Traffic-Oblivious Multi-Commodity Flow Network Design

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Abstract

We consider the Minimum Multi-Commodity Flow Subgraph (MMCFS) problem: given a directed graph G with edge capacities *cap* and a retention ratio $\alpha \in (0, 1)$, find an edge-wise minimum subgraph $G' \subseteq G$ such that for all traffic matrices T routable in G using a multi-commodity flow, $\alpha \cdot T$ is routable in G'. This natural yet novel problem is motivated by recent research that investigates how the power consumption in backbone computer networks can be reduced by turning off connections during times of low demand without compromising the quality of service. Since the actual traffic demands are generally not known beforehand, our approach must be traffic-oblivious, i.e., work for all possible sets of simultaneously routable traffic demands in the original network.

In this paper we present the problem, relate it to other known problems in literature, and show several structural results, including a reformulation, maximum possible deviations from the optimum, and NP-hardness (as well as a certain inapproximability) already on very restricted instances. The most significant contribution is a $\max(1/\alpha, 2)$-approximation based on a surprisingly simple LP-rounding scheme. We also give instances where this worst-case approximation ratio is met and thus prove that our analysis is tight.

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1 Introduction

We present the (suprisingly seemingly novel) network design problem Minimum Multi-Commodity Flow Subgraph (MMCFS): given a directed flow network G with edge capacities and a *retention ratio* $\alpha \in (0, 1)$, find a subnetwork $G' \subseteq G$ of minimum size such that G' still allows for a multi-commodity flow (MCF) routing of *any* traffic demands routable in G when they are scaled down by factor α.

The problem arises naturally in recent research concerning power saving in backbone (Tier 1) networks of Internet service providers. There, the overall amount of traffic has distinct peaks in the evenings (when people are, e.g., streaming videos) and lows late at night and in the early mornings [40]. Clearly, the networks are built to handle the peak times. This opens up the possibility to reduce the power consumption of the network by turning off some resources – e.g., connections, line cards, or servers – during low traffic periods [44, 14].

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So consider a computer network that allows for the simultaneous routing of the traffic at peak times. This traffic is comprised of a set of *commodities*, where each commodity is identified by a pair of nodes (s, t) in the network and has a *demand* d specifying that d flow units have to be sent from s to t. The entirety of the demands for each pair of nodes is encoded in a *traffic matrix* T. Commonly, one makes the simplifying assumption that during low traffic periods the traffic demands are upper bounded by a down-scaling of T using a factor $\alpha \in (0, 1)$; the most practically relevant scenarios concern $\alpha \geq \frac{1}{2}$ [34]. The task now is to minimize the size of the network by deactivating connections such that the reduced network still accommodates a routing of the scaled-down demands $\alpha \cdot T$. However, in practice the traffic matrix T may change from day to day (in fact, even within sampling windows of 15 minutes) and while we can assume the capacity of the network to be large enough for all occurring traffic at any given time, T is usually not known beforehand. Thus, our solution should be *traffic-oblivious*, i.e., independent of any specific traffic matrix.

Technically, routing in realistic scenarios is *not* done via fully general multi-commodity flows: while MCF would be optimal in terms of minimizing congestion [40], it would be too complicated and temporally unstable for the routing hardware. Instead, simpler techniques like *2-segment routing* are used (which, in contrast to trivial shortest path routing, routes flow along a sequence of two chained shortest subpaths) [19]. Interestingly, studies show that in realistic networks, these routing solutions are virtually identical to those achieved by MCF [40, 12]. At the same time, for a given fixed network and traffic matrix, an MCF can be computed in polynomial time while establishing an optimal routing table for 2-segment routing (which is then deployable on router hardware) is NP-hard [25]. Thus, we describe the feasibility of solutions in our problem setting in terms of routability via MCF, and assume that realistic (simpler) routing protocols will still be able to attain effective routability.

Let us give a formal description of our problem: Given a directed graph (or *digraph*) $G = (V, E)$ with positive edge capacities $cap\colon E \to \mathbb{Q}$ and a traffic matrix T, a *flow* $f_{s,t}\colon E \to \mathbb{Q}$ from a vertex $s \in V$ to a vertex $t \in V$ is a function satisfying the flow conservation constraints

$$\sum_{uv \in E} f_{s,t}(uv) - \sum_{vu \in E} f_{s,t}(vu) \;=\; \begin{cases} -T(s,t) & \text{if } v = s \\ T(s,t) & \text{if } v = t \\ 0 & \text{else} \end{cases} \qquad \forall v \in V.$$

A *multi-commodity flow (MCF)* is a set of flows $\mathcal{F} = \{f_{s,t} \mid (s,t) \in V^2\}$ satisfying

$$\sum_{(s,t) \in V^2} f_{s,t}(uv) \leq cap(uv) \qquad \forall uv \in E.$$

For an edge $e = st \in E$, we may use the shorthand notation $T(e) \coloneqq T(s,t)$. Further, we call a traffic matrix T *routable in an edge set* $A \subseteq E$ if there exists an MCF $\mathcal{F}$ for (G, cap, T) with $\sum_{f \in \mathcal{F}} f(e) = 0$ for all $e \notin A$. Based on this notion, we define the Minimum Multi-Commodity Flow Subgraph (MMCFS) problem as follows: given a digraph $G = (V, E)$ with edge capacities *cap* and a retention ratio $\alpha \in (0, 1)$, find the edge set $A \subseteq E$ with minimum cardinality such that for all traffic matrices T that are routable in E, $\alpha \cdot T$ is routable in A.

Throughout this paper, when mentioning a problem's name or the corresponding abbreviation in sans-serif typeface, e.g. MMCFS, we refer to the optimization question. To indicate that a subgraph is an optimal solution for the problem, we will denote it by the abbreviation in normal typeface, e.g. an MMCFS (V, A) with $A \subseteq E$.

Our contribution. We present the MMCFS problem, which has both a natural formulation and practical applicability. After discussing related problems from literature in Section 2, we give some structural results in Section 3: We show how the MMCFS problem, even though it is traffic-oblivious in nature, can be reformulated to consider a specific single "hardest" traffic matrix. We also establish how an MCF can be routed in an optimal MMCFS solution, and how the ratio between the values of a feasible MMCFS solution and an optimal one relates to their average edge capacities. In Section 4, we prove that MMCFS is NP-hard already with unit edge capacities. Additionally, we show that it is NP-hard (and a closely related problem cannot be approximated within a sublogarithmic factor) already on directed acyclic graphs (DAGs). Our most important contribution is given in Section 5, where we present a $\max(1/\alpha, 2)$-approximation algorithm for MMCFS: after modelling MMCFS as an ILP, we can deduce a surprisingly simple LP-rounding scheme, whose complexity is solely shifted to the correctness proof. Moreover, we show that our analysis of this algorithm is tight.

2 Related Work

There is a rich body of work on multi-commodity flows – see e.g. [3, Ch. 17] for a primer on this topic and [38] for a recent literature review. The ability to route an MCF in an MMCFS solution not only determines the latter's feasibility, the problem of routing an MCF also has many close ties to several other network design problems. These, however, involve constraints unrelated to MMCFS, are usually not traffic-oblivious, and mostly focus on undirected graphs. Concerning approaches on directed graphs, Foulds [20] minimizes the cost of an MCF in a bidirected network where the use of some unidirectional arcs is prohibited to reduce congestion. Gendron et al. [22, 23] discuss a directed MCF problem that considers costs for both the installation of an edge and the amount of flow routed over it. Further, in *buy-at-bulk network design* [8, 39, 13], capacity on edges must be bought as cheaply as possible such that a given traffic matrix becomes routable – with the caveat that larger amounts of capacity can be bought at a lower price per capacity unit.

In *robust network design* [10, 4, 9, 24], possible traffic matrices are given as an uncertainty set in the form of a polytope, and the objective is usually to minimize the cost of reserving capacity on the edges. While the *dynamic routing* variant considers a different MCF for every traffic matrix in the polytope, *static routing* specifies a fixed unit flow for each commodity that is only scaled with the respective demand value. For directed graphs, Al-Najjar et al. [5] show that an exact algorithm for static routing would yield an $\mathcal{O}(|V|)$-approximation for dynamic routing. Our result can be seen as a better approximation ratio for the special uncertainty set $\{\alpha \cdot T \mid \alpha \in (0,1),\ T \text{ is routable in } G\}$ under dynamic routing, but w.r.t. minimizing the number of edges in a subgraph rather than the cost of reserved capacity.

There are also several related graph construction and subgraph minimization problems: Khuller et al. [28] give an approximation algorithm for the construction of an undirected tree with constant degree that accommodates given traffic demands between its leaves such that the maximum load on any edge is minimized. Otten et al. [34] evaluate an integer linear program (ILP) and a heuristic for a green traffic engineering problem on digraphs – however, there a specific traffic matrix is also given (and rather than finding an edge subset of minimum cardinality, they minimize the number of "line cards", i.e., sets of 8 incident edges at each vertex). Another well-known topic in the realm of subgraph minimization problems is that of *spanners* [1], i.e., subgraphs that preserve the length of a shortest path within a given ratio (stretch factor) between each pair of vertices. There exists a correspondence between upper bounds on the stretch of shortest paths and the congestion of MCFs, however, this only applies to the existence of probabilistic mappings in undirected graphs [6, 36]. Nonetheless, this correspondence was used to find *flow sparsifiers* in undirected graphs G [18]. While a

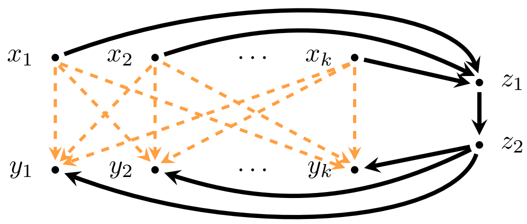

Figure 1 Digraph G constructed in the proof of Observation 1. Edges of the unique optimal MCPS-solution E' of (G, cap, β) are solid (black), and the remaining edges E_{XY} are dashed (orange).

MMCFS solution is similar to a flow sparsifier that preserves the congestion up to a factor $\frac{1}{\alpha}$, they differ in that a flow sparsifier is an entirely new (undirected) graph, not necessarily subgraph, containing a subset of the vertices of G but both old and new edges [32, 30, 7].

Closely related to MMCFS are classical Directed Survivable Network Design (DSND) problems, where, given a (possibly capacitated) input digraph $G = (V, E)$ and a requirement function $r \colon V^2 \to \mathbb{N}$, one aims to find an edge-wise minimum subgraph of G in which one can send a flow of value $r(s, t)$ from s to t for every $(s, t) \in V^2$. On undirected graphs with unit edge capacities, there exists a 2-approximation by Jain [26], which has been adapted to directed instances, but only for a very restricted set of requirement functions [31]. The DSND problem most similar to MMCFS is the Minimum Capacity-Preserving Subgraph (MCPS) problem [15], where the requirement value for a vertex pair (s, t) equals a fraction β of the value of a maximum flow from s to t. However, in all of the aforementioned DSND approaches, each routed commodity is considered in isolation from the others, whereas in the MMCFS setting, all commodities are routed simultaneously. For example, given a digraph $G = (V, E)$ with edge capacities *cap* and a traffic matrix T routable in E, the scaled-down matrix $\alpha \cdot T$ is not necessarily routable in any optimal MCPS-solution of (G, cap, β) since some edges may be congested by the simultaneous routing of multiple commodities. This is especially easy to see for $\alpha \geq \beta$ but even holds true when $\alpha \ll \beta$:

▶ **Observation 1.** *Given an arbitrarily small $\alpha \in (0, 1)$ and an arbitrarily large $\beta \in (0, 1)$, there exists a digraph $G = (V, E)$ with edge capacities cap and a traffic matrix T such that T is routable in E, but $\alpha \cdot T$ is not routable in any optimal MCPS-solution E' of (G, cap, β).*

Proof. Let $C := \left\lceil \frac{2\beta}{1-\beta} \right\rceil$ be a (high) edge capacity value and $k > \sqrt{\frac{C}{\alpha}}$ a number of vertices. Construct $G = (V, E)$ (visualized in Figure 1) as follows: Create two vertex sets X and Y with k vertices each, as well as two distinct vertices z_1, z_2, and let $V := X \cup Y \cup \{z_1, z_2\}$. Moreover, let $E_{XY} := X \times Y$, $E' := (X \times \{z_1\}) \cup \{z_1 z_2\} \cup (\{z_2\} \times Y)$, and $E := E_{XY} \cup E'$. Edge capacities and the traffic matrix are chosen as follows:

$$cap(uv) = \begin{cases} 1 & \text{if } uv \in E_{XY}, \\ C & \text{otherwise;} \end{cases} \qquad T(u, v) = \begin{cases} 1 & \text{if } uv \in E_{XY}, \\ 0 & \text{otherwise.} \end{cases}$$

Every non-zero demand $T(u, v)$ can be routed in (G, cap) using the respective edge $uv \in E_{XY}$.

The only optimal MCPS-solution of (G, cap, β) is E': For every vertex pair $(u, v) \in E'$, the only u-v-path in G is the one consisting of the edge uv – so the edges E' must be in the solution. E' also establishes a maximum flow of sufficient value for the remaining vertex pairs. In particular, for every vertex pair $(u, v) \in X \times Y$, there exists a maximum u-v-flow of value C in E', which is at least β times the value $(C + 1)$ of a maximum u-v-flow in E:

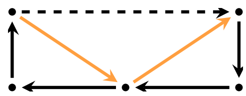

Figure 2 An MMCFS instance with an optimal solution (given by the solid edges) that does not contain the MED. All edge capacities are 1, and $\alpha = \frac{1}{2}$. The MED, which is unique in this example and drawn in black, is not a feasible MMCFS solution: for T with $T(s,t) = cap(st)$ if $e = st \in E$ and 0 otherwise, the dashed edge would have to accommodate a flow of $\frac{3}{2}$ to satisfy all demands, but it only has a capacity of 1.

$$C = \left\lceil \frac{2\beta}{1-\beta} \right\rceil \geq \beta \cdot \frac{2 \cdot (1+\beta-\beta)}{1-\beta} = \beta \left(\frac{2\beta}{1-\beta} + 2 \right) \geq \beta \left(\left\lceil \frac{2\beta}{1-\beta} \right\rceil + 1 \right) = \beta \cdot (C+1)$$

However, the scaled matrix $\alpha \cdot T$ is not routable in E': $|X| \cdot |Y| = k^2 > \frac{C}{\alpha}$ many demands of value α would have to be routed over the single edge $z_1 z_2$, exceeding its capacity C. ◀

Lastly, we want to highlight similarities of MMCFS to the well-established NP-hard Minimum Equivalent Digraph (MED) problem [2, 37, 21, 27], where, given a digraph G, one asks for the edge-wise minimum subgraph of G that preserves the reachability relation of G. In Section 4, we show that MED is a special case of MMCFS. Further, we observe:

▶ **Observation 2.** *In a simple DAG G, the unique minimum equivalent digraph (MED) of G must be contained in every feasible MMCFS solution of G, regardless of edge capacities and α.*

Proof. In any simple DAG G, the MED is unique and consists of exactly those edges st for which there is no s-t-path in $G - st$ [2]. A feasible MMCFS solution must also contain these edges st in order to allow for a non-zero amount of flow from s to t. ◀

However, in general digraphs, a feasible MMCFS solution may not always contain the MED, see Figure 2. Note that MED is not only polynomial-time solvable on DAGs, but there are also several polynomial approximation algorithms for general graphs [29, 45] with the currently best approximation ratio being 1.5 [11, 42].

3 Structural Results

We present some structural insights concerning MMCFS that give a deeper understanding of the problem. Most importantly, we give a reformulation of MMCFS that is used throughout the rest of the paper to obtain structural and algorithmic results: Recall that a feasible solution A for a given MMCFS instance $(G = (V, E), cap, \alpha)$ is defined as an edge set $A \subseteq E$ such that for *all* traffic matrices T that are routable in E, $\alpha \cdot T$ is routable in A. Interestingly, instead of explicitly considering all routable traffic matrices T, it suffices to consider the single specific traffic matrix $\mathbb{T}_E$, which forces each edge to be utilized to its full capacity but has no demands between non-adjacent vertices:

$$\mathbb{T}_E(s,t) := \begin{cases} cap(e) & \text{if } e = st \in E, \\ 0 & \text{otherwise.} \end{cases}$$

We show that an edge set $A \subseteq E$ is a feasible MMCFS solution iff $\alpha \cdot \mathbb{T}_E$ is routable in A:

▶ **Theorem 3.** *Given a digraph $G = (V, E)$ with edge capacities cap, a retention ratio $\alpha \in (0, 1)$, and an edge set $A \subseteq E$, the following statements are equivalent:*

- *For all traffic matrices T that are routable in E, the scaled matrix $\alpha \cdot T$ is routable in A.*
- *The scaled matrix $\alpha \cdot \mathbb{T}_E$ is routable in A.*

Proof. $\mathbb{T}_E$ is routable in E by definition. If every traffic matrix routable in E is also routable in A when scaled down by α, then so is $\alpha \cdot \mathbb{T}_E$. For the other direction, consider any arbitrary traffic matrix T routable in E. Let $\{f^T_{s,t} \mid (s,t) \in V^2\}$ be the MCF that routes T in E with the vector $\mathbf{f}^T := \sum_{(s,t)\in V^2} f^T_{s,t}$ specifying the total flow over each edge. Using this MCF, we can construct a new traffic matrix T':

$$T'(s,t) := \begin{cases} \mathbf{f}^T(st) & \text{if } e = st \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $T' \leq \mathbb{T}_E$ when using component-wise comparison. Thus, since $\alpha \cdot \mathbb{T}_E$ is routable in A, so is $\alpha \cdot T'$. But if $\alpha \cdot T'$ is routable in A using the flows $\{f^{\alpha\cdot T'}_{u,v} \mid (u,v) \in V^2\}$, then $\alpha \cdot T$ is also routable in A using the flows $\{f^{\alpha\cdot T}_{s,t} \mid (s,t) \in V^2\}$ constructed as follows: for each commodity $st \in E$ and each edge $uv \in E$, calculate the fraction of flow routed over uv that is used by $f^T_{s,t}$, and route this fraction over the path chosen by $f^{\alpha\cdot T'}_{u,v}$. In short,

$$f^{\alpha\cdot T}_{s,t}(e) := \sum_{uv\in E\colon \mathbf{f}^T(uv)>0} \frac{f^T_{s,t}(uv)}{\mathbf{f}^T(uv)} \cdot f^{\alpha\cdot T'}_{u,v}(e).$$ ◀

We note that a related but slightly different concept to Theorem 3 has been implicitly used previously in [35] (on undirected graphs). From this point onwards, we may refer to edges as commodities since $\mathbb{T}_E$ specifies a non-zero demand precisely for the edges in E. Moreover, given a flow $f_{\tilde{e}}$ for a commodity $\tilde{e} \in E$, we call $f_{\tilde{e}}(\tilde{e})$ the *direct flow* for $\tilde{e}$. We observe that in an optimal MMCFS solution A, for every commodity $\tilde{e} \in A$, the demand $\mathbb{T}_E(\tilde{e})$ can always be fully satisfied by a direct flow $f_{\tilde{e}}(\tilde{e})$:

▶ **Observation 4.** *Let $G = (V, E)$ be a digraph with edge capacities cap and T a traffic matrix routable in an edge set $A \subseteq E$ with $T(e) \leq cap(e)$ for all edges $e \in E$. Then, there exists an MCF $\mathcal{F} = \{f_{s,t} \mid (s,t) \in V^2\}$ in the graph $G' := (V, A)$ satisfying the demands T such that $f_{\tilde{e}}(\tilde{e}) = T(\tilde{e})$ for all edges $\tilde{e} \in A$.*

Proof. Among all MCFs that witness the routability of T in A, let $\mathcal{F}' = \{f'_{s,t} \mid (s,t) \in V^2\}$ be one with a maximum sum of direct flow values $\sum_{\tilde{e}\in E} f'_{\tilde{e}}(\tilde{e})$.

We give a proof by contradiction: Assume that there exists an edge $e' = uv \in A$ such that $f'_{e'}(e') < T(e')$ (if no such edge exists, $\mathcal{F} = \mathcal{F}'$ and we are done). There must exist at least one alternative u-v-path P that routes at least some of the remaining demand $T(e') - f'_{e'}(e')$. Further, the edge e' has residual capacity $cap(e') - \sum_{(s,t)\in V^2\colon} f'_{s,t}(e') = 0$ as otherwise we could increase $f_{e'}(e')$ (and decrease flow along P accordingly), which would contradict the selection of $\mathcal{F}'$. Thus, there exists an edge $e'' \in E$, $e'' \neq e'$, with $f_{e''}(e') > 0$. But then, we can exchange a non-zero amount $\varepsilon > 0$ of flow of commodity e' routed over P with an equally small amount of flow of commodity e'' routed over e'. This increases $f'_{e'}(e')$ without decreasing any other direct flow value – again a contradiction to the selection of $\mathcal{F}'$. ◀

Further, for any edge set A, we can compare its total edge capacity $\sum_{e\in A} cap(e)$ to the total flow needed to satisfy the demands $\mathbb{T}_E$. This not only gives us a necessary condition for an edge set A to be a feasible MMCFS solution, but, upon closer analysis, also allows us to relate its quality as a solution to its *mean capacity* $\overline{cap_A} := \frac{1}{|A|} \cdot \sum_{e\in A} cap(e)$. The following results apply both in the case of simple and non-simple graphs, but we can give better guarantees in the former case.

▶ **Theorem 5.** *Let O be an optimal solution and $A \neq O$ a feasible solution for an MMCFS instance (G, cap, α). Then, $\frac{|A|}{|O|} \leq \min\left\{\left(1 + \frac{1-\alpha}{\theta\alpha}\right) \cdot \frac{\overline{cap_O}}{\overline{cap_A}}, 1 + \frac{1-\alpha}{\theta\alpha} \cdot \frac{\overline{cap_O}}{\overline{cap_{A\setminus O}}}\right\}$ with $\theta = 2$ if G has no parallel edges and $\theta = 1$ otherwise.*

Proof. Let $X := A \setminus O$, and $Y := A \cap O$. The commodities of all $\tilde{e} \in O$ must be routed through O, requiring a total flow of at least $\alpha \sum_{e\in O} cap(e)$ and leaving a total remaining capacity in O of at most $(1-\alpha)\sum_{e\in O} cap(e)$. Every commodity $\tilde{e}' \in X$ has to be routed within this remaining capacity since O is feasible. This requires a total flow of at least $\theta\alpha \sum_{e\in X} cap(e)$; the θ is due to the fact that without parallel edges each such commodity $\tilde{e}'$ must be routed over at least two other edges in O since $\tilde{e}' \notin O$. We thus have

$$\theta\alpha \sum_{e\in X} cap(e) \leq (1-\alpha) \sum_{e\in O} cap(e) \ . \tag{1}$$

By adding $\theta\alpha \sum_{e\in Y} cap(e) \leq \theta\alpha \sum_{e\in O} cap(e)$ to this inequality, we obtain

$$\begin{aligned}
\theta\alpha \sum_{e\in A} cap(e) &\leq (1-\alpha) \sum_{e\in O} cap(e) + \theta\alpha \sum_{e\in O} cap(e) \\
\theta\alpha \cdot |A| \cdot \overline{cap_A} &\leq (1-\alpha+\theta\alpha) \cdot |O| \cdot \overline{cap_O} \\
\frac{|A|}{|O|} &\leq \left(1 + \frac{1-\alpha}{\theta\alpha}\right) \cdot \frac{\overline{cap_O}}{\overline{cap_A}} \ .
\end{aligned}$$

Alternatively, we can rewrite inequality (1) as $\theta\alpha \cdot |X| \cdot \overline{cap_X} \leq (1-\alpha) \cdot |O| \cdot \overline{cap_O}$ and obtain

$$\frac{|A|}{|O|} \leq \frac{|O| + |X|}{|O|} = 1 + \frac{|X|}{|O|} \leq 1 + \frac{1-\alpha}{\theta\alpha} \cdot \frac{\overline{cap_O}}{\overline{cap_X}} \ . \qquad ◀$$

▶ **Corollary 6.** *Any arbitrary feasible solution for MMCFS (including the trivial one, E itself), is a $(1 + \frac{1-\alpha}{\theta\alpha} \cdot \frac{\max_{e\in E} cap(e)}{\min_{e\in E} cap(e)})$-approximation.*

This ratio is met, e.g., on a bundle of parallel edges with $\ell := (\frac{1}{\alpha} - 1) \cdot k$ capacity-1 edges and one capacity-k edge (for any given $\alpha \in \mathbb{Q}_{(0,1)}$ and an arbitrary $k \in \mathbb{Q}$ s.t. $\ell \in \mathbb{N}$).

▶ **Corollary 7.** *For uniform capacities, any arbitrary feasible solution for MMCFS (including the trivial one, E itself) is a $(1 + \frac{1-\alpha}{\theta\alpha})$-approximation.*

4 Complexity

Given that even trivial MMCFS solutions satisfy an approximation guarantee according to Corollary 6, one might expect MMCFS to be polynomial-time solvable. However, in this section, we show that MMCFS is NP-hard already on DAGs and give a first inapproximability result. We begin by proving that MMCFS is NP-hard already with unit edge capacities using a reduction from MED that directly follows from Theorem 3:

▶ **Corollary 8.** *MED is the special case of MMCFS with unit edge capacities cap and $\alpha \leq \frac{1}{|E|}$.*

Proof. An optimal solution $A \subseteq E$ for an MMCFS instance (G, cap, α) with unit edge capacities is an edge set of minimum cardinality such that the demands $\alpha \cdot \mathbb{T}_E(s,t) = \alpha \cdot cap(st) \leq \frac{1}{|E|}$ for each edge $st \in E$ are routable in A. This is equivalent to ensuring that there exists an s-t-path in A for every edge $st \in E$ since the (unit) capacity of an edge can never be surpassed by the $|E|$ many flows of size at most $\frac{1}{|E|}$ each. ◀

Moreover, we can show that MMCFS is NP-hard already on DAGs using a reduction from the NP-hard decision variant of Set Cover [27], where one asks: given a universe U, a family of sets $\mathcal{S} = \{S_i \subseteq U\}_{1\leq i\leq k}$ with $k \in \mathcal{O}(\text{poly}(|U|))$, and a parameter φ, is there a subfamily $\mathcal{C} \subseteq \mathcal{S}$ of cardinality $|\mathcal{C}| \leq \varphi$ such that $\bigcup_{S\in\mathcal{C}} S = U$? The reduction is similar to that given in [15] to prove the NP-hardness of the Minimum Capacity-Preserving Subgraph problem.

▶ **Theorem 9.** *For any fixed $\alpha \in (0,1)$, MMCFS is NP-hard already on DAGs where the longest path has length 3.*

Proof. Given a Set Cover instance $(U, \mathcal{S}, \varphi)$ and a fixed retention ratio $\alpha \in (0,1)$, we construct an instance $I = (G = (V,E), \mathit{cap}, \alpha, \psi)$ for the decision variant of MMCFS: I is a yes-instance if and only if there exists a feasible MMCFS solution $E' \subseteq E$ for $(G, \mathit{cap}, \alpha)$ with cardinality $|E'| \leq \psi$. We construct I as follows (see Figure 3 for a visualization):

$$\begin{aligned}
V &\coloneqq V_U \cup V_{\mathcal{S}} \cup V_t^{\mathcal{S}} \cup \{t\} \text{ and } E \coloneqq E_U \cup E_{\mathcal{S}} \cup E_1 \\
V_U &\coloneqq \{v_u, v'_u \mid \forall u \in U\} \qquad V_{\mathcal{S}} \coloneqq \{v_S \mid \forall S \in \mathcal{S}\} \qquad V_{\mathcal{S}}^t \coloneqq \{z_S \mid \forall S \in \mathcal{S}\} \\
E_U &\coloneqq V_U \times \{t\} \qquad E_{\mathcal{S}} \coloneqq V_{\mathcal{S}} \times \{t\} \\
E_1 &\coloneqq \{v_u v_S, v'_u v_S \mid \forall S \in \mathcal{S}, u \in S\} \cup \{v_S z_S, z_S t \mid s \in \mathcal{S}\} \\
\mathit{cap}(e) &\coloneqq \Big\{ 1 \text{ if } e \in E_1; \qquad \tfrac{1-\alpha}{\alpha} \text{ if } e \in E_{\mathcal{S}}; \qquad \varepsilon \text{ if } e \in E_U, \text{ with } \varepsilon \leq \min\left(\tfrac{1-\alpha}{\alpha^2\cdot|U|}, \tfrac{1-\alpha}{\alpha}\right). \\
\psi &\coloneqq |E_1| + \varphi = 2 \cdot \textstyle\sum_{S\in\mathcal{S}} |S| + 2\cdot|\mathcal{S}| + \varphi
\end{aligned}$$

As G is a DAG, its MED is unique [2] and must be part of any feasible MMCFS solution, see Observation 2. This MED is formed by the edges $e \in E_1$; a flow of α must be routed over each of them in order to satisfy the demands $\mathbb{T}_E(e) = \alpha \cdot \mathit{cap}(e)$. The remaining capacity for each edge $e \in E_1$ is $1 - \alpha$.

So consider a single set $S \in \mathcal{S}$ and the corresponding two-path $\{v_S z_S, z_S t\} \in E_1$, whose remaining capacity can thus accommodate either arbitrarily many *item commodities* $\tilde{e}_U \in E_U$ (each one with the sufficiently small demand $\mathbb{T}_E(\tilde{e}_U) = \alpha\cdot\varepsilon$) or a single *set commodity* $\tilde{e}_{\mathcal{S}} \in E_{\mathcal{S}}$ (with the demand $\alpha \cdot \mathbb{T}_E(\tilde{e}_{\mathcal{S}}) = \alpha \cdot \frac{1-\alpha}{\alpha} = 1-\alpha$). In the former case, we can remove at least two *corresponding item edges* $v_u t$, $v'_u t$ with $u \in S$, which is more than the single *corresponding set edge* $v_S t$ we can remove in the latter case. Thus, for each item commodity $v_u t \in E_U$, an optimal MMCFS solution must contain one of the corresponding set edges $\{v_S t \mid S \ni u\} \subseteq E_{\mathcal{S}}$; the item commodity can then be routed over the path from v_u over v_S to t.

Given a Set Cover solution $\mathcal{C}$ with $|\mathcal{C}| \leq \varphi$, we can construct an MMCFS solution $E' = E_1 \cup \{v_S t \in E_{\mathcal{S}} \mid S \in \mathcal{C}\}$ with cardinality $|E'| = |E_1| + |\mathcal{C}| \leq |E_1| + \varphi = \psi$. Since every item is covered by the sets in $\mathcal{C}$, the constructed MMCFS solution includes at least one corresponding set edge for each item, ensuring its feasibility. Conversely, since a feasible MMCFS solution E' with $|E'| \leq \psi$ has at least one corresponding set edge for each item $u \in U$, the Set Cover solution $\mathcal{C} = \{S \mid v_S t \in E_{\mathcal{S}} \cap E'\}$ also contains at least one covering set for each item. Moreover, $|\mathcal{C}| = |E'| - |E_1| \leq \psi - |E_1| = \varphi$. ◀

Considering the optimization variants of Set Cover and MMCFS (and thus ignoring the additional input values φ and ψ), the reduction above also implies the inapproximability of the number of edges in an optimal MMCFS solution beyond the edges required for an MED: Consider an instance $I = (G = (V,E), \mathit{cap}, \alpha)$ for the optimization variant of MMCFS that is produced by the reduction above, and an arbitrary feasible solution $A \subseteq E$ for this instance. Let $\mu(I, A) \coloneqq |A| - \mathit{med}(G)$ where $\mathit{med}(G)$ denotes the number of edges in an MED of G. Then, A can be transformed into a feasible solution for the original Set Cover instance with objective value $\mu(I, A)$ in linear time. Further, let $\mu(I)$ be the minimum $\mu(I, A')$ over all

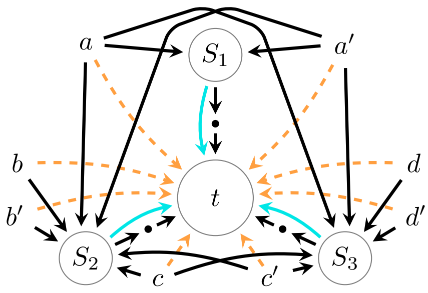

Figure 3 MMCFS instance constructed from a Set Cover instance with universe $U = \{a, b, c, d\}$ and family of subsets $\mathcal{S} = \{\{a\},\ \{a, b, c\},\ \{a, c, d\}\}$. An optimal solution contains the (solid black) MED as well as one (blue) corresponding set edge for each $u \in U$. Item edges are orange and dashed.

feasible solutions A' for I, and recall that the size $|E|$ of the MMCFS instance I is linear in the size $N \in \mathcal{O}(\text{poly}(U))$ of the Set Cover instance: if it was possible to approximate $\mu(I)$ within a factor in $o(\log |E|) = o(\log |U|)$, one could also approximate Set Cover within $o(\log |U|)$, which is NP-hard [17, 33]. This implies that any approximation algorithm for MMCFS (such as the one we present in Section 5) must leverage the existence of a comparatively high number of MED-edges in order to achieve its approximation ratio.

▶ **Observation 10.** *Given an MMCFS instance $I = (G = (V, E), cap, \alpha)$ and an MED of G, approximating $\mu(I)$ with a ratio in $o(\log |E|)$ is NP-hard. This already holds on DAGs where the longest path has length 3.*

5 LP-based Approximation

We present an extremely simple $\max(1/\alpha, 2)$-approximation for MMCFS based on LP rounding. This is a clear improvement over the default approximation guarantee of Corollary 6, which depends the instance's edge capacities and is thus not even polynomially bounded in α^{-1} or the instance size. Consider the following ILP formulation for MMCFS:

$$\min \sum_{e \in E} x_e \tag{1.a}$$

$$\sum_{u\colon uv \in E} f_{\tilde{e}}(uv) - \sum_{u\colon vu \in E} f_{\tilde{e}}(vu) = \begin{cases} -\alpha \mathbb{T}_E(\tilde{e}) & \text{if } v = s \\ \alpha \mathbb{T}_E(\tilde{e}) & \text{if } v = t \\ 0 & \text{else} \end{cases} \qquad \forall v \in V, \tilde{e} = st \in E \tag{1.b}$$

$$\sum_{\tilde{e} \in E} f_{\tilde{e}}(e) \leq x_e \cdot cap(e) \qquad \forall e \in E \tag{1.c}$$

$$f_{\tilde{e}}(e) \geq 0 \qquad \forall e \in E, \tilde{e} \in E \tag{1.d}$$

$$x_e \in \{0, 1\} \qquad \forall e \in E \tag{1.e}$$

A binary indicator variable x_e determines whether edge $e \in E$ is part of the solution subgraph or not. We minimize the sum of these variables in the objective function (1.a) to obtain a subgraph of minimum size. The non-negative variables $f_{\tilde{e}}(e)$ determine the amount of flow routed over edge e for commodity $\tilde{e} \in E$. Recall that $\mathbb{T}_E$ specifies a demand of $cap(\tilde{e})$ precisely for each edge $\tilde{e}$: the flow preservation constraints (1.b) guarantee that the f-variables represent proper flows that satisfy these demands. Lastly, the capacity constraints (1.c) ensure that the total sum of flow over any edge $e \in E$ does not surpass the capacity of e. This flow sum $\mathbf{f}(e) \coloneqq \sum_{\tilde{e} \in E} f_{\tilde{e}}(e)$ must be zero if e is not part of the solution.

We obtain the relaxation of ILP (1) by replacing the integrality constraints (1.e) on x_e by the inequalities $0 \leq x_e \leq 1$ for all $e \in E$. The only other type of constraint that bounds the x_e-variables is (1.c). Since the x_e-variables can assume fractional values in the LP relaxation, and the sum over all x_e is minimized, this lower bound of $\frac{\mathbf{f}(e)}{cap(e)}$ on x_e for all $e \in E$ will always be met with equality. Hence, the LP relaxation is equivalent to a standard MCF-LP that minimizes the sum of edge utilizations in the objective function $\sum_{e \in E} \frac{\mathbf{f}(e)}{cap(e)}$. Accordingly, we will refer to $cost(e) \coloneqq \frac{1}{cap(e)}$ as the *cost* that routing a single unit of flow over an edge e will add to this objective.

▶ Remark 11. The solution $x_e = \alpha$ for all $e \in E$ with objective value $\alpha|E|$ is always feasible for the relaxation of ILP (1). When the input graph is simple and all edge capacities are uniform, it is in fact optimal: the flow $f_{\tilde{e}}$ for commodity $\tilde{e}$ will always be routed completely over $\tilde{e}$ since all edges have the same cost and routing $f_{\tilde{e}}$ over an alternative path would cost more than routing it over a single edge. This implies that any arbitrary feasible solution for MMCFS on simple graphs with uniform edge capacities is an ($1/\alpha$)-approximation (which we already proved using a separate argument via Corollary 7). However, below we also use the LP relaxation to approximate MMCFS on non-simple graphs with non-uniform edge capacities.

For our approximation algorithm, we make use of the well-known fact that all standard LP solving algorithms will always return a *basic* optimal solution (if any solution exists) [43, p. 279]. A basic optimal solution – or equivalently, extreme point solution – is a vertex of the polyhedron defined as the convex hull of the set of feasible solutions. It does not lie on a higher-dimensional face of the polyhedron and thus cannot be expressed as a convex combination of two or more other feasible solutions [41, p. 100]. We prove that a basic optimal solution for the relaxation of ILP (1) will only have comparatively few variables with a positive value below α, allowing us to round up the solution to obtain an approximation.

▶ **Lemma 12.** *Let x^* be a basic optimal solution for the relaxation of ILP* (1)*, and h the number of edges e such that $x^*_e = 1$. There exist at most h many edges e' with $x^*_{e'} \in (0, \alpha)$.*

Proof. Let $H \coloneqq \{e \in E \mid x^*_e = 1\}$ be the h many edges that are saturated by the fractional multi-commodity flow, and $L \coloneqq \{e \in E \mid x^*_e \in (0, \alpha)\}$ with $\ell \coloneqq |L|$ the edges that are only used to a fraction less than α; in short, edges with *high* and *low* corresponding x^*-values. We show that an optimal fractional solution x^* with $\ell > h$ would allow us to construct a vector $p \in \mathbb{Q}^{|E|}$ such that both $(x^* + p)$ and $(x^* - p)$ are still feasible solutions for the LP relaxation – thus, x^* would not be basic.

For every edge $e = st \in L$, let P_e denote an arbitrary *alternative s-t-path* not using the edge e and $f_e(e') > 0$ for all edges $e' \in P_e$. Such a path must exist since at most $x^*_e \cdot cap(e) < \alpha \cdot cap(e)$ flow is routed over e, so an alternative s-t-path is necessary to satisfy the demand $\alpha \cdot \mathbb{T}_E(s,t) = \alpha \cdot cap(e)$. Further, the total cost of routing a unit of flow over P_e must be lower than or equal to the cost of routing it over e itself, i.e., $\sum_{e' \in P_e} \frac{1}{cap(e')} \leq \frac{1}{cap(e)}$, by the optimality of x^*. Based on the alternative paths, we construct a matrix $M \in \{0,1\}^{\ell \times h}$ indexed by pairs $(e, e'') \in L \times H$, with $M(e, e'') \coloneqq 1$ if $e'' \in P_e$, and 0 otherwise. Since $\ell > h$, the ℓ rows of M must be linearly dependent, i.e., there exists a vector of coefficients $q \in \mathbb{Q}^\ell$ with $q \neq \mathbf{0}$, such that $q^\top \cdot M = \mathbf{0}$ (and consequently, $-q^\top \cdot M = \mathbf{0}$).

We can obtain two new feasible solutions by modifying the MCF corresponding to the optimal basic solution x^* as follows: for each edge $e \in L$, and using a small positive value $\varepsilon \in \mathbb{Q}$, we send $\varepsilon \cdot q(e)$ less flow over e itself and $\varepsilon \cdot q(e)$ more flow over the alternative path P_e – or vice versa. Put formally, we argue that for a small enough $\varepsilon > 0$, the following vector $p \in \mathbb{Q}^{|E|}$ yields two feasible solutions $(x^* + p)$ and $(x^* - p)$ for the LP relaxation:

$$p(e') = \begin{cases} \varepsilon \cdot q(e') - \varepsilon \cdot \sum_{e \in L\colon e' \in P_e} q(e) & \text{if } e' \in L, \\ -\varepsilon \cdot \sum_{e \in L\colon e' \in P_e} q(e) & \text{otherwise.} \end{cases}$$

Clearly, the two solutions satisfy all demands and flow preservation constraints (1.b). Consider the capacity constraints (1.c): By construction of q, the flow difference on saturated edges is 0. For all non-saturated edges e' it holds: if we modified the flow routed over e', this flow was already non-zero before our modification, and ε can be chosen sufficiently small such that the flow will neither turn negative nor surpass $cap(e')$.

It remains to show that $p \neq \mathbf{0}$ given that $q \neq \mathbf{0}$. So, among the edges $e \in L$ with $q(e) \neq 0$, choose one with maximum cost $\frac{1}{cap(e)}$ and denote it by $e_{\max}$. We show that $p(e_{\max}) \neq 0$.

If $e_{\max}$ were contained in any of the alternative paths P_e with $e \in L$ and $q(e) \neq 0$, we would arrive at a contradiction even without using p: Recall that $\sum_{e' \in P_e} \frac{1}{cap(e')} \leq \frac{1}{cap(e)}$. So we must have $|P_e| = 1$, i.e., e and $e_{\max}$ are parallel edges with $cap(e) = cap(e_{\max})$. Since $e, e_{\max} \in L$ and thus $x^*_e, x^*_{e_{\max}} \in (0, \alpha)$, we could simply obtain two new solutions by shifting some small $\varepsilon > 0$ of flow from one to another, or vice versa, contradicting that x^* is basic.

Hence, $e_{\max}$ is only contained in alternative paths P_e with $e \in L$ s.t. $q(e) = 0$, and

$$p(e_{\max}) = \varepsilon \cdot q(e_{\max}) - \varepsilon \cdot \sum_{e \in L\colon e_{\max} \in P_e} q(e) = \varepsilon \cdot q(e_{\max}) - 0 \neq 0. \qquad \blacktriangleleft$$

The proof implicitly gives an intuitive explanation for the distribution of LP values:

▶ **Corollary 13.** *Let x^* be a basic optimal solution for the relaxation of ILP (1). For each edge $e = st$ with $x^*_e \in (0, \alpha)$ it holds: every* alternative *s-t-*path *P_e (not containing e) with $f_e(e') > 0$ for all of its edges $e' \in P_e$ must contain an edge e'' with $x^*_{e''} = 1$.*

Proof. Assume that, for some edge $e = st$ with $x^*_e \in (0, \alpha)$, there is a P_e that does not contain an edge e'' with $x^*_{e''} = 1$. Then, we can follow the proof of Lemma 12, choosing P_e as the alternative s-t-path for e. As a result, the matrix M constructed in the proof contains a row of zeroes, allowing us to route ε more flow over e and ε less flow over P_e, or vice versa. Thus, x^* cannot be basic, a contradiction. ◀

We now analyze the following LP-rounding algorithm: compute a basic optimal solution x^* for the relaxation of ILP (1) in polynomial time, and return the edge set $\{e \in E \mid x^*_e > 0\}$. Based on Lemma 12, one could use naïve techniques to prove approximation guarantees of $(2 + \frac{1}{\alpha})$ or $\frac{2}{\alpha}$ for this algorithm. However, we provide the stronger bound of $\max(1/\alpha, 2)$ based on the following intuition: the algorithm either "misses" the optimal solution mainly due to edges with $x^*_e \in (0, \alpha)$ and we can obtain a 2-approximation via Lemma 12, or due to edges with $x^*_e \in [\alpha, 1)$ and we can round up the variables for a $\frac{1}{\alpha}$-approximation.

▶ **Theorem 14.** *Let x^* be a basic optimal solution for the relaxation of ILP (1). The edge set $A \coloneqq \{e \in E \mid x^*_e > 0\}$ is a $\max(1/\alpha, 2)$-approximation for MMCFS. That is, rounding up x^* is a 2-approximation when $\alpha \geq \frac{1}{2}$, and an $1/\alpha$-approximation when $\alpha < \frac{1}{2}$.*

Proof. The solution A is clearly feasible. Let $z \coloneqq |A|$ be the number of edges e such that $x^*_e > 0$. Furthermore, let $\Delta = |\{e \in E \mid x^*_e \in [\alpha, 1)\}|$ be the number of these z edges whose x^*-variable is set to a value in the interval $[\alpha, 1)$. We know that the remaining $(z - \Delta)$ edges have a x^*-variable either set to 1 or a value lower than α. In particular, by Lemma 12, there exist at least as many edges e' with $x^*_{e'} = 1$ as there are edges e'' with $x^*_{e''} \in (0, \alpha)$. Thus, $|\{e \in E \mid x^*_e = 1\}| \geq \frac{z - \Delta}{2}$. Hence, the optimal fractional solution x^* has objective value

$$\begin{aligned} z^* &\geq |\{e \in E \mid x^*_e = 1\}| + \alpha \cdot |\{e \in E \mid x^*_e \in [\alpha, 1)\}| \\ &\geq \frac{z - \Delta}{2} + \alpha \cdot \Delta = \left(\frac{1}{2} + (\alpha - \frac{1}{2}) \cdot \frac{\Delta}{z}\right) \cdot z. \end{aligned}$$

Using the minimum fractional objective value z^* as a lower bound for the minimum integral objective value z^*_{int}, we can then give an upper bound for the approximation ratio:

$$r := \frac{z}{z^*_{int}} \leq \frac{z}{z^*} \leq \frac{1}{\frac{1}{2} + (\alpha - \frac{1}{2}) \cdot \frac{\Delta}{z}}$$

To obtain an upper bound on this ratio, we examine when its denominator is at its minimum. For $\alpha \geq \frac{1}{2}$, the term $(\alpha - \frac{1}{2}) \cdot \frac{\Delta}{z}$ is always non-negative and reaches its lowest value 0 when $\Delta = 0$, giving us the upper bound $2 \geq r$ in this case. In contrast, for $\alpha < \frac{1}{2}$, the term $(\alpha - \frac{1}{2}) \cdot \frac{\Delta}{z}$ is negative, and the minimum of the denominator is reached when $\Delta = z$, leading to the upper bound $\frac{1}{\alpha} \geq r$ in this second case. ◀

It is surprisingly hard to find instances (in particular, ones without parallel edges) where the worst-case approximation ratio given by Theorem 14 is actually met. However, we show that such instances exist for the most relevant $\alpha \in (0, 1)$, and hence our analysis is *tight*:

▶ **Lemma 15.** *The ratio* $1/\alpha$ *for the algorithm of Theorem 14 is tight for all* $\alpha = \frac{1}{q}$*,* $q \in \mathbb{N}_{>1}$*.*

Proof. Consider a complete bidirected graph $G = (V, E)$ on $q + 1$ vertices, and let $H \subset G$ denote an arbitrarily chosen directed Hamiltonian cycle in G (an example for $q = 4$ is visualized in Figure 4). For all edges $st \in E$, let $\eta(st)$ denote the number of edges of the unique s-t-path in H and set the capacity $cap(st) := \frac{1}{\eta(st)}$. $E(H)$ is a feasible integral solution for the MMCFS instance (G, cap, α) since the demands $\alpha \cdot \mathbb{T}_E$ are routable in it: A single edge $e \in H$ is able to satisfy its own demand $\alpha \cdot cap(e) = \frac{1}{q} \cdot 1$. Adding to this, for every $i \in \{2, \ldots, q\}$, there are i edges $uv \in E \setminus E(H)$ with $\eta(uv) = i$ whose unique u-v-path in H contains e. The edge e can accommodate all of these commodities uv because each of them has a demand of $\alpha \cdot cap(uv) = \frac{1}{q} \cdot \frac{1}{i}$, summing up to a total flow of $\sum_{i=1}^{q} i \cdot \frac{1}{i} \cdot \frac{1}{q} = 1 \leq \frac{1}{cap(e)}$. Lastly, $E(H)$ is not only feasible but optimal since we need at least $|E(H)|$ many edges to preserve the reachability relation of G in H.

The algorithm from Theorem 14 would potentially choose all edges of G, i.e., $|E| = (|V| - 1) \cdot |V| = q \cdot |E(H)| = \frac{1}{\alpha} \cdot |E(H)|$ many: As the cost $\frac{1}{cap(uv)} = \eta(uv)$ of an edge $uv \in E \setminus E(H)$ is equal to the total cost of its unique u-v-path in H, all u-v-paths are equal in cost. Thus, one optimal solution for the relaxation of ILP (1) simply routes all commodities $\tilde{e} \in E$ completely over their own edges $\tilde{e}$ and chooses $x^*_{\tilde{e}} = \alpha$. This solution is also basic: Let f denote the flow variables of this solution. If it were not basic, there would exist two other optimal solutions with flow variables f', f'' respectively, such that for some $\tilde{e}, e \in E$, $f'_{\tilde{e}}(e) < f_{\tilde{e}}(e) < f''_{\tilde{e}}(e)$. However, if $e = \tilde{e}$, f already routes the full demand of commodity $\tilde{e}$ over e and we cannot increase $f_{\tilde{e}}(e)$ without introducing a flow cycle, which would raise the objective value and thus be suboptimal. In contrast, if $e \neq \tilde{e}$, $f_{\tilde{e}}(e)$ is already 0 and cannot be decreased. ◀

▶ **Lemma 16.** *The ratio* 2 *for the algorithm of Theorem 14 is tight for all* $\alpha > \frac{1}{2}$*.*

Proof. We construct a family of MMCFS instances (visualized in Figure 5), each one consisting of a simple graph $G = (V, E)$ with edge capacities *cap* and the given retention ratio α, that meets the approximation ratio of 2 asymptotically with increasing size. Intuitively, the edge set of each instance contains two subsets that have a size quadratic in the number of vertices and two subsets of linear size; our algorithm chooses both quadratic-size subsets even though one could be replaced by a linear-size subset.

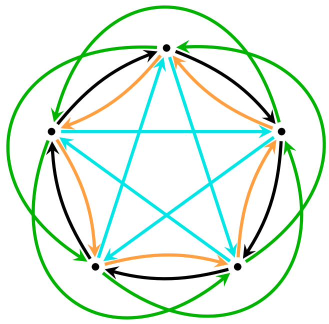

Figure 4 MMCFS instance for $\alpha = \frac{1}{4}$, constructed as a member of the family of instances in the proof of Lemma 15. The Hamiltonian cycle H is drawn as black. The value $\eta(st)$, denoting the distance from s to t in H for an edge st, is 2, 3, or 4 for blue, green, and orange edges, respectively.

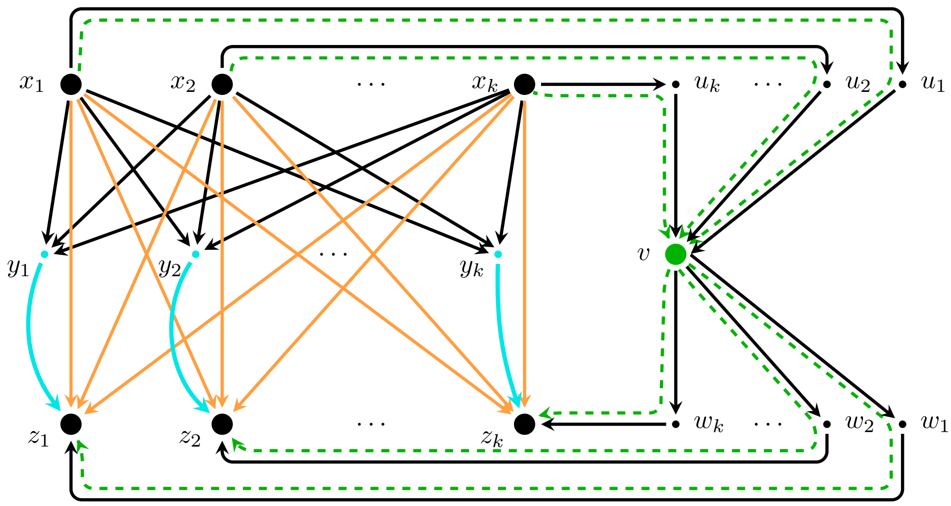

Figure 5 Family of MMCFS instances constructed in the proof of Lemma 16. Edges in E_v are drawn in green (and dashed), edges in E_{XZ} are orange, edges in E_1 are black, and edges in E_B are blue. Recall that edges E_1 and E_B (black and blue) form the unique MED, which is contained in every feasible solution. There exists a feasible MMCFS solution $E_1 \cup E_B \cup E_v$ without any (orange) edges from E_{XZ}, but the algorithm from Theorem 14 will include all edges of E_{XZ} in its solution.

Let V be comprised of five disjoint vertex sets X, Y, Z, U, W with $k \in \mathbb{N}$ vertices each (respectively named with the corresponding lowercase letter and indexed by $i \in \{1, \dots, k\}$) and a distinct vertex v. Further, let $\varepsilon \leq \frac{1-\alpha}{k \cdot \alpha}$ and $B \geq \frac{k \cdot \varepsilon}{1-\alpha}$ be sufficiently low and high positive values, respectively. Then, $E \coloneqq E_v \cup E_{XZ} \cup E_1 \cup E_B$ with

$$E_v \coloneqq (X \times \{v\}) \cup (\{v\} \times Z) \qquad\qquad E_{XZ} \coloneqq X \times Z$$

$$E_1 \coloneqq X \times Y \cup \{x_i u_i, u_i v, v w_i, w_i z_i \mid i \in \{1, \dots, k\}\} \qquad E_B \coloneqq \{y_i z_i \mid i \in \{1, \dots, k\}\}$$

$$\mathit{cap}(e) \coloneqq \Big\{ \tfrac{1-\alpha}{\alpha} \text{ if } e \in E_v, \qquad \tfrac{1-\alpha+\varepsilon}{\alpha} \text{ if } e \in E_{XZ}, \qquad 1 \text{ if } e \in E_1, \qquad B \text{ if } e \in E_B.$$

G is a DAG, thus its MED is unique [2] and must be part of any feasible solution for the MMCFS instance $(G, \mathit{cap}, \alpha)$, see Observation 2. The MED of G consists of the edges $E_1 \cup E_B$ and in particular can satisfy all demands $\alpha \cdot \mathbb{T}_E(e) = \alpha \cdot \mathit{cap}(e)$ with $e \in E_1 \cup E_B$. The "remaining" capacity of $1 - \alpha$ for each of the edges in E_1 (and $(1-\alpha) \cdot B$ for edges in E_B) is also sufficient to satisfy the demands $\alpha \cdot \mathbb{T}_E(e_v)$ of the commodities $e_v \in E_v$, and to *almost* satisfy the demands $\alpha \cdot \mathbb{T}_E(e_{XZ}) = 1 - \alpha + \varepsilon$ for the commodities $e_{XZ} \in E_{XZ}$: it only leaves a demand of ε for every such e_{XZ}. Thus, there exists a feasible solution $E_1 \cup E_B \cup E_v$ for $(G, \mathit{cap}, \alpha)$ that contains all of the $k^2 + 5k$ edges from $E_1 \cup E_B$ and additionally, to satisfy the remaining demands (of size ε each), the $2k$ edges from E_v. This feasible solution gives an upper bound on the objective value z^*_{int} of the optimal integral solution: $z^*_{\mathit{int}} \leq k^2 + 7k$.

In contrast, the algorithm from Theorem 14 also includes all $k^2 + 5k$ edges of the MED $E_1 \cup E_B$ in its solution, but must additionally choose (at least) the k^2 edges from E_{XZ}, resulting in an objective value $z \geq 2k^2 + 5k$. This is because every optimal solution for the relaxation of ILP (1) routes the commodities $st = e_{XZ} \in E_{XZ}$ over the edges e_{XZ} themselves: they have a lower cost than the alternative s-t-path of length 2 over the edges in E_v.

For increasing k, the ratio of the algorithmic solution's objective value to the optimum is

$$\lim_{k \to \infty} \frac{z}{z^*_{\mathit{int}}} \geq \lim_{k \to \infty} \frac{2k^2 + 5k}{k^2 + 7k} = 2. \qquad ◀$$

Combining Theorem 14 with Lemmas 15 and 16, we obtain:

▶ **Corollary 17.** *The approximation ratio* $\max(1/\alpha, 2)$ *for the algorithm given by Theorem 14 is tight for all* $\alpha > \frac{1}{2}$ *and all* $\alpha = \frac{1}{q}$ *with* $q \in \mathbb{N}_{>1}$.

▶ Remark 18. There are instances where the integrality gap of ILP (1) is $\frac{1}{\alpha}$ – e.g. trees, where the unique optimal integral solution contains every edge while the unique optimal fractional solution chooses $x^*_e = \alpha$ for all edges e of the input graph. Interestingly, the approximation ratio of our algorithm (for $\alpha \leq \frac{1}{2}$) equals the integrality gap exactly, but on completely different instances and not on trees (where the algorithm always finds the optimal solution).

6 Conclusion and Open Questions

We introduced the practically motivated Minimum Multi-Commodity Flow Subgraph (MMCFS) problem and paved the way for further research by giving several structural results, most importantly a reformulation of this traffic-oblivious problem that only needs to consider a single specific traffic matrix. Further, we showed that MMCFS is NP-hard (and a closely related problem cannot be approximated within a sublogarithmic factor) already on DAGs. Lastly, we gave an extremely simple LP-rounding scheme for MMCFS with a tight approximation guarantee of $\max(1/\alpha, 2)$.

Considering seemingly related problems (see Section 2), one observes that an approximation ratio of 2 (which we attain for the practically most relevant cases of $\alpha \geq \frac{1}{2}$ [34]) often arises as a seemingly "natural" limit for such ratios. Yet, it remains an open question whether there exists an approximation algorithm for MMCFS with a better quality guarantee, and whether there is a non-trivial lower bound on the approximation guarantee for any such algorithm (assuming P $\neq$ NP). Further, it might be of interest to explore several generalizations of MMCFS: this includes the non-traffic-oblivious variant where a specific traffic matrix is part of the input (which is also NP-hard via Theorem 3), and the variant where, given an additional cost function on the edges, one asks for a subgraph of minimum cost.

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