

Structural Parameterizations of Simultaneous Planarity

Thomas Depian ✉ 

TU Wien, Austria

Simon D. Fink ✉ 

TU Wien, Austria

Alexander Firbas ✉ 

TU Wien, Austria

Robert Ganian ✉ 

TU Wien, Austria

Matthias Pfretzschner ✉ 

University of Passau, Germany

Ignaz Rutter ✉ 

University of Passau, Germany

Abstract

Given a set of graphs on the same vertex set, the problem SIMULTANEOUS EMBEDDING WITH FIXED EDGES (SEFE) asks, whether there exist planar drawings of all input graphs, such that every pair of drawings coincides on their shared subgraph. It is known that SEFE is NP-complete [32], even in the so-called sunflower case, where all pairs of input graphs have the same shared graph $G_\cap$ [57]. Fink, Pfretzschner, and Rutter [26] recently initiated the study of the parameterized complexity of SEFE in the sunflower case, mainly focusing on structural parameters of $G_\cap$. In this work, we shift the focus towards parameters of the union graph $G_\cup$ that contains the edges of all input graphs. On the positive side, we establish fixed-parameter tractability for the problem with respect to the feedback edge set number of $G_\cup$. We complement this result by showing that it, surprisingly, remains NP-complete even if $G_\cup$ has constant vertex cover number. These results settle two open questions posed by Fink et al. [26].

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1 Introduction

The task of computing a plane drawing of a graph – or determining that none exists – has had a fundamental and broad impact on the field of theoretical computer science. An important reason why plane drawings are desirable is that they give rise to more easily readable visualizations of graphs [50, 51]. Nevertheless, in many scenarios we are faced with the issue that there isn’t a single graph we would need to visualize, but rather a (possibly small) set of planar graphs that do, however, share some of their edges and vertices. Examples where this occurs include, among others, the case of dynamic networks that change over time or can be the result of attempting to reconstruct the same underlying network using



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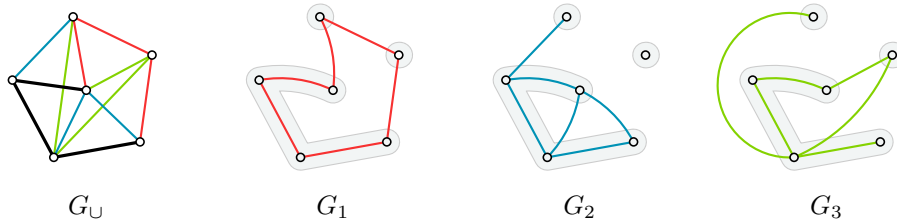
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different methods. Our input is thus a sequence of $k \geq 2$ graphs $G_1, \dots, G_k$, for which we want to compute a corresponding sequence of planar drawings $\Gamma_1, \dots, \Gamma_k$. It is, however, not enough to independently compute these drawings. In order to allow a user to compare the graphs by studying the sequence of drawings, it is important that the drawing is stable so that commonalities and difference can be easily observed. The problem **SIMULTANEOUS EMBEDDING WITH FIXED EDGES** models this by requiring that any vertex or edge that occurs in two of the graphs G_i, G_j is represented in the same way (i.e., by the same point resp. by the same curve) in Γ_i and Γ_j . This problem is known to be NP-complete for $k \geq 3$ [32] and despite significant research effort, the complexity of the problem is still open in the case $k = 2$.

An important special case of this problem is the *sunflower case*, where it is assumed that the intersection graph $G_i \cap G_j$ is the same for all $1 \leq i < j \leq k$. In this case, there is a single graph $G_\cap$, called the *shared graph*, that is contained in each input graph G_i , $1 \leq i \leq k$. Then, the vertices and edges of $G_\cap$ are called *shared*, whereas the vertices and edges that are contained in G_i but do not belong to $G_\cap$ are called *exclusive* and they cannot occur in any input graph G_j with $j \neq i$. We refer to instances with this property as *sunflower instances* and to the corresponding problem as *sunflower SEFE* (or S-SEFE for short). Observe that for $k = 2$, all instances of SEFE are sunflower instances. Without loss of generality, the exclusive vertices can be added to the shared graph (by adding them as isolated vertices to all other input graphs). We thus assume throughout that all our input graphs share the same vertex set V . A natural way to represent such sunflower instances of SEFE is by their *union graph*, $G_\cup$, whose vertex set is V and whose edge set is partitioned into $k + 1$ sets $E_\cap, E_1, \dots, E_k$, where $E_\cap$ contains the shared edges and E_i , $i \in [k]$ contains the exclusive edges of G_i . It is readily seen that a sunflower instance of SEFE admits a solution if and only if its union graph $G_\cup$ admits a drawing such that for each $i \in [k]$, no pair of curves representing edges from $E_\cap \cup E_i$ crosses. An illustration of the problem is provided in Figure 1.

Jünger and Schulz [41] showed that, in this setting, the problem can be formulated purely combinatorially. Namely, it is equivalent to finding (combinatorial) planar embeddings of the input graphs $G_1, \dots, G_k$ that induce the same planar embedding on the shared graph. Here, two planar embeddings are considered the same if they have the same rotation system and the same relative positions of their connected components. This establishes a link to the *partial drawing extension problem*, which asks whether a given drawing Γ_H (or equivalently a planar embedding) of a subgraph H of an input graph G can be extended to a planar drawing (or equivalently a planar embedding) of the complete graph G without changing Γ_H . In particular, S-SEFE is equivalent to deciding whether the shared graph $G_\cap$ admits a planar embedding that can be independently extended to each input graph G_i .



■ **Figure 1** An instance $\mathcal{I} = (G_1, G_2, G_3)$ of S-SEFE and the corresponding union graph $G_\cup$, where the shared edges are drawn in black. Note that the drawings of G_1 , G_2 , and G_3 coincide on the shared graph marked in gray.

The idea of simultaneous drawings and the extension of partial drawings readily generalize other types of graph representations such as geometric intersection and contact representations. This includes, among others, (several subclasses of) interval graphs [38, 54, 55, 56, 44, 43, 45], comparability and (circular) permutation graphs [39, 42, 49], circle graphs [19, 16], and circular-arc graphs [25], which have received considerable attention in the last years. Moreover, different drawing styles have been considered such as level-planar [3, 15] or orthogonal drawings [2, 6]. Moreover, fixed-parameter algorithms for drawing extension problems, often parameterized by the number of vertices and/or edges that still need to be added, have received considerable attention in recent years [23, 31, 9, 22] albeit the techniques developed there do not seem readily transferable to SEFE.

Since its introduction in 2005 [24], the SEFE problem has been the subject of substantial research efforts over the last two decades. SEFE is known to be NP-complete even for $k = 3$ input graphs [32] and this extends to the sunflower case [57] even in the case where the shared graph is a tree [4]. The case $k = 2$ is open and there is a plethora of work that aims at solving increasingly general special cases. Angelini, Di Battista, Frati, Jelínek, Kratochvíl, Patrignani and Rutter [5] show that S-SEFE can be solved in linear time if the embedding of the shared graph is fixed. For this case, also a full Kuratowski-type characterization of the positive instances has been given [40]. While there exist attempts to generalize such characterizations [36, 18], they seem to be inherently restricted to the biconnected case. More generally, when the embedding of the shared graph is not fixed, S-SEFE can be solved in linear time if the shared graph is biconnected [1, 35], if all connected components of the shared graph are cycles [14] and even if each connected component of the shared graph is biconnected or has maximum degree 3 [12, 57]. Most recently, it has been established that SEFE admits a polynomial-time algorithm for $k = 2$ if the shared graph is connected [28, 10, 27]. We refer to the surveys [13, 53] for a more detailed overview.

In this work, we target instances of S-SEFE beyond the case of $k = 2$. A major motivation for our results is the work of Fink, Pfretzschner and Rutter [26], who initiated the study of *parameterized* algorithms capable of solving such instances. The central question underlying their work – as well as a plethora of other studies within the domain of parameterized algorithmics – is whether we can identify one or several parameters $p_1, \dots, p_\ell$ of the input such that every instance $\mathcal{I}$ of S-SEFE can be solved in time $f(p_1 + \dots + p_\ell) \cdot |\mathcal{I}|^{\mathcal{O}(1)}$. Algorithms exhibiting such runtime behavior are called *fixed-parameter* (with respect to $p_1 + \dots + p_\ell$)¹. Their main results included fixed-parameter algorithms for S-SEFE parameterized (1) by k plus the *vertex cover number* (i.e., the minimum size of a vertex cover) of $G_\cup$, and (2) by k plus the *feedback edge number* (i.e., the minimum size of a feedback edge set) of $G_\cup$. In spite of these results – and a number of supplementary results targeting the structure of the shared graph – many prominent gaps remain in our understanding of the parameterized complexity of SEFE. Two questions explicitly identified by Fink, Pfretzschner and Rutter were [26, Section 5]:

- (A) Can the parametric dependence on k be avoided in fixed-parameter algorithm (1)?
- (B) Can the parametric dependence on k be avoided in fixed-parameter algorithm (2)?

Contributions. We expand our understanding of the complexity of S-SEFE by settling both open questions listed above. As our first result, we answer Question (A) in the negative and prove:

¹ We assume familiarity with the basic foundations of *parameterized complexity theory*, notably including the notions of *fixed-parameter tractability*, kernelization, and paraNP-hardness [21].

► **Theorem 1.** *S-SEFE remains NP-complete even when restricted to instances where $G_\cup$ has vertex cover number at most 15.*

The reduction underlying the proof of Theorem 1 is non-trivial and apart from a careful construction, it also relies on Vizing’s Theorem [58] and the edge coloring machinery introduced by Misra and Gries [48]. For our second result, we turn towards Question (B) which we answer positively:

► **Theorem 2.** *S-SEFE is fixed-parameter tractable when parameterized by the feedback edge number of $G_\cup$ alone, and in particular admits a linear kernel under this parameterization.*

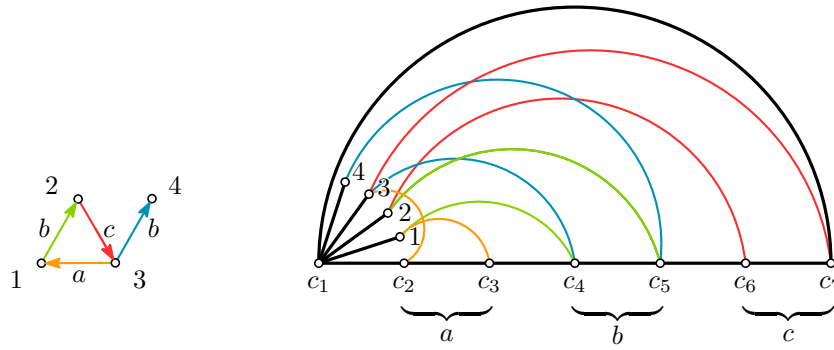
We note that here a *linear kernel* means that we provide a polynomial-time procedure that reduces an instance of S-SEFE to an equivalent one whose size is linear in the feedback edge number of the union graph $G_\cup$. The result strictly improves upon the previously established kernel of Fink, Pfretzschner and Rutter [26], whose size depended both on the feedback edge number and k .

In our concluding discussion, we also provide new insights into the complexity-theoretic behavior of S-SEFE (and also SEFE) under other graph parameters. Notably, we rule out parameterized algorithms w.r.t. the graph degeneracy of the union graph, and establish an equivalence between treewidth and clique-width in the sense that algorithms and lower bounds for the considered problems transfer between these two parameters.

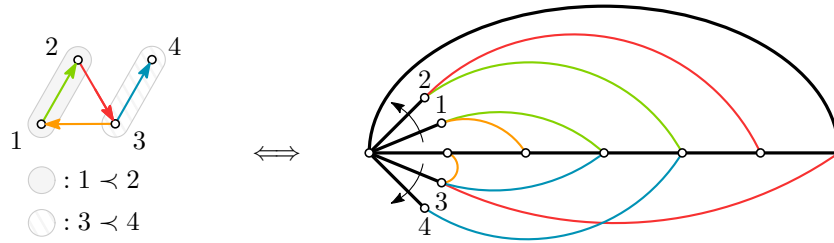
2 Preliminaries

Let $G = (V, E)$ be a simple graph. If there exists a path between every pair of vertices of G , then G is *connected*. A *connected component* of G is a maximal induced subgraph of G that is connected. A *cut-vertex* of G is a vertex whose removal disconnects G . A *drawing* Γ of G assigns every vertex $v \in V$ to a coordinate $\Gamma(v) \in \mathbb{R}^2$ and every edge $e \in E$ to a simple arc between $\Gamma(u)$ and $\Gamma(v)$. The drawing Γ is *planar* if no two distinct arcs intersect at an interior point. A graph is planar if it admits a planar drawing. Every planar drawing Γ of G uniquely defines a *rotation system*, i.e., the clockwise circular ordering of the incident edges (and, hence, also of the adjacent vertices) for each vertex. Moreover, Γ partitions the plane into regions called *faces*. If G is disconnected, Γ additionally defines *relative positions* for the connected components of G , that is an assignment of each connected component to a unique face of every other connected component. A *planar embedding* of G is an equivalence class of planar drawings of G that induce the same rotation system and the same relative positions.

Let $\mathcal{I} = (G_1, \dots, G_k)$ be an instance of S-SEFE. Recall that we consider the sunflower case, where all pairs of input graphs have the same shared graph $G_\cap$. A tuple $(\Gamma_1, \dots, \Gamma_k)$ of planar drawings of $(G_1, \dots, G_k)$ is a *simultaneous drawing* of $\mathcal{I}$ if each Γ_i induces the same drawing on $G_\cap$. Note that $\mathcal{I}$ is a positive instance of S-SEFE if and only if $(G_1, \dots, G_k)$ admits a simultaneous drawing. We refer to a tuple $\mathcal{E}_\cup = (\mathcal{E}_1, \dots, \mathcal{E}_k)$ of planar embeddings of $(G_1, \dots, G_k)$ as a *simultaneous embedding* of $\mathcal{I}$ if each $\mathcal{E}_i$ induces the same embedding on $G_\cap$. Jünger and Schulz [41, Theorem 4] showed that, for $k = 2$, the existence of a simultaneous drawing is equivalent to the existence of a simultaneous embedding. This can be straightforwardly extended to $k \geq 3$ in the sunflower case by successively applying their construction to each pair of input graphs. In this paper, we thus mainly work with simultaneous embeddings instead of drawings. We define $E_\cup := E(G_\cup)$ and $E_\cap := E(G_\cap)$. For a (not necessarily induced) subgraph H of $G_\cup$, we define $H_\cap := (V(H), E_\cap \cap E(H))$ and $H_i := (V(H), (E_\cap \cup E_i) \cap E(H))$.



■ **Figure 2** Example of the reduction from 2-DIGRAPH COLORING. For simplicity, the construction is carried out with three registers a, b, c instead of 7. Left: An instance of 2-DIGRAPH COLORING with vertices 1–4. Each color corresponds to one input graph of the S-SEFE instance. The letters a, b, c denote the register assignment for each arc. Right: The corresponding S-SEFE instance with registers a, b, c . Shared edges are drawn in black; exclusive edges are colored.



■ **Figure 3** A solution for the 2-DIGRAPH COLORING instance provided in Figure 2. Left: A 2-partition of the input graph's vertices and the corresponding topological orderings. Right: A simultaneous drawing of the corresponding S-SEFE instance. The black arrows indicate how to read off the 2-partition and the respective topological orderings.

For a graph G , a set $C \subseteq V$ is a *vertex cover* of G if every edge of G is incident to a vertex in C . The *vertex cover number* $vc(G)$ is the size of a minimum vertex cover of G . A *feedback edge set* of G is a set $F \subseteq E$ whose removal makes G acyclic. The *feedback edge set number* $fen(G)$ is the size of a minimum feedback edge set of G .

A directed graph $\vec{G}$ is acyclic precisely when its vertices admit an ordering σ such that every arc uv satisfies $\sigma(u) < \sigma(v)$. Such an ordering is referred to as a *topological ordering* of $\vec{G}$. The *underlying undirected graph* of a directed graph $\vec{G}$ is the simple graph obtained by removing the direction of every edge of $\vec{G}$.

3 ParaNP-hardness w.r.t. Vertex Cover of the Union Graph

In this section, we derive Theorem 1, that is, S-SEFE remains NP-complete even when restricted to instances where $vc(G_{\cup})$ has vertex cover number at most 15.

We achieve this via a reduction from 2-DIGRAPH COLORING. The problem asks whether, for a directed graph $\vec{G}$, there exists a 2-partition of its vertices $V_1 \cup V_2$, such that both $\vec{G}[V_1]$ and $\vec{G}[V_2]$ are acyclic. The problem remains NP-complete even if $\vec{G}$ has maximum degree 6 [37, Theorem 1]; this fact will be central to our construction. Let $\vec{G}$ be such an instance; we compute our reduction as follows.

We aim to construct an instance of S-SEFE $\mathcal{I} = (G_1, G_2, \dots, G_{|E(\vec{G})|})$. We assume without loss of generality that $\vec{G}$ contains no self-loops, for if $\vec{G}$ contains a self-loop, $\vec{G}$ is a trivial negative instance and we can output a trivial negative instance of S-SEFE. For each arc $uv \in \vec{G}$, $\mathcal{I}$ contains a separate input graph. To simplify the presentation, we will index the input graphs with the arcs of $\vec{G}$ rather than with the integers $[|E(\vec{G})|]$ in this section.

To build our instance, we will first describe how to construct the shared graph $G_\cap$ and then describe, for each input graph, how to extend $G_\cap$ to form the completed instance $\mathcal{I}$. (Technically, as the symbol $G_\cap$ is only well defined if the underlying instance fulfills the sunflower property, we first define an auxiliary graph $G_\cap^*$, extend this graph to obtain each input graph, and then argue that the so-obtained instance $\mathcal{I}$ fulfills the sunflower property, where in particular, $G_\cap = G_\cap^*$ (Lemma 3)). Note that in our construction, we require the edge coloring algorithm of Misra and Gries [48] specifically to enable the proof of Lemma 3.

We let $G_\cap^*$ be a cycle of length 15 with vertices $c_1, \dots, c_{15}$ (which will form a vertex cover of size 15 of $G_\cup$), where c_1 has $|V(\vec{G})|$ pendant vertices. The vertex set of the constructed instance is $\{c_1, \dots, c_{15}\} \cup V(\vec{G})$, where each $v \in V(\vec{G})$ identifies one pendant vertex of c_1 .

The intuition for our construction is as follows: Each pendant vertex of c_1 needs to be drawn either “inside” or “outside” the shared cycle. This models the 2-partition of 2-DIGRAPH COLORING. Moreover, for each arc ab , we define an input graph G_{ab} extending $G_\cap^*$ that forces that, if a and b are drawn both “inside” or both “outside” the cycle, in the cyclic order of vertices around c_1 , going from c_2 “towards” the respective face (“inside” vs “outside”), a needs to appear before b , as otherwise this input graph would need to be drawn with a crossing. This models that the subgraphs induced by either partition can be sorted topologically, that is, they are acyclic. See Figure 2 for an example of the reduction and Figure 3 for an example of how a solution of 2-DIGRAPH COLORING corresponds to a simultaneous embedding and vice versa.

Next, we construct the input graph G_{ab} for each $ab \in E(\vec{G})$. We first add all edges from $G_\cap^*$, and then proceed to add two additional edges as follows. We select two cycle vertices $c_{i(ab)}, c_{i(ab)+1}$ and add the edges $ac_{i(ab)}, bc_{i(ab)+1}$ to G_{ab} , where i is a map from $E(\vec{G})$ to $\{2, 4, 6, \dots, 14\}$ which we will define later. We call such an adjacent pair of cycle vertices, selected by $i(\cdot)$, a *register*. Note that there are 7 registers in total: $(c_2, c_3), (c_4, c_5), \dots, (c_{14}, c_{15})$.

To compute $i(\cdot)$, that is, assign each arc of $\vec{G}$ a register, we compute an edge coloring (that is, an assignment of edges to colors such that adjacent edges are assigned different colors) of the undirected graph underlying $\vec{G}$, which we will call G . In particular, this means that if two arcs share an endpoint, they are assigned different registers, except if they form a 2-cycle: then, they are assigned a common register.

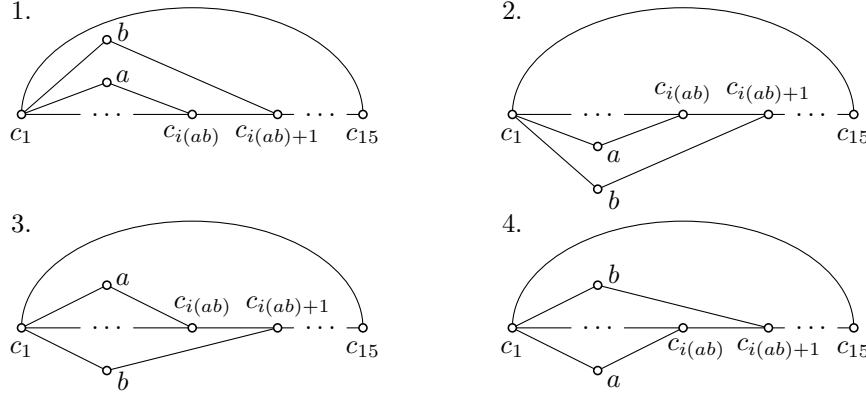
As $\vec{G}$ has a maximum degree of 6, so does G . By Vizing’s theorem [58], G admits an edge coloring using at most 7 colors. Using the algorithm of Misra and Gries [48], we can compute such a coloring in polynomial time. We identify each of the at most 7 colors with a unique register. Formally, if the edge coloring assigns arc ab register (c_j, c_{j+1}) , we set $i(ab) := j$.

This concludes the construction of our instance $\mathcal{I}$. Before we establish the correctness of our reduction, we show:

► **Lemma 3.** *Instance $\mathcal{I}$ obeys the sunflower property, where in particular, $G_\cap = G_\cap^*$.*

Proof. We need to show that $G_{xy} \cap G_{z\alpha} = G_\cap^*$ for each $xy, z\alpha \in E(\vec{G})$, implying that $G_\cap$ is well-defined and equal to $G_\cap^*$. Suppose not.

By construction, as $G_\cap^*$ is a subgraph of G_{xy} and $G_{z\alpha}$, and the codomain of $i(\cdot)$ consists of even integers, this means $xc_{i(xy)} = zc_{i(z\alpha)}$ or $yc_{i(xy)+1} = \alpha c_{i(z\alpha)+1}$.



■ **Figure 4** All 4 planar embeddings of G_{ab} for $ab \in E(\vec{G})$.

First, consider the case where $xy, z\alpha$ form a 2-cycle. Then, we have $\alpha = x$, $z = y$ and $j := i(xy) = i(z\alpha)$. In case we have $xc_{i(xy)} = zc_{i(z\alpha)}$, this means $x = y$, i.e., arc xy is a self-loop. As $\vec{G}$ does not have self-loops, this is a contradiction. Alternatively, in case we have $yc_{i(xy)+1} = \alpha c_{i(z\alpha)+1}$, this means $\alpha = z$, which again means $\vec{G}$ has a self-loop, a contradiction.

Otherwise $xy, z\alpha$ do not form a 2-cycle. First, if $xc_{i(xy)} = zc_{i(z\alpha)}$, we have $x = z$ and $i(xy) = i(z\alpha)$. But then, both y and α are neighbors of x in G . Additionally, $y \neq \alpha$, since otherwise, the two arcs would form a 2-cycle. Thus the edge coloring assigns xy and za different colors, which contradicts $i(xy) = i(z\alpha)$. The other case, $yc_{i(xy)+1} = \alpha c_{i(z\alpha)+1}$, is symmetric. ◀

To make the notion of how to “read off” a topological ordering of c'_1 s neighbors in a simultaneous embedding precise, we define:

► **Definition 4.** The x_i -oriented, x_j -delimited split of a cyclic permutation $(x_1, \dots, x_\ell)$ where $i \neq j$ is given by the left-face total order $x_{i-1} \prec x_{i-2} \prec \dots \prec x_{j+1}$, and the right-face total order $x_{i+1} \prec x_{i+2} \prec \dots \prec x_{j-1}$, where subscripts are interpreted modulo ℓ .

Next, we make an observation that will help us prove the correctness of our reduction. Essentially, we have that each possible embedding of G_{ab} tells us into which set of the 2-partition to assign a and b , and a guarantee that, if we assign them to the same set, the order in which we will “read off” their ordering will be consistent with there being an arc ab in the given 2-DIGRAPH COLORING instance.

► **Observation 5.** Let $ab \in E(\vec{G})$. The set of planar embeddings of G_{ab} is given by the set specified in Figure 4. Furthermore, for all planar embeddings of G_{ab} , we have that in both the left-face as well as the right-face total order of the c_2 -oriented c_{15} -delimited split of the rotation around c_1 , if both a and b are contained in the respective order, we have $a \prec b$.

Now, we can show:

► **Lemma 6.** The reduction is correct, that is, $\vec{G}$ is a positive instance of 2-DIGRAPH COLORING if and only if $(G_e)_{e \in E(\vec{G})}$ is a positive instance of S-SEFE.

Proof. ($\Rightarrow$): Let $L \cup R$ be a partition of $V(\vec{G})$ such that $\vec{G}[L]$ as well as $\vec{G}[R]$ are acyclic, and let $\ell_1 \prec \dots \prec \ell_{|L|}$ and $r_1 \prec \dots \prec r_{|R|}$ be a topological ordering of $G[L]$ and $G[R]$, respectively, where we call the former the *left order* and the latter the *right order*.

Towards finding a simultaneous embedding of the S-SEFE instance, we first fix a planar embedding of $G_\cap$. As $G_\cap$ is connected, it suffices to specify a rotation system. Note that, as $G_\cap$ is just a cycle with pendant vertices attached to c_1 , any rotation system for $G_\cap$ yields a planar embedding. For each c_i where $i \neq 1$, c_i has only two neighbors and hence, there is only one choice for c_i 's rotation. For c_1 , we define the following clockwise cyclic order of $N_{G_\cap}(c_1)$:

$$(\ell_{|L|}, \ell_{|L|-1}, \dots, \ell_1, c_2, r_1, r_2, \dots, r_{|R|}, c_{15}).$$

In other words, in the c_2 -oriented c_{15} -delimited split of the rotation around c_1 , the left-face order equals $\ell_1 \prec \dots \prec \ell_{|L|}$, and the right-face order equals $r_1 \prec \dots \prec r_{|R|}$.

It remains to find a planar embedding for each input graph G_{ab} , such that each embedding induces the same embedding on $G_\cap$. Consider G_{ab} for $ab \in E(\vec{G})$. We select a suitable embedding of G_{ab} 's four embeddings listed in Figure 4 as follows: If $a, b \in L$ and we have $a \prec b$, we use embedding 1. ($b \prec a$ is impossible since $\ell_1, \dots, \ell_{|L|}$ is a topological ordering of $\vec{G}[L]$ and $ab \in E(\vec{G}[L])$). If $a, b \in R$ and we have $a \prec b$, we use embedding 2. ($b \prec a$ is impossible since $r_1, \dots, r_{|R|}$ is a topological ordering of $\vec{G}[R]$ and $ab \in E(\vec{G}[R])$). If $a \in L$ and $b \in R$, we use embedding 3. Finally, if $b \in L$ and $a \in R$, we use embedding 4.

In each case, it is easy to verify that the clockwise cyclic order of c_1 in G_{ab} is a sub-order of the clockwise cyclic order of c_1 in $G_\cap$, the same holds for $c_{i(ab)}$ and $c_{i(ab)+1}$, and that the rotations for the remaining vertices coincide in the embedding of G_{ab} and the embedding of $G_\cap$. Thus, as required, we have constructed a simultaneous embedding of the S-SEFE instance.

($\Leftarrow$): Let a simultaneous embedding of the S-SEFE instance be given. Consider the c_2 -oriented c_{15} -delimited split of the clockwise order around c_1 in the planar embedding induced on $G_\cap$ by the simultaneous embedding. We denote the left-face order by $\ell_1, \dots, \ell_{|L|}$ (with vertex set L), and denote the right-face order by $r_1, \dots, r_{|R|}$ (with vertex set R). Observe that L and R form a partition of $V(\vec{G})$.

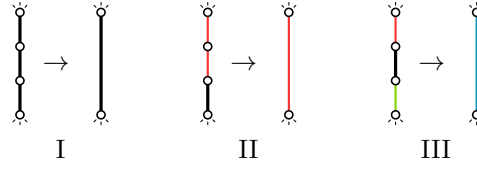
We claim that $\ell_1 \prec \dots \prec \ell_{|L|}$ is a topological ordering of $\vec{G}[L]$ and that $r_1 \prec \dots \prec r_{|R|}$ is a topological ordering of $\vec{G}[R]$, that is, $\vec{G}$ is a positive instance of 2-DIGRAPH COLORING. For this to be the case, whenever $ab \in E(\vec{G})$ and $a, b \in L$ (resp. $a, b \in R$), we need $a \prec b$ in $\ell_1 \prec \dots \prec \ell_{|L|}$ (resp. $r_1 \prec \dots \prec r_{|R|}$). Let $ab \in E(\vec{G})$ and $a, b \in L$. Consider the planar embedding of G_{ab} given by the simultaneous embedding of the S-SEFE instance. By Observation 5, we have $a \prec b$. The argument for the other order is symmetric. Hence, the proof is complete. $\blacktriangleleft$

Finally, to obtain Theorem 1, we observe that the reduction can be performed in polynomial time, as the Misra–Gries edge-coloring algorithm is polynomial-time computable. Furthermore, by Lemma 3, the construction yields an S-SEFE instance satisfying the sunflower property. Moreover, the set $\{c_1, \dots, c_{15}\}$ forms a vertex cover of $G_\cup$, so $\text{vc}(G_\cup) \leq 15$. Finally, S-SEFE is known to be in NP [32]. Thus, as our reduction is correct (Lemma 6), we obtain:

► **Theorem 1.** *S-SEFE remains NP-complete even when restricted to instances where $G_\cup$ has vertex cover number at most 15.*

4 A Linear Kernel via the Feedback Edge Number of the Union Graph

In this section, we derive Theorem 2, i.e., a linear kernel for S-SEFE when parameterized by the feedback edge number of $G_\cup$. To this end, let $\mathcal{I} = (G_1, G_2, \dots, G_k)$ be an instance of S-SEFE and let $F \subseteq E(G_\cup)$ be a minimum feedback edge set of $G_\cup$ of size $\text{fen}(G_\cup)$. In the



■ **Figure 5** Illustration for Rule I, II, and Rule III of Kernel 1 used to replace ID2-paths. Shared edges are drawn in black, exclusive edges in color. Blue represents a new color not occurring in the original instance.

following, we discuss a series of reduction rules that, if applied exhaustively, transform $\mathcal{I}$ into an instance $\mathcal{I}'$ of S-SEFE, such that the size of $\mathcal{I}'$ is bounded by a function in $\text{fen}(G_\cup)$. Moreover, $\mathcal{I}'$ has a simultaneous embedding if and only if $\mathcal{I}$ does.

Before we introduce said rules, let us first discuss some useful preprocessing steps that simplify the instance $\mathcal{I}$; see also [26]. Clearly, if $G_\cup$ is disconnected, we can solve each of its connected components individually. Furthermore, Bläsius, Karrer, and Rutter showed that $\mathcal{I}$ can be decomposed at the cut-vertices of $G_\cup$ into smaller, independent sub-instances, corresponding to the split components of $G_\cup$ [11, Lemma 1]. Observe that neither of these steps increases the size of $\text{fen}(G_\cup)$. Therefore, we can assume, without loss of generality, that $G_\cup$ is biconnected. Consequently, $G_\cup - F$ is a tree with at most $4\text{fen}(G_\cup)$ vertices of degree one or at least three. However, the number of degree 2 vertices is still unbounded.

In order to also bound the number of these, we introduce the concept of an *internal-degree-2 path*, or *ID2-path* for short, which is a maximal path P in $G_\cup$ between two vertices u and v such that every internal vertex of P has degree 2 in $G_\cup$ and u and v are not adjacent in $G_\cup$. Note that this implies that P has length at least three, i.e., consists of at least two edges. Our reduction rules that we introduce in the following kernel identify such ID2-paths and carefully replace them with a single edge. These rules are centered around the insight that, from the perspective of a single graph G_i , $i \in [k]$, the path P forces its endpoints u and v to share a face in $\mathcal{E}_i$ if $E(P) \subseteq E(G_i)$. However, we can also enforce this with a single edge $uv \in E(G_i)$. Otherwise, the path P does not exist in G_i , which removes the requirement for u and v to share a face in the embedding $\mathcal{E}_i$ of G_i . Hence, we can freely choose the placement of the internal vertices of P and edges $E(P) \cap E(G_i)$ in $\mathcal{E}_i$.

► **Kernel 1.** Consider an instance $\mathcal{I} = (G_1, G_2, \dots, G_k)$ of S-SEFE where $G_\cup$ is biconnected. We define the following rules, which, if applicable, map $\mathcal{I}$ to a new instance $\mathcal{I}'$ of S-SEFE. Figure 5 visualizes the rules.

Rule I: If there exists an ID2-path P in $G_\cup$ with endpoints u, v , such that $E(P) \subseteq E(G_\cap)$, then we remove all internal vertices of P from $\mathcal{I}$ and add the edge uv to every G_i , $i \in [k]$.

Rule II: If Rule I is not applicable and there exists an ID2-path P in $G_\cup$ with endpoints u, v , and some $i^* \in [k]$, such that $E(P) \subseteq E(G_{i^*})$, then we remove all internal vertices of P from $\mathcal{I}$ and add the edge uv to G_{i^*} .

Rule III: If Rules I and II are not applicable and there exists an ID2-path P in $G_\cup$ with endpoints u, v , then we remove all internal vertices of P from $\mathcal{I}$. If $E(G_i) \subseteq E(G_\cap) \cup E(P)$ for some $i \in [k]$, we also remove G_i from $\mathcal{I}$. Let $G'_\cup$ and $G'_\cap$ be the resulting union and shared graph, respectively. Afterwards, we add the graph G_P to $\mathcal{I}$ where $G_P = (V(G'_\cup), E(G'_\cap) \cup \{uv\})$.

We can readily verify that each reduction rule produces a valid instance $\mathcal{I}' = (G'_1, \dots, G'_{k'})$ of S-SEFE with union graph $G'_\cup$ and shared graph $G'_\cap$ that obeys the sunflower property; recall that, by definition, the endpoints u and v of the ID-2 path P are not adjacent in $G_\cup$.

In Lemmas 7–9, we establish the correctness of the above-introduced reduction rules.

► **Lemma 7.** *Rule I of Kernel 1 is safe.*

Proof. Let $\mathcal{I}$ be an instance of S-SEFE and $\mathcal{I}'$ the instance that we obtain after applying Rule I. Furthermore, let P be the ID2-path identified when applying the rule. We now show that $\mathcal{I}$ has a simultaneous embedding if and only if $\mathcal{I}'$ does.

($\Rightarrow$): Let $(\mathcal{E}_1, \dots, \mathcal{E}_k)$ be a simultaneous embedding of $\mathcal{I}$. As P consists solely of edges from $G_\cap$, it exists in every embedding $\mathcal{E}_i$, $i \in [k]$. We construct a simultaneous embedding of $\mathcal{I}'$ by contracting P into a single edge uv in each embedding $\mathcal{E}_i$, $i \in [k]$. This way, we ensure that all the resulting embeddings agree on $G'_\cap$. Finally, since planarity is preserved under edge contraction, we conclude that $(\mathcal{E}'_1, \dots, \mathcal{E}'_k)$ is a simultaneous embedding of $\mathcal{I}'$.

($\Leftarrow$): Let $(\mathcal{E}'_1, \dots, \mathcal{E}'_k)$ be a simultaneous embedding of $\mathcal{I}'$. Recall that we introduced the edge $uv \in E(G'_\cap)$ in $\mathcal{I}'$ that connects the two endpoints of P . Since $uv \in E(G'_\cap)$, the edge exists in every embedding $\mathcal{E}'_i$, $i \in [k]$. We now reintroduce P in each $\mathcal{E}'_i$ by subdividing the edge uv sufficiently many times and updating the embeddings accordingly, i.e., to reflect the subdivision process. Since the $\mathcal{E}'_i$ induce the same embedding on $G'_\cap$ before this operation, the obtained embeddings $\mathcal{E}_i$ do so on $G_\cap$ afterwards. Furthermore, subdividing an edge cannot destroy planarity, i.e., each $\mathcal{E}_i$ is a planar embedding and together they yield a simultaneous embedding of $\mathcal{I}$. ◀

► **Lemma 8.** *Rule II of Kernel 1 is safe.*

Proof. Let $\mathcal{I}$ be an instance of S-SEFE and $\mathcal{I}'$ the instance that we obtain after applying Rule II. Furthermore, let P denote the ID2-path and let G_{i^*} be the graph identified when applying the rule. We now show that $\mathcal{I}$ has a simultaneous embedding if and only if $\mathcal{I}'$ does.

($\Rightarrow$): Let $(\mathcal{E}_1, \dots, \mathcal{E}_k)$ be a simultaneous embedding of $\mathcal{I}$. We now modify each embedding $\mathcal{E}_i$, $i \in [k]$ and ensure that they yield the same embedding when restricted to $G'_\cap$. For the embedding $\mathcal{E}_{i^*}$, since $E(P) \subseteq E(G_{i^*})$, we can again contract the path P into the edge uv to obtain an embedding $\mathcal{E}'_{i^*}$ of G'_{i^*} . For all other embeddings $\mathcal{E}_j$, $j \in [k] \setminus i^*$, we obtain an embedding $\mathcal{E}'_j$ of G'_j by removing the internal vertices of P . Since planarity is preserved under edge contraction and vertex deletion, each $\mathcal{E}'_i$ is a planar embedding of G'_i , $i \in [k]$. Furthermore, as we remove the same edges from $G_\cap$ from each embedding $\mathcal{E}_i$, all $\mathcal{E}'_i$, $i \in [k]$, agree on $G'_\cap$ and witness the existence of a simultaneous embedding of $\mathcal{I}'$.

($\Leftarrow$): Let $(\mathcal{E}'_1, \dots, \mathcal{E}'_k)$ be a simultaneous embedding of $\mathcal{I}'$. We now re-insert the ID2-path P into each embedding $\mathcal{E}'_i$, $i \in [k]$. For $\mathcal{E}'_{i^*}$, we can re-insert P by subdividing the edge uv sufficiently many times and updating the embedding to reflect the subdivision. This yields a planar embedding $\mathcal{E}_{i^*}$ of G_{i^*} , which induces a (new) embedding $\mathcal{E}_\cap$ on $G_\cap$. For all other embeddings $\mathcal{E}'_j$, $j \in [k] \setminus i^*$, the fact that we could not apply Rule I tells us that there is at least one edge $e \in E(P) \setminus E(G_j)$. Hence, P is not a single path in G_j but a collection of paths and/or isolated vertices. Together with the fact that P is an ID2-path in $G_\cup$, this allows us to freely choose the embedding of the internal vertices of P and edges of $E(P) \cap E(G_\cap)$ when we insert them into $\mathcal{E}'_j$. In particular, we can choose the resulting embedding $\mathcal{E}_j$ such that its restriction to $G_\cap$ coincides with $\mathcal{E}_\cap$. Moreover, to see that $\mathcal{E}_j$ is a planar embedding, it suffices to observe that we can draw the internal vertices of P and edges of $E(G_\cap)$ arbitrarily close to u or v , respectively. We conclude that the obtained embeddings $\mathcal{E}_i$, $i \in [k]$, form a simultaneous embedding of $\mathcal{I}$. ◀

► **Lemma 9.** *Rule III of Kernel 1 is safe.*

Proof. Let $\mathcal{I}$ be an instance of S-SEFE and $\mathcal{I}'$ the instance that we obtain after applying Rule III. Furthermore, let P be the ID2-path identified when applying the rule. We now show that $\mathcal{I}$ has a simultaneous embedding if and only if $\mathcal{I}'$ does.

($\Rightarrow$): Let $(\mathcal{E}_1, \dots, \mathcal{E}_k)$ be a simultaneous embedding of $\mathcal{I}$. We now modify each $\mathcal{E}_i$, $i \in [k]$, to obtain a simultaneous embedding of $\mathcal{I}'$. First, we remove the internal vertices of P from each $\mathcal{E}_i$. If $E(G_i) \subseteq E(G_\cap) \cup E(P)$, i.e., if every edge of G_i belongs to $G_\cap$ or P , we can delete the entire embedding $\mathcal{E}_i$. Since we only deleted vertices (and their incident edges), the remaining embeddings $\mathcal{E}'_i$ yield a simultaneous embedding of the corresponding sub-instance of $\mathcal{I}$. Let $\mathcal{E}'_\cap$ be the resulting embedding of $G'_\cap$. What is missing from a simultaneous embedding of $\mathcal{I}'$ is the embedding $\mathcal{E}'_P$ of the graph G_P . Recall that $E(G_P) = E(G'_\cap) \cup \{uv\}$, where u and v are the end vertices of P . We initialize $\mathcal{E}'_P = \mathcal{E}'_\cap$. This ensures that all embeddings induce the same embedding when restricted to $G'_\cap$. What is missing from an embedding of G_P is the edge uv . We now show that vertices u and v must share a face in $\mathcal{E}'_P = \mathcal{E}'_\cap$, which allows us to insert the edge uv . Towards a contradiction, assume that this would not be the case. We recall that $\mathcal{E}'_\cap$ was obtained from $\mathcal{E}_\cap$ by vertex deletion. Hence, u and v do not share a face in $\mathcal{E}_\cap$ either. However, since $(\mathcal{E}_1, \dots, \mathcal{E}_k)$ is a simultaneous embedding of $\mathcal{I}$, and u and v are connected by the ID2-path P in $G_\cup$, there exists an edge $ab \in E(P) \cap E(G_i)$ for some $i \in [k]$ such that a and b do not share a face in $\mathcal{E}_\cap$. Since $\mathcal{E}_i$ is part of a simultaneous embedding of $\mathcal{I}$, this edge ab crosses another edge in any drawing Γ_i represented by $\mathcal{E}_i$, a contradiction to the assumption that $(\mathcal{E}_1, \dots, \mathcal{E}_k)$ is a simultaneous embedding. Therefore, u and v share a face in $\mathcal{E}'_P$, i.e., we can insert the edge uv in $\mathcal{E}'_P$. This completes the construction of the simultaneous embedding of $\mathcal{I}'$.

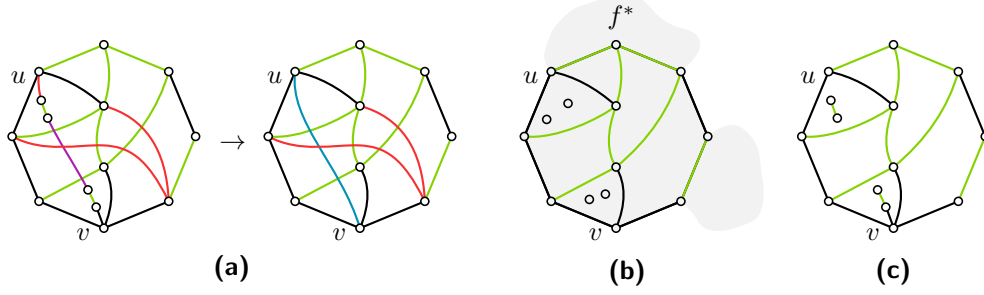
($\Leftarrow$): Let $(\mathcal{E}'_1, \dots, \mathcal{E}'_{k'})$ be a simultaneous embedding of $\mathcal{I}'$ and consider $\mathcal{E}'_P$. As $E(G_P) = E(G'_\cap) \cup \{uv\}$, the vertices u and v share a face in $\mathcal{E}'_P$. Moreover, we can identify a face f^* in $\mathcal{E}'_\cap$ which is shared by u and v and contains the edge uv (in $\mathcal{E}'_P$).

We now construct a simultaneous embedding $(\mathcal{E}_1, \dots, \mathcal{E}_k)$ of $\mathcal{I}$ as follows. Consider an input graph G_i , $i \in [k]$, and the embedding $\mathcal{E}'_i$. For now, we assume that this embedding exists but note that our reduction rule could have removed G_i entirely from $\mathcal{I}$. This case is handled at the end. Consider the face f^* in the embedding $\mathcal{E}'_i$ when restricted to $G'_\cap$. Observe that by above arguments, the face f^* contains both u and v . Recall that we apply Rule III only if Rules I and II are not applicable. Hence, similar to the proof of Lemma 8, we observe that when induced on G_i , the ID2-path P is not a single path between u and v but a collection of paths and/or isolated vertices. Hence, we can freely choose the embedding of the respective vertices and edges when extending $\mathcal{E}'_i$ into an embedding $\mathcal{E}_i$ of G_i . In particular, we can ensure that all internal vertices of P are inside the face f^* (when $\mathcal{E}_i$ is restricted to $G_\cap$). Moreover, we can include the edges $ux, vy \in E(P)$ in the rotation around u and v , respectively, such that they lie between the same shared edges as the edge uv in $\mathcal{E}'_P$; see Figure 6. This way we ensure that the constructed embeddings will agree on $G_\cap$. Furthermore, we can draw the internal vertices of P and edges of $E(G_\cap)$ arbitrarily close to u or v , i.e., $\mathcal{E}_i$ is a planar embedding of G_i .

Finally, recall that our reduction rule can remove a G_i from $\mathcal{I}$. However, since this only happens if $E(G_i) \setminus E(G_\cap) \subseteq E(P)$, we can initialize $\mathcal{E}'_i = \mathcal{E}'_\cap$ and proceed as above. Hence, we conclude that the constructed embeddings form a simultaneous embedding of $\mathcal{I}$. $\blacktriangleleft$

With Lemmas 7–9, we have all ingredients at hand to establish fixed parameter tractability of S-SEFE when parameterized by the feedback edge number of $G_\cup$.

► **Theorem 2.** *S-SEFE is fixed-parameter tractable when parameterized by the feedback edge number of $G_\cup$ alone, and in particular admits a linear kernel under this parameterization.*



■ **Figure 6** (a) An application of Rule III of Kernel 1, where we replace the ID2-path P from u to v with a new input graph G_P , visualized in blue. To construct the embedding $\mathcal{E}_i$, illustrated in green, the fact that $E(P) \not\subseteq E(G_i)$ allows us to freely choose the embedding of the internal vertices and respective edges of $E(P) \cap E(G_i)$. In particular, we can ensure that they (b) lie in the face f^* indicated in gray (when $\mathcal{E}_i$ is restricted to $G_\cap$) and that all constructed embeddings agree on $G_\cap$. (c) Moreover, $\mathcal{E}_i$ is a planar embedding since we can draw the inserted vertices and edges arbitrarily close to u and v , respectively.

Proof. Let $\mathcal{I} = (G_1, G_2, \dots, G_k)$ be an instance of S-SEFE. Recall that we can assume $G_\cup$ to be biconnected thanks to known steps that simplify instances of S-SEFE [11]. Let us exhaustively apply Reduction Rules I–III of Kernel 1 to $\mathcal{I}$ to obtain $\mathcal{I}' = (G'_1, G'_2, \dots, G'_{k'})$, $k' \geq 1$, with union graph $G'_\cup$. Note that the reduction rules maintain biconnectivity and the size of our parameter, i.e., the feedback edge number of $G_\cup$ – we only replace paths where every internal vertex is of degree 2 in $G_\cup$ by a single edge.

Let $F \subseteq E(G_\cup)$ be a minimum feedback edge set of $G'_\cup$ of size $\text{fen}(G'_\cup) = \text{fen}(G_\cup)$. We can compute F in polynomial time by, for example, computing a spanning tree T of $G'_\cup$ and setting $F := E(G'_\cup) \setminus E(T)$. As $G'_\cup$ is biconnected, the graph $T := G'_\cup - F$ is a tree. To ease argumentation, we assume T to be rooted at an arbitrary leaf. Every leaf is incident to at least one edge of F – otherwise the parent of this leaf would witness the existence of a cut vertex in $G'_\cup$ which is not possible. The tree T has therefore at most $2 \cdot \text{fen}(G_\cup)$ leaves and, consequently, at most $2 \cdot \text{fen}(G_\cup)$ internal vertices of degree at least three. Let C denote the set of vertices $v \in V(T)$ such that v is of degree at least three in T or we have $vu \in F$ for some $u \in G'_\cup$. Observe that C contains all leaves of T and, since $|F| = \text{fen}(G_\cup)$, its size is in $\mathcal{O}(\text{fen}(G_\cup))$. It remains to bound the number of vertices of degree 2 in T . We do that by bounding the length of a path in T between a vertex $u \in C$ and its (first) ancestor $v \in C$ in T . Let P be such a path. We observe that P can have length at most 4. Otherwise, i.e., if P consists of 5 or more vertices, there are at least 3 consecutive internal vertices that have degree 2 in $G'_\cup$. These form a path whose endpoints are not connected in $G'_\cup$. In particular, they form an ID2-path, a contradiction to the fact that we exhaustively applied the reduction rules of Kernel 1. As there are at most $\mathcal{O}(\text{fen}(G_\cup))$ such paths, one for each vertex in C (as T is a tree), each of which is of constant length, T (and thus $G_\cup$) consists of at most $\mathcal{O}(\text{fen}(G_\cup))$ vertices. Moreover, as T is a tree on $\mathcal{O}(\text{fen}(G_\cup))$ vertices and $\mathcal{I}$ has the sunflower property, we have for each edge $e \in E(T)$ either $e \in E(G'_\cap)$ or $e \in E(G'_i) \setminus E(G'_\cap)$ for a single $i \in [k']$. As T has $\mathcal{O}(\text{fen}(G_\cup))$ edges, $|F| = \text{fen}(G_\cup)$, and since we delete input graphs whose edge set coincides with that of $G'_\cap$, the number of remaining input graphs k' is in $\mathcal{O}(\text{fen}(G_\cup))$. Finally, as S-SEFE is known to be in NP [32], it is decidable. With the safeness of the Rules I–III of Kernel 1 established in Lemmas 7–9, the statement follows. ◀

5 Discussion and Concluding Remarks

While Theorems 1 and 2 improve upon the previous state of the art, the parameterized complexity of S-SEFE remains wide open under other parameterizations. A key milestone left for future work is to settle whether S-SEFE is fixed-parameter tractable, XP-tractable or paraNP-hard w.r.t. k plus the *treewidth* [52] of the union graph, i.e., $\text{tw}(G_\cup)$. Note that the question is only interesting if k is included in the parameterization – otherwise, we immediately obtain intractability by Theorem 1.

As a stepping stone towards understanding the problem’s behavior on graphs of bounded treewidth, below we show that any treewidth-based algorithm for S-SEFE (or SEFE) would immediately also transfer to the graph parameter *clique-width*, even though on general graphs the latter can be arbitrarily smaller than the former. We remark that the formal definitions of these notions are rather technical and in fact not needed to establish the desired relationship.

► **Proposition 10.** *(S-)SEFE parameterized by $\text{tw}(G_\cup) + k$ is in FPT (in XP) if and only if (S-)SEFE parameterized by $\text{cw}(G_\cup) + k$ is in FPT (in XP, respectively).*

Proof. An FPT (or XP) algorithm parameterized by $\text{cw}(G_\cup) + k$ yields an FPT (XP, respectively) algorithm parameterized by $\text{tw}(G_\cup) + k$ due to the well-known fact that $\text{cw}(G) \leq 3 \cdot 2^{\text{tw}(G)-1}$ for every graph G [20]. Conversely, assume we have an FPT (or XP) algorithm for (S-)SEFE parameterized by $\text{tw}(G_\cup) + k$. Consider an instance of (S-)SEFE parameterized by $\text{cw}(G_\cup) + k$. If one of the input graphs is not planar, we can reject the instance. Otherwise, k upper-bounds the thickness of $G_\cup$, i.e., the edges of $G_\cup$ can be partitioned into at most k edge sets such that each edge set induces a planar graph on $G_\cup$.

It is known that for every graph of thickness t' there exists an integer t which depends solely on the thickness of $G_\cup$ such that $G_\cup$ contains no $K_{t,t}$ as a subgraph [7]; notice here that t depends solely on k . This in turn ensures that $\text{tw}(G_\cup)$ is upper-bounded by a function of $\text{cw}(G_\cup) + t$ [34], and hence also by a function of $\text{cw}(G_\cup) + k$. Therefore, the existing FPT (or XP) algorithm would imply fixed-parameter (or XP, respectively) tractability of the problem w.r.t. $\text{cw}(G_\cup) + k$. ◀

A different structural parameterization that has been successfully used for some other problems is graph degeneracy [17, 46, 8]. A graph G is *d-degenerate* if every subgraph of G contains a vertex of degree at most d . The *degeneracy* of G is then the smallest d such that G is d -degenerate. Here, we provide a straightforward argument that rules out any parameterized algorithms w.r.t. graph degeneracy.

► **Proposition 11.** *(S-)SEFE is NP-hard even when restricted to instances where the union graph has degeneracy at most 2 and $k \leq 3$.*

Proof. We recall that SEFE and S-SEFE are known to remain NP-hard when restricted to instances satisfying $k \leq 3$ and, in particular, even $k = 3$. For each instance $\mathcal{I} = (G_1, G_2, G_3)$ of (S-)SEFE, we construct a new instance $\mathcal{I}'$ by subdividing every edge of each input graph once. Since $\mathcal{I}$ is a positive instance if and only if so is $\mathcal{I}'$ but the union graph of every constructed instance has degeneracy 2, the statement follows. ◀

Finally, we believe that a promising target for future work would be to settle the parameterized complexity of S-SEFE w.r.t. to k plus the *twin-cover number* [29, 30] (as has been asked in preceding work [26]) or k plus the *vertex integrity* of the union graph [33, 47].

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