Local Transformations of Bipartite Entanglement Are Rigid

John Bostanci
Department of Computer Science, Columbia University, New York, NY, USA

Tony Metger
Institute for Theoretical Physics, ETH Zurich, Switzerland

Henry Yuen
Department of Computer Science, Columbia University, New York, NY, USA

Abstract

Uhlmann's theorem is a fundamental result in quantum information theory that quantifies the optimal overlap between two bipartite pure states after applying local unitary operations (called *Uhlmann transformations*). We show that optimal Uhlmann transformations are *rigid* – in other words, they must be unique up to some well-characterized degrees of freedom. This rigidity is also *robust*: Uhlmann transformations achieving near-optimal overlaps must be close to the unique optimal transformation (again, up to well-characterized degrees of freedom). We describe two applications of our robust rigidity theorem: (a) we obtain better interactive proofs for synthesizing Uhlmann transformations and (b) we obtain a simple, alternative proof of the Gowers-Hatami theorem on the stability of approximate representations of finite groups.

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1 Introduction

Let $|C\rangle, |D\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ denote bipartite pure states; let A denote the first subsystem and B denote the second subsystem. What is the closest that one can get to $|D\rangle$ by performing a unitary on subsystem B of the state $|C\rangle$? Uhlmann's theorem [18] quantifies the optimal overlap achievable:

$$\mathrm{F}(\rho, \sigma) = \max_U |\langle D| \mathbb{1} \otimes U |C\rangle|, \tag{1.1}$$

where ρ and σ denote the reduced density matrices on subsystem A of $|C\rangle$ and $|D\rangle$ respectively, the function $\mathrm{F}(\rho, \sigma) = \mathrm{Tr}(\sqrt{\rho^{1/2}\sigma\rho^{1/2}})$ denotes the fidelity between the two states, and the maximization is over all unitary transformations acting on subsystem B. We call a unitary U achieving equality in Equation (1.1) an *Uhlmann transformation*.

Given the ubiquity of Uhlmann's theorem throughout quantum information science, it seems worthwhile to study the mathematical and computational properties of Uhlmann transforms. Many natural questions arise: how unique are Uhlmann transformations? How robust are they to perturbations of the underlying states $|C\rangle, |D\rangle$? What is the complexity of performing Uhlmann transformations on a quantum computer? Can difficult Uhlmann

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transformations be used for cryptography? The latter two questions were recently studied by Metger and Yuen [13] and Bostanci, et al. [2], who investigate a theory of state and unitary complexity, respectively.

This paper studies the first two questions concerning the uniqueness and robustness of Uhlmann transformations. At first glance, Uhlmann transformations are not generally unique. For example, suppose that the reduced density matrix of $|C\rangle$ on subsystem B does not have full support. Then any Uhlmann transformation can behave arbitrarily on the orthogonal complement of the support, while remaining optimal. What if we disregard these trivial degrees of freedom, however – could Uhlmann transformations be unique in some other meaningful way?

We provide an answer via *canonical Uhlmann transformations*, first defined by Metger and Yuen [13]. For every pair of bipartite states $(|C\rangle, |D\rangle)$, this is the operator

$$W := \mathrm{sgn}(\mathrm{Tr}_{\mathsf{A}}(|D\rangle\langle C|)) \tag{1.2}$$

where Tr_{A} denotes tracing out register A and $\mathrm{sgn}(\cdot)$ denotes the following function: for a matrix X with singular value decomposition $X = U\Sigma V^*$, we define $\mathrm{sgn}(X) := U\mathrm{sgn}(\Sigma)V^*$ where U, V are unitary operators and $\mathrm{sgn}(\Sigma)$ denotes the projection onto the eigenvectors of Σ with positive eigenvalues (i.e., the support of Σ). The canonical transformation W is a partial isometry[1]; it is unitary if and only if both reduced states ρ, σ (of $|C\rangle, |D\rangle$, respectively) are invertible.

The following was proven by Bostanci, et al. [2, Proposition 6.3] in their investigation of the computational complexity of implementing Uhlmann transformations:

▶ **Lemma 1.** *The canonical Uhlmann transformation W satisfies $|\langle D| \mathbb{1} \otimes W |C\rangle| = \mathrm{F}(\rho, \sigma)$, and furthermore for all partial isometries R such that $|\langle D| \mathbb{1} \otimes R |C\rangle| = \mathrm{F}(\rho, \sigma)$, we have that $W^*W \leq R^*R$ in the positive semidefinite ordering.*

In other words, the canonical Uhlmann transformation defined in Equation (1.2) achieves the optimal overlap between $|C\rangle$ and $|D\rangle$, and furthermore any other partial isometry achieving the optimal overlap must be supported on the domain of W. This is a rather weak statement, however: when R is unitary, then $W^*W \leq R^*R = \mathbb{1}$ is satisfied for *all* partial isometries W.

A stronger statement is the following:

▶ **Claim 2.** *For all partial isometries R such that $\langle D| \mathbb{1} \otimes R |C\rangle = \mathrm{F}(\rho, \sigma)$, we have that*

$$\mathbb{1} \otimes W |C\rangle = \mathbb{1} \otimes RW^*W |C\rangle \ .$$

This says that *any* optimal Uhlmann transformation, when restricted to the support of W, must behave identically to W on the state $|C\rangle$. This provides some justification in calling the W in Equation (1.2) the "canonical" Uhlmann transformation corresponding to $|C\rangle, |D\rangle$.

Claim 2 is in fact a special case of a more general *robust rigidity* theorem that we prove in this paper. Roughly speaking, the theorem (Theorem 6 below) states that any transformation R that achieves *approximately*-optimal fidelity (meaning that $|\langle D| \mathbb{1} \otimes R |C\rangle| \geq \mathrm{F}(\rho, \sigma) - \epsilon$) must be *approximately* the canonical Uhlmann transformation W. This is analogous to rigidity results for some quantum information processing tasks such as nonlocal games [10, 11, 17] and superdense coding [15]. These results show that the only way for a quantum operation

[1] A partial isometry can be thought of as the restriction of a unitary to a subspace. More formally, an operator W is a partial isometry if W^*W is a projection.

to achieve near-optimal performance according to some metric (e.g., winning probability in a nonlocal game, or decoding probability in superdense coding) is if, in fact, it is close to a canonical strategy or protocol.

First we give a way to quantify the rigidity of canonical Uhlmann transformations.

▶ **Definition 3.** *Let $|C\rangle, |D\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ be pure bipartite states with respective reduced density matrices ρ, σ on the subsystem* A*. Then we say the corresponding canonical Uhlmann transformation W defined in Equation* (1.2) *has $\delta(\epsilon)$-*robust rigidity *if, for all $\epsilon > 0$, for all unitaries R such that*

$$\langle D| \mathbb{1} \otimes R |C\rangle \geq \mathrm{F}(\rho, \sigma) - \epsilon ,$$

we have

$$\| \mathbb{1} \otimes (W - R) W^* W |C\rangle \|^2 \leq \delta(\epsilon) .$$

Thus, bounds on the function $\delta(\epsilon)$ of an Uhlmann transformation quantifies the extent to which the exact rigidity statement of Claim 2 can be made robust.

▶ Remark 4. The reader may notice an apparent asymmetry between $|C\rangle$ and $|D\rangle$ in Claim 2 and Definition 3. This is motivated by an operational interpretation of Uhlmann's theorem: starting with $|C\rangle$, how close can we get to $|D\rangle$ by acting on subsystem B? The choice of starting with $|C\rangle$ versus $|D\rangle$ is significant, as the canonical Uhlmann transformation can have different robustness functions depending on this choice (see [3, Section 4.2] for an example).

▶ Remark 5. The reader may also wonder about the role of W^*W in Definition 3, which is the projection onto the image of W. The image of W may not be fully contained in the support of subsystem B of $|C\rangle$ or $|D\rangle$. Interestingly, this projection is necessary in the statement of rigidity: *any* unitary completion of the partial isometry W achieves the optimal fidelity, as shown in [3, Claim 3.3]. In other words, it is possible to behave arbitrarily outside the image of W, and still attain the optimal fidelity.

Our main result is a general bound on the rigidity of Uhlmann transformations.

▶ **Theorem 6** (Robust rigidity of Uhlmann transformations)**.** *Let $|C\rangle, |D\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ be pure bipartite states with respective reduced density matrices ρ, σ on the subsystem* A*. The corresponding canonical Uhlmann transformation W satisfies the following:*

1. *(**Completeness**). For all unitary completions U of W, we have*

$$| \langle D| \mathbb{1} \otimes U |C\rangle | = \mathrm{F}(\rho, \sigma) .$$

2. *(**Rigidity**). W has $\delta(\epsilon)$-robust rigidity for $\delta(\epsilon) = \left(\frac{2\kappa}{\eta}\right)\epsilon$ where $\kappa = \|\rho^{-1/2} P \rho^{1/2}\|_\infty^2$ with P being the projection onto $\mathrm{Image}(\rho^{1/2}\sigma\rho^{1/2})$, and η is the smallest nonzero eigenvalue of the matrix geometric mean $\rho^{-1}\#\sigma$.*

For readers who are not familiar with the (beautiful notion of the) matrix geometric mean we provide a brief introduction in Section 2.

Thus, the canonical Uhlmann transformation is indeed robustly rigid, up to some blow-up that depends on two parameters η and κ (called the *spectral gap* and *obliqueness*, respectively) of the reduced density matrices ρ, σ. Intuitively, the obliqueness parameter κ is a measure of a combination of non-commutativity and non-invertibility of ρ, σ and the spectral gap parameter η is a measure of how "well-conditioned" the matrix geometric mean $\rho^{-1}\#\sigma$ is (which one can think of as a notion of "ratio" between σ and ρ).

▶ Remark 7. Suppose either

1. The density matrices ρ, σ commute, or
2. The density matrices ρ, σ are invertible.

Then the obliqueness parameter κ is equal to 1, and the robustness bound only depends on the spectral gap η of $\rho^{-1}\#\sigma$.

In [3, Sections 4.2 and 4.4] respectively we further show that some dependence on the spectral gap parameter η and the obliqueness parameter κ in Theorem 6 is necessary.

2 Matrix geometric mean

The matrix geometric mean is a noncommutative generalization of the geometric mean $\sqrt{ab}$ of two nonnegative numbers a, b. If A, B are commuting positive semidefinite matrices, then the matrix geometric mean $A\#B$ is defined as $A^{1/2}B^{1/2}$. For general positive definite (i.e., all eigenvalues are strictly positive) matrices A, B, the matrix geometric mean $A\#B$ is defined as

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} \ . \tag{2.1}$$

For positive definite matrices A, B the matrix geometric mean enjoys many pleasant properties, including

1. $A\#B$ is positive definite.
2. $A\#B = B\#A$.
3. $A\#B$ is the unique positive solution to the equation $XA^{-1}X = B$.
4. If X is invertible, then $X(A\#B)X^{-1} = (XAX^{-1})\#(XBX^{-1})$.
5. $A\#B \leq \frac{1}{2}(A + B)$, a noncommutative analogue of the arithmetic-geometric mean inequality.
6. $\Phi(A)\#\Phi(B) \leq \Phi(A\#B)$ for all positive maps Φ.

Proofs of these properties can be found in [1, Chapter 4]. For more applications of the matrix geometric mean in quantum information theory, see [4, 5, 9].

For noninvertible A, B (but still positive *semi*definite), the matrix geometric mean $A\#B$ is typically defined as a limit of geometric means of sequences of strictly positive matrices converging to A, B; however, in this case not all of the properties listed above are satisfied. For example, the symmetry property $A\#B = B\#A$ need not hold.

In this paper we do not use this limit definition, and instead stick to Equation (2.1) as the definition for the matrix geometric mean for *all* positive semidefinite matrices A, B, with the inverses now being Moore-Penrose pseudoinverses. Although it does not satisfy all the properties listed above, it satisfies a few important properties that are needed for the proof of Theorem 6; for example, as we will show in [3, Claim 3.1], when the density matrices ρ, σ are real, the canonical Uhlmann transformation can equivalently be expressed in terms of a matrix geometric mean:

$$W = Y^*(\rho^{1/2}\sigma^{1/2})^{-1}\rho^{1/2}(\rho^{-1}\#\sigma)\rho^{1/2}X$$

for some unitary operators X, Y (see the start of [3, Section 3] for their definitions). Furthermore, the fidelity between ρ and σ can also be written as $\mathrm{F}(\rho, \sigma) = \mathrm{Tr}((\rho^{-1}\#\sigma)\rho)$ (see e.g. [4]).

3 Applications

Given the centrality of Uhlmann transformations, the rigidity statement in Theorem 6 may be of interest in its own right, but it also turns out to be a useful technical tool for other applications. To illustrate this, we briefly discuss applications of our robust rigidity theorem to unitary complexity theory and approximate representation theory.

3.1 The complexity of the Uhlmann Transformation Problem

Bostanci, et al. [2] defined the Uhlmann Transformation Problem, a computational task associated to implementing canonical Uhlmann transformations corresponding to a pair $(|C\rangle, |D\rangle)$ whose circuit descriptions are given. They introduced a framework for unitary complexity theory in order to properly describe the complexity of performing Uhlmann transformations: for the special case that the pair $(|C\rangle, |D\rangle)$ have identical reduced density matrices (i.e., $\mathrm{F}(\rho, \sigma) = 1$), the Uhlmann Transformation Problem is complete for $\mathsf{avgUnitaryHVPZK}$, a unitary complexity class that captures *perfect zero knowledge* in the unitary synthesis setting [2, Theorem 6.1]. They left open the challenge of characterizing the complexity of canonical Uhlmann transformations for general values of $\mathrm{F}(\rho, \sigma)$.

In [3, Section 5.1] we present a simple 2-round quantum interactive synthesis protocol for the Uhlmann Transformation Problem (for all values of the fidelity of the reduced density matrices) – this improves upon the 8-round protocol that arises from the machinery of proving $\mathsf{avgUnitaryPSPACE} = \mathsf{avgUnitaryQIP}$ in [2]. The soundness of our 2-round protocol crucially depends on the robust Uhlmann rigidity theorem. We believe that this could be helpful for better understanding the complexity of the Uhlmann Transformation Problem in the future.

3.2 Approximate representation theory

In the mathematics literature, results such as Theorem 6 are known as *stability* results: if an object A approximately satisfies some constraints, then is it close (in the appropriate metric) to an object B that *exactly* satisfies those constraints [19]? In [3, Section 5.2], we show that our robust Uhlmann rigidity theorem is powerful enough to derive other stability results – in particular, we show that the Gowers-Hatami theorem on the stability of approximate representations of finite groups [6] is an easy consequence of Theorem 6. Our proof suggests a possible "mechanical template" for proving other kinds of stability results: first, define the appropriate pair of pure states $(|C\rangle, |D\rangle)$, show that the canonical Uhlmann transformation is the ideal, "exact" object, and then use robust Uhlmann rigidity to conclude that all approximate objects are close to the ideal, exact object. We note that our approach of proving the Gowers-Hatami theorem is reminiscent of Metger, Natarajan, Zhang's alternate proof of it [12].

4 Related work

4.1 A weaker rigidity theorem

Bostanci, et al. [2] proved the following rigidity theorem for Uhlmann transformations, where the robustness itself depends on the fidelity between the reduced states:

▶ **Theorem 8** (Weak Uhlmann rigidity). *Let $|C\rangle, |D\rangle$ be pure bipartite states with reduced density matrices ρ, σ on the first subsystem. Then for all unitaries R such that*

$$\langle D| \mathbb{1} \otimes R |C\rangle \geq \mathrm{F}(\rho, \sigma) - \epsilon \ ,$$

we have that

$$\| \mathbb{1} \otimes (W - R) |C\rangle \|^2 \leq 8\Big(1 - \mathrm{F}(\rho, \sigma) + \sqrt{\epsilon}\Big)$$

where W is the corresponding canonical Uhlmann transformation.

▶ Remark 9. Technically, the theorem is stated in greater generality in [2] for arbitrary channels rather than unitaries; we specialize the theorem statement for this paper.

Suppose the fidelity $\mathrm{F}(\rho, \sigma)$ is equal to 1. In our language, Theorem 8 implies that the canonical Uhlmann transformation W has robust rigidity $\delta(\epsilon) \leq 8\sqrt{\epsilon}$. (At first glance, it may seem rather nice that there is no dependence on the spectral gap η or the obliqueness parameter κ, but note that in this special case of $\mathrm{F}(\rho, \sigma) = 1$, the two density matrices are identical and therefore $\eta = \kappa = 1$.) Thus Theorem 8 implies a robust rigidity bound for the perfect fidelity setting.[2]

However, when $\mathrm{F}(\rho, \sigma)$ is strictly less than 1, then the rigidity bound of Theorem 8 becomes trivial as $\epsilon \to 0$; the upper bound on the closeness of R and W is always at least $8(1 - \mathrm{F}(\rho, \sigma))$, a quantity that is a constant compared to ϵ. Furthermore this gives trivial upper bounds whenever $\mathrm{F}(\rho, \sigma) \leq 7/8$.

Our main theorem (Theorem 6), on the other hand, gives a nontrivial rigidity bound no matter what $\mathrm{F}(\rho, \sigma)$ is.

4.2 Rigidity in quantum information theory

This paper is inspired by rigidity in nonlocal games (also known as *self-testing* in the nonlocal game literature), which is the phenomenon that for many nonlocal games of interest (such as the CHSH game or the Magic Square game), near-optimal strategies must be close to a canonical optimal strategy [10, 11, 14]. There is also a long line of work studying various aspects of rigidity in nonlocal games; we refer the reader to the extensive survey of [17]. Nonlocal game rigidity is a powerful tool in quantum cryptography and quantum complexity theory, with applications ranging from classical verification of quantum computations [16] to settling the complexity of quantum multiprover interactive proofs [8].

4.3 Stability of polar decompositions

The rigidity of Uhlmann transformations is loosely related to the *stability of polar decompositions*, a topic that has been studied extensively in numerical analysis [7]. Every square matrix A admits a polar decomposition $A = UP$ where U (the "polar factor" of A) is a partial isometry and P is a positive semidefinite matrix. How do the polar factors of a matrix A and a perturbation $A + \Delta A$ compare with each other, as a function of A and the perturbation ΔA? This is a central question to the study of numerical algorithms for computing the polar decomposition.

[2] We note that in the $\mathrm{F}(\rho, \sigma) = 1$ case Theorem 6 implies a quadratically-better robustness function $\delta(\epsilon) = 2\epsilon$.

The connection with the Uhlmann transformation is as follows. The canonical Uhlmann transformation W for a pair $(|C\rangle, |D\rangle)$ of states with corresponding density matrices ρ, σ can be derived from the polar decomposition of the matrix $A = \sqrt{\rho}\sqrt{\sigma}$. Perturbing the states ρ, σ (and consequently the states $|C\rangle, |D\rangle$) will perturb the canonical Uhlmann transformation W; this relationship is governed by the stability of the polar decomposition of A.[3]

However, the robust rigidity of Uhlmann transformations studied in this paper is a different notion of stability. Here, we do not consider perturbations of the states $|C\rangle, |D\rangle$; we are asking whether all *approximate* Uhlmann transformations R for a pair of states $(|C\rangle, |D\rangle)$ must be close to a unique *exact* Uhlmann transformation.

5 Summary

Uhlmann transformations, which are local transformations of bipartite (pure state) entanglement, are fundamental in quantum information theory. In this paper we showed that Uhlmann transformations possess a robust form of *rigidity*: near-optimal entanglement transformations must close (in a well-defined sense) to a unique optimal transformation. This unique optimal transformation is the canonical Uhlmann transformation introduced by [13], and our result gives further justification to calling it "canonical."

We showed that the robustness of the rigidity theorem inherently depends on two parameters called the spectral gap and obliqueness, which are functions of the underlying pure states $|C\rangle, |D\rangle$. An interesting open question is whether there is a general "rounding" procedure that converts any pair of states $(|C\rangle, |D\rangle)$ into a nearby pair $(|\tilde{C}\rangle, |\tilde{D}\rangle)$ with controlled spectral gap and controlled obliqueness. In [3, Section 4.3], we provide a rounding lemma that only controls the spectral gap.

Finally, we presented two applications of our robust rigidity theorem. The first is to unitary complexity theory, where we can improve the round complexity required for the task of synthesizing canonical Uhlmann transformations in the regime where the fidelity between the reduced states is not 1. Second, we demonstrate that the robust rigidity theorem is a very general form of robustness that can be used to derive other stability theorems. As an example, we re-derive the stability of approximate group representations by reducing to the robust rigidity of the canonical Uhlmann transformation between a specific pair of states. Given the ubiquity of Uhlmann transformations in protocols across quantum information theory and quantum complexity theory, developing an understanding of their rigidity properties should unlock further applications and deeper insight into the ways that bipartite entanglement can be locally transformed.

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[3] See [3, Section 4] for an illustration of how the canonical Uhlmann transformation is a sensitive function of the states $|C\rangle, |D\rangle$.

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