

Zero-Freeness Is All You Need: A Weitz-Type FPTAS for the Entire Lee–Yang Zero-Free Region

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Abstract

We present a Weitz-type FPTAS for the ferromagnetic Ising model across the entire Lee–Yang zero-free region, without relying on the strong spatial mixing (SSM) property. Our algorithm is Weitz-type for two reasons. First, it expresses the partition function as a telescoping product of ratios, with the key being to approximate each ratio. Second, it uses Weitz’s self-avoiding walk tree, and truncates it at logarithmic depth to give a good and efficient approximation. The key difference from the standard Weitz algorithm is that we approximate a carefully designed edge-deletion ratio instead of the marginal probability of a vertex being assigned a particular spin, ensuring our algorithm does not require SSM.

Furthermore, by establishing local dependence of coefficients (LDC), we indeed prove a novel form of SSM for these edge-deletion ratios, which, in turn, implies the standard SSM for the random cluster model. This is the first SSM result for the random cluster model on general graphs, beyond lattices. Our proof of LDC is based on a new division relation, and we show such relations hold quite universally. This leads to a broadly applicable framework for proving LDC across a variety of models, including the Potts model, the hypergraph independence polynomial, and Holant problems. Combined with existing zero-freeness results for these models, we derive new SSM results for them.

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1 Introduction

A *fully polynomial-time approximation scheme* (FPTAS) is a family of algorithms $\{A_\varepsilon\}$, where A_ε is a multiplicative $(1 \pm \varepsilon)$ -approximation algorithm with running time polynomial in both the input size and $1/\varepsilon$ for each $\varepsilon > 0$. For counting problems, a standard approach to designing FPTASes is based on complex zero-free regions of the associated *partition function*. Once such a zero-free region is established, Barvinok’s algorithm [2] provides an FPTAS for approximating the partition function in a slightly smaller region. Specifically, suppose the partition function Z has no zeros in a complex region that contains a computationally tractable point. Then, possibly after a complex conformal mapping, the zero-freeness property ensures that $\log Z$ can be well-approximated in a slightly smaller region by its



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Taylor expansion series truncated at degree $O(\log n)$ where n is the instance size. More precisely, the Taylor expansion series f_k of $\log Z$ truncated at degree k gives an approximation of $\log Z$ within additive error Cr^{-k} for some positive constant C and $r > 1$. The coefficients of f_k can be computed via subgraph counting in time $\Delta^{O(k)}$ [21] where Δ is the maximum degree. Clearly, the running time is polynomial on n when $k = O(\log n)$. This approach connects the long-standing study of complex zeros of the partition function in statistical physics to algorithmic studies.

Another (and earlier) approach for devising FPTASes, originating in the work of Weitz [31] and independently in Bandyopadhyay and Gamarnik [1], relies on the *correlation decay* property, or more precisely, *strong spatial mixing* (SSM). Roughly speaking, SSM asserts that correlations between spins decay exponentially with distance. Weitz’s algorithm approximates the partition function defined on a graph G by expressing it as a telescoping product of certain marginal probabilities. The key technical ingredient of Weitz’s algorithm is the *self-avoiding walk* (SAW) tree, which reduces the computation of a marginal probability on the original graph G to that on the SAW tree. However, the SAW tree may be exponentially large compared to the graph size n . The SSM property guarantees that the marginal probability can be well approximated by truncating the SAW tree at a depth of $O(\log n)$, making the evaluation efficient.

Both Barvinok’s algorithm and Weitz’s algorithm have been widely applied, especially to the study of 2-spin systems, which are among the most fundamental and well-studied models in statistical physics and counting problems. A 2-spin system is defined on a finite simple graph $G = (V, E)$ with parameters (β, γ, λ) : two edge activities β, γ representing edge interactions, and a vertex activity λ representing an external field. A partial configuration of this system refers to a mapping $\sigma : \Lambda \rightarrow \{+, -\}$ for some $\Lambda \subseteq V$ which may be empty. When $\Lambda = V$, it is a configuration and is assigned a weight $w(\sigma) = \beta^{m_+(\sigma)} \gamma^{m_-(\sigma)} \lambda^{n_+(\sigma)}$, where $m_+(\sigma)$ and $m_-(\sigma)$ count $(+, +)$ and $(-, -)$ edges respectively, and $n_+(\sigma)$ counts vertices with spin $+$. The associated partition function is $Z_G(\beta, \gamma, \lambda) = \sum_{\sigma: V \rightarrow \{+, -\}} w(\sigma)$. Many natural combinatorial problems reduce to evaluating $Z_G(\beta, \gamma, \lambda)$. For instance, the case $(\beta = 0, \gamma = 1)$ corresponds to the hard-core model (independence polynomial), while $\beta = \gamma$ gives the celebrated Ising model. Depending on whether $\beta\gamma > 1$ or $\beta\gamma < 1$, the model is classified as ferromagnetic or antiferromagnetic, respectively.

Although FPTASes for 2-spin systems have been obtained via both Barvinok’s algorithm [23, 6, 20, 19, 18, 24, 11, 22, 13, 27] and Weitz’s algorithm [33, 16, 17, 29], the applicability differs. While Barvinok’s algorithm covers broad regions including ferromagnetic systems, Weitz’s algorithm is mainly effective for antiferromagnetic systems where SSM holds. The SSM property crucially required by Weitz’s algorithm is often absent in the ferromagnetic regime. Recent work [25, 28] established a connection between these two frameworks by showing that zero-freeness implies SSM, provided zero-free results hold for graphs with pinned vertices. As a consequence, some new SSM results have been proved for 2-spin systems [28], which makes Weitz’s algorithm can be applied. However, some of the most celebrated zero-freeness results, such as the Lee–Yang theorem [32, 15] for the ferromagnetic Ising model, only hold for graphs without pinned vertices. Consequently, for the ferromagnetic Ising model on graphs of bounded degree, although Barvinok’s algorithm yields an FPTAS throughout the Lee–Yang zero-free region [19], i.e., $\lambda \in \mathbb{C}$ and $|\lambda| < 1$ or $|\lambda| > 1$ symmetrically, Weitz’s algorithm cannot be applied to the entire zero-free region due to the lack of SSM. So far, the best known SSM results hold for regions much smaller than the Lee–Yang zero-free region [28, 27], namely the union of the open disk centered at 0 with radius $\beta^{-\Delta}$ where Δ is the degree bound and a strip-shaped neighborhood of the real

segment $[0, \frac{1}{\beta^{\Delta-2}((\Delta-2)\beta^2-\Delta)})$. In fact, it is known that SSM does not hold throughout the entire zero-free region [3, 29]. For instance, for the three-dimensional Ising model at low temperatures where the Lee–Yang theorem holds, it is known that SSM does not hold [3], although weak spatial mixing does. Although these deterministic-approximation approaches face inherent limitations, it is worth noting that the ferromagnetic Ising model also admits classical polynomial-time approximate sampling algorithms via Markov chain Monte Carlo, most notably the Jerrum–Sinclair algorithm [14].

So far, for 2-spin systems, the regions where Barvinok’s algorithm applies strictly contain and are much larger than those accessible to Weitz’s algorithm. This raises a natural and interesting question: *Is Barvinok’s algorithm inherently more powerful than Weitz’s algorithm?* In this paper, we provide negative evidence for this question.

Our contributions

► **Theorem 1 (Informal).** *There is a Weitz-type FPTAS for the ferromagnetic Ising model throughout the entire Lee–Yang zero-free region. The algorithm does not require SSM.*

Our algorithm is a Weitz-type algorithm for two reasons. First, it expresses the partition function as a telescoping product of certain ratios and the key is to approximate each ratio. Secondly, in order to give a good approximation of the ratios, it uses the SAW tree and truncates it at logarithmic depth. However, crucial differences distinguish our algorithm from the standard Weitz algorithm, ensuring that our algorithm does *not* rely on SSM. First, instead of approximating the marginal probability $P_v = \frac{Z_{G, \sigma(v)=+}}{Z_G}$ of a vertex v being assigned spin $+$, we approximate a carefully designed edge-deletion ratio $P_e = \frac{Z_{G-e}}{Z_G}$ where $G - e$ denotes the graph obtained from G by removing an edge e . Second, since SSM is unavailable, we cannot argue that truncating the SAW tree yields a good approximation. Inspired by Barvinok’s algorithm, we show that each edge-deletion ratio viewed as a function on λ can be well approximated by truncating its Taylor expansion series at logarithmic degree.

A key difference from Barvinok’s algorithm is that, while in the Barvinok framework the low-order Taylor coefficients are computed via subgraph counting [21, 19], we compute these coefficients still via recursion on the SAW tree. The idea of using SAW-tree recursions to compute low-order coefficients has also been explored by Bencs and Regts [7], who compute low-order Taylor coefficients of $\log Z$. We note that their method remains Barvinok-type, as their goal is to approximate $\log Z$. In contrast, our approach is a genuinely Weitz-style telescoping algorithm that directly approximates each ratio using the SAW tree. Thus, although both approaches use recursive computation of Taylor coefficients, their algorithmic structures and objectives differ substantially.

The replacement of P_v by P_e is crucial for the above method to work. In fact, it is impossible to show that P_v could be approximated by logarithmic-degree Taylor truncation, since this would imply SSM throughout the Lee–Yang region contradicting known impossibility results [3]. The obstruction comes from the fact that, as shown in [28], the ratio P_v viewed as a function on λ and its SAW-tree version $P_v^{T_k}$ truncated at depth k share the same first k coefficients, known as *local dependence of coefficients* (LDC). Hence, if the Taylor expansion series f_k truncated at degree k approximates P_v well, i.e., $|P_v - f_k| \leq Cr^{-k}$ for some positive constants C and $r > 1$, then f_k also approximates $P_v^{T_k}$ well. Then, by a triangle inequality, one would have $|P_v - P_v^{T_k}| \leq 2Cr^k$, which is the standard SSM for the Ising model.

The above argument further implies that if LDC can be established for edge-deletion ratios, then together with zero-freeness, which guarantees good approximation by Taylor series of logarithmic degree, we obtain a form of SSM for these ratios. In other words, zero-freeness

plus LDC implies SSM. In this paper, we further show that such a LDC property indeed holds for the Ising model with arbitrary parameters β and λ , including regimes beyond the Lee–Yang zero-free region. Thus, together with the Lee–Yang theorem, we establish SSM for edge-deletion ratios¹ across the entire zero-free region. In summary, for Weitz-type FPTASes, zero-freeness alone suffices, while for edge-deletion SSM, one additionally needs LDC, a combinatorial property independent of zero-freeness.

► **Theorem 2** (SSM for edge-deletion ratios). *Fix $\beta > 1$ and $\lambda \in \mathbb{D}$. Then there exist constants $C > 0$ and $r > 1$ such that for every graph $G = (V, E)$, for any edge $e \in E$ and subsets $A, B \subseteq E \setminus \{e\}$, we have*

$$|P_{G-A,e} - P_{G-B,e}| \leq Cr^{-d_G(e, A \neq B)},$$

where $P_{G,e}$ denotes the edge-deletion ratio Z_{G-e}/Z_G and $d_G(e, A \neq B)$ is the distance in G between the edge e and the set of edges $(A \setminus B) \cup (B \setminus A)$.

Quite remarkably, the choice of the edge-deletion ratio is not only crucial for a Weitz-type FPTAS for the Ising model, but it also admits a probabilistic interpretation in the Ising related random cluster model. It corresponds exactly to the marginal probability that an edge is included in the random cluster model. Thus, our edge-deletion SSM implies standard SSM for the random cluster model. This is the first SSM result for the random cluster model on general graphs, whereas all previous results were confined to lattices. As brief background, the random cluster model associated the ferromagnetic Ising model is parameterized by edge parameters p and vertex parameters λ . For an edge $e \in E$ and a partial edge configuration σ_Λ specified on $\Lambda \subseteq E(G)$, let $P_{G,e}^{\sigma_\Lambda}$ denote the marginal probability under the random cluster distribution that edge e is excluded conditioned on σ_Λ .

► **Theorem 3** (SSM for the random cluster model). *Fix the parameters of the random cluster model $p \in [0, 1]$ and $\lambda \in [0, 1]$. Then there exist constants $C > 0$ and $r > 1$ such that for any $G = (V, E)$, any edge $e \in E$, any partial configuration σ_{Λ_1} and τ_{Λ_2} where $\Lambda_1, \Lambda_2 \subseteq E \setminus e$, we have*

$$|P_{G,e}^{\sigma_{\Lambda_1}} - P_{G,e}^{\tau_{\Lambda_2}}| \leq Cr^{-d_G(e, \sigma_{\Lambda_1} \neq \tau_{\Lambda_2})}.$$

where $(\sigma_{\Lambda_1} \neq \tau_{\Lambda_2}) = (\Lambda_1 \setminus \Lambda_2) \cup (\Lambda_2 \setminus \Lambda_1) \cup \{f \in \Lambda_1 \cap \Lambda_2 : \sigma_{\Lambda_1}(f) \neq \tau_{\Lambda_2}(f)\}$.

Beyond introducing the notion of edge-deletion SSM, a main technical contribution of this paper is a broadly applicable method for establishing LDC. The LDC property was first shown implicitly in [25] using cluster expansion, an analytic method applicable to the hard-core model, and was later formally introduced and extended to general 2-spin systems [28] using a combinatorial approach based on a Christoffel–Darboux type identity. In this paper, we substantially generalize the combinatorial framework by circumventing the need of explicit identities: a more applicable division relation proved via a delicate one-to-one correspondence already suffices for LDC. We show that such division relations hold broadly across diverse models including the Potts model, the hypergraph independence polynomial, and binary symmetric Holant problems. For each model, combining division relations with existing zero-free regions yields new SSM results, thereby broadening the applicability of correlation-decay techniques.

¹ We actually prove a more general form of SSM; see Theorem 13.

As an example, we prove the following LDC and SSM properties for the hypergraph independence polynomial. We define $P_{H,v}^{\sigma_{\Lambda}}(\lambda)$ as the standard marginal probability of the vertex v being occupied in hypergraph H under the partial configuration σ_{Λ} on $\Lambda \subseteq V(H)$.

► **Lemma 4** (LDC for the hypergraph independence polynomial). *Let $H = (V, E)$ be a hypergraph, $\sigma_{\Lambda_1}, \tau_{\Lambda_2}$ be two partial configurations on $\Lambda_1, \Lambda_2 \subseteq V$, v be a proper vertex to σ_{Λ_1} and τ_{Λ_2} , then the formal power series of $P_{H,v}^{\sigma_{\Lambda_1}}(\lambda)$ and $P_{H,v}^{\tau_{\Lambda_2}}(\lambda)$ agree in their first $d_H(v, \sigma_{\Lambda_1} \neq \tau_{\Lambda_2})$ coefficients.*

Combining this LDC property with the optimal zero-free region of [5], where $\lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}$ and $\lambda_s(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{\Delta^{\Delta}}$, we obtain the following new SSM result.

► **Theorem 5** (SSM for the hypergraph independence polynomial). *Fix $\Delta \geq 3$ and $\lambda \in \mathbb{D}_{\lambda_s(\Delta)} \cup (0, \lambda_c(\Delta))$. There exist constants $C > 0$ and $r > 1$ such that for any hypergraph $H = (V, E)$ with maximum degree at most Δ , any two feasible partial configurations σ_{Λ_1} and τ_{Λ_2} where Λ_1 may be different with Λ_2 , and any vertex v proper to σ_{Λ_1} and τ_{Λ_2} , we have*

$$|P_{H,v}^{\sigma_{\Lambda_1}}(\lambda) - P_{H,v}^{\tau_{\Lambda_2}}(\lambda)| \leq Cr^{-d_H(v, \sigma_{\Lambda_1} \neq \tau_{\Lambda_2})}$$

where $(\sigma_{\Lambda_1} \neq \tau_{\Lambda_2}) = (\Lambda_1 \setminus \Lambda_2) \cup (\Lambda_2 \setminus \Lambda_1) \cup \{v \in \Lambda_1 \cap \Lambda_2 : \sigma_{\Lambda_1}(v) \neq \tau_{\Lambda_2}(v)\}$.

Organization

The paper is organized as follows. In Section 3, to derive the Weitz-type algorithm, we show that the truncated series provides a good approximation of the edge-deletion ratio using zero-freeness, and then analyze the truncated series through the self-avoiding walk tree and operations on formal power series. This yields the FPTAS. In Section 4, we introduce the framework in which zero-freeness implies SSM via LDC, and establish the SSM property in terms of edge activities for the ferromagnetic Ising model, using the Lee–Yang theorem and the divisibility relation. This edge-based SSM property further implies the standard SSM of the corresponding random-cluster model. In Section 5, we extend the divisibility relation to various models and derive their SSM properties. Omitted proofs can be found in the full paper [26].

2 Preliminaries

2.1 Ising model

For a graph $G = (V, E)$, with edge activities $\beta = (\beta_e)_{e \in E}$ and vertex activities $\lambda = (\lambda_v)_{v \in V}$, the weight of a configuration $\sigma : V \rightarrow \{+, -\}$ is given by

$$w(\sigma) = \prod_{e \in m(\sigma)} \beta_e \prod_{v \in n(\sigma)} \lambda_v$$

where $m(\sigma) = \{e = (u, v) \in E \mid \sigma_u = \sigma_v\}$ is the set of edges whose endpoints have the same spin, and $n(\sigma) = \{v \in V \mid \sigma_v = +\}$ is the set of vertices assigned spin $+$. The partition function of the Ising model is defined by $Z_G(\beta, \lambda) = \sum_{\sigma: V \rightarrow \{+, -\}} w(\sigma)$. The celebrated Lee–Yang theorem states the zero-free region of the Ising model.

► **Theorem 6** (Lee–Yang theorem). *Let $G = (V, E)$ be a graph with parameters $\beta \in [1, \infty)^E$ and $\lambda \in \mathbb{D}^V$ where $\mathbb{D}$ is the unit open disk in the complex plane. Then the partition function of Ising model $Z_G(\beta, \lambda) \neq 0$.*

A partial configuration of the Ising model is a mapping $\sigma : \Lambda \rightarrow \{+, -\}$ for some $\Lambda \subseteq V$, which may be empty. The conditional partition function is defined as $Z_G^{\sigma_\Lambda}(\beta, \lambda) = \sum_{\sigma: V \rightarrow \{+, -\}} w(\sigma)$ where $\sigma|_\Lambda$ denotes the restriction of the configuration σ on Λ . Let $u, v \in V$, then define

$$Z_{G,v}^{\sigma_\Lambda, +}(\beta, \lambda) = \sum_{\substack{\sigma: V \rightarrow \{+, -\} \\ \sigma|_\Lambda = \sigma_\Lambda, \sigma(v) = +}} w(\sigma), \quad \text{and} \quad Z_{G,v,u}^{\sigma_\Lambda, +, +}(\beta, \lambda) = \sum_{\substack{\sigma: V \rightarrow \{+, -\} \\ \sigma|_\Lambda = \sigma_\Lambda, \sigma(v) = \sigma(u) = +}} w(\sigma).$$

Conditional partition functions for the remaining pinning choices $Z_{G,v}^{\sigma_\Lambda, -}(\beta, \lambda)$, $Z_{G,v,u}^{\sigma_\Lambda, +, -}(\beta, \lambda)$, $Z_{G,v,u}^{\sigma_\Lambda, -, +}(\beta, \lambda)$ and $Z_{G,v,u}^{\sigma_\Lambda, -, -}(\beta, \lambda)$ are defined analogously. Then the conditional marginal probability that v is pinned to $+$ given the partial configuration σ_Λ and the corresponding marginal ratio are defined as

$$P_{G,v}^{\sigma_\Lambda}(\beta, \lambda) = \frac{Z_{G,v}^{\sigma_\Lambda, +}(\beta, \lambda)}{Z_G^{\sigma_\Lambda}(\beta, \lambda)}, \quad \text{and} \quad R_{G,v}^{\sigma_\Lambda}(\beta, \lambda) = \frac{Z_{G,v}^{\sigma_\Lambda, +}(\beta, \lambda)}{Z_{G,v}^{\sigma_\Lambda, -}(\beta, \lambda)}.$$

2.2 Weitz's tree reduction

In the seminal work of Weitz [31], the self-avoiding walk (SAW) tree reduces the computation of marginal probabilities for 2-spin models on general graphs to the corresponding computation on trees. If the graph $G = (V, E)$ has maximum degree Δ , then the SAW tree $T_{\text{SAW}}(G, v)$ rooted at $v \in V$ also has maximum degree Δ . Moreover, the marginal probability and marginal ratio at v in G coincide with those at the root of $T_{\text{SAW}}(G, v)$ (with some leaves possibly pinned).

Consider a rooted tree $T = (V, E)$ with root $r \in V$. Suppose r has d children $v_1, \dots, v_d$. Removing r yields d subtrees $T_1, \dots, T_d$, where T_i is rooted at v_i . Let R_{T_i, v_i} denote the marginal ratio at v_i in T_i (e.g., $R_{T_i, v_i} = Z_{T_i, v_i}^+ / Z_{T_i, v_i}^-$), and let $R_{T, r}$ be the marginal ratio at r in T . Then the tree recursion expressing $R_{T, r}$ in terms of the R_{T_i, v_i} is given by a multivariate map $F_d : \hat{\mathbb{C}}^d \rightarrow \hat{\mathbb{C}}$:

$$R_{T, r} = F_d(R_{T_1, v_1}, \dots, R_{T_d, v_d}), \quad F_d(x_1, \dots, x_d) = \lambda \prod_{i=1}^d \frac{\beta x_i + 1}{x_i + \gamma},$$

where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the extended complex plane, β, γ are the edge-interaction parameters, and λ is the external field. For the ferromagnetic Ising model, we have $\beta = \gamma > 1$.

2.3 Complex analysis tools and truncated power series

A *region* is a connected open set in $\mathbb{C}$. In particular, an open disk with one interior point removed is also a region. We denote by $\mathbb{D}_\rho(z_0)$ the open disk centered at z_0 with radius ρ , and by $\partial\mathbb{D}_\rho(z_0)$ its boundary circle. If $z_0 = 0$, we simply write $\mathbb{D}_\rho$ and $\partial\mathbb{D}_\rho$; if $\rho = 1$, we omit the subscript ρ . Let $f(z) = \sum_{k \geq 0} a_k z^k$ be a (formal) power series, and write its truncation to degree m as

$$f^{[m]} := \sum_{k=0}^m a_k z^k \quad (= f \bmod z^{m+1}).$$

The following lemma is a standard result of complex analysis textbook.

► **Lemma 7.** *Let $f(z)$ be analytic in a neighborhood of $z = 0$, and suppose $|f(z)| \leq M$ on the circle $\partial\mathbb{D}_\rho$ for some $\rho > 0$. Then for any $z \in \mathbb{D}_\rho$, we have*

$$|f(z) - f^{[n]}(z)| \leq \frac{M}{\rho(r-1)r^n}, \quad \text{where } r = \rho/|z| > 1.$$

Applying Lemma 7 requires a uniform bound on a circle for a family of analytic functions. In [25], Regts employed Montel's theorem to obtain such bounds, leading to the following result.

► **Lemma 8.** *Let $\mathbb{U}$ be a region, and let $\mathcal{F}$ be a family of holomorphic functions $f : \mathbb{U} \rightarrow \mathbb{C}$ such that $f(\mathbb{U}) \subseteq \mathbb{C} \setminus \{0, 1\}$ for all $f \in \mathcal{F}$. If there exist a point $z_0 \in \mathbb{U}$ and a constant C such that $|f(z_0)| \leq C$ for all $f \in \mathcal{F}$, then for any compact subset $S \subseteq \mathbb{U}$, there exists another constant C' such that for all $f \in \mathcal{F}$ and all $z \in S$, we have $|f(z)| \leq C'$.*

Next, assume the first $m+1$ coefficients of f and g are given. We can compute $(kf)^{[m]}$ and $(f+g)^{[m]}$ in $O(m)$ arithmetic operations, and $(fg)^{[m]}$ as well as $(f/g)^{[m]}$ (assuming $g(0) \neq 0$) in $O(m \log m)$ operations using FFT-based multiplication and Newton iteration; see, e.g., [30, Secs. 8.2 and 9.1].

3 Weitz-type algorithm for the ferromagnetic Ising model

Our approach is a telescoping algorithm based on edge deletion. For a graph $G = (V, E)$ with parameters (β, λ) , we order the edges in E as $e_1, e_2, \dots, e_m$, denote $G_i = (V, E_i)$ where $E_i = \{e_1, e_2, \dots, e_i\}$ for $1 \leq i \leq m$ and $G_0 = (V, \emptyset)$. Then we have

$$\frac{1}{Z_G(\lambda)} = \prod_{i=1}^m \frac{Z_{G_{i-1}}(\lambda)}{Z_{G_i}(\lambda)} \frac{1}{Z_{G_0}(\lambda)} = (1 + \lambda)^{-|V|} \prod_{i=1}^m P_{G_i, e_i}(\lambda).$$

Thus, approximating $Z_G(\lambda)$ within a multiplicative error of ε reduces to approximating each ratio $P_{G_i, e_i}(\lambda)$ within a multiplicative error of at most $\varepsilon/(4m)$, for all $1 \leq i \leq m$.

3.1 Truncated series is a good approximation

To show the truncation series gives a good approximation via Lemma 7, we apply Lemma 8 to obtain a uniform bound of $P_{G, e}(\lambda)$ for all graph G and edge $e \in G$. This requires that $P_{G, e}(\lambda)$ avoid 0 and 1 for all graph G and edge $e \in G$.

► **Lemma 9.** *Let $G = (V, E)$ be a graph. For edge activity $\beta > 1$ and external field $\lambda \in \mathbb{D}$, the edge-deletion ratio $P_{G, e}(\beta, \lambda)$ omits the values 0 and 1.*

Then a uniform bound of $P_{G, e}(\lambda)$ for all graph G and edge $e \in G$ follows from Lemma 8, where the upper bound (for a circle) is used to establish the additive error in Lemma 7, and the lower bound (for a single point) is used to turn the additive error into a multiplicative error.

► **Lemma 10.** *Fix $\beta > 1$ and $S \subseteq \mathbb{D}$ be a compact set. There exist constants $M, b > 0$ such that $b \leq |P_{G, e}(\lambda)| \leq M$ for all graph G , edge $e \in G$ and $\lambda \in S$.*

Proof. Fix $\beta > 1$ and a compact $S \subseteq \mathbb{D}$. Let $\mathcal{F} = \{P_{G, e}(z) : G \text{ a graph, } e \in E(G)\}$. By Lemma 9, every $f \in \mathcal{F}$ omits $\{0, 1\}$ on $\mathbb{D}$, and $f(0) = 1/\beta$ is uniformly bounded. Hence, by Lemma 8, there exists $M > 0$ such that $|P_{G, e}(z)| \leq M$ for all $z \in S$, all G , and all e .

For the lower bound, apply the same argument to $\mathcal{F}^{-1} = \{1/P_{G, e}(z)\}$. Each $g \in \mathcal{F}^{-1}$ also omits $\{0, 1\}$ and $g(0) = \beta$ is uniformly bounded, so there exists $M' > 0$ with $|1/P_{G, e}(z)| \leq M'$ for all $z \in S$. Setting $b := 1/M'$ yields $b \leq |P_{G, e}(z)| \leq M$ for all $z \in S$, as claimed. ◀

► **Lemma 11.** Fix $\beta > 1$ and $\lambda \in \mathbb{D}$. Then there exists $k = O(\log(m/\varepsilon))$ such that for every graph $G = (V, E)$ with $m = |E|$ and every edge $e \in E$, the truncated series $P_{G,e}^{[k]}$ evaluated at λ , satisfies the relative bound

$$\frac{|P_{G,e}(\lambda) - P_{G,e}^{[k]}(\lambda)|}{|P_{G,e}(\lambda)|} \leq \frac{\varepsilon}{4m}.$$

Proof. Pick $\rho = \frac{1+|\lambda|}{2}$ so that $\lambda \in \mathbb{D}_\rho$. By Lemma 10, there exists $M > 0$ such that $|P_{G,e}(z)| \leq M$ for all $z \in \partial\mathbb{D}_\rho$ and all G, e . Applying Lemma 7 to $P_{G,e}$ yields

$$|P_{G,e}(\lambda) - P_{G,e}^{[k]}(\lambda)| \leq C r^{-k}, \quad r = \rho/|\lambda| > 1,$$

for some constant C independent of G, e . Moreover, Lemma 10 with $S = \{\lambda\}$ provides a uniform lower bound $b > 0$ such that $|P_{G,e}(\lambda)| \geq b$ for all G, e . Therefore,

$$\frac{|P_{G,e}(\lambda) - P_{G,e}^{[k]}(\lambda)|}{|P_{G,e}(\lambda)|} \leq \frac{C}{b} r^{-k}.$$

Thus choosing $k = O(\log(m/\varepsilon))$ suffices to ensure the right-hand side is at most $\varepsilon/(4m)$. ◀

Replacing each factor $P_{G_i, e_i}(\lambda)$ in the telescoping product with its truncated approximation $P_{G_i, e_i}^{[k]}(\lambda)$ in Lemma 11, we obtain $\hat{Z} = (1 + \lambda)^{|V|} / \prod_{i=1}^m P_{G_i, e_i}^{[k]}(\lambda)$, which is the desired approximation to $Z_G(\lambda)$.

3.2 Computing the truncated series via Weitz's tree reduction

Write $e_i = (u, v)$. By the definition of the Ising model, we have

$$\begin{aligned} \frac{Z_{G_{i-1}}}{Z_{G_i}} &= \frac{Z_{G_{i-1}, u, v}^{+,+} + Z_{G_{i-1}, u, v}^{-,-} + Z_{G_{i-1}, u, v}^{-,+} + Z_{G_{i-1}, u, v}^{+,-}}{Z_{G_i, u, v}^{+,+} + Z_{G_i, u, v}^{-,-} + Z_{G_i, u, v}^{-,+} + Z_{G_i, u, v}^{+,-}} \\ &= \frac{(Z_{G_i, u, v}^{+,+} + Z_{G_i, u, v}^{-,-})/\beta + Z_{G_i, u, v}^{-,+} + Z_{G_i, u, v}^{+,-}}{Z_{G_i, u, v}^{+,+} + Z_{G_i, u, v}^{-,-} + Z_{G_i, u, v}^{-,+} + Z_{G_i, u, v}^{+,-}}. \end{aligned}$$

It holds because if u and v have the same spin, the only extra contribution in Z_{G_i} (compared with $Z_{G_{i-1}}$) is the edge $e_i = (u, v)$, which contributes a factor of β . Dividing numerator and denominator by $Z_{G_i, u, v}^{-,-}$ and substituting the ratio identities

$$\frac{Z_{G_i, u, v}^{+,+}}{Z_{G_i, u, v}^{-,-}} = \frac{Z_{G_i, u, v}^{+,+}}{Z_{G_i, u, v}^{+,-}} \cdot \frac{Z_{G_i, u, v}^{+,-}}{Z_{G_i, u, v}^{-,-}} = R_{G_i, v}^{u+} R_{G_i, u}^{v-}, \quad \frac{Z_{G_i, u, v}^{-,+}}{Z_{G_i, u, v}^{-,-}} = R_{G_i, u}^{v-}, \quad \frac{Z_{G_i, u, v}^{+,-}}{Z_{G_i, u, v}^{-,-}} = R_{G_i, v}^{u-},$$

we have

$$P_{G_i, e_i} = \frac{(R_{G_i, v}^{u+} R_{G_i, u}^{v-} + 1)/\beta + R_{G_i, v}^{u-} + R_{G_i, u}^{v-}}{R_{G_i, v}^{u+} R_{G_i, u}^{v-} + 1 + R_{G_i, v}^{u-} + R_{G_i, u}^{v-}}.$$

To calculate $P_{G_i, e_i}(\lambda)^{[k]}$, it suffices to calculate the ratios $R_{G_i, v}^{u+}(\lambda)^{[k]}$, $R_{G_i, u}^{v-}(\lambda)^{[k]}$ and $R_{G_i, v}^{u-}(\lambda)^{[k]}$. This can be done by the tree recursion on the self-avoiding walk tree $T_{\text{SAW}}(G_i^{u+}, v)$, $T_{\text{SAW}}(G_i^{v-}, u)$ and $T_{\text{SAW}}(G_i^{u-}, v)$ respectively. Recall the tree recursion formula $R_{T, r}(\lambda) = \lambda \prod_{i=1}^d \frac{\beta R_{T_i, v_i}(\lambda) + 1}{R_{T_i, v_i}(\lambda) + \beta}$. To compute $R_{T, r}(\lambda)^{[k]}$, it suffices to compute the truncated series of $\prod_{i=1}^d \frac{\beta R_{T_i, v_i}(\lambda) + 1}{R_{T_i, v_i}(\lambda) + \beta}$ up to degree $k-1$; hence, for each child v_i we require $R_{T_i, v_i}(\lambda)^{[k-1]}$. Therefore $R_{T, r}(\lambda)^{[k]}$ can be obtained by traversing the truncated self-avoiding

walk tree to depth k and, for every node at depth $i \in [0, k]$, computing the truncated series of its ratio to degree $k - i$. Suppose T has maximum degree Δ . At each node, we multiply at most Δ series and perform one division, which costs $O(\Delta k \log k)$ using FFT-based series arithmetic. The truncated SAW tree to depth k has $O(\Delta^k)$ nodes. Hence the total running time is $O(\Delta^k \cdot \Delta k \log k) = O(\Delta^{k+1} k \log k)$.

If G has maximum degree Δ , then the self-avoiding walk tree $T_{\text{SAW}}(G_i^{u+}, v)$, $T_{\text{SAW}}(G_i^{v-}, u)$ and $T_{\text{SAW}}(G_i^{u-}, v)$ also have maximum degree Δ . Thus, the time complexity for computing $P_{G_i, e_i}(\lambda)^{[k]}$ is $O(k \log k \Delta^{k+1})$.

Our algorithm shows that SSM is unnecessary for a Weitz-type FPTAS, as we do not rely on SSM for the marginal probability of a vertex being assigned a particular spin. Instead, it is crucial for our algorithm to replace marginal probabilities of vertices by edge-deletion ratios, which eliminates the need for SSM. In the next section, we show that, even though the standard notion of SSM does not hold, by further proving the local dependence of coefficients (LDC), a combinatorial property independent of zero-freeness, we can indeed establish a new form of SSM for edge deletion ratios. Thus, zero-freeness alone gives Weitz-type FPTASes, while zero-freeness plus LDC gives new forms of SSM.

4 SSM for generalized edge ratios

Note that deleting an edge e in the Ising model can be viewed as setting the edge activity β_e to 1, we extend the edge-deletion ratio $P_{G,e}$ to a more general setting. For a partition function $Z_G(\beta, \lambda)$ viewed as a multivariate function on edge activities $(\beta_e)_{e \in E}$ and vertex external fields $(\lambda_v)_{v \in V}$, and a partial evaluation $m(V', E') : (\beta_e)_{e \in E'} \rightarrow [1, \infty), (\lambda_v)_{v \in V'} \rightarrow \mathbb{D}$ (i.e., substituting specific values for variables $(\beta_e)_{e \in E'}$ and $(\lambda_v)_{v \in V'}$), we consider the function

$$Z_G^{m(V', E')}((\beta_e)_{e \in E \setminus E'}, (\lambda_v)_{v \in V \setminus V'})$$

where the values of $(\lambda_v)_{v \in V'}$ and $(\beta_e)_{e \in E'}$ are assigned by $m(V', E')$. When context is clear, we may omit the subscript $e \in E \setminus E'$ and $v \in V \setminus V'$ in $Z_G^{m(V', E')}$. Besides setting $\beta_e \rightarrow 1$, several other partial evaluations have special meanings. For example, the assignment $m(u) : \lambda_u \rightarrow 0$ that assigns the external field λ_u of a particular vertex $u \in V$ to 0 gives the function $Z_G^{m(u)} = Z_{G,u}^-$ which is the partition function of the Ising model on the graph G with a pinned vertex u to the $-$ spin.

In this paper, we focus on the partial evaluation $m(\emptyset, E')$ that only assigns values to edge activities for edges in E' . For simplicity, we write $m(\emptyset, E')$ as $m(E')$. Then, as an extension of the marginal probability $P_{G,v}$, we can define the ratio $P_{G, m(E')}(\beta, \lambda) = Z_G^{m(E')}(\beta, \lambda) / Z_G(\beta, \lambda)$ for any partial evaluation $m(E')$. Moreover, we can define the ratio conditioning on a pre-specified partial evaluation $m_1(E_1)$ by $P_{G, m(E')}^{m_1(E_1)}(\beta, \lambda) = Z_G^{m_1(E_1), m(E')}(\beta, \lambda) / Z_G^{m_1(E_1)}(\beta, \lambda)$ for partial evaluation $m(E')$ satisfying $E' \cap E_1 = \emptyset$. If context is clear, we may omit the arguments (β, λ) and the specification of edge sets E_1 and E' , and write $P_{G, m(E')}^{m_1(E_1)}(\beta, \lambda)$ as $P_{G, m}^{m_1}$ for simplicity.

With these notations in hand, we are able to define the generalized form of edge-type SSM.

► **Definition 12** (Generalized edge-SSM). *Let $\mathcal{G}$ be a family of graphs with parameters (β, λ) and $C_2 \geq C_1 \geq 0$ be constants. The Ising model defined on $\mathcal{G}$ is said to satisfy generalized edge-type strong spatial mixing (GE-SSM) with exponential rate $r > 1$ if there exists a constant C such that for any $G = (V, E) \in \mathcal{G}$, any $e \in E$ with $m = \{\beta_e \rightarrow \beta'_e\}$ where $\frac{\beta'_e}{\beta_e} \in [C_1, C_2]$ and any partial evaluation m_1, m_2 defined on $A, B \subseteq E \setminus \{v\}$ respectively, then*

$$\left| P_{G, m}^{m_1} - P_{G, m}^{m_2} \right| \leq C r^{-d_G(e, m_1 \neq m_2)}.$$

Here, we denote $(m_1 \neq m_2) = (A \setminus B) \cup (B \setminus A) \cup \{f \in A \cap B : m_1(f) \neq m_2(f)\}$, which is the set of edges where m_1 and m_2 differ. The quantity $d_G(e, m_1 \neq m_2)$ is the shortest distance from any endpoint of e to any endpoint of an edge in $m_1 \neq m_2$.

Indeed, we establish that the Ising model satisfies the generalized edge-type SSM introduced in Definition 12 (in fact, in a slightly stronger form; see the full version of the paper).

► **Theorem 13.** *Fix $\delta \in (0, 1)$. Then there exist constants $C > 0$ and $r > 1$ such that, for every finite graph $G = (V, E)$ with parameters $\beta \in [1, \infty)^E$ and $\lambda \in (1 - \delta)\mathbb{D}^V$, the ferromagnetic Ising model satisfies GE-SSM.*

Such an edge-type SSM does not have an explicit probabilistic meaning in the Ising model. However, through the relationship between the Ising model and the random cluster model, we found that it can be interpreted as the standard SSM in the random cluster model.

4.1 LDC via a combinatorial bijection

By [28], a key ingredient in establishing SSM from zero-freeness is the local dependence of coefficients (LDC). For further details, we refer to the full version. In this section, we focus on proving LDC by proving a division relation using a combinatorial bijection.

► **Definition 14 (LDC).** *We say that the Ising model satisfies LDC if for all graphs $G = (V, E)$ with parameters $\beta \in [1, \infty)^E$ and $\lambda \in \mathbb{D}^V$, the following holds. For an edge $e \in E$ and subsets $A, B \subseteq E \setminus \{e\}$, define the partial evaluations:*

$$m = \{\beta_e \rightarrow \beta'_e\}, \quad m_1 = \{\beta_f \rightarrow \beta_f^A\}_{f \in A}, \quad m_2 = \{\beta_f \rightarrow \beta_f^B\}_{f \in B}$$

where the modified parameters satisfy $\beta'_e \in [1, \infty)$, $\beta_f^A \in [1, \infty)$ for $f \in A$ and $\beta_f^B \in [1, \infty)$ for $f \in B$. It holds that

$$z^{d_G(e, m_1 \neq m_2) + 1} \mid \left(P_{G, m}^{m_1}(\beta, \lambda z) - P_{G, m}^{m_2}(\beta, \lambda z) \right).$$

We establish a divisibility relation that implies the LDC.

► **Lemma 15.** *Let $G = (V, E)$ be a graph with parameters (β, λ) where $\beta \in [1, \infty)^E$ and $\lambda \in \mathbb{D}^V$. Let $A, B \subseteq E$ be two disjoint edge sets, define the partial evaluations:*

$$m_1 = \{\beta_f \rightarrow \beta_f^A\}_{f \in A}, \quad m_2 = \{\beta_f \rightarrow \beta_f^B\}_{f \in B}$$

where the modified parameters satisfy $\beta_f^A \in [1, \infty)$ for $f \in A$ and $\beta_f^B \in [1, \infty)$ for $f \in B$. Then

$$z^{d_G(A, B) + 1} \mid (Z_G(\beta, \lambda z) Z_G^{m_1, m_2}(\beta, \lambda z) - Z_G^{m_1}(\beta, \lambda z) Z_G^{m_2}(\beta, \lambda z))$$

where $d_G(A, B) = \min_{e_1 \in A, e_2 \in B} d_G(e_1, e_2)$.

Proof. For simplicity, we omit $(\beta, \lambda z)$ in the notation. Let $\mathcal{S} = V \rightarrow \{+, -\}$, then

$$\begin{aligned} & Z_G Z_G^{m_1, m_2} - Z_G^{m_1} Z_G^{m_2} \\ &= \sum_{\sigma \in \mathcal{S}} w_G(\sigma) \sum_{\sigma \in \mathcal{S}} w_G^{m_1, m_2}(\sigma) - \sum_{\sigma \in \mathcal{S}} w_G^{m_1}(\sigma) \sum_{\sigma \in \mathcal{S}} w_G^{m_2}(\sigma) \\ &= \sum_{\substack{(\sigma_1, \sigma_2) \in \\ (\mathcal{S} \times \mathcal{S})}} w_G(\sigma_1) w_G^{m_1, m_2}(\sigma_2) - \sum_{\substack{(\sigma_3, \sigma_4) \in \\ (\mathcal{S} \times \mathcal{S})}} w_G^{m_1}(\sigma_3) w_G^{m_2}(\sigma_4) \end{aligned}$$

Let $R = \{(\sigma, \tau) \in \mathcal{S} \times \mathcal{S} : n_+(\sigma) + n_+(\tau) < d(A, B) + 1\}$, where $n_+(\sigma)$ is the number of vertices with $+$ spin in σ . We will show that there exists an automorphism f on R such that if $(\sigma_3, \sigma_4) = f(\sigma_1, \sigma_2)$, then $w_G(\sigma_1)w_G^{m_1, m_2}(\sigma_2) = w_G^{m_1}(\sigma_3)w_G^{m_2}(\sigma_4)$.

Let $(\sigma_1, \sigma_2) \in R$, consider the subgraph $H = (V, E_+(\sigma_1 | \sigma_2))$, where $\sigma_1 | \sigma_2$ denotes the logical OR, interpreting $+$ as true. Since $n_+(\sigma_1) + n_+(\sigma_2) < d(A, B) + 1$, there are no paths connecting any edge between A and B in H . Let S be the minimal vertex set containing all connected components of H that intersect with $G[A]$ and $T = V \setminus S$. Swap the part at T of σ_1 and σ_2 , write it as $(\sigma_3, \sigma_4) = (\sigma_2|_S \cup \sigma_1|_T, \sigma_1|_S \cup \sigma_2|_T)$. Obviously, $(\sigma_3, \sigma_4) \in R$ and the process is reversible (note $\sigma_3|_S = \sigma_1|_S$, which is unchanged in the process), thus f is an automorphism.

Since there are no $(+, +)$ edges between S and T for $\sigma_1 | \sigma_2 = \sigma_3 | \sigma_4$, it follows that there are no $(+, +)$ edges between S and T for $\sigma_1, \sigma_2, \sigma_3$ and σ_4 . For an edge $e = (u, v) \in E$ between S and T , define $s(e, \sigma) = \mathbb{1}[e \in E_-(\sigma)]$. Recalling that e cannot be a $(+, +)$ edge in any σ_i ($i = 1, 2, 3, 4$), we obtain $s(e, \sigma) = 1 - \mathbb{1}[\sigma(u) = +] - \mathbb{1}[\sigma(v) = +]$. Moreover, note that $\mathbb{1}[\sigma_1(u) = +] + \mathbb{1}[\sigma_2(u) = +] = \mathbb{1}[\sigma_3(u) = +] + \mathbb{1}[\sigma_4(u) = +]$ and similarly for v . It follows that $s(e, \sigma_1) + s(e, \sigma_2) = s(e, \sigma_3) + s(e, \sigma_4)$.

Let $C = \{(u, v) \in E \mid u \in S, v \in T\}$ be the set of cut edges between S and T . By the definition of $w(\cdot)$, we have

$$\begin{aligned}
& w_G(\sigma_1)w_G^{m_1, m_2}(\sigma_2) \\
&= \prod_{e \in C} \beta_e^{s(e, \sigma_1)} w_{G[S]}(\sigma_1|_S) w_{G[T]}(\sigma_1|_T) \prod_{e \in C} \beta_e^{s(e, \sigma_2)} w_{G[S]}^{m_1}(\sigma_2|_S) w_{G[T]}^{m_2}(\sigma_2|_T) \\
&= \prod_{e \in C} \beta_e^{s(e, \sigma_1) + s(e, \sigma_2)} w_{G[S]}^{m_1}(\sigma_2|_S) w_{G[T]}(\sigma_1|_T) w_{G[S]}(\sigma_1|_S) w_{G[T]}^{m_2}(\sigma_2|_T) \\
&= \prod_{e \in C} \beta_e^{s(e, \sigma_3) + s(e, \sigma_4)} w_{G[S]}^{m_1}(\sigma_3|_S) w_{G[T]}(\sigma_3|_T) w_{G[S]}(\sigma_4|_S) w_{G[T]}^{m_2}(\sigma_4|_T) \\
&= \prod_{e \in C} \beta_e^{s(e, \sigma_3)} w_{G[S]}^{m_1}(\sigma_3|_S) w_{G[T]}(\sigma_3|_T) \prod_{e \in C} \beta_e^{s(e, \sigma_4)} w_{G[S]}(\sigma_4|_S) w_{G[T]}^{m_2}(\sigma_4|_T) \\
&= w_G^{m_1}(\sigma_3) w_G^{m_2}(\sigma_4).
\end{aligned}$$

Thus, the proof is complete. $\blacktriangleleft$

► **Lemma 16** (uniform bound). *Fix constant number $\delta \in (0, 1)$ and $C_2 \geq C_1 \geq 0$. Let S be a compact set of $\frac{1}{1-\delta}\mathbb{D}$. Then, there exists a constant $C > 0$ such that for any graph $G = (V, E)$ with parameters $\beta \in [1, \infty)^E$ and $\lambda \in (1-\delta)\mathbb{D}^V$, for any $e \in E$, any $\beta'_e \geq 1$ with $\frac{\beta'_e}{\beta_e} \in [C_1, C_2]$, we have $|P_{G, \{\beta_e \rightarrow \beta'_e\}}(\beta, \lambda z)| \leq C$ for all $z \in S$.*

LDC together with the uniform bound implies edge-type SSM for the Ising model, which in turn immediately yields SSM for the random cluster model.

4.2 SSM for the random cluster model

Let $G = (V, E)$ be a graph, $\mathbf{p} \in [0, 1]^E$, $\lambda \in [0, 1]^V$ be parameters. The weight of a configuration $S \subseteq E$ in the (weighted) random cluster model is defined by:

$$w_{G, \mathbf{p}, \lambda}^{\text{RC}}(S) = \prod_{e \in S} p_e \prod_{e \in E \setminus S} (1 - p_e) \prod_{C \in \kappa(V, S)} \left(1 + \prod_{j \in C} \lambda_j \right),$$

where $\kappa(V, S)$ denotes the set of connected components of graph (V, S) . The partition function of the random cluster model is given by $Z_G^{\text{RC}}(\mathbf{p}, \boldsymbol{\lambda}) = \sum_{S \subseteq E} w_{G, \mathbf{p}, \boldsymbol{\lambda}}^{\text{RC}}(S)$. The relationship between the Ising model with an external field and the random cluster model is given in the following lemma.

► **Lemma 17** ([10, Proposition 2.1]). *Let $G = (V, E)$ be a graph, and let $\beta \in [1, +\infty)^E$ and $\boldsymbol{\lambda} \in [0, 1]^V$ be parameters. Define $\mathbf{p} = 1 - \beta^{-1} = (1 - \beta_e^{-1})_{e \in E}$. Then*

$$Z_G^{\text{Ising}}(\beta, \boldsymbol{\lambda}) = \left(\prod_{e \in E} \beta_e \right) Z_G^{\text{RC}}(\mathbf{p}, \boldsymbol{\lambda}).$$

► **Remark 18.** Expressing it as $Z_G^{\text{RC}}(\mathbf{p}, \boldsymbol{\lambda}) = Z_G^{\text{Ising}}(\beta, \boldsymbol{\lambda}) / \prod_{e \in E} \beta_e$, we observe that setting $\beta_e = \infty$ is well-defined by taking the limit, which corresponds to setting $p_e = 1$ in the random cluster model.

When $\mathbf{p} \in [0, 1]^E$ and $\boldsymbol{\lambda} \in [0, 1]^V$, the random cluster model induces a distribution $\mu(S) = w(S)/Z$ on subsets $S \subseteq E$. For an edge $e \in E$, the marginal probabilities are $P_{G,e}^+(\mathbf{p}, \boldsymbol{\lambda}) = Z_{G,e}^+/Z_G$, $P_{G,e}^-(\mathbf{p}, \boldsymbol{\lambda}) = Z_{G,e}^-/Z_G$, where $Z_{G,e}^+ = \sum_{S \subseteq E, e \in S} w(S)$ and $Z_{G,e}^- = \sum_{S \subseteq E, e \notin S} w(S)$. For a partial configuration σ_A on $A \subseteq E$ (each $e \in A$ pinned in $(+)$ or out $(-)$), we can define the conditional partition function $Z_G^{\sigma_A}$ and the conditional marginal of e being unpicked is simply $P_{G,e}^{\sigma_A} = Z_{G,e}^{\sigma_A, -} / Z_G^{\sigma_A}$, exactly in the same way as in the Ising model.

► **Definition 19** (SSM for the random cluster model). *Let $\mathcal{G}$ be a family of graphs with parameters $(\mathbf{p}, \boldsymbol{\lambda})$. The random cluster model defined on $\mathcal{G}$ is said to satisfy strong spatial mixing with exponential rate $r > 1$ if there exists a constant $C > 0$ such that for any $G = (V, E) \in \mathcal{G}$, any edge $e \in E$, any partial configuration σ_{Λ_1} and τ_{Λ_2} where $\Lambda_1, \Lambda_2 \subseteq E \setminus e$, we have*

$$\left| P_{G,e}^{\sigma_{\Lambda_1}}(\mathbf{p}, \boldsymbol{\lambda}) - P_{G,e}^{\tau_{\Lambda_2}}(\mathbf{p}, \boldsymbol{\lambda}) \right| \leq C r^{-d_G(e, \sigma_{\Lambda_1} \neq \tau_{\Lambda_2})}.$$

► **Lemma 20.** *The conditional marginal probability of edge e under condition σ_A for $A \subseteq E \setminus e$ in the random cluster model can be translated to the edge-type ratio in the Ising model as*

$$P_{G,e}^{\sigma_A} = Z_{G-e}^{\text{Ising}, m(\sigma_A)} / Z_G^{\text{Ising}, m(\sigma_A)}$$

where $m(\sigma_A) = \{\beta_e \rightarrow 1 \mid \sigma_A(e) = -\} \cup \{\beta_e \rightarrow \infty \mid \sigma_A(e) = +\}$.

Thus, the GE-SSM or edge-deletion SSM of the Ising model will directly imply the SSM of the random cluster model.

► **Theorem 21.** *Fix a constant $\delta \in (0, 1)$. For any graph $G = (V, E)$ with parameters $\mathbf{p} \in [0, 1]^E$ and $\boldsymbol{\lambda} \in [0, 1 - \delta]^V$, SSM holds for the random cluster model.*

As showed in [12], Theorem 21 directly implies the following optimal mixing time for the random cluster model.

► **Theorem 22.** *Fix $1 < \beta_{\min} \leq \beta_{\max}$, $\delta \in (0, 1)$, and $d \in \mathbb{N}$. For any subgraph $G = (V, E)$ of the infinite d -dimensional lattice, with parameters $\beta \in [\beta_{\min}, \beta_{\max}]^E$ and $\boldsymbol{\lambda} \in [0, 1 - \delta]^V$, the mixing time of the Glauber dynamics for the corresponding random-cluster representation of the Ising model is $O(m \log m)$, where $m = |E|$.*

5 LDC and SSM for other models

5.1 SSM for the hypergraph independence polynomial

A hypergraph $H = (V, E)$ has vertex set V and edges E , where each edge is a nonempty subset of V . An *independent set* is a set of vertices containing no edge. The independence polynomial of H is

$$Z_H(\lambda) = \sum_{I \subseteq V \text{ indep.}} \lambda^{|I|}.$$

For a partial configuration σ_Λ , similarly to the random cluster model, the conditional probability that v is pinned $+$ is given by $P_{H,v}^{\sigma_\Lambda}(\lambda) = \frac{Z_{H,v}^{\sigma_\Lambda, +}(\lambda)}{Z_{H,v}^{\sigma_\Lambda}(\lambda)}$. A partial configuration σ_Λ is feasible if it is an independent set. We say v is proper to σ_Λ if $v \notin \Lambda$ and $\sigma_{\Lambda \cup \{v\}}^+$ is feasible.

Denote $\lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^\Delta}$ and $\lambda_s(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta}$. Let $d_H(\cdot, \cdot)$ the graph distance in $G(H)$. We directly state the new strong spatial mixing result for the hypergraph independence polynomial as a theorem, which is stronger than the definition in [8].

► **Theorem 23** (Theorem 5 restated). *Fix $\Delta \geq 3$ and $\lambda \in \mathbb{D}_{\lambda_s(\Delta)} \cup (0, \lambda_c(\Delta))$. There exist constants $C > 0$ and $r > 1$ such that for any hypergraph $H = (V, E)$ with maximum degree at most Δ , any two feasible partial configurations σ_{Λ_1} and τ_{Λ_2} where Λ_1 may be different with Λ_2 , and any vertex v proper to σ_{Λ_1} and τ_{Λ_2} , we have*

$$|P_{H,v}^{\sigma_{\Lambda_1}}(\lambda) - P_{H,v}^{\tau_{\Lambda_2}}(\lambda)| \leq Cr^{-d_H(v, \sigma_{\Lambda_1} \neq \tau_{\Lambda_2})}$$

where $\sigma_{\Lambda_1} \neq \tau_{\Lambda_2}$ is the set $(\Lambda_1 \setminus \Lambda_2) \cup (\Lambda_2 \setminus \Lambda_1) \cup \{v \in \Lambda_1 \cap \Lambda_2 : \sigma_{\Lambda_1}(v) \neq \tau_{\Lambda_2}(v)\}$.

The SSM result comes from the LDC property of the independence polynomial established in Lemma 24 and the optimal zero-free region established in [5].

► **Lemma 24** (Lemma 4 restated). *Let $H = (V, E)$ be a hypergraph, $\sigma_{\Lambda_1}, \tau_{\Lambda_2}$ be two partial configurations on $\Lambda_1, \Lambda_2 \subseteq V$, v be a proper vertex to σ_{Λ_1} and τ_{Λ_2} , then*

$$\lambda^{d_H(v, \sigma_{\Lambda_1} \neq \tau_{\Lambda_2})+1} \mid \left(P_{H,v}^{\sigma_{\Lambda_1}}(\lambda) - P_{H,v}^{\tau_{\Lambda_2}}(\lambda) \right).$$

► **Lemma 25** (Theorem 1.1 in [5]). *Let $\Delta \geq 2$. For any hypergraph $H = (V, E)$ with maximum degree at most Δ and $\lambda \in \mathbb{C}^V$ with $|\lambda_v| \leq \lambda_s(\Delta)$ for all $v \in V$ we have $Z_H(\lambda) \neq 0$.*

► **Lemma 26** (Theorem 1.2 in [5]). *Let $\Delta \geq 3$. There exists an open neighborhood U_Δ of the interval $(0, \lambda_c(\Delta))$ such that for any hypergraph $H = (V, E)$ with maximum degree at most Δ and $\lambda \in U$ we have $Z_H(\lambda) \neq 0$.*

5.2 SSM for binary symmetric Holant problems

Let $G = (V, E)$ be a graph of maximum degree Δ . We consider the Holant problem in the binary symmetric case, which we now describe. Let $\{f_v\}_{v \in V} : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ be a family of functions, one for each vertex $v \in V$ in the input graph. One should think of each f_v as representing a local constraint on the assignments to edges incident to v . Since we are restricting ourselves to the binary case, our configurations σ will map edges to $\{0, 1\}$ (or $-$ and $+$ spins). Furthermore, since we are restricting ourselves to the symmetric case, our local functions f_v will only depend on the number of edges incident to v which are mapped to 1. With these $\{f_v\}_{v \in V}$ in hand, we may write the multivariate partition function as

$$Z_G(\lambda) = \sum_{\sigma: E \rightarrow \{0,1\}} \prod_{v \in V} f_v(|\sigma_{E(v)}|) \prod_{e \in E, \sigma(e)=1} \lambda_e,$$

where $E(v)$ is the set of all edges adjacent to v , $\sigma_{E(v)}$ is the configuration restricted on $E(v)$, and $|\sigma_{E(v)}|$ is the number of edges in $E(v)$ with assignment 1.

Let $G = (V, E)$ be a graph, $e \in E$ an edge, and σ_Λ a partial configuration on $\Lambda \subseteq E \setminus \{e\}$. Similarly to the random cluster model, the conditional probability that e is pinned $+$ is given by $P_{G,e}^{\sigma_\Lambda}(\lambda) = Z_{G,e}^{\sigma_\Lambda,+}(\lambda) / Z_G^{\sigma_\Lambda}(\lambda)$. The strong spatial mixing can also be defined as below.

Using zero-free regions obtained by specializing the binary symmetric Holant under different vertex constraint functions, we derive SSM in two notable cases: (i) the Weighted Even Subgraphs specialization with $f_v(k) = 1$ when k is even and $f_v(k) = \rho$ when k is odd; and (ii) the specialization $f_v(k) = \beta^{\binom{k}{2}} \beta^{\binom{d-k}{2}}$ for $0 \leq k \leq d$ (and 0 otherwise), where d is the degree of v , which corresponds to the Ising model on the line graph. In each case, whenever the parameter λ lies in the associated zero-free region, the model exhibits SSM.

► **Definition 27** (SSM for binary symmetric Holant problems). *Fix f_v be the local function and λ . Let $\mathcal{G}$ be a family of graphs. The Holant problem defined on $\mathcal{G}$ with f_v and λ is said to satisfy strong spatial mixing with exponential rate $r > 1$ if there exists a constant C such that for any $G = (V, E) \in \mathcal{G}$, any edge $e \in E$, any partial configuration σ_{Λ_1} and τ_{Λ_2} where $\Lambda_1, \Lambda_2 \subseteq E \setminus e$, we have*

$$\left| P_{G,e}^{\sigma_{\Lambda_1}}(\lambda) - P_{G,e}^{\tau_{\Lambda_2}}(\lambda) \right| \leq C r^{-d_G(e, \sigma_{\Lambda_1} \neq \tau_{\Lambda_2})}.$$

► **Theorem 28.** *Fix $\rho \in (0, 1)$, $\Delta \in \mathbb{N}^+$ and $\lambda > 0$. Then the weighted even subgraph model for all graphs G with bounded degree Δ exhibits SSM.*

► **Theorem 29.** *Fix $\beta \in (0, 1)$, $\Delta \in \mathbb{N}^+$ and $\lambda > 0$. Then the Ising model for all line graphs G with bounded degree Δ exhibits SSM.*

The proofs of the above theorems follow directly from the LDC property of Holant established in Lemma 30 and the zero-free regions established in [9].

► **Lemma 30** (LDC for binary symmetric Holant problems). *Let $G = (V, E)$ be a graph, e be an edge in E , and $\sigma_{\Lambda_1}, \tau_{\Lambda_2}$ be two partial configurations on $\Lambda_1, \Lambda_2 \subseteq E \setminus \{e\}$, then*

$$\lambda^{d_G(e, \sigma_{\Lambda_1} \neq \tau_{\Lambda_2})+2} \mid \left(P_{G,e}^{\sigma_{\Lambda_1}}(\lambda) - P_{G,e}^{\tau_{\Lambda_2}}(\lambda) \right).$$

5.3 Edge-type SSM for the Potts model

The partition function of the Potts model (without external field) of a graph $G = (V, E)$ is defined as

$$Z_G(q, \mathbf{w}) = \sum_{\sigma: V \rightarrow [q]} \prod_{\substack{(u,v) \in E \\ \sigma(u) = \sigma(v)}} w_{(u,v)}.$$

where $[q] = \{1, 2, \dots, q\}$, q is the number of colors, $\mathbf{w} = (w_e)_{e \in E}$ is the edge activity vector.

Similar to the edge-deletion SSM for the Ising model in Theorem 2, define the ratio of the partition function of the Potts model as $P_{G,e}(w) = \frac{Z_{G-e}(q, w)}{Z_G(q, w)}$. We can prove the Potts model exhibits edge-deletion SSM, where the constant $\eta \geq 0.002$ is from best zero-free region [4].

► **Theorem 31** (Edge-type SSM for the Potts model). *Fix $\Delta \in \mathbb{N}$, $q \geq (2 - \eta)(2\Delta - 2)$ and $w \in [0, 1]$, then there exist constant $C > 0$ and $r > 1$ such that for any graph $G = (V, E)$ with maximum degree at most Δ , $e \in E$, $A, B \subseteq E \setminus \{e\}$, we have*

$$\left| P_{G-A,e}(q, w) - P_{G-B,e}(q, w) \right| \leq C r^{-d_G(e, A \neq B)}.$$

The SSM result comes from the LDC property of the Potts model established in Lemma 32 and the zero-free region established in [4].

► **Lemma 32** (LDC for the Potts model). *Let $G = (V, E)$ be a graph, $e \in E$, and $A, B \subseteq E \setminus \{e\}$, then the Taylor series near $w = 1$ of $P_{G-A,e}(q, w)$ and $P_{G-B,e}(q, w)$ satisfies*

$$(w - 1)^{d_G(e_1, A \neq B)} \mid (P_{G-A,e}(q, w) - P_{G-B,e}(q, w)).$$

5.4 Vertex-type SSM for the Ising model

Similar to the edge-type SSM of the Ising model in Theorem 13, we also establish a vertex-type SSM.

► **Theorem 33** (Vertex-type SSM for Ising model). *Fix edge activity $\beta \geq 1$ and uniform external $\lambda \in \mathbb{D}_{\frac{1}{\beta}}$ for Ising model, and $c \in [0, 1)$. Then there exist constant $C > 0$ and $r > 1$ such that for all graph $G = (V, E)$, $v \in V$, $A, B \subseteq V \setminus \{v\}$, let $m = \{\lambda_v \rightarrow c\lambda\}$, $m_1 = \{\lambda_u \rightarrow c\lambda\}_{u \in A}$, $m_2 = \{\lambda_u \rightarrow c\lambda\}_{u \in B}$, we have*

$$\left| P_{G,m}^{m_1} - P_{G,m}^{m_2} \right| \leq Cr^{-d_G(v, m_1 \neq m_2)}.$$

The SSM result comes from the LDC property of the Ising model established in Lemma 34 and the Lee-Yang theorem.

► **Lemma 34** (LDC for the Ising model). *For $\beta > 1$, $c \in [0, 1)$, $G = (V, E)$ be a graph, $\lambda \in \mathbb{D}^V$, $v \in V$, $A, B \subseteq V \setminus \{v\}$, $m = \{\lambda_v z \rightarrow c\lambda_v z\}$, $m_1 = \{\lambda_u z \rightarrow c\lambda_u z\}_{u \in A}$, $m_2 = \{\lambda_u z \rightarrow c\lambda_u z\}_{u \in B}$, then*

$$z^{d_G(v, m_1 \neq m_2) + 1} \mid \left(P_{G,m}^{m_1}(\beta, \lambda z) - P_{G,m}^{m_2}(\beta, \lambda z) \right)$$

where $m_1 \neq m_2$ is vertex set where m_1 and m_2 differ.

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