s = \sum_{i=1}^n d_i$ is TC-realizable if and only if one of the following holds*

- $s = 4(n-1) - 4$, $d_1 < n-1$, and $d_n \geq 2$
- $s \geq 4(n-1) - 2$ and (i) $n \leq 2$ (ii) $n > 2$, $d_{n-1} \geq 2$, and $d_n \geq 1$.

Proof. ($\Leftarrow$) If $\sum \mathbf{d} = 4(n-1) - 4$, $d_1 < n-1$, and $d_n \geq 2$, then due to Theorem 4.1, $\mathbf{d}$ has a realization G which is a C_4 -pivotal graph. By Harary and Schwenk [19] this implies that G is TC-realizable and thus that $\mathbf{d}$ is TC-realizable.

If $\sum \mathbf{d} \geq 4(n-1) - 2$ and (i) $n \leq 2$ or (ii) $n > 2$, $d_{n-1} \geq 2$, and $d_n \geq 1$, then $\mathbf{d}$ admits a realization G which has two spanning trees that share at most one edge (see Theorem 3.1). By Harary and Schwenk [19] this implies that G is TC-realizable and thus that $\mathbf{d}$ is TC-realizable.

($\Rightarrow$) Suppose that $\mathbf{d}$ is TC-realizable and let G be one of its realizations. By Hajnal et al. [17], G has at least $2n - 4$ edges, which implies $\sum \mathbf{d} \geq 4(n-1) - 4$.

First, consider the case that $\sum \mathbf{d} = 4(n-1) - 4$. That is, G has $2n - 4$ edges. Hence, by Göbel et al. [16], G is a C_4 -pivotal graph. Thus, by Theorem 4.1, $d_1 < n-1$, and $d_n \geq 2$.

Second, consider the case that $\sum \mathbf{d} \geq 4(n-1) - 2$. Since G is TC-realizable, G is connected. For $n \leq 2$, $\mathbf{d}$ is from $\{(0), (1, 1)\}$ as these are the only graphical sequences of length at most 2 fulfilling $\sum \mathbf{d} \geq 4(n-1) - 2$. Thus, consider $n > 2$. Since G is connected

and $n > 2$, G has a minimum degree of 1, which implies that $d_n \geq 1$. It remains to show that $d_{n-1} \geq 2$. To this end, assume towards a contradiction that $d_{n-1} = d_n = 1$. Then, G contains at least two vertices u and v of degree 1. These vertices are not adjacent in G , as otherwise, G would not be connected. Hence, the unique edge e_u incident with u and the unique edge e_v incident with v are distinct. Since G is TC-realizable, there is a TC-labeling of the edges of G . To ensure that there is a temporal path from u to v , this labeling has to assign a label to e_u which is strictly smaller than the label assigned to e_v . But simultaneously, to ensure that there is a temporal path from v to u , the label assigned to e_v has to be strictly smaller than the label assigned to e_u ; a contradiction. Consequently, G contains no two vertices of degree 1, which implies that $d_{n-1} \geq 2$ and $d_n \geq 1$ in the case that $n > 2$. ◀

5.2 Realizations of TC-realizable Multigraphs

We now prove our characterization for TC-realizable multigraphical sequences:

► **Theorem 5.2** ($\star$). *A multigraphical sequence $\mathbf{d} = (d_1, \dots, d_n)$ with $s := \sum_{i=1}^n d_i$ is TC-realizable if and only if one of the following holds*

- $s = 4(n-1) - 4$ and $d_n \geq 2$
- $s \geq 4(n-1) - 2$ and (i) $n \leq 2$ or (ii) $n > 2$, $d_{n-1} \geq 2$, and $d_n \geq 1$.

Note that the only difference from the characterization of (simple) graphical sequences (see Theorem 5.1) is the absence of the restriction $d_1 < n-1$ in the case $m = 4(n-1) - 4$.

We show the statement in the following similarly to the graphical case.

Intermediate characterization for spanning trees with one shared edge. First, we provide a characterization for multigraphical sequences that allow for a realization that has two spanning trees with at most one shared edge.

► **Theorem 5.3** ($\star$). *Let $\mathbf{d} = (d_1, \dots, d_n)$ be a multigraphical sequence. Then, $\mathbf{d}$ admits a realization with two spanning trees that share at most one edge if and only if $\sum_{i=1}^n d_i \geq 4(n-1) - 2$ and (i) $n \leq 2$ or (ii) $n > 2$, $d_{n-1} \geq 2$, and $d_n \geq 1$.*

Intermediate characterization for C_4 -pivotal multigraphs. Next, we provide a characterization for multigraphical sequences that allow for a realization which is a C_4 -pivotal multigraph.

► **Theorem 5.4.** *Let $\mathbf{d} = (d_1, \dots, d_n)$ be a multigraphical sequence. Then, $\mathbf{d}$ admits a realization which is a C_4 -pivotal multigraph if and only if $\sum_{i=1}^n d_i = 4(n-1) - 4$ and $d_n \geq 2$.*

Proof. ($\Rightarrow$) By definition, each C_4 -pivotal multigraph on n vertices has exactly $2n - 4$ multiedges. Hence, $\sum \mathbf{d} = 4(n-1) - 4$ holds. Next, we show that $d_n \geq 2$ also has to hold. The argument is identical to the one in the proof of Lemma 4.2: By definition, each vertex v of a the C_4 -pivotal multigraphs G is (i) part of the central cycle or (ii) a vertex that has at least one incident multiedge in each of the two spanning trees that only share edges of the central cycle. In both cases, vertex v has degree at least 2. Hence, the minimum degree of G is at least 2, which implies that $d_n \geq 2$ to allow for a realization which is a C_4 -pivotal multigraph. This completes this direction.

($\Leftarrow$) We have to show that $\mathbf{d}$ has a realization which is a C_4 -pivotal multigraph if $\sum \mathbf{d} = 4(n-1) - 4$ and $d_n \geq 2$. Note that this implies that $n \geq 4$. To show the statement, we distinguish between the possible values of d_{n-2} .

If $d_{n-2} \geq 3$, then $\sum \mathbf{d} - d_1 \geq 3(n-3) + d_{n-1} + d_n \geq 3(n-3) + 4 = 3(n-1) - 2$. By $\sum \mathbf{d} = 4(n-1) - 4$, this implies $d_1 \leq n-3$. Thus, by Corollary 2.2, $\mathbf{d}$ is also a graphical sequence, since $d_4 \geq 3$. This is due to the fact that $d_{n-2} \geq 3$ and $n \geq 8$, as otherwise, $\sum \mathbf{d} \geq 3n > 4(n-1) - 4$. Hence, Theorem 4.1 thus implies that $\mathbf{d}$ admits a realization which is a C_4 -pivotal graph (and thus a C_4 -pivotal multigraph).

Hence, we only need to consider the case that $d_{n-2} = 2 = d_{n-1} = d_n$. First, we consider special cases: $\mathbf{d} \in \{(2, 2, 2, 2), (3, 3, 2, 2, 2)\}$. Note that these are the only two options for $\mathbf{d}$ for which $\sum \mathbf{d} = 4(n-1) - 4$, $d_1 \leq 4$, and $d_{n-2} = d_{n-1} = d_n = 2$. Both these degree sequences are graphical and fulfill $d_1 < n-1$ and $d_n \geq 2$. Hence, by Theorem 4.1, $\mathbf{d}$ admits a realization which is a C_4 -pivotal graph (and thus a C_4 -pivotal multigraph).

For all other options of $\mathbf{d}$, we can thus assume that $d_1 \geq 4$. Let $\mathbf{d}' := (d'_1, \dots, d'_{n'})$ be the degree sequence with $n' := n-3$ obtained from $\mathbf{d}$ by (i) removing d_{n-2} , d_{n-1} , and d_n , and by (ii) reducing d_1 by 2 (and potentially reordering).

▷ Claim 5.5 (★). $\mathbf{d}'$ admits a realization G' with two edge-disjoint spanning trees T'_1 and T'_2 .

We obtain a realization G of $\mathbf{d}$ as follows: Let v_1 denote an arbitrary vertex of G' with degree $d_1 - 2$. We add three new vertices x, y , and z to G' and add the edges xy, yz, v_1x, v_1z . Note that G is a realization of $\mathbf{d}$, since we added three new vertices of degree 2 to G' and increased the degree of a vertex of degree $d_1 - 2$ by 2. It remains to show that G is a C_4 -pivotal multigraph. This is the case, since $C := (v_1, x, y, z)$ is an induced cycle of length 4 in G and $T_1 := T'_1 \cup \{v_1x, xy, yz\}$ and $T_2 := T'_2 \cup \{v_1z, xy, yz\}$ are two spanning trees of G that only share two edges and both these edges are from C (namely the edges xy and yz). Consequently, $\mathbf{d}$ admits a realization which is a C_4 -pivotal multigraph. ◀

With these characterizations, we can prove our characterization for TC-realizable multigraphical sequences. The proof is similar to Theorem 5.1 and deferred to the full version.

6 Linear-Time Algorithm for Realizing TC (Multi)graphs

In the previous section, we presented a full characterization of (multi)graphical degree sequences that are TC-realizable (see Theorems 5.1 and 5.2). These necessary and sufficient conditions can be checked in $\mathcal{O}(n)$ time in the graphical and in $\mathcal{O}(n+m)$ time in the multigraphical case, where $m := \frac{1}{2} \cdot \sum \mathbf{d}$ denotes the number of edges in each realization. Hence, in linear time, we can detect whether there is a TC-realizable graph for the given degree sequence. It is, however, not clear how efficiently we can build such a realization. In this section, we show that in both the graphical and multigraphical case, we can construct such a realization (together with a TC-labeling) in $\mathcal{O}(n+m)$ time. To this end, we observe that all existential proofs in the previous sections are constructive and allow for linear time implementations with the help of a simple data structure.

First, we describe the data structure and its properties, afterwards, we explain how all previous constructive algorithms can be implemented in linear time, and finally, we describe how to obtain a TC-labeling for the constructed realization in linear time.

The data structure. We assume that all graphs be stored via adjacency list. Since our arguments rely on spanning trees, we assign two Boolean flags to each edge to indicate whether this edge is part of the first tree, the second tree, both, or neither of them. This will allow to extract the two spanning trees in $\mathcal{O}(n+m)$ time, once the final realization is constructed.

We now come to the additional data structure for degree sequences and (multi)graphs. Instead of storing degree sequences as sequences of numbers, we store them as double-linked list that store entries of the form (x, c) . Such an entry indicates that the respective degree sequence contains exactly c entries of value x . The entries of the double-linked list are sorted decreasingly according to their x -values. In this way, both in the graphical and multigraphical case, we can lay off a degree d_i in $\mathcal{O}(d_i)$ time, since we only need to remove d_i and rearrange the c -values of the at most $d_i + 1$ first entries of the data structure. In this data structure, we can ensure that the degree sequence always stays sorted and no reordering is necessary.

Similarly, we use an additional data structure for (multi)graphs G . It is also a double-linked list and stores entries of the form (x, S) . Such an entry indicates that the vertices of S are exactly the vertices of G that have degree exactly x . These entries are also sorted decreasingly according to their x -values. This way, we can reattach a vertex of degree d_n to G by adding edges properly to vertices of the first d_n entries of the data structure. That is, we can obtain a realization for the degree sequence that, after laying off a vertex of degree d_n , leads to the degree sequence of G . In particular, this can be done in $\mathcal{O}(d_n)$ time.

In case of a realization for a C_4 -pivotal (multi)graph, we also output the central cycle separately.

Realizations for two edge-disjoint spanning trees. We now turn to the algorithmic aspects of finding a realization with two edge-disjoint spanning trees and show that such a realization can be computed in linear time.

For the graphical case, we used the result by Kundu [25]. While their proof is constructive in nature, the arguments are brief and incomplete. Thus, we give a complete construction in Lemma 6.1 and argue how it can be implemented in $\mathcal{O}(n + m)$ time using our data structure.

For the multigraphical case, we build on the result by Gu et al. [15], which was proven non-constructively. In Lemma 6.4, we give a complete inductive construction for such multigraphical sequences, and argue that this construction, too, can be carried out in $\mathcal{O}(n + m)$ time using our data structure.

► **Lemma 6.1** ($\star$). *Let $\mathbf{d} = (d_1, \dots, d_n)$ be a graphical sequence with $\sum \mathbf{d} \geq 4(n - 1)$ and $d_n \geq 2$. We can compute a realization G of $\mathbf{d}$ and two disjoint spanning trees of G in $\mathcal{O}(n + m)$ time.*

Proof. To prove the statement, we show that for each n and for each graphical sequence $\mathbf{d}$ of length n with $\sum \mathbf{d} \geq 4(n - 1)$ and $d_n \geq 2$, we can construct a realization with two edge-disjoint spanning trees. We show this statement via induction over n .

For the base case, note that for $n \in \{1, 2, 3\}$, there is no graphical sequence of length n that sums up to $4(n - 1)$. Moreover, for $n = 4$, $(3, 3, 3, 3)$ is the only graphical sequence of length n with $\sum \mathbf{d} \geq 4(n - 1)$ and $d_n \geq 2$. Since the complete graph on four vertices, which obviously contains two edge-disjoint spanning trees, is the only realization of $\mathbf{d}$, the base case holds for $n \leq 4$.

For the inductive step, let $n \geq 5$ and assume that the statement holds for $n - 1$. Let $\mathbf{d}$ be an arbitrary graphical sequence of length n fulfilling $\sum \mathbf{d} \geq 4(n - 1)$ and $d_n \geq 2$. We show that $\mathbf{d}$ admits a realization with two edge-disjoint spanning trees. Observe that whenever $\sum \mathbf{d} = 4(n - 1)$, it holds that $d_n \in \{2, 3\}$ as otherwise $\sum \mathbf{d} \geq 4n$.

If $\sum \mathbf{d} = 4(n - 1)$ and $d_n = 2$, or $\sum \mathbf{d} > 4(n - 1)$ and $d_n \geq 2$, consider the residual sequence $\mathbf{d}' = (d'_1, \dots, d'_{n'})$ of length $n' = n - 1$ obtained from laying off the entry d_n according to Definition 2.4. In the full version, we show that this degree sequence fulfills the conditions of the induction hypothesis and thus admits a realization with two edge-disjoint spanning trees. Moreover, we can then obtain a desired realization for $\mathbf{d}$ by reattaching a vertex of degree d_n .

Otherwise, $\sum \mathbf{d} = 4(n-1)$ and $d_n = 3$. In that case, consider the sequence $\mathbf{d}'$ obtained from $\mathbf{d}$ by reducing d_1 by 1 and removing the entry d_n .

▷ **Claim 6.2** ($\star$). $\mathbf{d}'$ admits a realization G' with two edge-disjoint spanning trees T'_1 and T'_2 .

Now consider the graph G obtained from G' by adding a degree-3 vertex v_n , connecting it to a vertex v_1 of G' of degree $d_1 - 1$, and inserting it on an arbitrary edge $v_i v_j$ of G' with $v_1 \notin \{v_i, v_j\}$ (replacing $v_i v_j$ with $v_n v_i$ and $v_n v_j$). Then G is a realization of $\mathbf{d}$. Moreover, G contains the edge-disjoint spanning trees $T_1 := T'_1 \cup v_1 v_n$ and $T_2 := (T'_2 - v_i v_j) \cup \{v_n v_i, v_n v_j\}$.

We defer the analysis of the running time to the full version. ◀

For multigraphical sequences, we split the proof in two parts: we first show how to construct the desired realization for a multigraphical sequence $\mathbf{d}$ with $\sum \mathbf{d} = 4(n-1)$ and $d_n \geq 2$ in Lemma 6.3, and then describe how this can be used to obtain a realization for every multigraphical sequence fulfilling $\sum \mathbf{d} > 4(n-1)$ and $d_n \geq 2$.

► **Lemma 6.3** ($\star$). *Let $\mathbf{d} = (d_1, \dots, d_n)$ be a multigraphical sequence with $\sum \mathbf{d} = 4(n-1)$ and $d_n \geq 2$. We can compute a realization G of $\mathbf{d}$ and two disjoint spanning trees of G in $\mathcal{O}(n+m)$ time.*

Sketch. Intuitively, this result is again obtained via a constructive induction over the of n . Two cases are distinguished in the inductive step: If $d_n = 2$, then simply laying off d_n and reattaching it to the provided solution for the sequence of length $n-1$ provides a desired realization. By our data structure, this laying off and reattaching can be done by a constant time overhead. If $d_n = 3$, then the degree sequence actually is graphical and the algorithm behind Lemma 6.1 provides the desired result. ◀

The construction for sequences with a degree sum larger than $4(n-1)$ follows directly from the laying-off process for multigraphical sequences (Theorem 2.7), which reduces the degree sum by two while preserving the length of the sequence.

► **Lemma 6.4** ($\star$). *Let $\mathbf{d} = (d_1, \dots, d_n)$ be a multigraphical sequence with $\sum \mathbf{d} \geq 4(n-1)$ and $d_n \geq 2$. We can compute a realization G of $\mathbf{d}$ and two disjoint spanning trees of G in $\mathcal{O}(n+m)$ time.*

Realizations with spanning trees that share at most one edge and C_4 -pivotal graph realizations. By carefully recalling the existential proofs of the previous sections, one can observe that these proofs imply linear-time constructions by using the data structure.

► **Lemma 6.5** ($\star$). *Let $\mathbf{d}$ be a (multi)graphical degree sequence. If $\mathbf{d}$ is TC-realizable, then we can compute a realization of $\mathbf{d}$ in $\mathcal{O}(n+m)$ time.*

Computing a TC-labeling. We recall the labeling schemes for TC-labelings (that can be implemented in linear time) in the full version. We conclude the following algorithmic main result of our work.

► **Theorem 6.6** ($\star$). *Let $\mathbf{d}$ be a (multi)graphical degree sequence. We can decide in $\mathcal{O}(n+m)$ time whether $\mathbf{d}$ is TC-realizable. If this is the case, then in $\mathcal{O}(n+m)$ time, we can compute a (multi)graph G which is a realization of $\mathbf{d}$ together with a TC-labeling for G .*

7 Conclusion

In this work, we presented complete characterizations of graphical and multigraphical degree sequences that admit temporally connected realizations under proper labelings. All characterizations presented in this paper are constructive and lead to linear-time algorithms for constructing TC realizations (including the corresponding labelings), provided that the input satisfies the necessary conditions.

As explained in introduction, the fact that we only considered proper labelings is not a limitation, as the remaining cases involving non-proper labelings are either trivial or covered by our techniques. Thus, these results cover directly or indirectly all the possible combinations of settings among proper/non-proper, simple/non-simple, and strict/non-strict.

Several natural questions could follow from the present work. For instance, one may ask to enumerate all TC-realizable graphs of a given degree sequence, or ask for realizations with additional constraints such as bounded lifetime, bounded diameter, or the existence of temporal spanners of a certain size. Another promising avenue is to optimize over these criteria while searching for a realization. Finally, the realizability question studied in this paper could be generalized to temporal k -connectivity or to directed temporal graphs.

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