Minimum Partition of Polygons Under Width and Cut Constraints

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Abstract

We study the problem of partitioning a polygon into the minimum number of subpolygons using cuts in predetermined directions such that each resulting subpolygon satisfies a given width constraint. A polygon satisfies the unit-width constraint for a set of unit vectors if the length of the orthogonal projection of the polygon on a line parallel to a vector in the set is at most one. We analyze structural properties of the minimum partition numbers, focusing on monotonicity under polygon containment. We show that the minimum partition number of a simple polygon is at least that of any subpolygon, provided that the subpolygon satisfies a certain orientation-wise convexity with respect to the polygon. As a consequence, we prove a partition analogue of the Bang's conjecture about coverings of convex regions in the plane: for any partition of a convex body in the plane, the sum of relative widths of all parts is at least one. For any convex polygon, there exists a direction along which an optimal partition is achieved by parallel cuts. Given such a direction, an optimal partition can be computed in linear time.

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1 Introduction

Most works in partitioning polygons have primarily focused on maximizing geometric measures, such as fatness or the minimum side length of resulting pieces [11, 13, 20]. In this paper, we study an opposite objective: partitioning a polygon into subpolygons whose

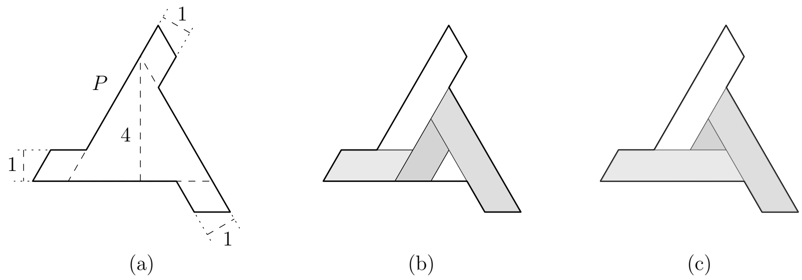

Figure 1 (a) The polygon P has a windmill shape with three arms extending from an equilateral triangle of height 4. (b–c) The minimum partitions of P under constraints $U = \{\mathbf{v}_{0^\circ}, \mathbf{v}_{60^\circ}, \mathbf{v}_{120^\circ}\}$ and $W = \{\mathbf{v}_{30^\circ}, \mathbf{v}_{90^\circ}, \mathbf{v}_{150^\circ}\}$, where $\mathbf{v}_\theta = (\cos\theta, \sin\theta) \in \mathbb{S}^+$. (b) A guillotine partition of five trapezoids. (c) A non-guillotine partition of four trapezoids.

widths are bounded above in certain directions. Such a width constraint commonly arises in manufacturing and recycling industries, where materials must be cut or processed within certain width limits. For example, wood chipping and metal shredding require pieces to fit within the machine's inlet. In some materials, cut directions are critical for preserving structural strength; for example, fabric is typically cut along the fiber direction [17].

Our problem is rooted in classical questions in convex geometry, notably Tarski's plank problem and its affine-invariant extension by Bang [7]. The original conjecture asserts that any covering of a convex body in $\mathbb{R}^d$ by strips must have a total width at least the minimal width of the body, which was proven by Bang. Bang further proposed the *affine plank problem*, in which strip widths are measured relative to the body's width in the same direction. The affine version remains open in general, with only partial results [16, 4, 19, 6]. It is known to be equivalent to the Davenport conjecture [5], which concerns partitions. Several partition analogues have been studied [9, 3, 10].

Our work can also be viewed as a partition analogue of these problems in the plane, where width constraints replace strips, and simple polygons substitute convex bodies.

1.1 Problem definition and results

We consider the problem of partitioning polygons into the minimum number of pieces satisfying both a unit-width constraint and a cut constraint. Let P be a simple polygon. Let Q be a piece in a partition of P satisfying unit-width constraint $W \subseteq \mathbb{S}^+$, where $\mathbb{S}^+$ is the set of unit vectors $\{(\cos\theta, \sin\theta) \mid 0 \le \theta < \pi\}$. Let $\omega_{\mathbf{v}}(Q)$ denote the *width* of Q in $\mathbf{v} \in \mathbb{S}^+$, that is, the length of the orthogonal projection of Q on a line parallel to $\mathbf{v}$. We say Q satisfies the *unit-width constraint* W if $\omega_{\mathbf{v}}(Q) \le 1$ for some vector $\mathbf{v} \in W$.

The process of partitioning P must satisfy a cut constraint $U \subseteq \mathbb{S}^+$. A *cut* is defined as a line segment within P whose relative interior lies in the interior of P. A unit vector is often used to represent a direction, such as the orientation of a cut, meaning that the cut lies on a line parallel to the vector. We require that every cut must be in a direction in U. If a cut extends from one edge of P to the other edge, it divides P into two distinct pieces; such a cut is called a *guillotine* cut. A *guillotine partition* of P (also called a *binary partition*) is obtained by a finite sequence of guillotine cuts; it starts from P and recursively partitions each piece into two subpieces using a guillotine cut. A non-guillotine partition of P is an arbitrary partition of P using cuts that are not necessarily guillotine. Figure 1 shows a guillotine partition and a non-guillotine partition.

Given a simple polygon P, our objective is to partition P into the minimum number of pieces using cuts constrained by U such that each piece in the partition satisfies unit-width constraint W. We denote this minimum partition problem by $\mathsf{wPartition}(P, W, U)$ and its optimum value by $\mathsf{opt}(P, W, U)$. Throughout this paper, we use W and U exclusively to refer to the unit-width constraint and the cut constraint, respectively.

Related Work. Damian and Pemmaraju [14] and Damian-Iordache [15] gave a polynomial-time algorithm for partitioning a simple polygon into the minimum number of subpolygons without using Steiner points such that each subpolygon has diameter at most α, for $\alpha > 0$. Later, Buchin and Selbach [11] showed that this problem becomes NP-hard for polygons with holes. Worman [22] proved NP-completeness for a variant in which each subpolygon must be contained in an axis-aligned square of side length α. Abrahamsen and Stade [2] showed that allowing Steiner points leads to NP-hardness for the partition problem under axis-aligned unit-square containment, even for simple polygons without holes. This marks the first known NP-hardness result of the minimum partition problems for hole-free polygons. Abrahamsen and Rasmussen [1] studied the problem of partitioning simple polygons into the minimum number of pieces such that each piece satisfies a bounded-size constraint (e.g., unit area, perimeter, diameter, or containment within unit disks or squares).

Our Results. Our main contribution is an analysis of the minimum partition number under constraints $W, U \subseteq \mathbb{S}^+$. First, we provide necessary and sufficient conditions for the existence of feasible partitions in $\mathsf{wPartition}(P, W, U)$, along with a decision algorithm for testing feasibility

Second, we study the monotonicity of the minimum partition number under polygon containment $Q \subseteq P$ (Sections 3, 4, and 5). We show that this monotonicity does not hold in general, and identify a sufficient condition based on a restricted-orientation convexity, called $\mathcal{O}$*-convexity*, where $\mathcal{O} \subseteq \mathbb{S}^+$. Theorem 3 states that $\mathsf{opt}(Q, W, U) \le \mathsf{opt}(P, W, U)$ holds if Q is U-convex with respect to P for guillotine partitions, or $\overline{W}$-convex with respect to P for non-guillotine partitions, where $\overline{W}$ is the set of all unit vectors perpendicular to those in W.

Finally, we prove a partition analogue of Bang's conjecture (Section 6). The statement of Bang's conjecture is as follows: if a convex body $K \subset \mathbb{R}^d$ is covered by strips $H_1, \dots, H_m$, then $\sum_{i=1}^m \inf_{\mathbf{v} \in \mathbb{S}^+} \frac{\omega_{\mathbf{v}}(H_i)}{\omega_{\mathbf{v}}(K)} \ge 1$. Our theorem replaces strip coverings with arbitrary partitions and extends the direction set to any subset $W \subseteq \mathbb{S}^+$.

▶ **Theorem 1** (Bang-Type Partition Analogue). *Let $K \subset \mathbb{R}^2$ be a convex body, and let $K_1 \cup \dots \cup K_m = K$ be its arbitrary partition. Then, for any subset $W \subseteq \mathbb{S}^+$,*

$$\sum_{i=1}^{m} \inf_{\mathbf{v} \in W} \frac{\omega_{\mathbf{v}}(K_i)}{\omega_{\mathbf{v}}(K)} \ge 1.$$

To the best of our knowledge, this is the first partition analogue that allows non-convex pieces. We also show that, for $U \subseteq \overline{W}$, an optimal partition of a convex polygon can be computed in linear time using equally spaced parallel cuts (See Corollary 17). The omitted proofs will be found in the full version of the paper at `https://arxiv.org/abs/2509.09981`.

2 Preliminaries

Let P be a simple polygon with n vertices in the plane. We assume that the vertices of P are given in a list sorted in counterclockwise order along its boundary. A partition of a simple polygon is a set of connected pieces with pairwise disjoint interiors whose union equals the polygon. The cardinality of a partition is the number of its pieces.

For a set $X \subseteq \mathbb{R}^2$, we denote by ∂X the boundary of X, by $\mathsf{int}(X)$ the interior of X, and by $\mathsf{cl}(X)$ the closure of X. We treat a polygon as the union of its interior and boundary; $\mathsf{cl}(P) = P$, ∂P is the boundary, and $\mathsf{int}(P)$ is the interior.

For a point $p \in \mathbb{R}^2$, let $x(p)$ and $y(p)$ be its x- and y-coordinates, respectively. For any two points p and q in $\mathbb{R}^2$, we use $\overline{pq}$ to denote the line segment connecting p and q with length $|\overline{pq}|$. We call $\overline{pq}$ a cut in P if it lies entirely in the interior of P, excluding its endpoints. If both endpoints lie on ∂P, we call it a guillotine cut.

The inner product of any two vectors $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\langle \mathbf{u}, \mathbf{v} \rangle$. The Euclidean norm of a vector $\mathbf{v}$ is denoted by $\|\mathbf{v}\|$. A vector with norm 1 is called a *unit vector*. We use $\mathbb{S}^+$ to denote the set of unit vectors $\{(\cos\theta, \sin\theta) \mid 0 \le \theta < \pi\}$. For a subset $V \subseteq \mathbb{S}^+$, we define $\overline{V} = \{\mathbf{v} \in \mathbb{S}^+ \mid \langle \mathbf{u}, \mathbf{v} \rangle = 0 \text{ for some } \mathbf{u} \in V\}$.

For a compact set $X \subseteq \mathbb{R}^2$ and a vector $\mathbf{v} \in \mathbb{S}^+$, let $\omega_{\mathbf{v}}(X)$ denote the length of the orthogonal projection of X on a line parallel to $\mathbf{v}$. We say X satisfies unit-width constraint $W \subseteq \mathbb{S}^+$ if and only if $\omega_{\mathbf{v}}(X) \le 1$ for some $\mathbf{v} \in W$.

A *strip* is the region in the plane bounded by two parallel lines. The distance between the bounding lines is the *width* of the strip, and the direction in $\mathbb{S}^+$ orthogonal to the bounding lines is called the *normal* vector of this strip. If a strip has width 1, we call it a *unit* strip.

We use the notation $[m] := \{1, 2, \ldots, m\}$ for a positive integer m. For a finite set A, we use $|A|$ to denote the cardinality of A which is the number of its elements.

3 Monotonicity of minimum partition numbers

In this section, we assume that $\mathsf{wPartition}(P, W, U)$ has a partition satisfying the constraints. The necessary and sufficient condition for feasibility is presented in the full version. We show $\mathsf{opt}(Q, W, U) \le \mathsf{opt}(P, W, U)$ for any subpolygon Q of P that satisfies a certain condition. For both guillotine and non-guillotine partitions, we identify sufficient conditions on Q that ensure this monotonicity. When the constraints W and U are clear from context, we abbreviate $\mathsf{opt}(P, W, U)$ as $\mathsf{opt}(P)$.

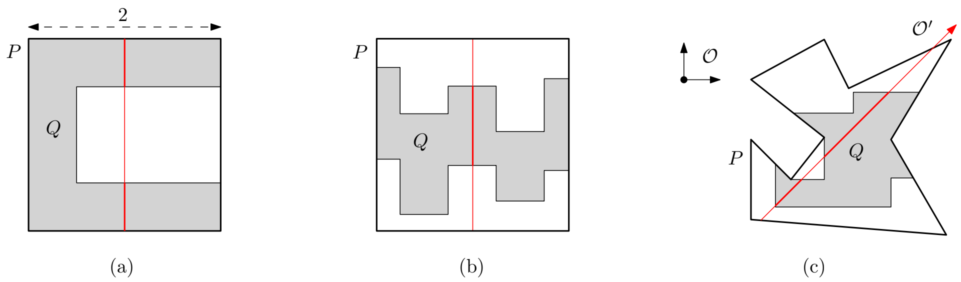

Figure 2 (a) A polygon P and its subpolygon Q (gray) with $\mathsf{opt}(P, W, U) = 2$ and $\mathsf{opt}(Q, W, U) > 2$ for $W = \{(1, 0)\}$ and $U = \{(0, 1)\}$. (b) Q (gray) is U-convex (or $\overline{W}$-convex) with respect to P, resulting in $\mathsf{opt}(Q, W, U) \le \mathsf{opt}(P, W, U)$. (c) Q (gray) of P is $\mathcal{O}$-convex, but not $\mathcal{O}'$-convex with respect to P, where $\mathcal{O} = \{(1, 0), (0, 1)\}$ and $\mathcal{O}' = \{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\}$.

Let P be a square of side length 2, and let Q be a subpolygon of P as shown in Figure 2(a). Consider the instance $\mathsf{wPartition}(P, W, U)$ with $W = \{(1, 0)\}$ and $U = \{(0, 1)\}$. Since U is a singleton, every cut must be a guillotine cut. Observe that neither P nor Q satisfies unit-width constraint W. A vertical cut halving P yields a feasible partition of two pieces. No vertical cut, however, in Q partitions Q into two pieces, each with horizontal width at most 1. Thus, $\mathsf{opt}(Q) > 2 = \mathsf{opt}(P)$, implying that the inclusion $Q \subseteq P$ alone is not sufficient to ensure $\mathsf{opt}(Q) \le \mathsf{opt}(P)$.

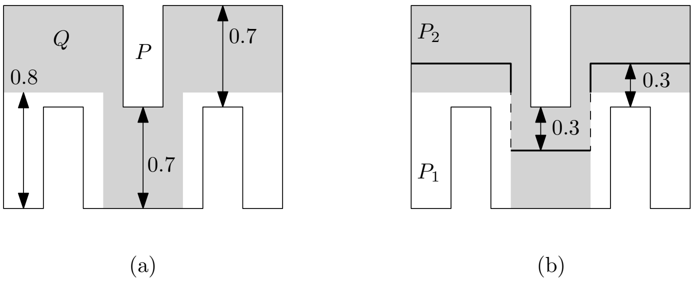

Figure 3 A non-guillotine partition of P, where $W = \{(0,1)\}$ and $U = \{(0,1),(1,0)\}$. (a) The polygon P and a subpolygon Q, both having vertical widths greater than 1. (b) An optimal non-guillotine partition of P into two pieces, but four when restricted to Q.

Such failures are common in minimum partitioning problems, as the optimal partition number depends on the geometric complexity of Q. Two typical approaches are restricting the geometry of Q relative to P, and constraining the partition class to specific families, such as the guillotine (binary) class [9]. These alter the structural properties of feasible solutions.

3.1 Restricted-orientation convexity

Rawlins [21] introduced the *restricted-orientation convexity* as a generalization of standard convexity in Euclidean space. For $\mathcal{O} \subseteq \mathbb{S}^+$, a set X is $\mathcal{O}$*-convex* if, for every line ℓ parallel to a vector in $\mathcal{O}$, $X \cap \ell$ is connected (we regard an empty set as being connected). When $\mathcal{O} = \mathbb{S}^+$, $\mathcal{O}$-convexity coincides with standard convexity in $\mathbb{R}^2$. We extend this concept to subpolygons of simple polygons with respect to guillotine cuts.

▶ **Definition 2.** *Let Q be a subpolygon of a polygon P, and $\mathcal{O}$ be a set of unit vectors. Then Q is $\mathcal{O}$-convex with respect to P if its intersection with every guillotine cut in P parallel to a vector in $\mathcal{O}$ is connected.*

Figure 2(c) shows a subpolygon Q that is $\mathcal{O}$-convex with respect to P for $\mathcal{O} = \{(0,1),(1,0)\}$, but not $\mathcal{O}'$-convex for $\mathcal{O}' = \{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\}$ since a guillotine cut in P parallel to $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ intersects Q in at least two connected components.

Observe that $\mathcal{O}$-convexity with respect to a polygon P is equivalent to the *geodesic convexity* within P when $\mathcal{O} = \mathbb{S}^+$. The following theorem gives sufficient conditions for monotonicity of the minimum partition numbers.

▶ **Theorem 3** (Monotonicity in Polygon Containment)**.** *Let U and W be sets of unit vectors, and let Q be a subpolygon of a polygon P. Assume that* wPartition(P, W, U) *has a solution.*

- opt$(Q) \leq$ opt(P) *for guillotine partitions if Q is U-convex with respect to P.*
- opt$(Q) \leq$ opt(P) *for non-guillotine partitions if Q is $\overline{W}$-convex with respect to P.*

Revisit the subpolygons Q in Figure 2(a,b). In (a), Q is neither U-convex nor $\overline{W}$-convex with respect to P, whereas in (b), Q satisfies both conditions, and Theorem 3 implies that opt$(Q) \leq$ opt$(P) = 2$, for both guillotine and non-guillotine partitions.

3.2 Monotonicity in guillotine partitions

We prove Theorem 3 for the guillotine case. Let $\Pi = \{P_1, \dots, P_m\}$ be a guillotine partition of P feasible to wPartition(P, W, U), and let $\Pi[Q] = \{P_i \cap Q\}_{i=1,\dots,m}$ be its restriction to a subpolygon Q that is U-convex with respect to P. Then, $\bigcup_{i=1}^{m}(P_i \cap Q) = Q$, and thus, $\Pi[Q]$

also forms a partition of Q. Note that each piece in $\Pi[Q]$ satisfies unit-width constraint W, as every piece is a subset of some P_i that satisfies the constraint. To show $\mathsf{opt}(Q) \leq \mathsf{opt}(P)$, it suffices to verify two aspects: (1) $\Pi[Q]$ is a guillotine partition of Q and (2) $|\Pi[Q]| \leq m$.

Since Π is a guillotine partition, the cuts used for Π are partially ordered by their precedence in the partitioning process. Let $\mathcal{C} = \langle c_1, \ldots, c_{m-1} \rangle$ denote a sequence of guillotine cuts that produces Π. Then each $c_i \cap Q$ is connected as c_i is a cut in P with a direction in U and Q is U-convex with respect to P. Let $\mathcal{D} = \langle d_1, \ldots, d_{m'} \rangle$ be the subsequence of $\langle c_1 \cap Q, \ldots, c_{m-1} \cap Q \rangle$ consisting of those satisfying $c_i \cap \mathsf{int}(Q) \neq \emptyset$ for $i = 1, \ldots, m-1$. Note that each d_j for $j \in [m']$ is a cut in Q and $m' \leq m - 1$. Also, observe that $\Pi[Q]$ is induced by $\mathcal{D}$. It suffices to show that $\mathcal{D}$ is indeed a sequence of guillotine cuts in Q: each $d_j \in \mathcal{D}$ is a guillotine cut in one subpolygon obtained by applying $\langle d_1, \ldots, d_{j-1} \rangle$ to Q.

Let i be the smallest index in $[m-1]$ such that $c_i \cap \mathsf{int}(Q) \neq \emptyset$. Since no cut in $\langle c_1, \ldots, c_{i-1} \rangle$ intersects $\mathsf{int}(Q)$, there is a subpolygon in the partition of P by $\langle c_1, \ldots, c_{i_1 - 1} \rangle$ that contains Q. Observe that c_i is a guillotine cut in this subpolygon that is aligned with a direction in U. Since Q is U-convex with respect to P, $d_1 = c_i \cap Q$ is a line segment, and thus, d_1 is a guillotine cut in Q.

We proceed by induction on j with $1 < j \leq m'$, that $\langle d_1, \ldots, d_{j-1} \rangle$ forms a sequence of guillotine cuts in Q. Among the pieces of Q partitioned by the sequence, let Q_j denote the one containing d_j, where d_j is a cut in Q_j. By definition, there exists an index $k \in [m]$ such that $d_j = c_k \cap Q$. Let P_k be the piece in the partition of P by $\langle c_1, \ldots, c_{k-1} \rangle$ that contains c_k.

To see that d_j is a guillotine cut of Q_j, recall that $d_j = c_k \cap Q_j$, where c_k is a guillotine cut of P_k and $Q_j \subseteq P_k$. Let c'_k be the guillotine cut in P obtained by extending c_k until it touches ∂P. Since Q is U-convex, $c'_k \cap Q$ is connected, and hence so is its subsegment $c_k \cap Q = d_j$. Moreover, as c_k spans ∂P_k, its restriction to Q_j necessarily touches ∂Q_j at both endpoints. Thus, d_j is a guillotine cut in Q_j. We conclude that $\mathcal{D} = \langle d_1, \ldots, d_{m'} \rangle$ induces a guillotine partition of Q. Since $m' \leq m$, we have $|\Pi[Q]| \leq m$.

▶ **Lemma 4.** *Let P be a polygon and let $\Pi = \{P_1, \ldots, P_m\}$ be a solution to the problem $\mathsf{wPartition}(P, W, U)$ for guillotine cuts. If a subpolygon Q of P is U-convex with respect to P, the restricted partition $\{Q \cap P_i\}_{i=1,\ldots,m}$ is a solution to $\mathsf{wPartition}(Q, W, U)$ for guillotine cuts with at most m pieces.*

4 Reconfiguration of restricted non-guillotine partitions

Let Q be a subpolygon of P that is $\overline{W}$-convex with respect to P. The monotonicity $\mathsf{opt}(Q) \leq \mathsf{opt}(P)$ trivially holds when $\mathsf{opt}(Q) = 1$. Also, any feasible partition is guillotine when U is a singleton. Assume that $\mathsf{opt}(Q) > 1$ and U contains at least two distinct vectors.

Let $\Pi = \{P_1, \ldots, P_m\}$ be any feasible partition of P to $\mathsf{wPartition}(P, W, U)$. Its restriction to Q, defined as $\Pi[Q] = \{P_i \cap Q\}_{i=1,\ldots,m}$, is a feasible partition of Q to $\mathsf{wPartition}(Q, W, U)$. However, some regions $P_i \cap Q$ may be disconnected, even when Q is $\overline{W}$-convex with respect to P (See Figure 3(a–b)). To address this, we modify the cuts in $\Pi[Q]$ to reconnect disjoint fragments into connected regions while preserving feasibility for $\mathsf{wPartition}(Q, W, U)$.

Consider any element P_i in the partition Π of P such that $\mathsf{int}(P_i) \cap \mathsf{int}(Q) \neq \emptyset$. The intersection $\mathsf{int}(P_i) \cap \mathsf{int}(Q)$ consists of open connected components. We define R_i as

$$R_i := \{\, \mathsf{cl}(X) \mid X \text{ is a connected component of } \mathsf{int}(P_i) \cap \mathsf{int}(Q) \text{ with } \mathsf{cl}(X) \cap \partial Q \neq \emptyset \,\}.$$

Let $R_i = \{C_1, C_2, \ldots, C_t\}$. Note that each C_j is a subpolygon of Q with positive area that touches ∂Q. To connect elements in R_i into a single piece, we reconfigure $\Pi[Q]$ by iteratively performing a process, called *reallocation*. Let X be a connected region in Q. We

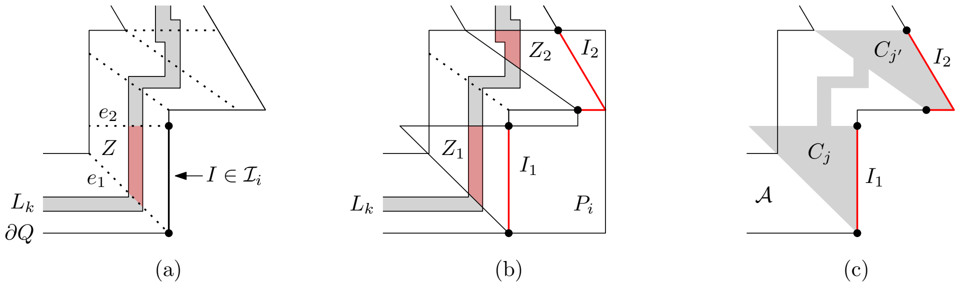

Figure 4 (a) A layer segment Z of R_i is bounded by $e_1, e_2 \in E_i^{\text{bd}}$, and it corresponds to $I \in \mathcal{I}_i$. (b) A link instruction $\lambda = (I_1, I_2, k)$ reallocates the layer segments in $L_k[Z_1, Z_2]$ to P_i. (c) Applying λ merges C_j and $C_{j'}$ into a single piece through layer segments in L_k, where $I_1 \in \mathcal{I}_{ij}$ and $I_2 \in \mathcal{I}_{ij'}$ for some $j, j' \in [t]$.

define the reallocation of X to P_i as the operation that modifies $\Pi[Q]$ by expanding the region assigned to P_i so that it includes X. This can be done by adding new boundaries along $\partial X \setminus P_i$ and removing those along $\partial P_i \setminus X$. If X and P_i intersect in their interiors, or share a boundary segment of positive length, then $X \cup P_i$ appears as a single piece in the resulting partition. We say that the region X is *reallocated* to P_i.

4.1 Construction and layering of corridor of Q

We first construct a narrow corridor along ∂Q. This corridor lies entirely within Q, closely following ∂Q, and provides the space needed to link elements of R_i.

The corridor is an annular region bounded by two simple closed curves: the outer curve ∂Q and an inner curve that lies at a small distance δ inward. The region between the two curves is called the *δ-corridor* of Q, where δ denotes its width. In our construction, we set δ to be a sufficiently small positive value and denote the resulting corridor by $\mathcal{A}$.

Consider the edges of each $C_j \in R_i$ that have one endpoint on ∂Q and intersect $\mathsf{int}(Q)$. Let E_{ij}^{bd} denote the set of such edges of C_j. We define $E_i^{\text{bd}} \coloneqq \bigcup_{j=1}^{t} E_{ij}^{\text{bd}}$ and $E_Q^{\text{bd}} \coloneqq \bigcup_{i=1}^{m} E_i^{\text{bd}}$. The corridor $\mathcal{A}$ is subdivided into a sequence of nested subregions, which we refer to as *layers*. Let $h \in \mathbb{Z}_{>0}$ denote the number of layers which will be determined in Section 4.4. For each $k \in [h]$, we construct a simple closed curve Γ_k lying entirely within $\mathsf{int}(\mathcal{A})$, such that the curves $\Gamma_1, \ldots, \Gamma_h$ are pairwise disjoint and arranged sequentially inward from ∂Q. Each Γ_k follows a zigzag pattern using the two directions in U, and intersects every edge in E_Q^{bd} once or twice, depending on whether the edge is a guillotine cut in Q. Details on the construction of the corridor $\mathcal{A}$ and simple curves are provided in the full version.

These h curves subdivide $\mathcal{A}$ into h nested subregions, called *layers*: for each $k \in [h]$, the k-th layer is the annular region bounded by Γ_{k-1} and Γ_k, where we define $\Gamma_0 = \partial Q$.

4.2 Link instructions with circular intervals and layers

Let C_j and $C_{j'}$ be two elements in R_i with $j \neq j'$. For $k \in [h]$, let L_k denote the k-th layer. Each connected component of $L_k \cap C_j$ for $C_j \in R_i$ is a subregion of Q referred to as a *layer segment* of C_j (and of R_i) in L_k. Each layer segment is bounded by four parts: continuous portions of the inner and outer boundaries of L_k, and two edges from E_{ij}^{bd}. Removing any layer segment from L_k alters its topological structure from an annulus to a weakly simple polygon. All layer segments in L_k can be arranged in cyclic order along L_k. Figure 4(a) illustrates a layer segment of R_i in L_k.

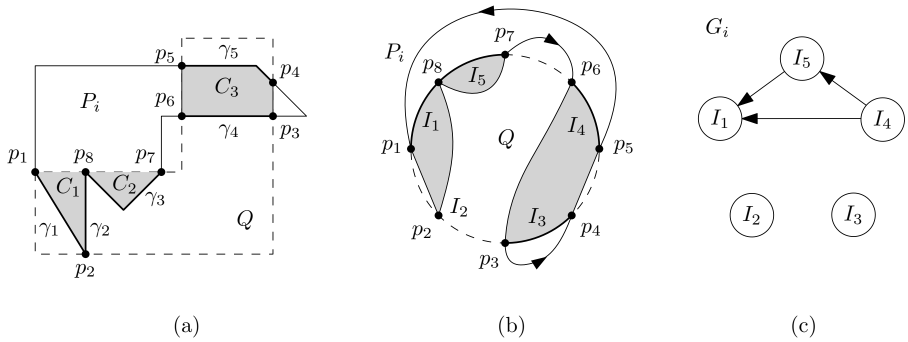

Figure 5 (a) $\mathsf{int}(P_i) \cap \mathsf{int}(Q)$ induces $R_i = \{C_1, C_2, C_3\}$, and each $\partial C_j \cap \mathsf{int}(Q)$ induces simple paths $\{\gamma_1, \ldots, \gamma_5\}$. (b) p_8 serves as both an entry and exit point for the transition $C_1 \to C_2$. The simple paths outside $\mathsf{int}(Q)$, from p_7 to p_6 and from p_5 to p_1, represents the transitions $C_2 \to C_3$ and $C_3 \to C_1$, respectively. (c) The directed graph G_i, constructed based on P_i and Q shown in (a).

We use an interval notation over layer segments that are cyclically ordered. For two layer segments Z_1, Z_2 in L_k, we denote by $L_k[Z_1, Z_2]$ the set of layer segments encountered in counterclockwise traversal from Z_1 to Z_2 in L_k, where $Z_1, Z_2 \in L_k[Z_1, Z_2]$. Similarly, $L_k(Z_1, Z_2)$, $L_k[Z_1, Z_2)$, and $L_k(Z_1, Z_2]$ denote the open and half-open intervals over layer segments between Z_1 and Z_2 in L_k.

Assume that $Z_1 \subseteq C_j$ and $Z_2 \subseteq C_{j'}$ are layer segments of R_i in L_k, for $j, j' \in [t]$ with $j \neq j'$. The reallocation of $L_k[Z_1, Z_2]$ to P_i merges C_j and $C_{j'}$ into a single piece within L_k. We refer to such an ordered pair (Z_1, Z_2) as a *link instruction* of R_i. See Figure 4(b–c).

Decomposition of ∂Q into circular intervals. For $C_j \in R_i$, C_j is a subpolygon of Q that touches ∂Q. The intersection $\partial C_j \cap \partial Q$ consists of continuous paths on ∂Q. Specifically, it can be the loop ∂Q itself if $\partial Q \subseteq \partial C_j$, implying that Q satisfies unit-width constraint W. However, assuming $\mathsf{opt}(Q) > 1$, this case does not occur. Thus, every connected component of $\partial C_j \cap \partial Q$ must be a continuous path on ∂Q that is not a loop. We define $\mathcal{I}_{ij}$ as

$$\mathcal{I}_{ij} \coloneqq \{I \subseteq \partial Q \mid I \text{ is a connected component of } \partial C_j \cap \partial Q\}.$$

We then define the aggregate sets $\mathcal{I}_i \coloneqq \bigcup_{j=1}^{t} \mathcal{I}_{ij}$ and $\mathcal{I}_Q \coloneqq \bigcup_{i=1}^{m} \mathcal{I}_i$.

Let $I \in \mathcal{I}_{ij}$ be a continuous path along ∂Q with endpoints p and q such that I corresponds to the portion of ∂Q from p to q in counterclockwise order. Since ∂Q is a simple closed curve, topologically equivalent to a circle, we regard each path I as a circular interval on ∂Q. As $\mathcal{I}_{ij}$ contains no loops, the case $p = q$ occurs only when ∂C_j touches ∂Q at a single point p, and no other portion of ∂C_j near p intersects ∂Q. Such intervals are called *degenerate* intervals.

For $i \in [m]$, $\mathcal{I}_i$ may contain both degenerate and non-degenerate intervals. Since P_i is simple, all intervals in $\mathcal{I}_i$ are pairwise interior-disjoint. Figure 5(a) shows a piece P_i and the subpolygon Q, from which the circular intervals in $\mathcal{I}_i$ are defined. These intervals are illustrated in Figure 5(b).

Note that a degenerate interval at some point $p \in \partial Q$ may appear in multiple $\mathcal{I}_i$'s. In such cases, each occurrence is treated as a distinct element in $\mathcal{I}_Q$. Since $\Pi = \{P_1, \ldots, P_m\}$ is a partition of P and $Q \subseteq P$, every point on ∂Q belongs to some interval of $\mathcal{I}_Q$, and no two intervals in $\mathcal{I}_Q$ intersect each other in their interiors. Thus, $\mathcal{I}_Q$ forms a decomposition of ∂Q.

Finally, we relate these circular intervals to layer segments so as to define a link instruction formally. For any $j \in [t]$ and any $k \in [h]$, consider the component $C_j \in R_i$ and the layer L_k. Each connected component of $\mathcal{A} \cap C_j$ (or $L_k \cap C_j$) is bounded by four parts: continuous portions of the inner and outer boundaries of $\mathcal{A}$ (or L_k), and two edges from E_{ij}^{bd}. Each connected component of $L_k \cap C_j$ is entirely contained within a unique connected component of $\mathcal{A} \cap C_j$. Moreover, the portion of ∂Q that bounds each connected component of $\mathcal{A} \cap C_j$ corresponds to a circular interval in $\mathcal{I}_{ij}$. Thus, each layer segment of C_j in L_k is uniquely associated with a circular interval in $\mathcal{I}_{ij}$, in a one-to-one correspondence. The number of layer segments in $\mathcal{A} \cap C_j$ (or $L_k \cap C_j$) is $|\mathcal{I}_{ij}| \cdot h$ (or $|\mathcal{I}_{ij}|$).

We revisit the link instruction that merges two layer segments $Z_1 \subseteq L_k \cap C_j$ and $Z_2 \subseteq L_k \cap C_{j'}$ for $j, j' \in [t]$ along layer L_k. The link instruction reallocates the region in L_k spanned counterclockwise from Z_1 to Z_2. Since Z_1 and Z_2 correspond to circular intervals in $\mathcal{I}_{ij}$ and $\mathcal{I}_{ij'}$, respectively, each link instruction can be represented by a triple (I_1, I_2, k), where $I_1, I_2 \in \mathcal{I}_Q$ and $k \in [h]$ (See Figure 4). Note that $(I_1, I_2, k) \neq (I_2, I_1, k)$, as the counterclockwise span from Z_1 to Z_2 differs from that in the reverse order.

4.3 Graph for encoding link instructions

Recall that $R_i = \{C_1, C_2, \ldots, C_t\}$ is the set of closures of connected pieces of $\mathsf{int}(P_i) \cap \mathsf{int}(Q)$ such that each C_j has a positive area and touches ∂Q. We construct a graph for each R_i, which specifies how the elements of R_i are to be connected into a single piece within Q.

For each $j \in [t]$, $\mathcal{I}_{ij}$ consists of circular intervals, each corresponding to a continuous path in $\partial C_j \cap \partial Q$ along ∂Q. Let $\mathcal{I}_{ij}^+$ denote the subset of $\mathcal{I}_{ij}$ consisting of only those intervals with positive length. We define $\mathcal{I}_i^+ = \bigcup_{j=1}^t \mathcal{I}_{ij}^+$ and $\mathcal{I}_Q^+ = \bigcup_{i=1}^m \mathcal{I}_i^+$. Unlike $\mathcal{I}_Q$, the set $\mathcal{I}_Q^+$ contains only non-degenerate intervals, which are pairwise interior-disjoint. Note that $\{\mathcal{I}_i^+\}_{i=1,\ldots,m}$ forms a partition of ∂Q.

Sequencing subpaths of ∂P_i. For each $C_j \in R_i$, $\partial C_j \cap \mathsf{int}(Q)$ consists of connected components, each forming a continuous path connecting two points on ∂Q. By the definition of $\mathcal{I}_{ij}$, the endpoints of these paths correspond to the endpoints of the circular intervals in $\mathcal{I}_{ij}$. The total number of such paths in $\partial C_j \cap \mathsf{int}(Q)$ is $|\mathcal{I}_{ij}|$. See Figure 5(a).

We traverse ∂P_i counterclockwise, starting at any point on $\partial C_1 \cap \mathsf{int}(Q)$ and completing a full circuit. During the traversal, each path in $\partial C_j \cap \mathsf{int}(Q)$ is visited exactly once for each $j \in [t]$, except for the path containing the starting point. The order in which the continuous paths are visited is denoted by $\gamma_1 \to \gamma_2 \to \cdots \to \gamma_l$, where $\gamma_1 = \gamma_l$ is the path in $\partial C_1 \cap \mathsf{int}(Q)$ containing the starting point, and $l = \sum_{j=1}^t |\mathcal{I}_{ij}| + 1 = |\mathcal{I}_i| + 1$.

Consider two consecutive paths γ_k and γ_{k+1} for any $k \in [l-1]$. Each path is derived from some piece in R_i; there exist $j, j' \in [t]$ such that $\gamma_k \subseteq \partial C_j \cap \mathsf{int}(Q)$ and $\gamma_{k+1} \subseteq \partial C_{j'} \cap \mathsf{int}(Q)$. As we traverse from γ_k to γ_{k+1}, we exit $\mathsf{int}(Q)$ through an endpoint of γ_k and re-enter $\mathsf{int}(Q)$ through an endpoint of γ_{k+1}.

▶ **Lemma 5.** *Let (γ_k, γ_{k+1}) be a pair of consecutive paths for $k \in [l-1]$, where $\gamma_k \subseteq \partial C_j \cap \mathsf{int}(Q)$ and $\gamma_{k+1} \subseteq \partial C_{j'} \cap \mathsf{int}(Q)$ for some $j, j' \in [t]$. The traversal from γ_k to γ_{k+1} exits and re-enters $\mathsf{int}(Q)$ through intervals $I_1 \in \mathcal{I}_{ij}$ and $I_2 \in \mathcal{I}_{ij'}$, respectively. If $j = j'$, then $I_1 = I_2$. Otherwise, I_1 and I_2 are interior-disjoint, non-degenerate intervals.*

By Lemma 5, a pair of distinct intervals in $\mathcal{I}_i^+$ provides an exit-entry pair for each transition between distinct pieces in R_i in the counterclockwise traversal of ∂P_i.

Construction of G_i. Based on the counterclockwise traversal of ∂P_i, we construct a directed graph $G_i = (V_i, E_i)$ as follows. Each vertex $v \in V_i$ corresponds to an interval in $\mathcal{I}_i^+$, so $|V_i| = |\mathcal{I}_i^+|$. Let γ_1 and γ_2 be consecutive paths in the traversal where $\gamma_1 \subseteq \partial C_j \cap \mathsf{int}(Q)$

and $\gamma_2 \subseteq \partial C_{j'} \cap \mathsf{int}(Q)$ with $j \neq j'$. The traversal from γ_1 to γ_2 encounters intervals I_1 and I_2 in $\mathcal{I}_i^+$ that contain the exit and entry points, respectively. By Lemma 5, these intervals I_1 and I_2 are uniquely determined. We add a directed edge e between vertices v_1 and v_2 corresponding to I_1 and I_2, respectively.

Let p and q denote the exit and entry points of the traversal, respectively; p is the endpoint of I_1 and q is the endpoint of I_2. The counterclockwise traversal on ∂P_i between γ_1 and γ_2 follows a simple path along ∂P_i outside Q that starts from p and ends at q. Let $\gamma_{pq} \subseteq \partial P_i \setminus \mathsf{int}(Q)$ denote this path from p to q. Observe that the exit and entry points may coincide, i.e., $p = q$, if I_1 and I_2 are adjacent at p and I_1 lies counterclockwise from I_2 along ∂Q. In this case, the direction of e is assigned from v_2 to v_1.

Given that $p \neq q$, the simple path γ_{pq} can be viewed as a simple path connecting two distinct points on the boundary of the circle and lying outside the circle. The path γ_{pq} can be classified into one of two types, depending on how it winds around the circle. Formally, γ_{pq} is homotopic to a directed path in $\mathbb{R}^2 \setminus \mathsf{int}(Q)$ that winds around ∂Q in either counterclockwise or clockwise direction. Note that it cannot wind around the boundary more than once since γ_{pq} is simple. We assign the direction of e from v_1 to v_2 if the path is of the counterclockwise type, and from v_2 to v_1, otherwise. Figure 5(b) illustrates γ_{pq} for both cases where $p = q$ and $p \neq q$. The same rule is applied to assign directions to all other edges in G_i.

In summary, we construct the directed graph G_i for $i \in [m]$ to represent how the disjoint components of R_i are to be connected into a single piece within Q. Each vertex corresponds to an interval in $\mathcal{I}_i^+$. Each directed edge $e = (v_1, v_2)$ of G_i represents a link instruction (I_1, I_2, k), but k is not specified yet. See Figure 5(c) for an illustration of the directed graph G_i that defines three link instructions, where P_i and Q are as shown in Figure 5(a).

4.4 Layer assignments for link instructions

For $i \in [m]$, let G_i be the directed graph for $R_i = \{C_1, C_2, \ldots, C_t\}$. Each edge in G_i represents a link instruction that reallocates layer segments within a specific layer. The goal is to assign layers to link instructions so that no redundant reallocation of layer segments is allowed and all elements in R_i are eventually merged into a single piece for every $i = 1, \ldots, m$.

The underlying graph of a digraph is its undirected version, obtained by ignoring the directions of all edges. Consider a connected component T of the underlying graph of G_i. We assume that T contains more than one vertex, as a component of size 1 does not indicate any link instruction. Each vertex of T corresponds to a circular interval in $\mathcal{I}_i^+$. Let $\mathcal{I}_T$ be the subset of $\mathcal{I}_i^+$ such that $\mathcal{I}_T = \{I \in \mathcal{I}_i^+ \mid I \text{ corresponds to a vertex in } T\}$.

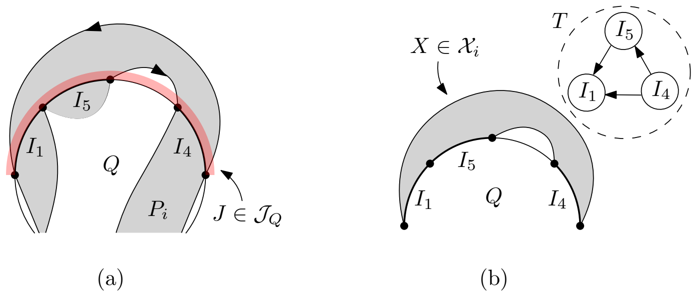

Figure 6 (a) The combine step merges I_1, I_5, and I_4 to a circular interval $J \in \mathcal{J}_Q$. (b) The interval J is derived from a connected component T of the underlying graph of G_i. This corresponds to a connected region $X \in \mathcal{X}_i$ that is bounded by the circular intervals associated with T.

We iteratively perform a combine step to merge all circular intervals in $\mathcal{I}_T$ into a single circular interval on ∂Q. The process begins with an initial set $S = \{u\}$ and a circular interval J_S, where u is any vertex of T and J_S is the circular interval in $\mathcal{I}_T$ corresponding to u. At each step, any vertex $v \in T \setminus S$ is chosen if there exists an edge between v and some vertex $w \in S$. Let $I_v \in \mathcal{I}_T$ be the circular interval corresponding to v. We add v to S and update J_S to be the smallest circular interval on ∂Q that spans from I_v to J_S in counterclockwise direction if the edge is directed from v to w, or in clockwise direction otherwise. Once $S = T$, the process returns a single circular interval on ∂Q, denoted by J_T, corresponding to the connected component T of G_i. See Figure 6(a) for an illustration of this process.

$$\mathcal{J}_Q \coloneqq \{J_T \mid T \text{ is a connected component of } G_i \text{ with } |T| > 1, \text{ for } i = 1, 2, \ldots, m\}.$$

Throughout, we use I to denote intervals in $\mathcal{I}_Q$ and J for those in $\mathcal{J}_Q$ to emphasize their distinct roles. Each $J \in \mathcal{J}_Q$ is non-degenerate. For any $J_1, J_2 \in \mathcal{J}_Q$ with $\mathsf{int}(J_1) \cap \mathsf{int}(J_2) \neq \emptyset$, there exists a proper containment between J_1 and J_2. To prove this, we present a lemma relating connectivity in the underlying graph of G_i to the existence of a path contained in a connected component of $P_i \setminus \mathsf{int}(Q)$, as illustrated in Figure 6(b).

▶ **Lemma 6.** *The vertices corresponding to I_1 and I_2 in $\mathcal{I}_i^+$ are connected in the underlying graph of G_i if and only if there is a simple path connecting $p \in I_1$ to $q \in I_2$ contained in $P_i \setminus \mathsf{int}(Q)$.*

Let $\mathcal{X}_i$ be the set of connected components in $P_i \setminus \mathsf{int}(Q)$. Then each element in $\mathcal{X}_i$ is a closed connected set. Moreover, it is a (weakly) simple polygon.

For $p, q \in \partial Q$, let $\Lambda(p, q)$ denote the set of all simple paths from p to q contained in $P_i \setminus \mathsf{int}(Q)$. For $I_1, I_2 \in \mathcal{I}_i^+$, we define $\Lambda(I_1, I_2) = \bigcup_{p \in I_1, q \in I_2} \Lambda(p, q)$. Lemma 6 guarantees the existence of a path between I_1 and I_2 whenever their corresponding vertices are connected in the underlying graph of G_i. Furthermore, all such paths share a consistent topological behavior, such as winding around ∂Q in the same direction.

▶ **Corollary 7.** *Let $I_1, I_2 \in \mathcal{I}_i^+$ with $I_1 \neq I_2$. If $\Lambda(I_1, I_2) \neq \emptyset$, then all paths in $\Lambda(I_1, I_2)$ wind around ∂Q in the same direction, and there is a unique component $X \in \mathcal{X}_i$ that contains all such paths.*

Using Lemma 6 and Corollary 7, we prove a proper containment relation.

▶ **Lemma 8.** *For any distinct $J_1, J_2 \in \mathcal{J}_Q$ with $\mathsf{int}(J_1) \cap \mathsf{int}(J_2) \neq \emptyset$, $J_1 \subsetneq J_2$ or $J_2 \subsetneq J_1$.*

By Lemma 8, we construct the circular interval graph $G_{\mathcal{J}}$, where each node corresponds to a circular interval in $\mathcal{J}_Q$, and a directed edge from J_1 to J_2 is added if $J_1 \subsetneq J_2$ for $J_1, J_2 \in \mathcal{J}_Q$. Then, $G_{\mathcal{J}}$ is acyclic as each edge corresponds to a strict containment.

Transitive reduction of $G_{\mathcal{J}}$. The graph $G_{\mathcal{J}}$ represents the transitive closure of the proper containment relation. That is, for $J_1, J_2 \in \mathcal{J}_Q$, there is a directed edge from J_1 to J_2 in $G_{\mathcal{J}}$ if there exists some $J' \in \mathcal{J}_Q$ such that (J_1, J') and (J', J_2) are directed edges in $G_{\mathcal{J}}$. A *transitive reduction* of $G_{\mathcal{J}}$ is a directed graph on the same vertex set with the minimum number of edges that preserves all reachability relations of $G_{\mathcal{J}}$. Since the transitive reduction of a DAG is unique [8], we denote the transitive reduction of $G_{\mathcal{J}}$ by $G_{\mathcal{J}}^{\mathrm{tr}}$. Let $UG_{\mathcal{J}}^{\mathrm{tr}}$ denote the underlying graph of $G_{\mathcal{J}}^{\mathrm{tr}}$.

▶ **Lemma 9.** *Any vertex of $G_{\mathcal{J}}^{\mathrm{tr}}$ has out-degree at most one, and $UG_{\mathcal{J}}^{\mathrm{tr}}$ is acyclic.*

By Lemma 9, $UG_{\mathcal{J}}^{\mathrm{tr}}$ is a forest. Each directed tree T of $G_{\mathcal{J}}^{\mathrm{tr}}$ has total out-degrees $|T| - 1$, implying that exactly one vertex has out-degree zero and all others have out-degree one. This structure corresponds to an *in-branching* tree [8].

Layer assignment of link instructions. In each in-branching tree of $G^{\text{tr}}_{\mathcal{J}}$, the unique vertex with out-degree zero is called the root. The level of a vertex is defined as the number of edges on the path from the root to that vertex plus one, where the level of the root is one. The level of each interval $J \in \mathcal{J}_Q$ is defined to be the level of the corresponding vertex in $G^{\text{tr}}_{\mathcal{J}}$.

Recall that each circular interval $J \in \mathcal{J}_Q$ is formed by merging intervals in $\mathcal{I}^+_i$ for some $i \in [m]$. These intervals correspond to the vertices of a connected component T of G_i, where each edge of T encodes a link instruction that merges two elements in R_i. Let $\texttt{INST}(J)$ denote the set of link instructions corresponding to the edges of T. We assign the k-th layer to every link instruction in $\texttt{INST}(J)$ if J is at level k in $G^{\text{tr}}_{\mathcal{J}}$. Note that the maximum level in $G^{\text{tr}}_{\mathcal{J}}$ is at most $\lfloor |\mathcal{I}^+_Q|/2 \rfloor$ if each connected component of G_i has size two for all $i = 1, 2, \ldots, m$. Thus, we set the number of layers in the corridor $\mathcal{A}$ to this maximum level: $h = \lfloor |\mathcal{I}^+_Q|/2 \rfloor$.

For each $J \in \mathcal{J}_Q$, all link instructions in $\texttt{INST}(J)$ reallocate layer segments within a single layer L_k to a common piece P_i, where $i \in [m]$ and $k \in [h]$ are determined by J. Let $I_a, I_b \in \mathcal{I}^+_i$ such that J is the smallest circular interval that spans from I_a to I_b in counterclockwise order. Let $Z_a, Z_b \subseteq L_k$ be the layer segments corresponding to I_a and I_b, respectively. Applying $\texttt{INST}(J)$ is equivalent to reallocating every $Z \in L_k[Z_a, Z_b]$ to P_i.

In summary, $\texttt{INST}_Q = \bigcup_{J \in \mathcal{J}_Q} \texttt{INST}(J)$ defines the set of all link instructions for reconfiguration. Applying $\texttt{INST}_Q$ to $\Pi[Q]$ yields a partition $Q = Q^*_1 \cup \cdots \cup Q^*_m$, where each Q^*_i is a region assigned to P_i within Q. When Q is $\overline{W}$-convex with respect to P, each Q^*_i is connected and satisfies both constraints W and U.

5 Analysis of reconfigured non-guillotine partitions

Assuming that the link instructions in $\texttt{INST}_Q$ have been applied to $\Pi[Q]$ in an arbitrary order, we verify the following statements in order to prove Theorem 3.

(1) For each $i = 1, \ldots, m$, the region allocated to P_i within Q forms a single connected piece.
(2) The reconfiguration of $\Pi[Q]$ is a solution to the problem $\mathsf{wPartition}(Q, W, U)$ if Q is $\overline{W}$-convex with respect to P.

It follows from statements (1) and (2) that the reconfigured partition of Q satisfies the constraints W and U while ensuring that the number of pieces does not exceed m.

5.1 Connectivity in reconfigured partitions

For $i \in [m]$, let $\texttt{INST}_i$ denote the subset of $\texttt{INST}_Q$ consisting of instructions of the form (I_1, I_2, k), where $I_1, I_2 \in \mathcal{I}^+_i$ and $k \in [h]$. Then, $\texttt{INST}_Q = \bigcup_{i \in [m]} \texttt{INST}_i$. We first show that the link instructions in $\texttt{INST}_i$ merge all elements of R_i into a single connected piece. We then verify that applying the link instructions in $\texttt{INST}_Q \setminus \texttt{INST}_i$ does not disconnect the merged piece. Furthermore, the resulting reconfigured partition of Q is invariant under the order in which the link instructions in $\texttt{INST}_Q$ are applied to $\Pi[Q]$.

Merging R_i via $\texttt{INST}_i$. We show that any two pieces $C_j, C_{j'} \in R_i$ are connected by some link instructions in $\texttt{INST}_i$. During the traversal on ∂P_i, each continuous path of $\partial C_j \cap \mathsf{int}(Q)$ is encountered at least once for every $j \in [t]$, and we denote the sequence of these paths in the order they are visited as $(\gamma_1, \gamma_2, \ldots, \gamma_l)$, where $\gamma_1 = \gamma_l$. Then, there exist indices $k, k' \in [l]$ such that $\gamma_k \subseteq \partial C_j \cap \mathsf{int}(Q)$ and $\gamma_{k'} \subseteq \partial C_{j'} \cap \mathsf{int}(Q)$. Since it is a cyclic sequence with $\gamma_1 = \gamma_l$, we assume without loss of generality that $k < k'$.

Recall that an edge of G_i is added whenever two consecutive paths in the sequence are derived from distinct elements in R_i. This indicates that a transition between them occurs during the traversal. The link instruction associated with this edge merges the corresponding

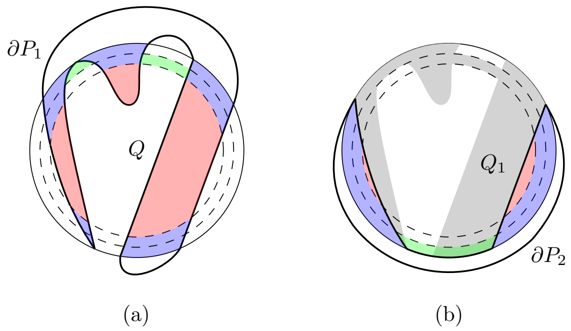

Figure 7 The corridor of Q has two layers, with $|R_1| = 3$ and $|R_2| = 2$. (a) Red regions represent $\{\mathrm{core}(C) \mid \text{for } C \in R_1\}$, while blue and green regions indicate layer segments in $\mathcal{Z}_1^{\text{in}}$ and $\mathcal{Z}_1^{\text{out}}$, respectively. (b) Q_1 is formed by merging R_1 according to $\texttt{INST}_1$, and some layer segments in $\mathcal{Z}_1^{\text{in}}$ are reallocated by $\texttt{INST}_2$. (c) Q_1^* is obtained by applying $\texttt{INST}_Q \setminus \texttt{INST}_1$ to Q_1.

elements in R_i. The subsequence $(\gamma_k, \gamma_{k+1}, \ldots, \gamma_{k'})$ contains multiple transitions between distinct elements in R_i, starting from C_j and eventually reaching $C_{j'}$. Applying all link instructions associated with the transitions in $(\gamma_k, \ldots, \gamma_{k'})$ results in C_j and $C_{j'}$ being merged into a single connected piece.

Let Q_i denote the subregion of Q resulting from merging elements of R_i via $\texttt{INST}_i$, with no other instructions applied. Then, Q_i consists of all elements in R_i and layer segments reallocated from other pieces: $Q_i = \left(\bigcup_{j\in[t]} C_j\right) \cup \mathcal{Z}_i^{\text{out}}$, where $\mathcal{Z}_i^{\text{out}}$ is the set of layer segments reallocated to P_i by $\texttt{INST}_i$.

The region $\mathbf{L} = L_1 \cup \cdots \cup L_h$ denotes the union of all layers in Q. For each $C_j \in R_i$, $\mathbf{L} \cap C_j$ consists of layer segments within C_j. By construction of layers, $C_j \setminus \mathbf{L}$ is connected and non-empty. We refer to this region as a *core* of C_j, denoted by $\mathrm{core}(C_j) = C_j \setminus \mathbf{L}$. Each C_j consists of its core together with the layer segments it contains. Let $\mathcal{Z}_i^{\text{in}}$ denote the set of all layer segments in $\mathbf{L} \cap C_j$ over all $C_j \in R_i$. Therefore, for $i \in [m]$, we have $Q_i = \left(\bigcup_{j\in[t]} \mathrm{core}(C_j)\right) \cup \mathcal{Z}_i^{\text{in}} \cup \mathcal{Z}_i^{\text{out}}$. Figure 7(a) illustrates the parts of Q_i: cores and layer segments. The shapes are drawn schematically to reflect the topological structure, rather than an exact polygonal description.

We now turn to the link instructions in $\texttt{INST}_Q \setminus \texttt{INST}_i$ that are applied to Q_i. Let Q_i^* be a subregion of Q_i that is obtained by applying all link instructions in $\texttt{INST}_Q \setminus \texttt{INST}_i$ to Q_i. Note that any part of Q_i reallocated by $\texttt{INST}_Q \setminus \texttt{INST}_i$ lies within $\mathcal{Z}_i^{\text{in}} \cup \mathcal{Z}_i^{\text{out}}$, and thus the cores remain unchanged. We show that Q_i^* is well-defined, meaning that Q_i^* is invariant under the order in which the instructions in $\texttt{INST}_Q$ are applied.

▶ **Lemma 10.** *Let $(I_1, I_2, k) \in \texttt{INST}_i$ be a link instruction, and let Z_1 and Z_2 be the layer segments in L_k corresponding to I_1 and I_2, respectively. Then, the layer segments $L_k[Z_1, Z_2]$ remain assigned to P_i under any link instruction in $\texttt{INST}_Q \setminus \texttt{INST}_i$.*

Lemma 10 implies that no layer segment is reassigned by more than one link instruction. As a consequence, we obtain the following corollary, which states that Q_i^* is well-defined.

▶ **Corollary 11.** *In the reconfiguration of $\Pi[Q]$, each layer segment in Q is reallocated at most once.*

Corollary 11 ensures that each layer segment may be reallocated to a different piece at most once, and no chains of reallocations such as $P_{i_1} \to P_{i_2} \to P_{i_3}$ with $i_1 \neq i_2$ and $i_2 \neq i_3$ occur.

This lemma further implies that layer segments in $\mathcal{Z}_i^{\text{out}}$ are preserved, while only those in $\mathcal{Z}_i^{\text{in}}$ are reallocated by $\texttt{INST}_Q \setminus \texttt{INST}_i$. Let $\mathcal{Z}_i^{\text{in}*}$ be the subset of $\mathcal{Z}_i^{\text{in}}$ that consists of layer segments preserved under $\texttt{INST}_Q \setminus \texttt{INST}_i$. Then, Q_i^* is the subpolygon that is obtained from Q_i by removing those layer segments in $\mathcal{Z}_i^{\text{in}} \setminus \mathcal{Z}_i^{\text{in}*}$. Then $Q_i^* = \left(\bigcup_{j\in[t]} \text{core}(C_j)\right) \cup \mathcal{Z}_i^{\text{in}*} \cup \mathcal{Z}_i^{\text{out}}$, where $\mathcal{Z}_i^{\text{in}*} \subseteq \mathcal{Z}_i^{\text{in}}$. Figure 7(b–c) illustrates the construction of Q_i^*, in which only layer segments in $\mathcal{Z}_i^{\text{in}}$ are reallocated by $\texttt{INST}_Q \setminus \texttt{INST}_i$.

Path-Connectivity of Q_i^*. To prove that Q_i^* forms a connected piece, it suffices to verify two types of path-connectivity among its constituent parts, which must be preserved during the reallocation induced by $\texttt{INST}_Q \setminus \texttt{INST}_i$.

- Each layer segment in $\mathcal{Z}_i^{\text{in}*} \cup \mathcal{Z}_i^{\text{out}}$ is path-connected to some $\text{core}(C_j)$ within Q_i^* for $C_j \in R_i$.
- The cores $\{\text{core}(C_j) \mid C_j \in R_i\}$ are mutually path-connected within Q_i^*.

Here, two sets $A, B \subseteq X$ are said to be path-connected within a region X if there exists a path in X joining some $a \in A$ and $b \in B$.

▶ **Lemma 12.** *Each layer segment in $\mathcal{Z}_i^{\text{in}*} \cup \mathcal{Z}_i^{\text{out}}$ is path-connected within Q_i^* to the core of some $C_j \in R_i$.*

By Lemma 12, every layer segment in $\mathcal{Z}_i^{\text{in}*} \cup \mathcal{Z}_i^{\text{out}}$ has a path to some core within Q_i^*. It remains to show that $\text{core}(C_1), \ldots, \text{core}(C_t)$ are mutually path-connected within Q_i^*.

▶ **Lemma 13.** *All cores of elements in R_i are mutually path-connected within Q_i^*.*

By Lemmas 12 and 13, each Q_i^* is connected. Thus, applying $\texttt{INST}_Q$ to $\Pi[Q]$ yields the partition $\Pi^*[Q] = \{Q_1^*, Q_2^*, \ldots, Q_m^*\}$, where each Q_i^* is a connected subregion of Q.

Remarks. Applying link instructions in $\texttt{INST}_Q$ may induce holes within merged pieces in the reconfigured partition of Q. When Q_i^* contains holes, reallocating them to P_i does not increase $\omega_{\mathbf{v}}(Q_i^*)$ for every $\mathbf{v} \in \mathbb{S}^+$. Therefore, each Q_i^* can be regarded as a simple polygon.

5.2 Feasibility of reconfigured partitions

We first observe that the reconfigured partition $\Pi^*[Q] = \{Q_1^*, Q_2^*, \ldots, Q_m^*\}$ remains as a valid partition of Q. By construction, the cut constraint U is also preserved: each layer segment is bounded by the boundaries of layers and the cuts from $\Pi[Q]$, all aligned with directions in U. It remains to check the unit-width constraint W.

Since $\Pi = \{P_1, \ldots, P_m\}$ is a solution to $\mathsf{wPartition}(P, W, U)$, there exists a vector $\mathbf{v_i} \in W$ such that $\omega_{\mathbf{v_i}}(P_i) \leq 1$. In other words, there exists a unit strip H_i with normal vector $\mathbf{v_i}$ that contains P_i. If every layer segment reallocated by $\texttt{INST}_i$ is contained within H_i, then Q_i^* also satisfies unit-width constraint W.

Assume that $\lambda = (I_1, I_2, k')$ is a link instruction associated with a directed edge from v_1 to v_2 in G_i for $I_1, I_2 \in \mathcal{I}_i^+$ and $k' \in [h]$. Let $j, j' \in [t]$ be two distinct indices such that $I_1 \in \mathcal{I}_{ij}^+$ and $I_2 \in \mathcal{I}_{ij'}^+$. Recall that the edge between v_1 and v_2 is added to G_i if and only if there is a transition between ∂C_j and $\partial C_{j'}$ during the counterclockwise traversal on ∂P_i.

For each $k \in [h]$, let Z_1^k and Z_2^k denote the layer segments in L_k corresponding to I_1 and I_2, respectively. The link instruction λ reallocates the layer segments in $L_{k'}(Z_1^{k'}, Z_2^{k'})$ to P_i, where $Z_1^{k'}$ and $Z_2^{k'}$ are already assigned to P_i in $\Pi[Q]$. We define $\mathbf{Z}_\lambda \coloneqq \bigcup_{k\in[h]} \bigcup_{Z\in\mathcal{Z}^k} Z$, where $\mathcal{Z}^k \coloneqq L_k(Z_1^k, Z_2^k)$. It suffices to show that $\mathbf{Z}_\lambda \subseteq H_i$. If this inclusion holds, every layer segment reallocated by λ lies within H_i, and thus the unit-width constraint W is preserved.

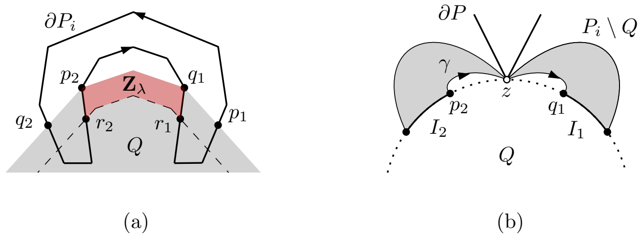

Figure 8 (a) The link instruction $\lambda = (I_1, I_2, k)$ is derived from the traversal of ∂P_i from p_1 to q_2 or from p_2 to q_1, where $I_1 = \partial Q[p_1, q_1]$ and $I_2 = \partial Q[p_2, q_2]$. $\partial \mathbf{Z}_\lambda$ consists of $\partial Q[q_1, p_2], \partial Q^\phi[r_1, r_2], \overline{q_1 r_1}$, and $\overline{p_2 r_2}$. (b) The path γ from p_2 to q_1 must pass through z when ∂P intersects ∂Q at z.

Decomposition of $\partial \mathbf{Z}_\lambda$. The region $\mathbf{Z}_\lambda$ is connected and bounded by four parts: two continuous portions of the inner and outer boundaries of $\mathcal{A}$, and two subsegments of edges from C_j and $C_{j'}$. The outer boundary of $\mathcal{A}$ is ∂Q and its inner boundary is ∂Q^ϕ, where Q^ϕ is the inner ϕ-offset polygon of Q.

For the sake of clarity, we introduce interval notation to represent subpaths of ∂Q and ∂Q^ϕ. For any two points $x, y \in \partial Q$, we denote by $\partial Q[x, y]$ the portion of ∂Q from x to y in counterclockwise order, including both endpoints. Let $\partial Q(x, y] = \partial Q[x, y] \setminus \{x\}$, $\partial Q[x, y) = \partial Q[x, y] \setminus \{y\}$, and $\partial Q(x, y) = \partial Q[x, y] \setminus \{x, y\}$. Similarly, we define $\partial Q^\phi[x, y], \partial Q^\phi(x, y], \partial Q^\phi[x, y)$, and $\partial Q^\phi(x, y)$ as portions of ∂Q^ϕ.

Since I_1 and I_2 are non-degenerate intervals on ∂Q, let $I_1 = \partial Q[p_1, q_1]$ and $I_2 = \partial Q[p_2, q_2]$ for some points $p_1, q_1, p_2, q_2 \in \partial Q$ with $p_1 \neq q_1$ and $p_2 \neq q_2$. The portion of $\partial \mathbf{Z}_\lambda$ contained in ∂Q is $\partial Q[q_1, p_2]$. Let $r_1, r_2 \in \partial Q^\phi$ be the endpoints of the portion of $\partial \mathbf{Z}_\lambda$ lying on ∂Q^ϕ, which we denote by $\partial Q^\phi[r_1, r_2]$. Finally, the parts of $\partial \mathbf{Z}_\lambda$ along the edges of C_j and $C_{j'}$ correspond to the segments $\overline{q_1 r_1}$ and $\overline{p_2 r_2}$, respectively. Thus, $\partial \mathbf{Z}_\lambda$ decomposes into the four parts $\partial Q[q_1, p_2], \partial Q^\phi[r_1, r_2], \overline{q_1 r_1}$, and $\overline{p_2 r_2}$. This decomposition is shown in Figure 8(a).

Since H_i is convex, $\mathbf{Z}_\lambda \subseteq H_i$ if and only if all parts of $\partial \mathbf{Z}_\lambda$ are contained in H_i. Note that $\overline{q_1 r_1}$ and $\overline{p_2 r_2}$ lie in H_i, as both C_j and $C_{j'}$ are contained in P_i. To prove $\mathbf{Z}_\lambda \subseteq H_i$, it remains to show that $\partial Q[q_1, p_2]$ and $\partial Q^\phi[r_1, r_2]$ are contained in H_i.

Containment of $\partial Q[q_1, p_2]$ in $\mathsf{int}(P)$. During the transition between ∂C_j and $\partial C_{j'}$ in the counterclockwise traversal on ∂P_i, it follows a simple path, denoted by γ, which lies outside $\mathsf{int}(Q)$. The path γ exits and re-enters $\mathsf{int}(Q)$ through the endpoints of I_1 and I_2.

Let v_1 and v_2 denote the vertices in G_i corresponding to I_1 and I_2, respectively. The link instruction $\lambda = (I_1, I_2, k')$ is derived from the directed edge (v_1, v_2) in G_i. The direction of the edge is determined by how γ winds around ∂Q (either clockwise or counterclockwise) and whether the path exits $\mathsf{int}(Q)$ through the endpoint of I_1 or that of I_2. If γ winds around ∂Q counterclockwise, it exits $\mathsf{int}(Q)$ from p_1 and re-enters at q_2. Otherwise, it exits $\mathsf{int}(Q)$ from p_2 and re-enters at q_1. Figure 8(a) illustrates both cases.

Up to this point, we consider both clockwise- and counterclockwise-type instructions. However, counterclockwise ones can be omitted in the reconfiguration. Assume that γ winds around ∂Q counterclockwise. By Corollary 7, there exists a unique component $X \in \mathcal{X}_i$ whose boundary contains I_1, I_2, and the path γ. Since X is a weakly simple polygon, we can traverse ∂X in counterclockwise order. This traversal encounters a sequence of circular intervals in $\mathcal{I}_i^+$, and, from the construction of G_i, each consecutive pair of intervals in this sequence corresponds to an edge in G_i whose direction is determined by whether the subpath of ∂X between the intervals winds around ∂Q clockwise or counterclockwise.

Since the traversal of ∂X follows the path γ from I_1 to I_2 that winds around ∂Q counterclockwise, the other subpath of ∂X runs from I_2 back to I_1, and winds around ∂Q clockwise. This path encounters a sequence of intervals in $\mathcal{I}_i^+$ starting from I_2 to I_1, where each consecutive pair of intervals induces a clockwise-type edge in G_i. In Figure 6(b), the counterclockwise-type path from I_4 to I_1 corresponds to the sequence of clockwise-type paths $I_1 \to I_5$ and $I_5 \to I_4$. Thus, link instructions associated with counterclockwise-type edges can be omitted without affecting the resulting partition $\Pi^*[Q]$.

Without loss of generality, we restrict our analysis to clockwise type instructions. In this case, γ is a path from p_2 to q_1. The path γ can be continuously deformed into a path $\tilde{\gamma} = \partial Q[q_1, p_2]$ on ∂Q while preserving its endpoints. As illustrated in Figure 8(b), if ∂P intersects Q at a point $z \in \partial Q[q_1, p_2]$, then γ must pass through z. This implies that ∂P_i intersects itself at z, contradicting that P_i is a simple polygon.

▶ **Lemma 14.** *All points on $\partial Q[q_1, p_2]$ lie within the interior of P.*

Note that a point $p \in \mathsf{int}(P)$ if and only if there exists a sufficiently small ball $B(p)$ centered at p such that $B(p) \subseteq \mathsf{int}(P)$. We slightly extend the polygonal chain $\partial Q[q_1, p_2]$ beyond its endpoints and denote the resulting chain by $\partial Q[q_1^-, p_2^+]$ for $q_1^- \in \partial Q(p_1, q_1)$ and $p_2^+ \in \partial Q(p_2, q_2)$. Since $\partial Q[q_1, p_2] \subseteq \mathsf{int}(P)$ by Lemma 14, $\partial Q[q_1^-, p_2^+]$ also lies in $\mathsf{int}(P)$. Furthermore, as I_1 and I_2 are non-degenerate, the slight extension guarantees that $\partial Q[q_1^-, p_2^+] \subseteq \partial Q[p_1, q_2]$.

Turning points on $\partial Q[q_1^-, p_2^+]$. Without loss of generality, assume H_i is a vertical strip, meaning that its boundary consists of two vertical lines. A portion of ∂Q is a polygonal chain, consisting of a sequence of line segments. As we traverse a polygonal chain from one endpoint to the other, these line segments are encountered sequentially. A point $z \in \partial Q(q_1^-, p_2^+)$ is called a *turning point* of $\partial Q[q_1^-, p_2^+]$ if the traversal changes its horizontal direction (from leftward to rightward or vice versa) at z. Since q_1^- and p_2^+ lie sufficiently close to q_1 and p_2 along ∂Q, every turning must occur on $\partial Q[q_1, p_2]$.

When $\partial Q[q_1^-, p_2^+]$ contains vertical edges, the above definition of turning points is not sufficient. Consider a path that initially moves in the positive (or negative) x-direction, then follows a vertical segment, and subsequently moves in the negative (or positive) x-direction. We define the lowest point on the vertical segment as the unique turning point on that segment. See Figure 9(a).

Note that the number of turning points is invariant under the traversal direction; that is, it remains unchanged whether we traverse $\partial Q[q_1^-, p_2^+]$ from q_1^- to p_2^+ or in the reverse direction. As illustrated in Figure 9(b), if there are two turning points on $\partial Q[q_1^-, p_2^+]$, we can draw a vertical guillotine cut c in P such that $c \cap Q$ is disconnected, since all points on $\partial Q[q_1^-, p_2^+]$ lie within $\mathsf{int}(P)$. This implies that Q cannot be $\overline{W}$-convex with respect to P.

▶ **Lemma 15.** *If Q is $\overline{W}$-convex with respect to P, then the number of turning points on $\partial Q[q_1^-, p_2^+]$ is at most one.*

Since both $\partial Q[p_1, q_1]$ and $\partial Q[p_2, q_2]$ lie in P_i, $q_1^-, p_2^+ \in H_i$. If there are no turning points on $\partial Q[q_1^-, p_2^+]$, then the points with the largest and smallest x-coordinates along $\partial Q[q_1^-, p_2^+]$ appear at q_1^- and p_2^+. Since H_i is a vertical slab, all points on $\partial Q[q_1^-, p_2^+]$ lie in H_i.

Consider the case that $\partial Q[q_1^-, p_2^+]$ has a turning point at $z \in \partial Q[q_1, p_2]$, and it is unique by Lemma 15. Then, the point with the largest or smallest x-coordinate along $\partial Q[q_1^-, p_2^+]$ may appear at z. Without loss of generality, we assume that z is the point with the largest x-coordinate. The point with the smallest x-coordinate lies at $q_1^- \in H_i$ or $p_2^+ \in H_i$.

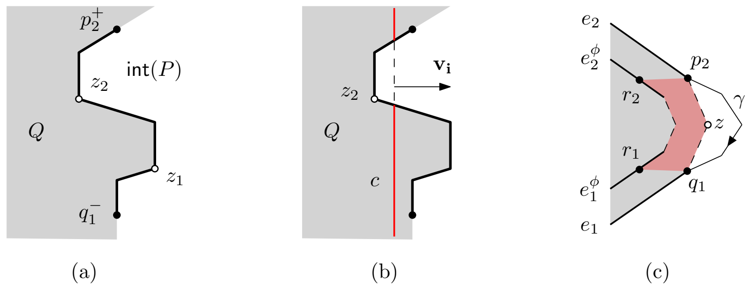

Figure 9 (a) The turning points on $\partial Q[q_1^-, p_2^+]$ are z_1 and z_2. (b) For the guillotine cut c along $x = x(z) + \varepsilon$, the intersection $c \cap Q$ consists of at least two maximal line segments. (c) γ contains a point with x-coordinate larger than that of the turning point z. The sequence of edges on $\partial Q^\phi[r_1, r_2]$ from e_1^ϕ to e_2^ϕ corresponds to a subsequence of those of $\partial Q[p_1, q_2]$.

Recall that the path γ follows from p_2 to q_1 outside $\mathsf{int}(Q)$ and is deformed into the path $\tilde{\gamma}$ that traverses $\partial Q[q_1, p_2]$. Note that z is a convex vertex of Q with locally largest x-coordinate, and γ encloses z from outside $\mathsf{int}(Q)$. It follows that γ must pass through a point with x-coordinate at least $x(z)$. Since $\gamma \subseteq P_i \subseteq H_i$, the x-coordinate of z is smaller than that of the right boundary of H_i. Thus, $\partial Q[q_1^-, p_2^+] \subseteq H_i$. See Figure 9(c).

Containment of $\mathbf{Z}_\lambda$ within H_i. Revisiting the boundary of $\mathbf{Z}_\lambda$, we have shown that three parts, $\overline{r_1q_1}, \overline{r_2p_2}$, and $\partial Q[q_1, p_2]$, are contained in H_i. The remaining part is $\partial Q^\phi[r_1, r_2]$ which is the portion of the inner ϕ-offset polygon of Q. By definition of the offset polygon, each edge of Q^ϕ is parallel to its corresponding edge in Q and the edges of Q^ϕ appear in the same cyclic order along its boundary as the edges of Q.

We traverse the chain $\partial Q[q_1^-, p_2^+]$ from q_1^- to p_2^+. Let e_1 and e_2 be the edges of Q that contain the first and last segments of this chain, respectively. Likewise, traversing $\partial Q^\phi[r_1, r_2]$ from r_1 to r_2 gives edges e_1^ϕ, e_2^ϕ of Q^ϕ incident to r_1 and r_2, respectively. We claim that the edges of Q corresponding to e_1^ϕ and e_2^ϕ appear along the traversal of $\partial Q[q_1^-, p_2^+]$.

The segments $\overline{r_1q_1}$ and $\overline{r_2p_2}$ lie on edges of C_j and $C_{j'}$, respectively. Since $\phi < \phi_{ij}$ and $\phi < \phi_{ij'}$, the edges of Q corresponding to e_1^ϕ and e_2^ϕ are incident to q_1 and p_2, respectively. Recall that we work on the extended chain $\partial Q[q_1^-, p_2^+]$, obtained by slightly extending $\partial Q[q_1, p_2]$. This guarantees that the corresponding edges of Q appear along $\partial Q[q_1^-, p_2^+]$. Consequently, the sequence of line segments forming $\partial Q^\phi[r_1, r_2]$ corresponds to a subsequence of those forming $\partial Q[q_1^-, p_2^+]$. See Figure 9(c) for an illustration of this correspondence. Thus, by Lemma 15, the number of turning points of $\partial Q^\phi[r_1, r_2]$ is also at most one.

Assuming that the largest x-coordinate of $\partial Q[q_1^-, p_2^+]$ occurs at its turning point, the largest x-coordinate of $\partial Q^\phi[r_1, r_2]$ is smaller than that of $\partial Q[q_1^-, p_2^+]$. The argument is symmetric when the smallest x-coordinate is attained at the turning point. It follows that $\partial Q^\phi[r_1, r_2]$ is also contained in H_i, which completes the proof that $\partial \mathbf{Z}_\lambda \subseteq H_i$. Hence, the reconfigured partition $\Pi^*[Q] = \{Q_1^*, \ldots, Q_m^*\}$ is a solution to the problem $\mathsf{wPartition}(Q, W, U)$ with at most m connected pieces; thus, by definition, $\mathsf{opt}(Q) \leq m = \mathsf{opt}(P)$.

6 Bang-type theorem for partitions of a convex body

We adapt the reconfiguration technique in Section 4 to prove Theorem 1. We then show that, when $\overline{W} \subseteq U$, an optimal partition of a convex polygon P is achieved by equally spaced parallel cuts, which can be computed in linear time.

Let K be a convex body in $\mathbb{R}^2$, and let $P_1 \cup P_2 \cup \cdots \cup P_m$ be its arbitrary partition. Note that each P_i is compact and possibly non-convex. Let $\mathrm{CH}(X)$ denote a convex hull of a set X in $\mathbb{R}^2$. A pocket of $\mathrm{CH}(X)$ is defined as a closure of a connected component of $\mathrm{CH}(X) \setminus X$. Each pocket is bounded by a subpath along ∂P_i and a unique line segment lying outside P_i. We refer to this segment as the *hull-edge* of the pocket.

To convexify P_i, we iteratively reallocate its pockets to P_i. However, such a reallocation may split other pieces P_j with $j \neq i$. Accordingly, we perform a reconfiguration step to merge such fragments into a single piece in K, ensuring that each piece remains connected.

This configuration mirrors Section 4, but with spatial roles reversed. Previously, we considered the restriction of a partition to a subpolygon Q, and reconnected the fragments of other pieces within Q. Here, we restrict the partition to the complement of a pocket, and consider the fragments of other pieces that lie outside the pocket. The circular intervals in the earlier setting now correspond to the intervals along the hull-edge of the pocket.

▶ **Lemma 16.** *Let $\{P_1, \ldots, P_m\}$ be a partition of a convex body $K \subseteq \mathbb{R}^2$. Then, there exists a convex partition $K = P_1^* \cup \cdots \cup P_m^*$, such that $\omega_{\mathbf{v}}(P_i) \geq \omega_{\mathbf{v}}(P_i^*)$ for all $\mathbf{v} \in \mathbb{S}^+$ and $i \in [m]$.*

By Lemma 16, we have a convex partition $K = P_1^* \cup \cdots \cup P_m^*$ such that $\omega_{\mathbf{v}}(P_i) \geq \omega_{\mathbf{v}}(P_i^*)$ for all $\mathbf{v} \in \mathbb{S}^+$ and $i \in [m]$. Given this convex partition, Akopyan [4] showed that $\sum_{i=1}^m r_K(P_i^*) \geq 1$, where $r_K(P_i^*) = \sup\{h \geq 0 \mid \exists t \in \mathbb{R}^2$ such that $hK + t \subseteq P_i^*\}$. For any direction $\mathbf{v} \in \mathbb{S}^+$, we have $r_K(P_i^*) \leq \omega_{\mathbf{v}}(P_i^*)/\omega_{\mathbf{v}}(K)$. Thus, for any subset $W \subseteq \mathbb{S}^+$, $\sum_{i=1}^m \inf_{\mathbf{v}\in W} \frac{\omega_{\mathbf{v}}(P_i)}{\omega_{\mathbf{v}}(K)} \geq \sum_{i=1}^m \inf_{\mathbf{v}\in W} \frac{\omega_{\mathbf{v}}(P_i^*)}{\omega_{\mathbf{v}}(K)} \geq \sum_{i=1}^m r_K(P_i^*) \geq 1$.

Optimal partition for a convex polygon. Let P be a convex polygon with n vertices, and let $W, U \subseteq \mathbb{S}^+$ such that $\overline{W} \subseteq U$. Choose an arbitrary vector $\mathbf{u} \in W$; without loss of generality, assume that $\mathbf{u} = (1, 0)$. Let q be the leftmost vertex of P, and partition P by vertical lines along $x_i = x(q) + i$ for all $i = 1, \ldots, \lceil \omega_{\mathbf{u}}(P) \rceil - 1$. This partitions P into $\lceil \omega_{\mathbf{u}}(P) \rceil$ pieces, each of horizontal width at most 1, and it is a feasible solution to $\mathsf{wPartition}(P, W, U)$. Since $\mathbf{u}$ is chosen arbitrarily, $\mathsf{opt}(P) \leq \min_{\mathbf{v}\in W} \lceil \omega_{\mathbf{v}}(P) \rceil$.

Suppose, for the sake of contradiction, that the optimal partition has fewer than $\min_{\mathbf{v}\in W} \lceil \omega_{\mathbf{v}}(P) \rceil$ pieces. Let $P = P_1 \cup \cdots \cup P_m$ be an optimal partition for $\mathsf{wPartition}(P, W, U)$, with $m = \mathsf{opt}(P)$. By Theorem 1, we have $1 \leq \sum_{i=1}^m \inf_{\mathbf{v}\in W}(\omega_{\mathbf{v}}(P_i)/\omega_{\mathbf{v}}(P))$. Since each P_i satisfies unit-width constraint W, there exists $\mathbf{u}_i \in \mathbb{S}^+$ such that $\omega_{\mathbf{u}_i}(P_i) \leq 1$. Thus, $1 \leq \sum_{i=1}^m \inf_{\mathbf{v}\in W}(\omega_{\mathbf{v}}(P_i)/\omega_{\mathbf{v}}(P)) \leq \sum_{i=1}^m (\omega_{\mathbf{u}_i}(P_i)/\omega_{\mathbf{u}_i}(P)) \leq \sum_{i=1}^m (1/\omega_{\mathbf{u}_i}(P))$.

We analyze two cases depending on whether $\omega_{\mathbf{v}}(P)$ attains a minimum over W. If it does, we have $1 \leq \sum_{i=1}^m (1/\min_{\mathbf{v}\in W} \omega_{\mathbf{v}}(P))$, and $\min_{\mathbf{v}\in W} \omega_{\mathbf{v}}(P) \leq m$. As m is an integer, $\min_{\mathbf{v}\in W} \lceil \omega_{\mathbf{v}}(P) \rceil \leq m$. If no minimum is attained over W, $1 < \sum_{i=1}^m (1/\inf_{\mathbf{v}\in W} \omega_{\mathbf{v}}(P))$ and $\inf_{\mathbf{v}\in W} \omega_{\mathbf{v}}(P) < m$. Then $\min_{\mathbf{v}\in W} \lceil \omega_{\mathbf{v}}(P) \rceil = \inf_{\mathbf{v}\in W} \lceil \omega_{\mathbf{v}}(P) \rceil \leq m$. Both cases contradict our assumption, and thus $\mathsf{opt}(P) = \min_{\mathbf{v}\in W} \lceil \omega_{\mathbf{v}}(P) \rceil$.

Let $\mathbf{u} \in W$ be a vector minimizing $\lceil \omega_{\mathbf{u}}(P) \rceil$, computed in $O(n)$ time for $W = \mathbb{S}^+$ [18]. Assume that $\mathbf{u}$ is given. For the vertices of P given in counterclockwise order, such $m - 1$ parallel cuts can be computed in $O(\min\{n, m \log \frac{n}{m}\})$ time [12]. Since $x/(1+x) < \log(1+x) < x$ for all $x > 0$, $\min\{n, m \log \frac{n}{m}\} = \Theta(m \log(1 + \frac{n}{m}))$. For fixed n, $f(m) = m \log(1 + \frac{n}{m})$ starts at $\Theta(\log n)$ when $m = 1$ and increases monotonically, approaching $\Theta(n)$ as $m \to \infty$.

▶ **Corollary 17.** *Let P be a convex polygon with n vertices, and let $U, W \subseteq \mathbb{S}^+$ be sets of unit vectors such that $\overline{W} \subseteq U$. Then an optimal partition for the problem $\mathsf{wPartition}(P, W, U)$ is achieved by equally spaced parallel cuts orthogonal to $\mathbf{u} \in W$ that minimizes $\lceil \omega_{\mathbf{u}}(P) \rceil$. Given such a direction $\mathbf{u}$, the partition can be computed in $O(\omega_{\mathbf{u}}(P) \log(1 + \frac{n}{\omega_{\mathbf{u}}(P)}))$ time.*

Remarks. We work in the Real RAM model, which supports unit-cost arithmetic $(+, -, \times, \div)$ and comparisons on real numbers. Our algorithms perform integer rounding via comparisons.

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