

A General Framework for Finding Diverse Solutions via Network Flow and Its Applications

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Abstract

In this paper, we present a general framework for efficiently computing diverse solutions to combinatorial optimization problems. Given a problem instance, the goal is to find k solutions that maximize a specified diversity measure – the sum of pairwise Hamming distances or the size of the union of the k solutions. Our framework applies to problems satisfying two structural properties: (i) All solutions are of equal size and (ii) the family of all solutions can be represented by a surjection from the family of ideals of some finite poset. Under these conditions, we show that the problem of computing k diverse solutions can be reduced to the minimum cost flow problem and the maximum s - t flow problem. As applications, we demonstrate that both the unweighted minimum s - t cut problem and the stable matching problem satisfy the requirements of our framework. By utilizing the recent advances in network flows algorithms, we improve the previously known time complexities of the diverse problems, which were based on submodular function minimization.

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1 Introduction

In modeling real-world problems as optimization problems, many factors are inevitably omitted from the model due to the limited information and the need for tractability. Consequently, finding a single optimal solution to the optimization problem may not be enough. Motivated by this issue, developing algorithms to find *diverse* multiple solutions has recently gained attention in the field of combinatorial optimization [3, 4, 7, 8, 9, 10, 11, 15, 17].

We formally define the k -diverse problem for a positive integer k . Let d_{div} be a function that measures the “diversity” for a k -tuple of subsets of some arbitrary fixed ground set E . For a combinatorial problem **PROB**, the k -diverse problem of **PROB** asks, given an instance **I** of **PROB**, to find a k -tuple $(S_1, S_2, \dots, S_k)$ of solutions of **I** that maximizes the fixed measure d_{div} . Typical examples of the measure of diversity include the following:



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$$d_{\text{sum}}(S_1, S_2, \dots, S_k) := \sum_{1 \leq i < j \leq k} |S_i \Delta S_j|, \quad d_{\text{cov}}(S_1, S_2, \dots, S_k) := \left| \bigcup_{1 \leq i \leq k} S_i \right|.$$

If we use d_{sum} (resp. d_{cov}) as the measure, then we refer to the corresponding k -diverse problem as SUM- k -DIVERSE (resp. COV- k -DIVERSE).

In this paper, we present a general framework for efficiently finding diverse solutions to combinatorial optimization problems. Namely, we focus on SUM- k -DIVERSE/COV- k -DIVERSE of a combinatorial problem PROB having two properties (S) and (R) introduced below. The first property (S) states that all solutions of each instance of PROB have the same size:

(S) For any instance $\mathbf{I}$ of PROB, there exist a finite ground set E and a positive integer q such that the family $\mathcal{S}(\mathbf{I})$ of solutions of $\mathbf{I}$ consists of subsets of E of size q .

The second property (R) indicates that $\mathcal{S}(\mathbf{I})$ admits a lattice structure and $\mathcal{S}(\mathbf{I})$ has a compact representation using the lattice. To state this property precisely, we first introduce some notation. Let $(P, \preceq)$ be a finite poset with minimum element $\perp$ and maximum element $\top$. For such a poset P , let P^* denote the poset obtained from P by removing $\perp$ and $\top$. For a map $r: E \rightarrow P^2$ that assigns each element $e \in E$ to an ordered pair

$$r(e) := (e^+, e^-)$$

of elements in P with $e^+ \preceq e^-$, we define a set $\text{sup}_r(I)$ by

$$\text{sup}_r(I) := \{e \in E \mid e^+ \in I \cup \{\perp\} \not\preceq e^-\}$$

for each ideal $I \in \mathcal{I}(P^*)$. Here, an *ideal* of a poset is a subset $I \subseteq P^*$ that is closed downward under $\preceq$, that is, $y \preceq x \in I$ implies $y \in I$ for all $x, y \in P^*$, and $\mathcal{I}(P^*)$ denotes the family of ideals of the poset P^* . With this notation, the property (R) can be stated as follows:

(R) For any instance $\mathbf{I}$ of PROB, there exist a finite poset $(P, \preceq)$ with distinct minimum element $\perp$ and maximum element $\top$ and a map $r: E \rightarrow P^2$ given by $e \mapsto (e^+, e^-)$ with $e^+ \preceq e^-$ such that

$$\mathcal{S}(\mathbf{I}) = \{\text{sup}_r(I) \mid I \in \mathcal{I}(P^*)\}.$$

We refer to a map r appearing in the property (R) as a *reduction map*. In other words, the property (R) requires that every instance $\mathbf{I}$ of PROB admits a reduction map r such that sup_r forms a surjection from the family $\mathcal{I}(P^*)$ of ideals of P^* to the family $\mathcal{S}(\mathbf{I})$ of solutions of $\mathbf{I}$. A toy example of a reduction map is found in Example 1.

► **Example 1.** Let $E = \{a, b, c, d\}$ be the ground set. Suppose that the set of solutions is $\mathcal{S}(\mathbf{I}) = \{\{a, b\}, \{b, c\}, \{c, d\}\}$. We consider a poset $(P = \{\perp, u, v, \top\}, \preceq)$, where $\perp$ is the minimum element, $\top$ is the maximum element, and $u \prec v$. We define a map $r: E \rightarrow P^2$ by $r(a) = (\perp, u)$, $r(b) = (\perp, v)$, $r(c) = (u, \top)$, $r(d) = (v, \top)$. Since $\mathcal{I}(P^*) = \{\emptyset, \{u\}, \{u, v\}\}$, we can see that r is a reduction map. Indeed, we have $\text{sup}_r(\emptyset) = \{e \in E \mid e^+ \in \{\perp\} \not\preceq e^-\} = \{a, b\}$, $\text{sup}_r(\{u\}) = \{b, c\}$, and $\text{sup}_r(\{u, v\}) = \{c, d\}$.

We show that if the problem PROB has the properties (S) and (R), then $\mathcal{S}(\mathbf{I})$ with a partial order defined from a reduction map forms a distributive lattice (see Theorem 11).

Our framework exploits network flow techniques to efficiently compute diverse solutions. To utilize them, we will construct a directed acyclic graph (DAG) to represent a poset. Here, we say that a DAG G represents a poset $(P, \preceq)$ with distinct minimum element $\perp$ and maximum element $\top$ if the vertex set of G is P , $\top$ is a source and $\perp$ is a sink in G , and for any $u, v \in P^*$, there is a u - v path in G if and only if $u \succeq v$.

Our main result is stated as follows. We focus on deterministic algorithms in this paper.

► **Theorem 2.** *Suppose that a combinatorial problem PROB has the properties (S) and (R). Then, for any instance $\mathbf{I}$ of PROB, the problems SUM- k -DIVERSE and COV- k -DIVERSE of PROB can be solved in $O(T_P(\mathbf{I}) + T_r(\mathbf{I}) + (|A| + k|E|)^{1+o(1)})$ time and in $O(T_P(\mathbf{I}) + T_r(\mathbf{I}) + (|A| + |E|)^{1+o(1)} \log^2 k + kq)$ time, respectively. Here, E is the ground set of $\mathbf{I}$, q is the size of each solution of $\mathbf{I}$, $T_P(\mathbf{I})$ is the time required to construct a DAG representing the poset P in (R), $T_r(\mathbf{I})$ is the time required to construct a reduction map r in (R), and A is the set of arcs of a DAG G that represents P .*

We obtain Theorem 2 via the reduction of SUM- k -DIVERSE/COV- k -DIVERSE of PROB to a classical network flow problem, called the *minimum cost flow problem*. This enables us to utilize the state-of-the-art (deterministic) algorithm [25] for the minimum cost flow problem in solving SUM- k -DIVERSE/COV- k -DIVERSE of PROB. To this end, we introduce an intermediate problem called the *minimum k -potential problem*, and reduce the k -diverse problems to the minimum k -potential problem. Then we further reduce the minimum k -potential problem to the minimum cost flow problem by utilizing the idea of Ahuja, Hochbaum, and Orlin [1], who dealt with a more general problem. Our novelty is to identify the properties (S) and (R) as a sufficient condition to reduce the k -diverse problem to the minimum k -potential problem.

We also develop a reduction of SUM- k -DIVERSE/COV- k -DIVERSE of PROB to the *maximum s - t flow problem* via the minimum k -potential problem. We describe this in the full version. While the running-time of this reduction is slightly worse than that of the reduction to the minimum cost flow problem, its practical performance may be superior thanks to the simplicity of the reduction and the maximum s - t flow problem.

As applications of our framework, we demonstrate that two classical combinatorial problems, UNWEIGHTED MINIMUM s - t CUT and STABLE MATCHING, have properties (S) and (R). Here, UNWEIGHTED MINIMUM s - t CUT is to find an s - t cut of a given digraph G with the minimum size, and STABLE MATCHING is to find a matching of two parties such that no unmatched pair both prefer each other to their current partners (see Sections 5.2 and 5.3 for the definitions). In fact, both problems are known to possess a distributive lattice structure. By applying Theorem 2 to SUM- k -DIVERSE/COV- k -DIVERSE of these problems, we obtain the following results.

► **Theorem 3.**

1. *The problems SUM- k -DIVERSE and COV- k -DIVERSE of UNWEIGHTED MINIMUM s - t CUT can be solved in $O(n + (km)^{1+o(1)})$ time and in $O(n + m^{1+o(1)} \log^2 k + kq)$ time, respectively, where n denotes the number of vertices of the input digraph G , m the number of arcs of G , and q the size of a minimum s - t cut of G .*
2. *The problems SUM- k -DIVERSE and COV- k -DIVERSE of STABLE MATCHING can be solved in $O((kn^2)^{1+o(1)})$ time and in $O(n^{2+o(1)} \log^2 k + kn)$ time, respectively, where n denotes the size of the ground set U (or V) of the input instance $(U, V; (\leq_u)_{u \in U}, (\leq_v)_{v \in V})$.*

The polynomial-time solvability of SUM- k -DIVERSE/COV- k -DIVERSE of UNWEIGHTED MINIMUM s - t CUT has already been shown by De Berg, Martínez, and Spieksma [7]. Very recently, independently of our work, the same authors [8] developed a framework for solving

SUM- k -DIVERSE/COV- k -DIVERSE of a combinatorial problem having a certain property, and they show that SUM- k -DIVERSE/COV- k -DIVERSE of STABLE MATCHING can be solved in polynomial time. Both of their tractability results in [7, 8] are based on the polynomial-time solvability of the *submodular function minimization (SFM)* [2, 12]. Even if we use the state-of-the-art algorithm for SFM, given in [18], the running-time of their algorithms is (polynomial but) not very fast; it takes $O(k^5 n^5)$ time for SUM- k -DIVERSE/COV- k -DIVERSE of UNWEIGHTED MINIMUM s - t CUT, and $O(k^5 n^9)$ time for that of STABLE MATCHING. As in Theorem 3, our proposed algorithms for SUM- k -DIVERSE/COV- k -DIVERSE of UNWEIGHTED MINIMUM s - t CUT/STABLE MATCHING are much faster, thanks to recent advance in network flow algorithms. Moreover, we show that the framework of [8] for SUM- k -DIVERSE and COV- k -DIVERSE can be captured by our framework (see Section 5.4). This allows us to improve the time complexity.

Due to the space limitation, all the proofs (marked with $\star$) are omitted, and they can be found in the full version.

Related work. There exists a vast body of literature on the problem of finding diverse solutions; here, we mention only a few representative papers. Finding diverse solutions is generally harder than finding a single one. The k -diverse problems with respect to d_{sum} of some polynomial-time solvable problems – such as the maximum matching problem [10] and the (global) minimum cut problem [15] – are known to be NP-hard. Thus, in recent years, there has been a growing body of work on fixed-parameter tractable (FPT) algorithms for NP-hard diverse problems [3, 4, 9, 10, 11, 17]. Meanwhile, polynomial-time algorithms are known for several other k -diverse problems with respect to the diversity measures d_{sum} or d_{cov} , including the k -diverse variants of the spanning-tree problem [17], the shortest s - t path problem [16], as well as UNWEIGHTED MINIMUM s - t CUT [7] and STABLE MATCHING [8], which have already been mentioned.

2 Preliminaries

Let $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{R}$, and $\mathbb{R}_+$ denote the set of integers, nonnegative integers, real numbers, and nonnegative real numbers, respectively. For a nonnegative integer k , let $[k] := \{1, 2, \dots, k\}$ and $[0, k] := \{0, 1, \dots, k\}$. The symmetric difference $(S \setminus T) \cup (T \setminus S)$ of sets S and T is denoted by $S \triangle T$. For a finite set E and a nonnegative integer $q \in \mathbb{Z}_+$, let $\binom{E}{q}$ denote the family of all subsets of E of size q , i.e., $\binom{E}{q} := \{S \subseteq E \mid |S| = q\}$. For any finite set E , element $e \in E$, and k -tuple $\mathbf{S} = (S_1, S_2, \dots, S_k)$ of subsets of E , the *multiplicity* of e with respect to $\mathbf{S}$, denoted by $\mu_e(\mathbf{S})$, is defined as the number of subsets S_i that contain e , i.e., $\mu_e(\mathbf{S}) := |\{i \in [k] \mid e \in S_i\}|$.

Posets. A *partially ordered set* (or *poset*) is a pair $(P, \preceq)$ of a set P and a binary relation $\preceq$ over P satisfying, for $x, y, z \in P$, that $x \preceq x$ (reflexivity), $x \preceq y$ and $y \preceq x$ imply $x = y$ (antisymmetry), and $x \preceq y$ and $y \preceq z$ imply $x \preceq z$ (transitivity). By $x \prec y$ we mean $x \preceq y$ and $x \neq y$. Such a binary relation is called a *partial order*. If no confusion arises, we denote by P a poset and its underlying set interchangeably. In this paper, we consider only a finite poset, i.e., posets whose underlying sets are finite. Hence, by a poset we mean a finite poset. A partial order $\preceq$ on P is called a *total order* if $x \preceq y$ or $y \preceq x$ holds for any $x, y \in P$. For a poset $(P, \preceq)$, a subset $I \subseteq P$ is called an *ideal* if I is closed under $\preceq$, i.e., for any $v \in P$ and $u \preceq v$, we have $u \in I$. Let $\mathcal{I}(P)$ denote the set of all ideals of P . If P has the minimum element $\perp$ and the maximum element $\top$, then we denote by P^* the poset obtained from P by removing $\perp$ and $\top$.

Network flows. Let $G = (V, A)$ be a digraph. For a vertex subset $X \subseteq V$, let $\Delta_G^+(X)$ (resp. $\Delta_G^-(X)$) denote the set of outgoing arcs from (resp. incoming arcs to) X . If no confusion arises, we omit the subscript G from $\Delta_G^+(X)$ and $\Delta_G^-(X)$. If X consists of a single vertex v , i.e., $X = \{v\}$, we simply write $\Delta^+(v)$ and $\Delta^-(v)$ instead of $\Delta^+(\{v\})$ and $\Delta^-(\{v\})$, respectively. A function $f: A \rightarrow \mathbb{Z}_+$ from the arc set A to the nonnegative integers is called a *flow* of G . In the case where the digraph G has an arc capacity $c: A \rightarrow \mathbb{Z}_+$, a flow $f: A \rightarrow \mathbb{Z}_+$ is said to be *feasible* (with respect to c) if $f(a) \leq c(a)$ for all $a \in A$. For a feasible flow f with respect to an arc capacity c , its *residual graph*, denoted by G_f , is the digraph whose vertex set is V and arc set is $\{a \mid a \in A, f(a) < c(a)\} \cup \{\bar{a} \mid \bar{a} \in A, 0 < f(\bar{a})\}$, where $\bar{a}$ denotes the reverse arc (v, u) of $a = (u, v)$. This plays an important role in algorithms for network flow problems (and our algorithms). For a flow $f: A \rightarrow \mathbb{Z}_+$, its *boundary* $\partial f: V \rightarrow \mathbb{Z}$ is defined by $\partial f(v) := \sum_{a \in \Delta^+(v)} f(a) - \sum_{a \in \Delta^-(v)} f(a)$ for each $v \in V$.

In the following (except for Section 5.4), we assume that PROB is a combinatorial problem having the properties (S) and (R). For an instance $\mathbf{I}$ of PROB, we denote by $\mathcal{S}(\mathbf{I})$ the family of solutions of $\mathbf{I}$, $T_P(\mathbf{I})$ the time required to construct a DAG representing the poset P in (R), and $T_r(\mathbf{I})$ the time required to construct a reduction map r in (R).

3 Reduction to the minimum k -potential problem

In this section, we provide a reduction from the k -diverse problem of PROB to the *minimum k -potential problem*, which we introduce later.

In SUM- k -DIVERSE and COV- k -DIVERSE of PROB, we can regard the diversity measures d_{sum} and d_{cov} as the functions over $(\mathcal{S}(\mathbf{I}))^k$ for each instance $\mathbf{I}$ of PROB. Since $\mathcal{S}(\mathbf{I}) \subseteq \binom{E}{q}$ holds by the property (S), we have $\sum_{e \in E} \mu_e(\mathbf{S}) = \sum_{i=1}^k |S_i| = kq$ for any $\mathbf{S} = (S_1, S_2, \dots, S_k) \in (\mathcal{S}(\mathbf{I}))^k$, which is a constant. Hence, the functions d_{sum} and d_{cov} are representable as

$$\begin{aligned} d_{\text{sum}}(\mathbf{S}) &= \sum_{e \in E} \mu_e(\mathbf{S})(k - \mu_e(\mathbf{S})) = \text{Const.} - \sum_{e \in E} \mu_e(\mathbf{S})^2, \\ d_{\text{cov}}(\mathbf{S}) &= \sum_{e \in E} \min\{1, \mu_e(\mathbf{S})\} = \text{Const.} - \sum_{e \in E} \max\{0, \mu_e(\mathbf{S}) - 1\} \end{aligned}$$

for each $\mathbf{S} \in (\mathcal{S}(\mathbf{I}))^k$. Thus, the problems SUM- k -DIVERSE and COV- k -DIVERSE of PROB, namely, the problems of maximizing the functions d_{sum} and d_{cov} over $(\mathcal{S}(\mathbf{I}))^k$, are equivalent to those of minimizing

$$d_{\text{sum}}^*(\mathbf{S}) := \sum_{e \in E} \mu_e(\mathbf{S})^2, \quad d_{\text{cov}}^*(\mathbf{S}) := \sum_{e \in E} \max\{0, \mu_e(\mathbf{S}) - 1\},$$

respectively.

By using the concept of discrete convex functions, we can uniformly handle these functions d_{sum}^* and d_{cov}^* as follows. A function $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}$ is said to be *discrete convex* [20, Chapter 3.4] if $\varphi(x-1) + \varphi(x+1) \geq 2\varphi(x)$ for all $x \in \mathbb{Z}$, and said to be *non-decreasing on $\mathbb{Z}_+$* if $\varphi(x) \leq \varphi(x+1)$ for all $x \in \mathbb{Z}_+$. For a discrete convex function φ with $\varphi(0) = 0$ that is non-decreasing on $\mathbb{Z}_+$, we define

$$d_\varphi^*(\mathbf{S}) := \sum_{e \in E} \varphi(\mu_e(\mathbf{S}))$$

for $\mathbf{S} \in (\mathcal{S}(\mathbf{I}))^k$. Since the functions $x \mapsto x^2$ and $x \mapsto \max\{0, x-1\}$ are discrete convex functions that are non-decreasing on $\mathbb{Z}_+$ and satisfy $0 \mapsto 0$, both of d_{sum}^* and d_{cov}^* admit such representations.

Our framework can be applied to the k -diverse problem with respect to the diversity measure of the form

$$d_\varphi(\mathbf{S}) := \text{Const.} - d_\varphi^*(\mathbf{S})$$

for $\mathbf{S} \in (\mathcal{S}(\mathbf{I}))^k$, where φ is a discrete convex function with $\varphi(0) = 0$ that is non-decreasing on $\mathbb{Z}_+$. In the following, we consider the k -diverse problem with respect to d_φ of PROB, or equivalently, the problem of minimizing d_φ^* over $(\mathcal{S}(\mathbf{I}))^k$ for a discrete convex function φ .

We then introduce the *minimum k -potential problem*, to which we reduce the problem of minimizing d_φ^* later. Let $G = (V, A)$ be a DAG having unique source vertex $\top$ and sink vertex $\perp$ with $\top \neq \perp$. We refer to an assignment $p: V \rightarrow \mathbb{Z}_+$ of integers to vertices as a k -potential if p satisfies the following conditions:

(P1) $p(\perp) = k$ and $p(\top) = 0$.

(P2) $0 \leq p(v) \leq k$ for each $v \in V$.

(P3) p is monotone non-increasing with respect to A , i.e., $p(u) \leq p(v)$ for each $(u, v) \in A$.

In the *minimum k -potential problem*, we are given a DAG $G = (V, A)$ with unique source $\top$ and sink $\perp$, an arc weight $w: A \rightarrow \mathbb{Z}_+$, and a discrete convex function $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}$ with $\varphi(0) = 0$ that is non-decreasing on $\mathbb{Z}_+$, and asked to find a k -potential p of G that minimizes

$$H(p) := \sum_{a=(u,v) \in A} w(a) \varphi(p(v) - p(u)).$$

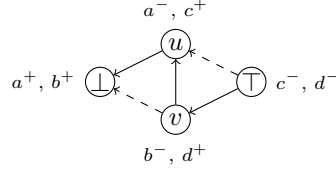
We note that the function H does not change even if we remove vertices $v \in V \setminus \{\perp, \top\}$ with $\Delta^+(v) = \Delta^-(v) = \emptyset$, called *isolated vertices*, from G .

Finally, we reduce, for an instance $\mathbf{I}$ of PROB, the problem of minimizing d_φ^* over $(\mathcal{S}(\mathbf{I}))^k$ to the minimum k -potential problem by utilizing the property (R). Let $(P, \preceq)$ be a poset having minimum element $\perp$ and maximum element $\top$, and let $r: E \rightarrow P^2$ be a reduction map as in the property (R). Then, we construct a DAG $G_{\mathbf{I}}$ whose vertex set is $V = P$ and whose arc set is $A = A_P \cup A_E$, where A_P is an arc set such that a DAG (P, A_P) represents the poset $(P, \preceq)$, and $A_E := \{(e^-, e^+) \mid e \in E\}$. The resulting $G_{\mathbf{I}}$ is still a DAG that represents P and has unique source $\top$ and sink $\perp$; each arc $(e^-, e^+) \in A_E$ is compatible with the partial order $\preceq$ of P , since $e^- \succeq e^+$. An arc weight $w_{\mathbf{I}}: A \rightarrow \mathbb{Z}_+$ is defined by $w_{\mathbf{I}}(a) := |\{e \in E \mid a = (e^-, e^+)\}|$ for $a \in A$. Note that $w_{\mathbf{I}}(a) = 0$ for each $a \in A \setminus A_E$. We set φ , which satisfies the non-decreasing property on $\mathbb{Z}_+$ and $\varphi(0) = 0$, as the input discrete convex function of the minimum k -potential problem. Then, the triple $(G_{\mathbf{I}}, w_{\mathbf{I}}, \varphi)$ is an instance of the minimum k -potential problem; its construction time is $|E|$.

► **Example 4.** Recall the toy example in Example 1. A graph $G = (P = \{\perp, u, v, \top\}, A_P)$ with $A_P = \{(u, \perp), (v, u), (\top, v)\}$ is a DAG representing $(P, \preceq)$. By construction, $A_E = \{(u, \perp), (v, \perp), (\top, u), (\top, v)\}$. Then, the DAG $G_{\mathbf{I}} = (P, A_P \cup A_E)$ is illustrated in Figure 1. The arc weights are $w_{\mathbf{I}}(v, u) = 0$, and $w_{\mathbf{I}}(a) = 1$ for all other arcs $a \in (A_P \cup A_E) \setminus \{(v, u)\}$.

Intuitively, any k -potential represents the direct sum of k ideals $I_1, I_2, \dots, I_k \in \mathcal{I}(P^*)$ as a multiset. Conversely, for any k ideals in $\mathcal{I}(P^*)$, there exists a k -potential that represents their direct sum as a multiset. Furthermore, by the property (R), each ideal of P^* corresponds to a solution of $\mathbf{I}$ via sup_r . The following lemma verifies this intuition.

► **Lemma 5** ($\star$). *For each k -tuple $\mathbf{S} \in (\mathcal{S}(\mathbf{I}))^k$, there is a k -potential $p_{\mathbf{S}}$ of $G_{\mathbf{I}}$ such that $H(p_{\mathbf{S}}) = d_\varphi^*(\mathbf{S})$. Conversely, for each k -potential p of $G_{\mathbf{I}}$, there is a k -tuple $\mathbf{S}_p \in (\mathcal{S}(\mathbf{I}))^k$ of solutions of $\mathbf{I}$ such that $H(p) = d_\varphi^*(\mathbf{S}_p)$, and we can construct $\mathbf{S}_p$ from p in $O(|E| + kq)$ time.*



■ **Figure 1** A DAG G_I for the toy example. Arcs in A_P and $A_E \setminus A_P$ are shown as solid and dashed lines, respectively.

Lemma 5 immediately implies that we can construct in $O(|E| + kq)$ time a minimizer $\mathbf{S}_p$ of d_φ^* over $(\mathcal{S}(\mathbf{I}))^k$ from a minimum k -potential p for the instance (G_I, w_I, φ) of the minimum k -potential problem. Therefore, we obtain the following.

► **Theorem 6** (★). *We can solve the k -diverse problem with respect to d_φ of PROB in $O(T_P(\mathbf{I}) + T_r(\mathbf{I}) + T_{\text{mp}}(\mathbf{I}) + |E| + kq)$ time, where $\mathbf{I}$ is a given instance of PROB and $T_{\text{mp}}(\mathbf{I})$ is the time of solving the instance of the minimum k -potential problem reduced from $\mathbf{I}$.*

4 Solving the minimum k -potential problem via minimum cost flow

Here, we reduce the minimum k -potential problem to the minimum cost flow problem.

Let us first recall the *minimum cost flow problem* (see e.g., [19, Chapter 9] and [23, Chapter 12] for details). In this problem, we are given a digraph $G = (V, A)$, arc cost $\gamma: A \rightarrow \mathbb{Z}$, arc capacity $c: A \rightarrow \mathbb{Z}_+$, and vertex demand $d: V \rightarrow \mathbb{Z}$, and asked to find a feasible flow $f: A \rightarrow \mathbb{Z}_+$ (i.e., $f(a) \leq c(a)$ for $a \in A$) that minimizes $\sum_{a \in A} \gamma(a)f(a)$ subject to $\partial f(v) = d(v)$ for all $v \in V$. An optimal solution of the minimum cost flow problem is called a *minimum cost flow*. We denote by $T_{\text{mcf}}(n, m, \Gamma, C, D)$ the time required to solve the minimum cost flow problem for a network of n vertices and m arcs with cost at most Γ in absolute values, capacity at most C , and a demand vector with values at most D in absolute values. Using the state-of-the-art algorithm for the minimum cost flow problem given in [25], we have $T_{\text{mcf}}(n, m, \Gamma, C, D) = O(m^{1+o(1)} \log(\max\{C, D\}) \log \Gamma)$.

Our reduction follows the work by Ahuja, Hochbaum, and Orlin [1], who dealt with a more general problem called the *convex cost integer dual network flow problem*. They showed that the Lagrangian dual of their problem is reduced to the minimum cost flow problem. For the complexity analysis, we explicitly provide the reduction in our case and the construction of a minimum k -potential from a minimum cost flow.

For a discrete convex function $\psi: \mathbb{Z} \rightarrow \mathbb{Z}$, an integer $x \in \mathbb{Z}$ is called a *breakpoint* of ψ if $\psi(x+1) + \psi(x-1) > 2\psi(x)$, i.e., the *left slope* $\psi(x) - \psi(x-1)$ and *right slope* $\psi(x+1) - \psi(x)$ of ψ at x are different. We can observe that, for each $x \in \mathbb{Z}$, the left slope of ψ at x is at most the right slope of ψ at x . Let $B(\psi)$ denote the set of breakpoints of ψ .

Let $(G = (V, A), w: A \rightarrow \mathbb{Z}_+, \varphi: \mathbb{Z} \rightarrow \mathbb{Z})$ be an instance of the minimum k -potential problem. We define $B_k(\varphi) := (B(\varphi) \cap [0, k]) \cup \{0, k\}$ and suppose that $B_k(\varphi) = \{b_0, b_1, \dots, b_z\}$ with $(0 =) b_0 < b_1 < \dots < b_z (= k)$. Let s_i^- (resp. s_i^+) denote the left (resp. right) slope of φ at $b_i \in B(\varphi) \cap [k-1]$; note that $s_i^+ = s_{i+1}^- < s_{i+1}^+$. We set $M \in \mathbb{Z}_+$ as a sufficiently large integer satisfying $M > H(p)$ for any k -potential p of G , e.g., $M = \sum_{a \in A} w(a)\varphi(k) + 1$.

We construct an instance of the minimum cost flow problem. The vertex set of the input digraph $\bar{G}$ is $\bar{V} := V \cup \{0\}$. We set the arc set $\bar{A}$ of $\bar{G}$, arc cost $\bar{\gamma}: \bar{A} \rightarrow \mathbb{Z}$, and capacity $\bar{c}: \bar{A} \rightarrow \mathbb{Z}_+$ by creating

- $|B_k(\varphi)| = z+1$ copies of each $a \in A$ satisfying $w(a) > 0$ with costs $b_0, b_1, \dots, b_{z-1}, b_z$ and capacities $w(a)s_1^- + M, w(a)(s_1^+ - s_1^-), w(a)(s_2^+ - s_2^-), \dots, w(a)(s_{z-1}^+ - s_{z-1}^-), M - w(a)s_{z-1}^+$, respectively,

- two copies of each $a \in A$ satisfying $w(a) = 0$ or each $a = (0, u)$ for $u \in V \setminus \{\perp, \top\}$ with costs $0, k$ and capacities M, M , respectively,
- arcs $(0, \perp)$ and $(0, \top)$ with cost k and 0 , respectively, and capacity $2M$.

The vertex demand $\bar{d}: \bar{V} \rightarrow \mathbb{Z}$ is set as $\bar{d}(v) := M(|\Delta_{G'}^+(v)| - |\Delta_{G'}^-(v)|)$ for each $v \in \bar{V}$, where $G' := (\bar{V}, A')$ and $A' := A \cup \{(0, v) \mid v \in V\}$.

Let f^* be a minimum cost flow of the resulting instance $(\bar{G}, \bar{\gamma}, \bar{c}, \bar{d})$. Then we construct the residual graph $\bar{G}_{f^*}$ of $\bar{G}$ with respect to f^* , and set the arc length ℓ of $\bar{G}_{f^*}$ as $\ell(a) := \gamma(a)$ if $a \in \bar{A}$, and $\ell(a) := -\gamma(a)$ if $\bar{a} \in \bar{A}$. Let us define $\bar{p}^*: \bar{V} \rightarrow \mathbb{Z}$ as a feasible potential with $\bar{p}^*(0) = 0$ in $\bar{G}_{f^*}$ with respect to arc length ℓ , i.e., an assignment $\bar{p}^*: \bar{V} \rightarrow \mathbb{Z}$ satisfying $\bar{p}^*(0) = 0$ and $\ell(a) \geq \bar{p}^*(v) - \bar{p}^*(u)$ for each arc $a = (u, v) \in \bar{A}$. The following lemma justifies our reduction.

► **Lemma 7** ($\star$). *The restriction $p^*: V \rightarrow \mathbb{Z}$ of $\bar{p}^*$ to V forms a minimum k -potential of (G, w, φ) .*

The validity of Lemma 7 follows from exactly the same argument given in [1]. For the sake of completeness, we provide a direct proof in the full version.

We then show the time complexity of solving the minimum k -potential problem via our reduction. The digraph $\bar{G}$ has $|V| + 1 = O(|V|)$ vertices, and at most $2(|V| - 1 + |A_0|) + |B_k(\varphi)||A_+| = O(|V| + |A_0| + |B_k(\varphi)||A_+|)$ arcs. Here, A_0 and A_+ denote the set of arcs a with $w(a) = 0$ and $w(a) > 0$, respectively. Thus, we can construct the minimum cost flow instance $(\bar{G}, \bar{\gamma}, \bar{c}, \bar{d})$ in $O(|V| + |A_0| + |B_k(\varphi)||A_+|)$ time. The costs and capacities are nonnegative integers at most k and $\bar{C} := \max\{2M, \max_{a \in A} w(a)\varphi(k) + M\}$, respectively. The demands are at most $M|V|$ in absolute values. Hence, we can find a minimum cost flow $f^*: \bar{A} \rightarrow \mathbb{Z}_+$ of $(\bar{G}, \bar{\gamma}, \bar{c}, \bar{d})$ in $O(T_{\text{mcf}}(|V|, |V| + |A_0| + |B_k(\varphi)||A_+|, k, \bar{C}, M|V|))$ time. The residual graph $\bar{G}_{f^*}$ of $\bar{G}$ with respect to f^* is constructed in $O(|\bar{A}|) = O(|V| + |A_0| + |B_k(\varphi)||A_+|)$ time. A feasible potential $\bar{p}^*$ in $\bar{G}_{f^*}$ with respect to ℓ is found in $O(T_{\text{mcf}}(|V|, |V| + |A_0| + |B_k(\varphi)||A_+|, k, |V|, |V|))$ time by computing shortest distances from a supernode to vertices in $\bar{V}$. Therefore, by Lemma 7, we can obtain a minimum k -potential p^* , which is a restriction of $\bar{p}^*$ to V , in $O(T_{\text{mcf}}(|V|, |V| + |A_0| + |B_k(\varphi)||A_+|, k, \bar{C}, M|V|))$ time.

By applying the algorithm in [25], we obtain the following, which implies Theorem 2.

► **Theorem 8** ($\star$). *We can solve the k -diverse problem with respect to d_φ of PROB for an instance $\mathbf{I}$ in $O(T_P(\mathbf{I}) + T_r(\mathbf{I}) + (|A_P| + |B_k(\varphi)||E|)^{1+o(1)} \log(\varphi(k)) \log k + kq)$ time. In particular, the problems SUM- k -DIVERSE of PROB and COV- k -DIVERSE of PROB can be solved in $O(T_P(\mathbf{I}) + T_r(\mathbf{I}) + (|A_P| + k|E|)^{1+o(1)})$ time and in $O(T_P(\mathbf{I}) + T_r(\mathbf{I}) + (|A_P| + |E|)^{1+o(1)} \log^2 k + kq)$ time, respectively.*

5 Applications

In this section, we introduce two applications of our framework; one is the k -diverse problems of UNWEIGHTED MINIMUM s - t CUT and the other is that of STABLE MATCHING.

In order to apply our framework to the k -diverse problem of a concrete combinatorial problem, we need to construct a poset and a reduction map appearing in the property (R) for each instance. In Section 5.1, we develop a common strategy of building these components. In fact, it is known that both families of minimum s - t cuts and stable matchings are naturally identified with set families called *ring families*. A ring family with inclusion order forms a poset, particularly a *distributive lattice*. We will utilize these facts to construct a reduction map for UNWEIGHTED MINIMUM s - t CUT and STABLE MATCHING in Sections 5.2 and 5.3,

respectively. In Section 5.4, we briefly describe a framework for the k -diverse problem recently developed by De Berg, Martínez, and Spieksma in [8], and show that our framework can capture theirs in the case of SUM- k -DIVERSE/COV- k -DIVERSE.

5.1 How to construct a poset and a reduction map

We start this subsection with introducing terminology on lattices (see e.g., [6] for details). A poset $(L, \preceq)$ is called a *lattice* if, for any two elements $x, y \in L$, their least upper bound (*join*) $x \vee y$ and greatest lower bound (*meet*) $x \wedge y$ exist in L . A lattice L is said to be *distributive* if the distributive law $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ holds for any $x, y, z \in L$.

A typical example of a distributive lattice is the family $\mathcal{I}(P)$ of ideals of a poset over P with inclusion order $\subseteq$. In this case, the join of two ideals I and I' is their union $I \cup I'$, and the meet is their intersection $I \cap I'$; we can easily see that they are also ideals. More generally, a *ring family*, which is a nonempty family $\mathcal{R} \subseteq 2^R$ of subsets of a nonempty finite set R such that it is closed under the union and intersection, endowed with inclusion order $\subseteq$ forms a distributive lattice $(\mathcal{R}, \subseteq)$, where $X \vee Y = X \cup Y$ and $X \wedge Y = X \cap Y$ for $X, Y \in \mathcal{R}$.

The celebrated *Birkhoff's representation theorem* [5] states that every distributive lattice $(L, \preceq)$ is isomorphic to the distributive lattice $(\mathcal{I}(P), \subseteq)$ for some poset P . Here, two lattices $(L, \preceq)$ and $(L', \preceq')$ are said to be *isomorphic* if there is a bijection $h: L \rightarrow L'$ such that $x \preceq y$ if and only if $h(x) \preceq' h(y)$ for any $x, y \in L$. For a distributive lattice $(L, \preceq)$, we refer to a poset P such that $(L, \preceq)$ and $(\mathcal{I}(P), \subseteq)$ are isomorphic as a *Birkhoff representation* of $(L, \preceq)$.

A Birkhoff representation of the distributive lattice $(\mathcal{R}, \subseteq)$ over a ring family $\mathcal{R}$ can be obtained as follows. The ring family $\mathcal{R}$ has the unique minimal set $X_\perp := \bigcap_{X \in \mathcal{R}} X$ and unique maximal set $X_\top := \bigcup_{X \in \mathcal{R}} X$. Take any maximal chain $X_\perp =: X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n =: X_\top$ from $X_\perp$ to $X_\top$ in $\mathcal{R}$; namely, there are no $X \in \mathcal{R}$ and $i \in [n]$ with $X_{i-1} \subsetneq X \subsetneq X_i$. Then, $\Pi^*(\mathcal{R}) := \{X_i \setminus X_{i-1} \mid i \in [n]\}$ forms a partition of $X_\top \setminus X_\perp$. We define a partial order $\preceq$ on $\Pi^*(\mathcal{R})$ by setting $\hat{X} \preceq \hat{Y}$ if and only if every $Z \in \mathcal{R}$ with $Z \supseteq \hat{Y}$ also includes $\hat{X}$. The resulting $(\Pi^*(\mathcal{R}), \preceq)$ is actually a poset and is independent of the choice of a maximal chain from $X_\perp$ to $X_\top$. It is known (see e.g., [21, Chapter 2.2.2]) that $(\Pi^*(\mathcal{R}), \preceq)$ is a Birkhoff representation of $(\mathcal{R}, \subseteq)$; more precisely, the map $\mathcal{Y} \mapsto (\bigcup_{\hat{X} \in \mathcal{Y}} \hat{X}) \cup X_\perp$ is an isomorphism from $(\mathcal{I}(\Pi^*(\mathcal{R})), \subseteq)$ to $(\mathcal{R}, \subseteq)$. In this paper, we refer to $(\Pi^*(\mathcal{R}), \preceq)$ as *the* Birkhoff representation of $(\mathcal{R}, \subseteq)$.

We are ready to develop a strategy to construct a reduction map by using a ring family. Let $\mathcal{R} \subseteq 2^R$ be a ring family over a nonempty finite set R . We may assume that the minimal set $X_\perp$ is nonempty and the maximal set $X_\top$ is a proper subset of R ; otherwise we add two elements $\perp$ and $\top$ to R and update each subset $X \in \mathcal{R}$ as $X \cup \{\perp\}$, which makes $\mathcal{R}$ satisfy $\perp \in X_\perp$ and $\top \in \overline{X_\top} := R \setminus X_\top$. Let $\Pi(\mathcal{R})$ denote the partition $\Pi^*(\mathcal{R}) \cup \{X_\perp, \overline{X_\top}\}$ of R . We extend the partial order $\preceq$ on $\Pi^*(\mathcal{R})$ to that on $\Pi(\mathcal{R})$ by setting $X_\perp \prec \hat{X} \prec \overline{X_\top}$ for any $\hat{X} \in \Pi^*(\mathcal{R})$. For a map $\hat{r}: E \rightarrow R^2$ given by $e \mapsto (\hat{e}^+, \hat{e}^-)$, we define

$$\sup_{\hat{r}}(X) := \{e \in E \mid \hat{e}^+ \in X \not\prec \hat{e}^-\}$$

for $X \in \mathcal{R}$. We say that $\hat{r}: E \rightarrow R^2$ is a *pre-reduction map* if $\hat{e}^- \in X$ implies $\hat{e}^+ \in X$ for all $X \in \mathcal{R}$ and $e \in E$, and $\mathcal{S}(\mathbf{I}) = \{\sup_{\hat{r}}(X) \mid X \in \mathcal{R}\}$. We show that we can construct a reduction map as long as we have a pre-reduction map.

► **Lemma 9** ($\star$). *Suppose that $\hat{r}: E \rightarrow R^2$ given by $e \mapsto (\hat{e}^+, \hat{e}^-)$ is a pre-reduction map. Then, the map $r: E \rightarrow \Pi(\mathcal{R})^2$ defined by*

$$r(e) := (\Pi(\hat{e}^+), \Pi(\hat{e}^-)) \quad (e \in E)$$

is a reduction map, where $\Pi(x)$ ($x \in R$) denotes the unique member of $\Pi(\mathcal{R})$ that contains x .

Thus, we obtain the following:

► **Proposition 10.** *Let R be a nonempty finite set, and let $\mathcal{R}$ be a ring family over R . Suppose we are given a pre-reduction map $\hat{r}: E \rightarrow R^2$ and the partition $\Pi(\mathcal{R})$ of R . Then, we can construct a reduction map $r: E \rightarrow \Pi(\mathcal{R})^2$ in constant time.*

The property (R) requires that, for each instance $\mathbf{I}$ of PROB, the family $\mathcal{S}(\mathbf{I})$ of its solutions is closely related to the family $\mathcal{I}(P^*)$ of ideals of the poset P^* . Hence, it is natural to expect that $\mathcal{S}(\mathbf{I})$ admits the distributive lattice structure. We conclude this subsection by showing that this indeed holds, which might be viewed as a necessary condition on PROB for our framework to be applicable.

Recall the notation in the property (R). We may assume that $e^+ \prec e^-$ for each $e \in E$, since if $e^+ = e^-$, then no member S in $\mathcal{S}(\mathbf{I})$ contains e , i.e., $\mathcal{S}(\mathbf{I}) \subseteq \binom{E \setminus \{e\}}{q}$, and hence we can remove e from E . Let $\leq$ denote a partial order on E such that $e \leq e'$ if and only if $e = e'$ or $e^- \preceq e'^+$. This $\leq$ can be extended for $\mathcal{S}(\mathbf{I})$ by setting $S \leq T$ if and only if there exists a bijection $\pi: S \rightarrow T$ such that $e \leq \pi(e)$ for all $e \in S$. Then, the following holds.

► **Theorem 11** (*). *Suppose that a combinatorial problem PROB has the properties (S) and (R). For any instance $\mathbf{I}$ of PROB, the poset $(\mathcal{S}(\mathbf{I}), \leq)$ forms a distributive lattice.*

5.2 The k -diverse unweighted minimum s - t cut problem

In this subsection, we consider the k -diverse problem of UNWEIGHTED MINIMUM s - t CUT. Here, this is the minimum s - t cut problem for a digraph $G = (V, A)$ with unit arc capacity $c: A \rightarrow \mathbb{Z}_+$, i.e., $c(a) = 1$ for all $a \in A$. Clearly, UNWEIGHTED MINIMUM s - t CUT has the property (S) by setting the ground set E as the arc set A and the integer q as the size of minimum s - t cuts. Our aim is to construct, for some ring family $\mathcal{R}$, a DAG D representing the Birkhoff representation $(\Pi^*(\mathcal{R}), \preceq)$, the partition $\Pi(\mathcal{R})$, and a pre-reduction map $\hat{r}$ based on $\mathcal{R}$, which implies that UNWEIGHTED MINIMUM s - t CUT has the property (R) by Lemma 9.

Let $G = (V, A)$ with $s, t \in V$ be a digraph that is an instance of UNWEIGHTED MINIMUM s - t CUT. Then, the family $\mathcal{S}(G) \subseteq 2^A$ of solutions of the instance G is the set of all minimum s - t cuts, which is a subset of the family $\{\Delta^+(X) \mid X \subseteq V \text{ with } s \in X \not\preceq t\}$. Let $\mathcal{R}_{st}$ denote the family of vertex subsets $X \subseteq V$ with $s \in X \not\preceq t$ such that its outgoing arcs form a minimum s - t cut of G , i.e.,

$$\mathcal{R}_{st} := \{X \subseteq V \mid s \in X \not\preceq t, \Delta^+(X) \in \mathcal{S}(G)\}.$$

It is well-known that $\mathcal{R}_{st}$ forms a ring family, which directly follows from Lemma 12 below. Moreover, since the unique minimal set $X_\perp$ in $\mathcal{R}_{st}$ contains s and the unique maximal set $X_\top$ in $\mathcal{R}_{st}$ excludes t , we have $X_\perp \neq \emptyset \neq \overline{X_\top} := V \setminus X_\top$. We will construct a pre-reduction map based on $\mathcal{R}_{st}$.

We utilize the representation introduced by Picard and Queyranne [22], which is used for enumerating all minimum s - t cuts of a digraph (with arbitrary positive capacities). In our case, we consider the unit capacity case, and use this representation to construct a DAG representing the Birkhoff representation $(\Pi^*(\mathcal{R}_{st}), \preceq)$ of the distributive lattice $(\mathcal{R}_{st}, \subseteq)$, as well as the partition $\Pi(\mathcal{R}_{st})$ of V .

To begin, we briefly describe the representation proposed by Picard and Queyranne. Let f be an arbitrary maximum s - t flow of G , where each arc has unit capacity. Let G_f denote the residual graph of f . Then, the following characterization is known.

► **Lemma 12** ([22]). *An s - t cut $\Delta^+(X)$ is a minimum s - t cut in G if and only if X is a closed set under reachability in G_f , containing s but not t .*

Using this, Picard and Queyranne constructed a DAG $D = (V', A')$ from G through the following steps: (1) Contract each strongly connected component in G_f into a single vertex, (2) remove the component containing s along with all vertices reachable from s , and (3) remove the component containing t along with all vertices reachable to t . For each vertex $v' \in V'$, let $R(v')$ denote the set of vertices in G_f contracted into v' . Similarly, let $R(s')$ and $R(t')$ denote the sets of vertices in G_f removed in steps (2) and (3), respectively. By Lemma 12, the DAG D represents all minimum s - t cuts in G . Specifically, for any $X \subseteq V'$ that is closed under reachability in D , the set of outgoing arcs from $R(X) := (\bigcup_{v' \in X} R(v')) \cup R(s')$ in G forms a minimum s - t cut in G . Conversely, for any minimum s - t cut $\Delta^+(X)$ in G , the set of components corresponding to $X \setminus R(s')$ is closed under reachability in D .

In fact, D is the desired DAG that represents the Birkhoff representation $(\Pi^*(\mathcal{R}_{st}), \preceq)$ of $(\mathcal{R}_{st}, \subseteq)$. Let $(V', \preceq)$ be a poset where reachability in D defines the partial order i.e., for any $u, v \in V'$, $u \preceq v$ if and only if u is reachable from v in D . From the above discussion, the map $R: \mathcal{I}(V') \rightarrow \mathcal{R}_{st}$ is an isomorphism between $(\mathcal{I}(V'), \subseteq)$ and $(\mathcal{R}_{st}, \subseteq)$, because for all $X, X' \in \mathcal{I}(V')$, we have $X \subseteq X'$ if and only if $R(X) \subseteq R(X')$. Moreover, choose an arbitrary maximal chain $\emptyset = X_0 \subsetneq X_1 \subsetneq \dots \subsetneq X_n = V'$ in $\mathcal{I}(V')$. Since $R(X_0) \subsetneq R(X_1) \subsetneq \dots \subsetneq R(X_n)$ forms a maximal chain from $X_\perp$ to $X_\top$ in $\mathcal{R}_{st}$, and $|X_i \setminus X_{i-1}| = 1$ for all $i \in [n]$, we obtain an isomorphism that maps each $v \in V'$ to $R(v)$, implying the equivalence between $(V', \preceq)$ and $(\Pi^*(\mathcal{R}_{st}), \preceq)$.

Thus, as all strongly connected components in G can be found in linear time by [24], the partition $\Pi(\mathcal{R}_{st})$ and a DAG representing $(\Pi^*(\mathcal{R}_{st}), \preceq)$ can be constructed from G_f in $O(|V| + |A|)$ time. Furthermore, the number of vertices and arcs representing $(\Pi^*(\mathcal{R}_{st}), \preceq)$ are at most $|V|$ and $|A|$, respectively.

We then define a map $\hat{r}: A \rightarrow V^2$ by

$$\hat{r}(a) := \begin{cases} a & \text{if } f(a) = 1, \\ (t, t) & \text{if } f(a) = 0. \end{cases} \quad (1)$$

We confirm that $\hat{r}$ is a pre-reduction map.

► **Lemma 13** (*). *The map $\hat{r}: A \rightarrow V^2$ defined as (1) is a pre-reduction map.*

This together with Proposition 10 and the above argument implies that $T_P(G) + T_r(G) = O(|V| + |A|)$. Thus, by Theorem 8, we obtain the following, which implies Theorem 3 (1).

► **Theorem 14** (*). *The k -diverse problem with respect to d_φ of UNWEIGHTED MINIMUM s - t CUT can be solved in $O(n + (|B_k(\varphi)|m)^{1+o(1)} \log(\varphi(k)) \log k + kq)$ time, where n denotes the number of vertices in the input digraph G , m the number of arcs, and q the size of any minimum s - t cut in G . In particular, SUM- k -DIVERSE and COV- k -DIVERSE of UNWEIGHTED MINIMUM s - t CUT can be solved in $O(n + (km)^{1+o(1)})$ time and in $O(n + m^{1+o(1)} \log^2 k + kq)$ time, respectively.*

5.3 The k -diverse stable matching problem

In this subsection, we consider the k -diverse problem of STABLE MATCHING. Let us first introduce the problem STABLE MATCHING (see e.g., [14] for details). Let U and V be disjoint finite sets with the same size n endowed with total orders $\leq_u$ on V ($u \in U$) and $\leq_v$ on U ($v \in V$). Intuitively, the total order $\leq_u$ (resp. $\leq_v$) represents the preference of u on V (resp. v

on U); $v <_u v'$ means that “ u prefer v to v' .” A subset $M = \{(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)\} \subseteq U \times V$ is called a *matching* if M provides the one-to-one correspondence between U and V , i.e., all u_i, v_j are different, $U = \{u_1, u_2, \dots, u_n\}$, and $V = \{v_1, v_2, \dots, v_n\}$. For a matching M , we denote by $p_M(u)$ (resp. $p_M(v)$) the partner of $u \in U$ (resp. $v \in V$) in M , i.e., $(u, p_M(u)) \in M$ (resp. $(p_M(v), v) \in M$). A matching M is said to be *stable* if there is no pair $(u, v) \in U \times V$ such that $v <_u p_M(u)$ and $u <_v p_M(v)$. In STABLE MATCHING, we are given a tuple $(U, V; (\leq_u)_{u \in U}, (\leq_v)_{v \in V})$ of finite sets U, V with the same size n and the total orders $(\leq_u)_{u \in U}$ on V and $(\leq_v)_{v \in V}$ on U , and asked to find a stable matching M . A stable matching always exists and can be found in $O(n^2)$ time by the *Gale–Shapley algorithm* [13].

Clearly, STABLE MATCHING has the property (S), in which $q = |U| = |V|$. Similarly to Section 5.2, our aim is to construct, for some ring family $\mathcal{R}$, a DAG representing the Birkhoff representation $(\Pi^*(\mathcal{R}), \preceq)$, the partition $\Pi(\mathcal{R})$, and a pre-reduction map $\hat{r}$ based on $\mathcal{R}$.

Let $\mathbf{I} := (U, V; (\leq_u)_{u \in U}, (\leq_v)_{v \in V})$ be an instance of STABLE MATCHING, and let $n := |U| = |V|$. Then, the family $\mathcal{S}(\mathbf{I})$ of solutions of the instance $\mathbf{I}$ is the set of all stable matchings of $\mathbf{I}$. It is known [14, Theorem 1.3.2] that $\mathcal{S}(\mathbf{I})$ forms a distributive lattice with the partial order $\preceq$ defined by setting $M \preceq M'$ if and only if $p_M(u) \leq_u p_{M'}(u)$ for all $u \in U$. For each stable matching $M \in \mathcal{S}(\mathbf{I})$, the *P-set* of M , denoted by $P(M)$, is the set of pairs (u, v) such that v is at least as preferred as $p_M(u)$ by $u \in U$, i.e., $P(M) := \{(u, v) \in U \times V \mid v \leq_u p_M(u)\}$. Let $\mathcal{R}_{\mathbf{I}}$ denote the family of all P-sets for $\mathbf{I}$, i.e.,

$$\mathcal{R}_{\mathbf{I}} := \{P(M) \mid M \in \mathcal{S}(\mathbf{I})\}.$$

The map that sends a stable matching to its P-set is an isomorphism between $(\mathcal{S}(\mathbf{I}), \preceq)$ and $(\mathcal{R}_{\mathbf{I}}, \subseteq)$. Hence, $\mathcal{R}_{\mathbf{I}}$ forms a ring family, which serves the base of our pre-reduction map. We note that the unique minimal set $X_{\perp}$ of $\mathcal{R}_{\mathbf{I}}$ is nonempty. To ensure that $X_{\top}$ is a proper subset of the ground set, we add a new element $\top$ to V , and we regard $U \times (V \cup \{\top\})$ as the ground set in the following.

We can construct the Birkhoff representation $(\Pi^*(\mathcal{R}_{\mathbf{I}}), \preceq)$ of the distributive lattice $(\mathcal{R}_{\mathbf{I}}, \subseteq)$ following the work of Gusfield and Irving [14, Chapter 3]. For each stable matching M and each $u \in U$, let $s_M(u)$ denote the most preferable element $v \in V$ for u such that $u <_v p_M(v)$, if such a v exists. A rotation is an ordered list $\rho = ((u_0, v_0), (u_1, v_1), \dots, (u_{c-1}, v_{c-1}))$ of pairs in some stable matching M , satisfying $s_M(u_i) = v_{i+1}$ for all $i \in [0, c-1]$ (where $i+1$ is taken modulo c). Let $\Lambda_{\mathbf{I}}$ be the set of all rotations of $\mathbf{I}$. It is shown that the map d defined by

$$d(\rho) = \{(u_i, v) \in U \times V \mid i \in [0, c-1], v_i <_{u_i} v \leq_{u_i} v_{i+1}\}$$

is a bijection between $\Lambda_{\mathbf{I}}$ and $\Pi^*(\mathcal{R}_{\mathbf{I}})$. A partial order on $\Lambda_{\mathbf{I}}$ is induced by $\Pi^*(\mathcal{R}_{\mathbf{I}})$ through d . Gusfield and Irving present an $O(n^2)$ -time algorithm that constructs $\Lambda_{\mathbf{I}}$ along with a DAG representing the poset over $\Lambda_{\mathbf{I}}$ having $O(n^2)$ arcs [14, Lemma 3.3.2 and Corollary 3.3.1]. In particular, their algorithm computes the unique minimal and maximal stable matchings.

We can construct $\Pi^*(\mathcal{R}_{\mathbf{I}})$, which is a partition of $X_{\top} \setminus X_{\perp}$, from $\Lambda_{\mathbf{I}}$ in $O(n^2)$ time using d . Clearly, $X_{\perp}$ is the P-set of the unique minimal stable matching. Thus, we can construct the partition $\Pi(\mathcal{R}_{\mathbf{I}})$ and a DAG with $O(n^2)$ arcs representing $(\Pi^*(\mathcal{R}_{\mathbf{I}}), \preceq)$ in $O(n^2)$ time.

We then construct a pre-reduction map. For each $u \in U$ and $v \in V$, we denote by v_u^- the element in V that is the cover of v with respect to $\leq_u$, i.e., $v <_u v_u^-$ and there is no element v' in V with $v <_u v' <_u v_u^-$. If v is the maximum element with respect to $\leq_u$, then we define $v_u^- := \top$. We define a map $\hat{r}: U \times V \rightarrow (U \times (V \cup \{\top\}))^2$ by

$$\hat{r}(u, v) := ((u, v), (u, v_u^-)). \quad (2)$$

We show that this is a pre-reduction map.

► **Lemma 15** (★). *The map $\hat{r}: U \times V \rightarrow ((U \times V) \cup \{\top\})^2$ defined as (2) is a pre-reduction map.*

This together with Proposition 10 and the above argument implies that we have $T_P(\mathbf{I}) + T_r(\mathbf{I}) = O(n^2)$. Thus, by Theorem 8, we obtain the following, which implies Theorem 3 (2).

► **Theorem 16** (★). *The k -diverse problem with respect to d_φ of STABLE MATCHING can be solved in $O((|B_k(\varphi)|n^2)^{1+o(1)} \log(\varphi(k)) \log k + kn)$ time. In particular, SUM- k -DIVERSE of STABLE MATCHING and COV- k -DIVERSE of STABLE MATCHING can be solved in $O((kn^2)^{1+o(1)})$ time and in $O(n^{2+o(1)} \log^2 k + kn)$ time, respectively.*

5.4 The k -diverse problem on the product of total orders

Very recently, De Berg, Martínez, and Spieksma [8] introduce a framework for the k -diverse problem on the product of total orders, which leads to the polynomial-time solvability of SUM- k -DIVERSE/COV- k -DIVERSE of UNWEIGHTED MINIMUM s - t CUT/STABLE MATCHING. In this subsection, we briefly introduce their framework and show that our framework can capture theirs for SUM- k -DIVERSE and COV- k -DIVERSE. Here, we do *not* assume that PROB has the properties (S) and (R).

We first introduce some terminology. For a distributive lattice $(\mathcal{L}, \preceq)$, an element $x \in \mathcal{L}$ is said to be *join-irreducible* if $x \neq y \vee z$ for any $y, z \in \mathcal{L} \setminus \{x\}$. Let $\mathcal{L}_{\text{ir}}$ denote the set of join-irreducible elements in $\mathcal{L}$. Then, it is known [5] that the subposet $(\mathcal{L}_{\text{ir}}, \preceq)$ of $\mathcal{L}$ induced by $\mathcal{L}_{\text{ir}}$ forms a Birkhoff representation of $\mathcal{L}$; the map $I \mapsto \bigvee_{x \in I} x$ is an isomorphism from $(\mathcal{I}(\mathcal{L}_{\text{ir}}), \subseteq)$ to $(\mathcal{L}, \preceq)$. We refer to this Birkhoff representation $(\mathcal{L}_{\text{ir}}, \preceq)$ as the *join-irreducible representation* of $\mathcal{L}$. For total orders $(E_1, \leq_1), (E_2, \leq_2), \dots, (E_q, \leq_q)$, their *product* $(\mathcal{E}, \leq)$ is the poset such that its ground set $\mathcal{E}$ is the product $E_1 \times E_2 \times \dots \times E_q$ of $E_1, E_2, \dots, E_q$ and the partial order $\leq$ is defined by setting $(e_1, e_2, \dots, e_q) \leq (e'_1, e'_2, \dots, e'_q)$ if and only if $e_i \leq_i e'_i$ for all $i \in [q]$. Actually, this $\mathcal{E}$ forms a lattice; the meet $(e_1, e_2, \dots, e_q) \wedge (e'_1, e'_2, \dots, e'_q)$ is given by $(\min\{e_i, e'_i\})_{i \in [q]}$ and the join $(e_1, e_2, \dots, e_q) \vee (e'_1, e'_2, \dots, e'_q)$ is given by $(\max\{e_i, e'_i\})_{i \in [q]}$, where

$$\min\{e_i, e'_i\} := \begin{cases} e_i & \text{if } e_i \leq_i e'_i, \\ e'_i & \text{if } e'_i <_i e_i, \end{cases} \quad \max\{e_i, e'_i\} := \begin{cases} e'_i & \text{if } e_i \leq_i e'_i, \\ e_i & \text{if } e'_i <_i e_i \end{cases}$$

for each $i \in [q]$. We say that a subset $\mathcal{L} \subseteq \mathcal{E}$ is a *sublattice* if $\mathcal{L}$ is closed under the meet $\wedge$ and join $\vee$, i.e., $x, y \in \mathcal{L}$ implies $x \wedge y, x \vee y \in \mathcal{L}$. We can observe that a sublattice of the product of total orders is distributive (see the paragraph after Theorem 17).

De Berg, Martínez, and Spieksma [8] impose the following property on PROB:

(T) For any instance $\mathbf{I}$ of PROB, there are q total orders $(E_1, \leq_1), (E_2, \leq_2), \dots, (E_q, \leq_q)$ such that the family $\mathcal{S}(\mathbf{I})$ of solutions of $\mathbf{I}$ is a sublattice of the product of those total orders. Then they show that, if PROB has the property (T) and we can construct (a DAG representing) the join-irreducible representation $(\mathcal{S}(\mathbf{I})_{\text{ir}}, \preceq)$ of $\mathcal{S}(\mathbf{I})$ in polynomial time, then we can solve SUM- k -DIVERSE/COV- k -DIVERSE of PROB in polynomial time by using an algorithm for the SFM problem over the distributive lattice $\mathcal{I}(\mathcal{S}(\mathbf{I})_{\text{ir}})$. The problem STABLE MATCHING has the property (T). Indeed, for an instance $\mathbf{I} := (U, V; (\leq_u)_{u \in U}, (\leq_v)_{v \in V})$ of STABLE MATCHING, we define $E_u := \{(u, v) \mid v \in V\}$ for each $u \in U$ and extend the total order $\leq_u$ on V to that on E_u by setting $(u, v) \leq_u (u, v')$ if and only if $v \leq_u v'$. Then, $\mathcal{S}(\mathbf{I})$ forms a sublattice of the product of $(E_u, \leq_u)$ for all $u \in U$. Similarly, UNWEIGHTED MINIMUM s - t CUT also has the property (T) by introducing the *left-right order* to q arc disjoint paths, where q denotes the size of a minimum s - t cut (or the maximum number of arc disjoint s - t paths); see [7, 8] for details.

We can show the following, whose proof is almost the same as that in Section 5.3.

► **Theorem 17** (★). *If a combinatorial problem PROB has the property (T), then it also has the properties (S) and (R).*

Here, for a solution $X = (e_1, e_2, \dots, e_q) \in \mathcal{S}(\mathbf{I})$, we define its *P-set* $P(X)$ by $P(X) := \{(e'_1, e'_2, \dots, e'_q) \mid e'_i \leq_i e_i \ (i \in [q])\}$ and denote by $\mathcal{R}_{\mathbf{I}}$ the family of all P-sets of $\mathbf{I}$, as in Section 5.3. Then we can easily see that $(\mathcal{S}(\mathbf{I}), \leq)$ and $(\mathcal{R}_{\mathbf{I}}, \subseteq)$ are isomorphic, both of which are distributive.

Let us see the result of De Berg, Martínez, and Spieksma [8] in detail to compare the time complexity of their algorithm with ours. Let E be the disjoint union (or direct sum) of $E_1, E_2, \dots, E_q$. We denote by $T_{\text{ir}}(\mathbf{I})$ the time required to construct a DAG representing the join-irreducible representation $(\mathcal{S}(\mathbf{I})_{\text{ir}}, \leq)$, and by $T_{\text{SFM}}(n, \text{EO})$ the time required to minimize a submodular function f with n variables such that one value evaluation of f takes EO time. Then, it is shown in [8] that we can solve SUM- k -DIVERSE/COV- k -DIVERSE of PROB having the property (T) in $O(T_{\text{ir}}(\mathbf{I}) + T_{\text{SFM}}(k|E|, k^2|E|q))$ time. Using the state-of-the-art algorithm for SFM given in [18] with we have $T_{\text{SFM}}(n, \text{EO}) = O(n^3 \text{EO})$. Hence, the running-time of the algorithms of De Berg, Martínez, and Spieksma is $O(T_{\text{ir}}(\mathbf{I}) + k^5|E|^4q)$.

Our framework provides much faster algorithms for SUM- k -DIVERSE and COV- k -DIVERSE of PROB as follows.

► **Theorem 18** (★). *Suppose that a combinatorial problem PROB has the property (T). Then we can solve SUM- k -DIVERSE and COV- k -DIVERSE of PROB in $O(T_{\text{ir}}(\mathbf{I}) + |E|q + (|A| + k|E|)^{1+o(1)})$ time and in $O(T_{\text{ir}}(\mathbf{I}) + (|E| + k)q + (|A| + |E|)^{1+o(1)} \log^2 k)$ time for an instance $\mathbf{I}$ of PROB, respectively, where A denotes the arc set of a constructed DAG representing $(\mathcal{S}(\mathbf{I})_{\text{ir}}, \leq)$.*

Since $|\mathcal{S}(\mathbf{I})_{\text{ir}}|$ is upper-bounded by the length of a maximal chain in the product of total orders $(E_1, \leq_1), (E_2, \leq_2), \dots, (E_q, \leq_q)$, we have $|\mathcal{S}(\mathbf{I})_{\text{ir}}| = O(|E|)$. Thus, $|A|$ is upper-bounded by $|\mathcal{S}(\mathbf{I})_{\text{ir}}|(|\mathcal{S}(\mathbf{I})_{\text{ir}}| - 1) = O(|E|^2)$. Even in the worst case of $|A| = \Theta(|E|^2)$, the running-time of our algorithms for SUM- k -DIVERSE and COV- k -DIVERSE of PROB are $O(T_{\text{ir}}(\mathbf{I}) + (|E|^2 + k|E|)^{1+o(1)})$ time and $O(T_{\text{ir}}(\mathbf{I}) + kq + |E|^{2+o(1)} \log^2 k)$ time, respectively, which are much faster than the previous ones that take $O(T_{\text{ir}}(\mathbf{I}) + k^5|E|^4q)$ time.

We also mention that the framework of [8] works for a diversity measure based on lengths of the maximal chains. It may be interesting to generalize our result to treat such a measure.

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