

Enumerating the Irreducible Closed Sets of an Acyclic Implicational Base of Bounded Degree

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Abstract

We consider the problem of enumerating the irreducible closed sets of a closure system given by an implicational base. To date, the complexity status of this problem is widely open, and it is further known to generalize the notorious hypergraph dualization problem, even in the case of acyclic convex geometries, i.e., closure systems admitting an acyclic implicational base. This paper studies this case with a focus on the degree, which corresponds to the maximal number of implications in which an element occurs. We show that the problem is tractable for bounded values of this parameter, even when relaxed to the notions of premise- and conclusion-degree. Our algorithms rely on a sequential approach leveraging from acyclicity, combined with the solution graph traversal technique for the case of premise-degree. They are shown to perform in incremental-polynomial time. These results are complemented in the long version of this document by showing that the dual problem of constructing the implicational base can be solved in polynomial time. Finally, we argue that our running times cannot be improved to polynomial delay using the standard framework of flashlight search.

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1 Introduction

Finite closure systems appear in disguise in several areas of mathematics and computer science, including database theory [29, Chapter 13], Horn logic [12, Chapter 6], or lattice theory [13, Chapter 2], to mention but a few.

A closure system is a family of subsets of some finite ground set being closed by intersection and containing the ground set. The study of the various representations of closure systems and the algorithmic task of translating between them have received lots of attention in these respective fields, and have been the topic of the Dagstuhl Seminar 14201 [1]. Most importantly, the translation task we consider in this paper is known for being a considerable generalization of the notorious and widely open hypergraph dualization problem, while



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also not being known to be intractable, contrary to analogous other natural generalizations such as lattice dualization [4, 14, 26]. We refer the reader to the surveys [7, 33, 39] for more information on this topic.

Among the several representations of closure systems, two of them occupy a central role, namely, *irreducible closed sets* and *implicational bases*, whose formal definitions are postponed to Section 2. Informally, irreducible closed sets are elementary sets of the closure system from which every other can be recovered. On the other hand, implicational bases are sets of implications of the form $A \rightarrow B$, where A , the premise, and B , the conclusion, are subsets of the ground set. As noted in [28], these two representations are generally incomparable in size and usefulness. Notably, there exist instances where the minimum cardinality of an implicational bases is exponential in the number of irreducible closed sets, and vice versa. Moreover, the computational complexity of some algorithmic tasks such as reasoning, abduction, or recognizing lattice properties can differ significantly depending on the representation at hand [3, 25, 29]. These observations motivate the problem of translating between the two representations, in which one representation is given, and the other is to be computed. In this extended abstract, a focus is made on the problem of enumerating irreducible closed sets, denoted by ICS-ENUM; the dual problem of constructing the implicational base is only addressed in the long version of this work.

Because of the possible exponential gap between the sizes of the two representations, input sensitive complexity is not a meaningful efficiency criterion when analyzing the performance of an algorithm solving ICS-ENUM. Instead, the framework of algorithmic enumeration in which an enumeration algorithm has to list without repetitions a set of solutions, is usually adopted. The complexity of an algorithm is then analyzed from the perspective of *output-sensitive complexity*, which estimates the running time of the algorithm using both input and output sizes. In that context, an algorithm is considered tractable when it runs in *output-polynomial* time, that is, if its execution time is bounded by a polynomial in the combined sizes of the input and the output. This notion can be further constrained to guarantee some regularity in the enumeration. Namely, we say that an algorithm runs in *incremental-polynomial time* if it outputs the i^{th} solution in a time which is bounded by a polynomial in the input size plus i . It is said to run with *polynomial delay* if the time spent before the first output, after the last output, and between consecutive outputs is bounded by a polynomial in the input size only. We refer the reader to [24, 35] for more details on enumeration complexity.

In this work, we investigate *acyclic convex geometries*, a particular class of closure systems. They are the convex geometries – closure systems with the anti-exchange property – that admit an implicational base with acyclic implication-graph. Acyclic convex geometries have been extensively studied [8, 15, 23, 34, 37]. Even in this class though, the problem ICS-ENUM was shown in [15] to be harder than distributive lattices dualization, a generalization of hypergraph dualization. On the other hand, the authors in [15] show that if, in addition to acyclicity, the implication graph admits a rank function, then the problem can be solved using hypergraph dualization. Using this result, they derive a tractable algorithm for so-called ranked convex geometries admitting an implicational base of bounded premise size. It should be noted that, even under these restrictions, the problem generalizes the one of enumerating all maximal independent sets of a graph (in fact of hypergraphs of bounded edge size) a central and notorious problem in enumeration [24, 36]. These results have then been extended to a more general setting in [32]. Other related results in the literature include the work [5] which deals with k -meet-semidistributive lattices, and closure systems of bounded Caratheodory number, see e.g., [39].

In this paper, we continue the line of research of characterizing tractable cases of acyclic convex geometries, by identifying for which parameters k one can derive tractable algorithms for ICS-ENUM for fixed values of k . We note that in this research, finding an algorithm running in quasi-polynomial time should already be considered a success, since no sub-exponential time algorithms are known to date. We focus on the degree, which is defined as the maximum number of implications in which an element participates, and its two obvious relaxations which are the premise- and conclusion-degree; these parameters are defined in Section 2. The principal message of this work is that bounding one of these two relaxed notions of degree suffices to provide not only quasi-polynomial but polynomial algorithms for the translation. More specifically, as the first of our two main theorems, we obtain the following, whose quantitative formulation is postponed to Sections 4 and 5.

► **Theorem 1.** *There is an incremental-polynomial time algorithm enumerating the irreducible closed sets of an acyclic convex geometry given by an acyclic implicational base of bounded premise- or conclusion-degree.*

Our algorithms for acyclic convex geometries of bounded degree rely on several steps which combine structure and enumeration techniques. We give a rough description which highlight the techniques used, and refer to the appropriate sections for more details.

First, we use the fact that each irreducible closed set is *attached* to an element of the ground set, and that this element is unique in convex geometries; this is detailed in Section 2. Then, leveraging from acyclicity, our algorithm relies on a sequential procedure which consists of enumerating, incrementally and in a top-down fashion according to a topological ordering $x_1, \dots, x_n$ of the elements of the ground set, all the irreducible closed sets attached to x_i . This step can be seen as enumerating the irreducible closed set attached to x_i with additional information as an input, namely, the irreducible closed sets of ancestor elements. Finally, it remains to solve this later task, which we can do either by brute forcing minimal transversals of a carefully designed hypergraph in case of bounded conclusion-degree, or performing a solution graph traversal in case of bounded premise-degree. In the end, the resulting algorithms run in incremental-polynomial time, where the dependence in the number of solutions principally arises from the sequential top-down procedure.

Let us note that Theorem 1 also has implications on the dual problem of constructing an implicational base of minimum cardinality. Namely it allows us to obtain the following result whose description is detailed in the long version of this work [16] and included here for the reader familiar with closure systems.

► **Theorem 2.** *There is a polynomial-time algorithm constructing a minimum implicational base of an acyclic convex geometry of bounded premise- or conclusion-degree given by its irreducible closed sets.*

We note that in this direction, our algorithm is (input) polynomial thanks to the fact that the output has polynomial cardinality because of the bounded degree.

The remainder of the paper is organized as follows. Section 2 introduces the concepts and notations used throughout the paper. Section 3 presents the top-down approach that we use as the key ingredient in our algorithms. Sections 4 and 5 are devoted to proving Theorem 1 in the respective cases of bounded conclusion- and premise-degree. In Section 6 we discuss possible generalizations and open problems related to the current work, and also argue that our running times cannot be improved to polynomial delay using the standard framework of flashlight search.

Due to space constraints, the proofs of our different statements are not included in the present extended abstract: the interested reader is referred to [16] for more details.

2 Preliminaries

All the objects we consider in this paper are finite. If X is a set, $\mathcal{P}(X)$ is its powerset. The size of X is denoted $|X|$. A *set system*, or *hypergraph*, over a ground set X is a pair $(X, \mathcal{F})$ where $\mathcal{F}$ is a (non-empty) collection of subsets of X , that is $\mathcal{F} \subseteq \mathcal{P}(X)$. Its *size* is $|X| + |\mathcal{F}|$; note that the binary length of any standard encoding of a hypergraph is polynomially related to its size. Thus, for simplicity, we shall consider this measure for families as input sizes in the different problems we will consider. If X is clear from the context, we may identify $\mathcal{F}$ with $(X, \mathcal{F})$. In examples, if there is no ambiguity, we may write a set as the concatenation of its elements, for instance 123 instead of $\{1, 2, 3\}$. Similarly, to avoid cumbersome notations, we sometimes withdraw brackets when dealing with singletons, e.g., we may write say $\phi(x)$ instead of $\phi(\{x\})$.

Independent and hitting sets, hypergraph parameters. Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph. We call *vertices* of $\mathcal{H}$ the elements in X , and *edges* the sets in $\mathcal{E}$. We say that $\mathcal{H}$ is *Sperner* if any two of its edges are pairwise incomparable by inclusion. An *independent set* of $\mathcal{H}$ is a set $I \subseteq X$ that contains no edges of $\mathcal{H}$, i.e., $E \not\subseteq I$ for every E in $\mathcal{E}$. The family of inclusion-wise maximal independent sets of $\mathcal{H}$ is denoted $\text{MIS}(\mathcal{H})$. Formally, $\text{MIS}(\mathcal{H}) = \max_{\subseteq} \{I : I \subseteq X, E \not\subseteq I \text{ for every } E \text{ in } \mathcal{E}\}$. Dually, a *hitting set*, or *transversal*, of $\mathcal{H}$ is a subset T of X that intersects each edge of $\mathcal{H}$. The collection of inclusion-wise minimal hitting sets of $\mathcal{H}$ is $\text{MHS}(\mathcal{H}) = \min_{\subseteq} \{T : T \subseteq X, T \cap E \neq \emptyset \text{ for every } E \text{ in } \mathcal{E}\}$. The hypergraph $\text{MHS}(\mathcal{H})$ is usually called the *dual hypergraph* of $\mathcal{H}$. Observe that $I \subseteq X$ is independent if and only if $T := X \setminus I$ is a hitting set. Consequently, $\text{MHS}(\mathcal{H}) = \{X \setminus I : I \in \text{MIS}(\mathcal{H})\}$. Moreover, if $\mathcal{H}$ is not Sperner, then its minimal hitting sets (resp. maximal independent sets) are precisely those of the hypergraph formed by the inclusion-wise minimal edges of $\mathcal{H}$. The two equivalent generation problems associated to maximal independent sets and minimal hitting sets read as follows:

Maximal Independent Sets Enumeration (MIS-ENUM)

Input: A (Sperner) hypergraph $\mathcal{H} = (X, \mathcal{E})$.

Output: the family $\text{MIS}(\mathcal{H})$.

Minimal Hitting Sets Enumeration (MHS-ENUM)

Input: A (Sperner) hypergraph $\mathcal{H} = (X, \mathcal{E})$.

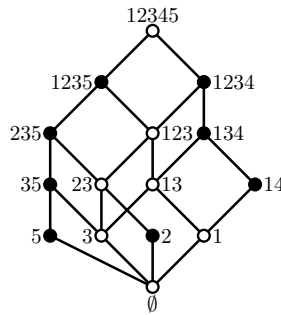
Output: the family $\text{MHS}(\mathcal{H})$.

To date, the best algorithm for these tasks runs in output-quasi-polynomial time and is due to Fredman and Khachiyan [21]. However, a number of output-polynomial time algorithms have been proposed for particular cases, especially when bounding structural parameters [19], in particular the *dimension* of $\mathcal{H}$ being the size of its largest edge, and the *dual dimension*, being the dimension of the dual hypergraph of $\mathcal{H}$.

Closure systems, closure spaces. A *closure system* is a set system $(X, \mathcal{C})$ that satisfies $X \in \mathcal{C}$, and $C \cap C' \in \mathcal{C}$ whenever $C, C' \in \mathcal{C}$. The sets in $\mathcal{C}$ are called *closed (sets)*. A *closure space* is a pair (X, ϕ) where $\phi: \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is a mapping that satisfies for every $A, B \subseteq X$: (1) $A \subseteq \phi(A)$, (2) $\phi(A) \subseteq \phi(B)$ if $A \subseteq B$, and (3) $\phi(\phi(A)) = \phi(A)$. The map ϕ is a *closure operator*. Closure spaces and closure systems are in one-to-one correspondence. More precisely, if (X, ϕ) is a closure space, the family $\mathcal{C} = \{C : C \subseteq X, \phi(C) = C\}$ defines a closure system. Dually, if $(X, \mathcal{C})$ is a closure system, the map ϕ defined by $\phi(A) = \min_{\subseteq} \{C : C \in \mathcal{C}, A \subseteq C\} = \bigcap \{C : C \in \mathcal{C}, A \subseteq C\}$ is a closure operator whose

associated closure system is precisely $(X, \mathcal{C})$. In this paper, if (X, ϕ) is a closure space, we call $(X, \mathcal{C})$ the corresponding closure system and vice-versa. Moreover, we assume without loss of generality that $\emptyset \in \mathcal{C}$, a standard assumption.

Let $(X, \mathcal{C})$ be a closure system. The poset $(\mathcal{C}, \subseteq)$ of closed sets ordered by inclusion is a (*closure*) *lattice* where $C \wedge C' = C \cap C'$ and $C \vee C' = \phi(C \cup C')$. Let $M \in \mathcal{C}$, $M \neq X$. The closed set M is *irreducible* if it is not the intersection of other distinct closed sets, that is, $M = C \cap C'$ for some $C, C' \in \mathcal{C}$ only if $C = M$ or $C' = M$. We denote by $\text{irr}(\mathcal{C})$ the family of irreducible closed sets of $(X, \mathcal{C})$. For every $C \in \mathcal{C}$, we have $C = \bigcap \{M : M \in \text{irr}(\mathcal{C}), C \subseteq M\}$. Moreover, $\text{irr}(\mathcal{C})$ is the unique minimal subset of $\mathcal{C}$ from which $\mathcal{C}$ can be reconstructed using intersections. Let $x \in X$. We denote by $\text{irr}(x)$ the inclusion-wise maximal closed sets not containing x , i.e., $\text{irr}(x) = \max_{\subseteq} \{M : M \in \mathcal{C}, x \notin M\}$. Observe that $\text{irr}(x) \subseteq \text{irr}(\mathcal{C})$ for every $x \in X$. An irreducible closed set M in $\text{irr}(x)$ is said to be *attached* to x . In fact, it is well known that $\text{irr}(\mathcal{C}) = \bigcup_{x \in X} \text{irr}(x)$ (see, e.g., [29, 38]) and this union will even be disjoint for us (see Proposition 3). An example of a closure system and its irreducible closed sets is depicted in Figure 1.



■ **Figure 1** A closure system over $X = \{1, 2, 3, 4, 5\}$ represented via the Hasse diagram of its closure lattice. Black dots are irreducible closed sets which can be seen by the fact that they have a unique successor. We have, for instance, $\phi(25) = 235$ and $\phi(4) = 14$, $\text{irr}(3) = \{5, 2, 14\}$ and $\text{mingen}(3) = \{25, 15, 24, 45\}$. An implicational base for this closure system is for instance $\Sigma = \{4 \rightarrow 1, 25 \rightarrow 3, 12 \rightarrow 3, 15 \rightarrow 23\}$.

Implicational bases. An *implication* over X is a statement $A \rightarrow B$ where A, B are subsets of X . In $A \rightarrow B$, A is the *premise* and B the *conclusion*. An *implicational base* (IB) is a pair (X, Σ) where Σ is a collection of implications over X . Given an IB (X, Σ) , $|\Sigma|$ is the cardinality of Σ , that is the number of implications in Σ . The *size* of the IB is the value $|X| + |\Sigma|$. The *degree* of an element $x \in X$, denoted $\deg(x)$, is the number of implications of Σ in which x appears. We refine this notion by defining the *premise-degree* of x as $\text{pdeg}(x) := |\{A : A \rightarrow B \in \Sigma, x \in A\}|$, and its *conclusion-degree* as $\text{cdeg}(x) := |\{A \rightarrow B : A \rightarrow B \in \Sigma, x \in B\}|$. Then $\deg(x) \leq \text{pdeg}(x) + \text{cdeg}(x)$. Equality may not hold in general because an element can belong to both premise and the conclusion of an implication. In this paper though, we avoid this degenerate case by always considering the premise and conclusion of an implication are disjoint. Moreover, we will mainly use the *degree*, *premise-degree*, and *conclusion-degree* of Σ defined by $\text{pdeg}(\Sigma) := \max_{x \in X} \text{pdeg}(x)$, $\text{cdeg}(\Sigma) := \max_{x \in X} \text{cdeg}(x)$, $\deg(\Sigma) := \max_{x \in X} \deg(x)$. Note that $\deg(\Sigma) \leq \text{pdeg}(\Sigma) + \text{cdeg}(\Sigma)$ for any Σ . As an example, for the implicational base given in Figure 1, we have $\text{pdeg}(2) = 2$ and $\text{cdeg}(2) = 1$.

We now relate IBs to closure systems. Let (X, Σ) be an IB. A set $C \subseteq X$ is *closed* for (X, Σ) if for every implication $A \rightarrow B \in \Sigma$, $A \subseteq C$ implies $B \subseteq C$. The pair $(X, \mathcal{C})$ where $\mathcal{C}$ is the family of closed sets of (X, Σ) is a closure system. An IB (X, Σ) thus induces a

closure operator $\phi(\cdot)$ that can be computed in polynomial time using the *forward chaining* algorithm. This procedure starts from a set Y and produces a sequence of sets $Y = Y_0 \subseteq Y_1 \subseteq \dots \subseteq Y_k = \phi(Y)$ such that $Y_i = Y_{i-1} \cup \bigcup \{B : A \rightarrow B \in \Sigma, A \subseteq Y_{i-1}, B \not\subseteq Y_{i-1}\}$ for $1 \leq i \leq k$. On the other hand, any closure system $(X, \mathcal{C})$ can be represented by several *equivalent* IBs. For instance, $\Sigma = \{A \rightarrow \phi(A) : A \subseteq X\}$ constitutes an IB for $(X, \mathcal{C})$ as well as $\Sigma' = \{C \cup \{x\} \rightarrow \phi(C \cup \{x\}) : C \in \mathcal{C}, x \notin C\}$ (recall that we assume $\emptyset \in \mathcal{C}$). If (X, Σ) is an IB, we use the notation $\text{irr}(\Sigma)$ to denote the irreducible closed sets of the associated closure system. An implicational base for the closure system of Figure 1 is given in Figure 2.

The *degree* of a closure system is the least integer k such that it admits an implicational bases of degree k . The premise- and conclusion-degree are defined analogously.

Convex geometries, acyclic convex geometries. A closure system $(X, \mathcal{C})$ is a *convex geometry* if for every closed set C distinct from X there exists $x \notin C$ such that $C \cup \{x\}$ is closed. Equivalently, $(X, \mathcal{C})$ is a convex geometry if the corresponding closure operator ϕ has the *anti-exchange* property: for every closed set C and each distinct $x, y \notin C$, $x \in \phi(C \cup \{y\})$ entails $y \notin \phi(C \cup \{x\})$. The closure system depicted in Figure 1 is a convex geometry. Convex geometries have been rediscovered in several fields and, as such, enjoy several characterizations [31]. In particular, they can be characterized by their irreducible closed sets, as recalled in the subsequent statement:

► **Proposition 3** (see e.g., [18, Theorem 2.4]). *A closure system $(X, \mathcal{C})$ is a convex geometry if and only if each irreducible closed set is attached to a unique element, i.e., $\{\text{irr}(x) : x \in X\}$ is a partition of $\text{irr}(\mathcal{C})$.*

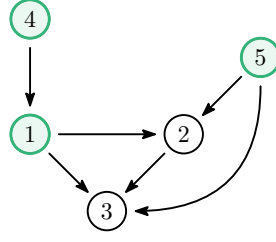
Let $T \subseteq X$. We call *irreducible selection* for T a family of sets obtained by choosing, for each $t \in T$, one $M \in \text{irr}(t)$. Note that since we are dealing with convex geometries, and by Proposition 3, the closed sets M are all distinct and the resulting family has the same cardinal as T . In the following, we denote by $\mathcal{S}(T)$ the family of all irreducible selections for a given set T . We will use irreducible selections as a way to forbid elements, as stated in the following observation.

► **Observation 4.** *For any $T \subseteq X$ and any selection $\mathcal{S} \in \mathcal{S}(T)$, the intersection $\bigcap \mathcal{S}$ of all sets in $\mathcal{S}$ is disjoint from T , i.e., it contains no $t \in T$.*

Let us now turn our attention to acyclic convex geometries. They have been studied under various names such as poset-type convex geometries [34], G -geometries [37], or as acyclic Horn functions from the perspective of Horn CNFs [8, 23]. Let (X, Σ) be an IB. The *implication-graph* [8, 23] of (X, Σ) is the directed graph $G(\Sigma) = (X, E)$ where there is an arc ab in E if there exists $A \rightarrow B$ in Σ such that $a \in A$ and $b \in B$. The IB (X, Σ) is *acyclic* if $G(\Sigma)$ is. In this case, $G(\Sigma)$ can be seen as a poset, and naturally induces an *ancestor* relation over X defined for all $x \in X$ as follows:

$$\text{anc}(x) = \{y : y \in X, x \neq y, \text{ and there exists a directed path from } y \text{ to } x \text{ in } G(\Sigma)\}$$

Equivalently, $\text{anc}(x)$ is the set of in-neighbors of x in the transitive closure of $G(\Sigma)$. An element x of X is thus maximal if it has no ancestors, and minimal if it has no descendants. As an example, Figure 2 illustrates the implication-graph of the IB given in Figure 1. A closure system that admits an acyclic IB is *acyclic* and it is a well-known fact that acyclic closure systems are convex geometries [34, 37].



■ **Figure 2** The implication-graph $G(\Sigma)$ of the IB (X, Σ) of Figure 1. This directed graph is acyclic, which makes the closure system an acyclic convex geometry. The ancestors of 2, highlighted in green, are 1, 4, 5, i.e., $\text{anc}(2) = \{1, 4, 5\}$. Maximal elements are 4, 5 and the unique minimal element is 3. Here, an example of *topological order* that we will refer to in the next section is 4, 1, 5, 2, 3.

Irreducible closed sets enumeration. To conclude the preliminaries, we formally present the enumeration problem we study in this paper and review its main aspects.

Irreducible Closed Sets Enumeration (ICS-ENUM)

Input: An implicational base (X, Σ) .

Output: The irreducible closed sets of the associated closure system.

Let us stress the fact that in ICS-ENUM, only an implicit (and possibly compact) description of the closure system is provided, namely, the implicational base. Typically, the output of ICS-ENUM may be of exponential size with respect to the input size, as illustrated in Example 5 below. Hence, the problem is studied through the lens of output-sensitive complexity.

► **Example 5.** Let $X = \{a_1, \dots, a_n\} \cup \{b_1, \dots, b_n\} \cup \{x\}$, and $\Sigma = \{a_i b_i \rightarrow x : 1 \leq i \leq n\}$. Denoting $(X, \mathcal{C})$ the associated closure system, we have $\text{irr}(\mathcal{C}) = \{X \setminus \{y\} : y \in X, y \neq x\} \cup \{\{c_1, \dots, c_n\} : c_i \in \{a_i, b_i\}, 1 \leq i \leq n\}$. Therefore, $|\text{irr}(\mathcal{C})| = 2^n + 2n$ while $|\Sigma| = n$.

As explained in the introduction, this problem has been studied in different fields [3, 17, 22, 28, 38] and its complexity remains unsettled to date. Yet it is harder than MIS-ENUM, in the sense that an output-polynomial time algorithm for one of ICS-ENUM implies one for MIS-ENUM. Moreover, ICS-ENUM reduces to the problem of enumerating the irreducible closed sets attached to a point, which we call ACS-ENUM, and that will be considered in this paper:

Attached Irreducible Closed Set Enumeration (ACS-ENUM)

Input: An implicational base (X, Σ) and $x \in X$.

Output: The family $\text{irr}(x)$ of irreducible closed sets attached to x .

Indeed, an algorithm solving ACS-ENUM in output-polynomial time can be applied to each element x to produce all irreducible closed sets of a closure system, with at most a linear number of repetitions in general, and no repetitions in convex geometries by Proposition 3. Unfortunately, it was shown by Kavvadias et al. [26] in the language of Horn CNFs that the problem ACS-ENUM admits no output-polynomial time algorithm unless $P = NP$.

Nevertheless, in the remaining of the paper, we will show that a variation of the problem taking profit of the acyclicity of the input allows us to obtain tractable algorithms for acyclic convex geometries of bounded degree.

3 Top-down procedure

We describe the general sequential procedure that we use to solve ICS·ENUM. We use a similar approach to tackle the dual problem of constructing the implicational base in the long version of this work [16]. Intuitively, it consists in enumerating, incrementally and in a top-down fashion according to a topological ordering $x_1, \dots, x_n$ of the implicational base, all the irreducible closed sets attached to x_i .

Recall that the ancestors of an element x in an acyclic implicational base are the elements that participate in a path of implications to x , and which can be identified in polynomial time; see Section 2. More formally, we show that ICS·ENUM reduces to the following version of ACS·ENUM in which we are given ancestors' irreducible closed sets as additional information.

Attached Closed Sets Enumeration with Ancestors' solutions (ACS+A·ENUM)

Input: An acyclic implicational base (X, Σ) , an element $x \in X$, and the family $\text{irr}(y)$ for every ancestor y of x in Σ .

Output: The set $\text{irr}(x)$.

Clearly, an algorithm for ACS·ENUM can be used to solve ACS+A·ENUM, while the reverse may not be true. We argue that by iteratively using an algorithm for ACS+A·ENUM on a maximal element of the implicational base among those that have not been selected yet, we obtain an algorithm for ICS·ENUM. Note that the base case corresponds to computing the irreducible closed set of maximal elements x verifying $\text{irr}(x) = \{X \setminus \{x\}\}$. Moreover, this procedure preserves incremental-polynomial bounds. In terms of difficulty, ACS+A·ENUM may thus be seen as standing as an intermediate problem between ACS·ENUM and ICS·ENUM in acyclic convex geometries. More formally, we obtain the following.

► **Theorem 6.** *Let $f: \mathbb{N}^2 \rightarrow \mathbb{N}$ be a function. Let us assume that there is an algorithm that, given an instance of ACS+A·ENUM of size N , produces its i^{th} solution in $f(N, i)$ -time. Then there is an algorithm that, given an acyclic instance of ICS·ENUM of size N' , produces its j^{th} solution in $O(\text{poly}(N') + f(N' + j, j))$ time.*

We note that an analogous statement to the one in Theorem 6 holds by relaxing the required and obtained time bounds to output-polynomial times. On the other hand, improving the complexity of an algorithm for ACS+A·ENUM to polynomial delay (or even input-polynomial as we will provide later for the case of bounded conclusion-degree) does not generally ensure polynomial delay (or incremental-linear) for ICS·ENUM, since the size of an instance of ACS+A·ENUM grows with the number of already obtained solutions for ICS·ENUM, which typically becomes exponential in the size of the implicational base.

Further limitations related to the top-down approach – in particular on the hardness of ACS·ENUM on implicational bases of height at most two – are discussed in the long version of this work [16]. Nevertheless, we shall argue in the remaining of the paper that this approach yields tractable algorithms when bounding structural parameters such as the premise- and conclusion-degree.

4 Bounded conclusion-degree

We show that the top-down approach successfully provides an incremental-polynomial time algorithm for ICS·ENUM whenever the implicational base has bounded conclusion-degree, proving one of the two cases of Theorem 1.

In the following, let us consider an instance of ACS+A·ENUM, i.e., we fix some acyclic implicational base (X, Σ) , an element $x \in X$, and assume that we are given $\text{irr}(y)$ for all ancestors y of x in Σ . Let us consider the hypergraph defined by

$$\mathcal{E}_x := \{A : A \rightarrow B \in \Sigma, x \in B\}, \text{ and} \\ \mathcal{H}_x := \left(\bigcup \mathcal{E}_x, \mathcal{E}_x \right).$$

The intuition behind the following lemma is that, in order to get an irreducible closed set attached to x , one should remove at least one element a in the premise of each implication having x in the conclusion, which can be done by considering the intersection with an irreducible closed set attached to a . Recall that $\mathcal{S}(T)$ denotes the family of irreducible selections; see Section 2.

► **Lemma 7.** *For every $M \in \text{irr}(x)$ there exists a minimal transversal T of $\mathcal{H}_x$ and an irreducible selection $\mathcal{S} \in \mathcal{S}(T)$ such that $M = (\bigcap \mathcal{S}) \setminus \{x\}$.*

Note that, by definition, the number of edges in $\mathcal{H}_x$ is exactly $\text{cdeg}(x)$, which is at most $\text{cdeg}(\Sigma)$. Moreover, since the ground set of $\mathcal{H}_x$ is a subset of the ancestors of x , the selections of Lemma 7 are part of the input we consider for ACS+A·ENUM. By brute-forcing the irreducible selections for every transversal, we derive a polynomial-time algorithm for ACS+A·ENUM whenever the conclusion-degree of x is bounded by a constant.

► **Theorem 8.** *There is an algorithm solving ACS+A·ENUM in $N^{O(k)}$ time on inputs of size N and maximum conclusion-degree k .*

We conclude to the following, as a corollary of Theorems 6 and 8, and proving the case of bounded-conclusion degree in Theorem 1.

► **Theorem 9.** *There is an incremental-polynomial time algorithm solving ICS·ENUM on acyclic implicational bases of bounded conclusion-degree.*

Let us remark that in our algorithm, we actually do not need the knowledge of $\text{irr}(a)$ for all ancestors of x , but only for those in a premise which has x in conclusion. This however makes no asymptotic difference on the final complexity we get for ICS·ENUM.

Also, it appears in the proof of Theorem 8 that we only need a constant upper bound on the size k of a minimal transversal of $\mathcal{H}_x$, to derive an input-polynomial time algorithm solving ACS+A·ENUM. We immediately derive the following generalization of Theorem 9 dealing with hypergraphs of bounded dual-dimension (admitting minimal transversals of bounded size) who define complements of conformal hypergraphs [6].

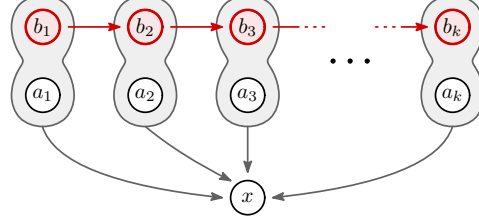
► **Theorem 10.** *There is an incremental-polynomial time algorithm solving ICS·ENUM if, for any element x of the ground set, the hypergraph of premises having x in their conclusion has bounded dual-dimension.*

We end the section by showing the limitations of this approach in order to get output-polynomial time bounds whenever we drop the condition of bounded conclusion-degree.

First, note that the key step of the approach is to compute all possible intersections as provided in Lemma 7. However, there are cases where the number of possible such intersections is exponential in $|\text{irr}(\Sigma)|$, which typically occurs if minimal transversals of $\mathcal{H}_x$ are of unbounded size, or if $\text{MHS}(\mathcal{H}_x)$ is of exponential size in $\text{irr}(x)$.

To get one such example, consider $k \in \mathbb{N}$ and the implicational base defined by $X = \{a_1, b_1, \dots, a_k, b_k, x\}$ and $\Sigma = \{a_i b_i \rightarrow x : 1 \leq i \leq k\} \cup \{b_i \rightarrow b_{i+1} : 1 \leq i < k\}$; see Figure 3 for an illustration. Then $\text{irr}(x) = \{(a_i)_{1 \leq i \leq j} \cup (b_i)_{j+1 \leq i \leq k} : 0 \leq j \leq k\}$ so $|\text{irr}(x)| = k+1$, and

the other elements in $X \setminus \{x\}$ are easily seen to have a single attached closed set. Therefore, $\text{irr}(\Sigma)$ is of polynomial size in $|X| = 2k + 1$ and $|\Sigma| = 2k - 1$. However, $|\text{MHS}(\mathcal{H}_x)| = 2^k$ and computing this set would fail in providing an output-polynomial time algorithm for enumerating $\text{irr}(x)$.



■ **Figure 3** An instance where $|\text{MHS}(\mathcal{H}_x)| = 2^k$ but $|\text{irr}(\Sigma)| = 3k + 1$ and $|\Sigma| = 2k - 1$.

At last, we note that this example has bounded premise-degree. We will show however in the next section that such instances can actually be efficiently solved by solution graph traversal.

5 Bounded premise-degree

We provide an incremental-polynomial time algorithm solving ICS-ENUM whenever the given implicational base has bounded premise-degree. Our algorithm relies on the top-down approach from Section 3 and solves ACS+A-ENUM relying on the solution graph traversal technique.

In the *solution graph traversal* framework, one has to define a digraph called *solution graph*, whose nodes are the solutions we aim to enumerate, and whose arcs (F, F') consists of ways of reconfiguring a given solution F into another solution F' . Then, the goal is not to build the solution graph entirely, but rather to traverse all its nodes efficiently and avoiding repetitions of nodes. This general technique has been fruitfully applied to numerous enumeration problems, see e.g., [9, 27, 40], and generalized to powerful frameworks [2, 10, 11]. In the following we will use the next folklore result which states conditions for such a technique to be efficient; see [20] for a quantitative statement and a proof.

► **Theorem 11 (Folklore).** *Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a solution set over a ground set X , and let us denote by n the size of the input. Then there is a $\text{poly}(n)$ -delay algorithm enumerating $\mathcal{F}$ whenever:*

1. *one member of $\mathcal{F}$ can be found in $\text{poly}(n)$ -time;*
2. *there is a function $\mathcal{N} : \mathcal{F} \rightarrow \mathcal{P}(\mathcal{F})$ called neighboring function mapping every solution to a set of solutions that can be computed in $\text{poly}(n)$ time given $F \in \mathcal{F}$;*
3. *the directed solution graph $(\mathcal{F}, \{(F, F') : F' \in \mathcal{N}(F)\})$ is strongly connected.*

In the following, let us consider an instance of ACS+A-ENUM, i.e., we fix some acyclic implicational base (X, Σ) , an element $x \in X$, and we assume that we are given $\text{irr}(y)$ for all ancestors y of x .

We first argue that Condition 1 holds for ACS+A-ENUM. Let GC_x be a function which maps any $E \subseteq X$ such that $x \notin \phi(E)$ to a maximal set $M \supseteq E$ with this property. We call the resulting set the *greedy completion* of E . Note that $\text{GC}_x(E)$ can be computed in polynomial time in the size of (X, Σ) , and that the obtained set lies in $\text{irr}(x)$; see Section 2. We derive the following.

► **Lemma 12.** *A first solution $\text{GC}_x(\emptyset) \in \text{irr}(x)$ can be obtained in $\text{poly}(|X| + |\Sigma|)$ time.*

We now turn on showing the existence of a polynomial-time computable neighboring function satisfying Conditions 2 and 3. Given some irreducible closed set $M \in \text{irr}(x)$, and some element $y \in X \setminus M$, we define

$$\mathcal{E}_{M,y} := \{A : A \rightarrow B \in \Sigma \text{ with } A \setminus M = \{y\}, B \not\subseteq M\},$$

$$V_{M,y} := \bigcup \mathcal{E}_{M,y} \text{ and } \mathcal{H}_{M,y} := (V_{M,y}, \mathcal{E}_{M,y}).$$

In other words, $\mathcal{H}_{M,y}$ is the hypergraph of premises of Σ not included in M , triggered by adding y , and with a conclusion not in M . Note that $|\mathcal{E}_{M,y}| \leq \text{pdeg}(y) \leq \text{pdeg}(\Sigma)$. Also, because every premise in $\mathcal{E}_{M,y}$ implies a $b \notin M$, hence an ancestor of x , we derive the following observation by transitivity.

► **Observation 13.** *Every vertex in $V_{M,y}$ is an ancestor of x .*

Regarding Condition 2, we first need the following statement, which holds the same idea as Lemma 7.

► **Lemma 14.** *Let M, M^* be two distinct sets in $\text{irr}(x)$. Then there exist $y \in M^* \setminus M$, $T \in \text{MHS}(\mathcal{H}_{M,y})$, and $\mathcal{S} \in \mathcal{S}(T)$ such that $M^* \subseteq \bigcap \mathcal{S}$.*

We are now ready to define the aforementioned neighboring function. Given $M \in \text{irr}(x)$ and $y \notin M$, we define

$$\mathcal{N}(M, y) := \left\{ \text{GC}_x \left((M \cap \bigcap \mathcal{S}) \cup \{y\} \right) : \begin{array}{l} \mathcal{S} \in \mathcal{S}(T) \text{ and } T \in \text{MHS}(\mathcal{H}_{M,y}) \\ \text{such that } x \notin \phi((M \cap \bigcap \mathcal{S}) \cup \{y\}) \end{array} \right\}$$

and put $\mathcal{N}(M) := \bigcup_{y \notin M} \mathcal{N}(M, y)$. We prove that Conditions 2 and 3 hold for this function.

► **Observation 15.** *By definition, the greedy completion in the definition of $\mathcal{N}(M, y)$ is well defined and $\mathcal{N}(M) \subseteq \text{irr}(x)$.*

► **Lemma 16.** *Let N be the sum of the sizes of X , Σ , and $\text{irr}(a)$ for all $a \in \text{anc}(x)$. Then, for all $M \in \text{irr}(x)$, $\mathcal{N}(M)$ can be computed in $N^{O(k)}$ time where k is the premise-degree of Σ .*

Note that Condition 2 is ensured by Observation 15 and Lemma 16. We are thus left to prove Condition 3, which we obtain by proving that for any pair of solutions, there is a neighbor of the first one with greater intersection with the second.

► **Lemma 17.** *Let M, M^* be two distinct sets in $\text{irr}(x)$. Then there exists $M' \in \mathcal{N}(M)$ such that $M \cap M^* \subsetneq M' \cap M^*$.*

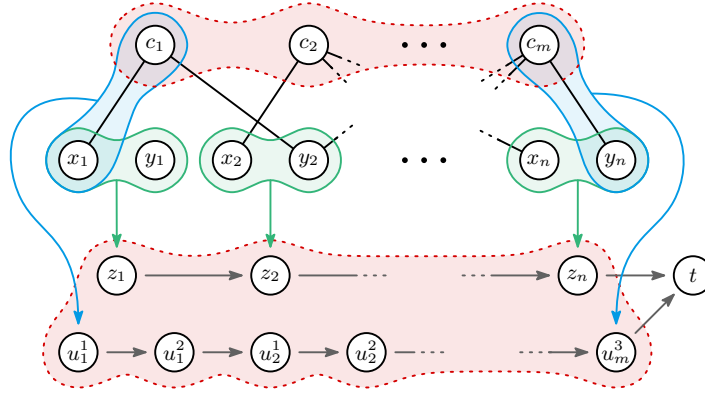
► **Corollary 18.** *The solution graph $(\text{irr}(x), \{(M, M') : M' \in \mathcal{N}(M)\})$ is strongly connected.*

Combining Theorem 11, Lemma 12, Lemma 16, and Corollary 18, we obtain a polynomial-delay algorithm for ACS+A·ENUM by solution graph traversal for acyclic implicational bases of bounded premise-degree. This, together with Theorem 6, in turn implies the following theorem, proving the remaining case of bounded premise-degree in Theorem 1.

► **Theorem 19.** *There is an incremental-polynomial time algorithm solving ICS·ENUM on acyclic implicational bases of bounded premise-degree.*

As in the previous section, we only needed to bound the size of a minimal transversal of $\mathcal{H}_{M,y}$ in the proof of previous theorem to derive an input-polynomial time algorithm solving ACS+A·ENUM. This gives the following result, analogous to Theorem 10.

► **Theorem 20.** *There is an incremental-polynomial time algorithm solving ICS·ENUM if, for any irreducible closed set M and $y \notin M$, the hypergraph $\mathcal{H}_{M,y}$ has bounded dual-dimension.*



■ **Figure 4** An illustration of the construction from Theorem 21. Black edges correspond to the incidence bipartite graph of a given CNF with clauses' size at most 3, and variable occurring in at most 3 clauses. There is a blue implication for each such edge (partially represented). The green implications represent conflicting literals. With this reduction, there is an irreducible closed set attached to t and avoiding the red set if and only if the formula is satisfiable.

6 Conclusion

In this paper, we described two incremental-polynomial time algorithms for the enumeration of irreducible closed sets in acyclic convex geometries of bounded (premise- or conclusion-) degree. This solves one of the two translation task for this class, the other one being addressed in the long version of this work [16]. We note that the delay of our algorithms is intrinsically not polynomial since our algorithm rely on a sequential top-down procedure which reduces the enumeration to an auxiliary enumeration problem whose instance grows with the total number of solutions output so far. This suggests the following question.

► **Question 1.** *Can ICS-ENUM be solved with polynomial delay for acyclic convex geometries of bounded degree?*

Toward this direction, we now argue that the classic flashlight search framework may not be used in a straightforward way to obtain a positive answer to this question. In the *flashlight search technique*, the goal is to construct solutions one element at a time, according to a linear order $x_1, \dots, x_n$ on the ground set, and deciding at each step whether x_i should or should not be included in the solution. For this approach to be tractable, the framework requires to solve the so-called “extension problem” in polynomial time: given two disjoint subsets K, F of the ground set, decide whether there exists a solution including K and avoiding F . We refer to, e.g., [9, 30] for a more detailed description of this folklore technique. More formally, this technique reduces the enumeration to the following decision problem.

Irreducible Closed Set Extension (ICS-EXT)

Input: An implicational base (X, Σ) and $K, F \subseteq X$.

Question: Does there exist $M \in \text{irr}(\mathcal{C})$ such that $K \subseteq M$ and $M \cap F = \emptyset$?

We show that this problem is NP-complete even for acyclic implicational bases of bounded degree with $K = \emptyset$, suggesting that this framework may not be used in a straightforward way to improve our results. Let us however remark that this does not rule out the possibility of a polynomial-delay algorithm for the problem, using a different approach or relying on particular vertex orderings. We refer to [16] for the proof of this result, the construction of which is hinted in Figure 4.

► **Theorem 21.** *The extension problem for ICS-ENUM is NP-complete even for acyclic implicational bases with degree, premise-degree, conclusion-degree and dimension (i.e., the maximum size of a premise) simultaneously bounded by 4, 4, 2 and 2, respectively.*

Note that, as an intermediate step, one may wonder whether there exists an incremental-polynomial time algorithm for ICS-ENUM where the degree of the polynomial in the number of solutions is independent of the premise degree.

Among other directions, it is natural to ask whether our results extend to broader classes of closure systems or to parameters that generalize the degree. We include a thorough discussion on the relevant classes and parameters in the long version of this work [16] as well as some considerations related to parameterized enumeration. In particular, we argue that without acyclicity the degree alone is not helpful anymore as ICS-ENUM for instances with bounded degree becomes as general as ICS-ENUM.

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